

# Positive Bound States to Nonlinear Choquard Equations in the Presence of Nonsymmetric Potentials

Liliane Maia

*Departamento de Matemática, Universidade de Brasília, Brazil*  
lilimaia@unb.br

Benedetta Pellacci, Delia Schiera

*Dip. di Matematica e Fisica, Università della Campania “Luigi Vanvitelli”, Caserta, Italy*  
benedetta.pellacci@unicampania.it, delia.schiera@unicampania.it

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The existence of a positive solution to a class of Choquard equations with potential going at a positive limit at infinity possibly from above or oscillating is proved. Our results include the physical case and do not require any symmetry assumptions on the potential.

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## 1. Introduction

In this paper we will prove the existence of a positive solution of the following problem

$$(P_V) \quad \begin{cases} -\Delta u + V(x)u = (I_\alpha * u^2)u & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

where  $\alpha \in ((N-4)^+, N-1]$  (1)

and  $I_\alpha$  represents the Riesz operator of order  $\alpha$ , defined for each  $x \in \mathbb{R}^N \setminus \{0\}$  by

$$I_\alpha(x) = \frac{A_\alpha}{|x|^{N-\alpha}}, \quad \text{where} \quad A_\alpha = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\alpha/2)2^\alpha \pi^{N/2}},$$

and the potential  $V$  is such that

$$V \in C^0, \quad \inf_{x \in \mathbb{R}^N} V(x) > 0, \quad \text{and} \quad \lim_{|x| \rightarrow \infty} V(x) = V_\infty \in (0, +\infty). \quad (2)$$

This equation appears in the context of various physical models and we refer to [22, Section 2] for an extensive introduction on the physical context.

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When  $V(x) \equiv V_\infty$  problem  $(P_V)$  reduces to the autonomous problem

$$(P_\infty) \quad -\Delta u + V_\infty u = (I_\alpha * |u|^2)u \quad \text{in } \mathbb{R}^N,$$

and in the physical case,  $N = 3$ ,  $\alpha = 2$ , the first result goes back to Lieb ([14]) who proves the existence of a normalized solution, corresponding to the unique minimum point of the energy functional on the  $L^2(\mathbb{R}^3)$  sphere. The existence of infinitely many radial symmetric solutions has been obtained by Lions in [15] again for  $N = 3$  and  $\alpha = 2$ . These existence results have been extended by Moroz and Van Schaftingen ([21]) to different exponents  $\alpha$ , different dimensions  $N$  and even to powers different from the square on the function  $u$ , starting to the fact that, by the Hardy-Littlewood-Sobolev inequality, the right hand side of  $(P_\infty)$  (and of  $(P_V)$ ) is well defined on  $H^1(\mathbb{R}^N)$  when

$$\frac{N-2}{N+\alpha} < \frac{1}{p} < \frac{N}{N+\alpha} \quad (3)$$

More precisely, in [21] it is proved that  $(P_\infty)$  has a positive radially symmetric least action solution  $\omega \in C^2(\mathbb{R}^N)$ . The question of the validity of the uniqueness of positive solution has been already addressed by Lieb in [14] who proves it for normalized solutions; this result has been extended to any positive solution by Ma and Zhao ([17]) who prove that there exists a unique positive solution of problem  $(P_\infty)$  for  $\alpha = 2$  and  $N = 3$ ; this results has been extended for  $N = 4, 5$  in [27]. Let us also mention that the uniqueness property of the least action solution has been proved in [28] for powers  $p \neq 2$  belonging in a suitable range.

Precise decay estimates for  $\omega$  are proved in [21, 20]; in particular the decay turns out to be exponential in our case for  $\alpha < N - 1$  and a polynomial perturbation of an exponential decay for  $\alpha = N - 1$  (see for more details Theorem 2.1 in Section 2).

Coming back to the non-autonomous problem  $(P_V)$ , when  $V(x) \leq V_\infty$  the existence of a least action solution is due to [16] (see also [22, 24]) and can be obtained by minimizing the associated action functional

$$\mathcal{I}_V(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) - \frac{1}{4} \int_{\mathbb{R}^N} (I_\alpha * u^2)u^2.$$

on the Nehari manifold

$$\mathcal{N}_V = \{u \in H^1(\mathbb{R}^N) \setminus \{0\} : \langle \mathcal{I}'_V(u), u \rangle = 0\}.$$

But, when  $V(x)$  approaches  $V_\infty$  from above or oscillating, one is forced to look for higher action level solutions, and as a consequence, a deeper comprehension of the possible lack of compactness of a bounded Palais-Smale sequence is needed.

To this aim, a well-known tool is the so-called Splitting Lemma (see [6, 23]), whose application requests the uniqueness of positive solutions of  $(P_\infty)$ , which, as above observed, is known for  $p = \alpha = 2$  and  $N = 3, 4, 5$ .

The use of the Splitting Lemma allows to detect an action level's interval where compactness is recovered, so that the existence of a critical point can be obtained

by constructing a minimax level in this interval. In this construction, a precise knowledge of the decay of  $\omega$  is needed, and it is crucial a meticulous comparison between the asymptotical decay of the solution of the limit problem and the decay of the potential acting in the problem.

Following this path we will prove the following result

**Theorem 1.1.** *Assume  $\alpha = 2$ ,  $N = 3, 4, 5$  and that (2) holds. We also assume that the potential  $V(x)$  satisfies*

$$V(x) \leq V_\infty + A_0(1 + |x|)^\sigma e^{-\beta|x|}, \quad \text{with } A_0 > 0, \forall x \in \mathbb{R}^N, \tag{4}$$

and the exponent  $\sigma$  is such that

$$\begin{cases} \sigma \in \mathbb{R} & \text{if } \beta > 2\sqrt{V_\infty} \\ \sigma < -2 & \text{if } \beta = 2\sqrt{V_\infty} \text{ and } N = 5, \\ \sigma < -\frac{3}{2} & \text{if } \beta = 2\sqrt{V_\infty} \text{ and } N = 4, \\ \sigma < -1 & \text{if } \beta = 2\sqrt{V_\infty} \text{ and } N = 3. \end{cases} \tag{5}$$

Then there exists a positive solution to  $(P_V)$ .

Theorem 1.1 will be a direct consequence of an abstract result stated in Section 2 (Theorem 2.2). Let us point out that in Theorem 1.1 it is admitted the possibility that  $V(x)$  approaches  $V_\infty$  from above or oscillating. Moreover, we believe that the decay assumptions on  $V(x)$  are optimal in this type of argument and are naturally strictly related to the decay estimates of  $\omega$  which varies when  $N = 3$  and  $N = 4, 5$  (see for more details Theorem 2.1).

Other strategies to find nontrivial solutions, avoiding the use of the uniqueness properties of the limit problem  $(P_\infty)$ , have been implemented in the last years (see [2, 8, 11, 12, 13] and the references therein). In particular, Clapp and Salazar ([11]) take advantage of the use of symmetries to increase the minimum action level and to show the existence of a positive (or even sign-changing) solution for potentials enjoying the same symmetries. This has been possible requiring an enough high level of symmetries in order to construct a minimax level into the range of compactness. Moreover, the decay of the potential is assumed to be of exponential type with a negative exponent that naturally depends on the symmetries and it is sufficiently large in modulus. The use of symmetries has been also adopted by Cingolani, Clapp and Secchi in [8] to obtain existence results for a class of magnetic nonlinear Choquard equations.

Here, taking advantage of the uniqueness property of the limit problem  $(P_\infty)$ , we will not exploit any symmetries' action, so that our existence result does not require any invariance property of the potential. In addition, our decay assumptions on  $V(x)$  include the possibility that the exponential part in the decay is equal to the exponential decay of  $\omega^2$ .

Let us conclude the introduction mentioning the existence result of positive solutions contained in [25], where a Choquard equation with competing potentials is studied in the case  $p = \alpha = 2$ ,  $N = 3$  under some stronger decay assumptions on the potentials than the one assumed here.

This paper is organized as follows: in Section 2 we give the variational setting of the problem and some preliminary results, whereas in Section 3 we get the fundamental asymptotic estimates we need in the proof of the main results. The proof of Theorems 2.2 and 1.1 is performed in Section 4.

## 2. Setting of the problem and preliminaries

We will work in the functional space  $H^1(\mathbb{R}^N)$  endowed, thanks to (2), with the scalar product and norm, equivalent to the usual one

$$(u, v)_V = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x)uv), \quad \|u\|_V^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2).$$

Every solution to  $(P_V)$  is a critical point of the action functional  $\mathcal{I}_V : H^1(\mathbb{R}^N) \mapsto \mathbb{R}$  defined by

$$\mathcal{I}_V(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) - \frac{1}{4} \int_{\mathbb{R}^N} (I_\alpha * u^2)u^2.$$

where  $\alpha$  satisfies (1).

Hypothesis (1) and the Hardy-Littlewood-Sobolev inequality imply that  $\mathcal{I}_V$  is a  $C^1$  functional on  $H^1(\mathbb{R}^N)$ , (see [22, Proposition 3.1]), so that we can define

$$\mathcal{N}_V = \{u \in H^1(\mathbb{R}^N) \setminus \{0\} : \langle \mathcal{I}'_V(u), u \rangle = 0\}, \quad c_V = \inf_{u \in \mathcal{N}_V} \mathcal{I}_V(u). \quad (6)$$

In an analogous way, we can define  $\mathcal{I}_\infty : H^1(\mathbb{R}^N) \mapsto \mathbb{R}$  by

$$\mathcal{I}_\infty(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V_\infty v^2) - \frac{1}{4} \int_{\mathbb{R}^N} (I_\alpha * v^2)v^2,$$

where  $H^1(\mathbb{R}^N)$  is endowed with the norm and the scalar product in

$$(u, v) = \int_{\mathbb{R}^N} (\nabla u \nabla v + V_\infty uv), \quad \|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + V_\infty u^2). \quad (7)$$

Accordingly  $\mathcal{N}_\infty(u)$  and  $c_\infty$  are defined as follows

$$\mathcal{N}_\infty = \{u \in H^1(\mathbb{R}^N) \setminus \{0\} : \langle \mathcal{I}'_\infty(u), u \rangle = 0\}, \quad c_\infty = \inf_{u \in \mathcal{N}_\infty} \mathcal{I}_\infty(u). \quad (8)$$

As already mentioned in the Introduction, the existence of a least action solution to  $(P_\infty)$  is proved, under assumption (1), in Theorem 3.2 in [22]. Moreover, weak solutions are classical, and, up to translation and inversion of the sign, positive and radially symmetric, see [21, 14]. In addition, precise decay asymptotic for solutions to  $(P_\infty)$  are given in Propositions 6.3, 6.5 and Remark 6.1 in [21], (see also [20]), and they are summarized in the following result.

**Theorem 2.1.** (Theorem 4 pg. 157, Remark 6.1 pg. 177 in [21]) *Let  $\omega$  a least action solution to  $(P_\infty)$ . Then the following asymptotic estimates hold:*

*If  $(N - 4)^+ < \alpha < N - 1$  it results*

$$\omega(x) = (c + o(1)) |x|^{-\frac{N-1}{2}} e^{-\sqrt{V_\infty}|x|} \quad \text{with } c > 0 \text{ and as } |x| \rightarrow \infty. \quad (9)$$

If  $\alpha = N - 1$ ,  $\omega$  decays at infinity as follows

$$\omega(x) = (c + o(1)) |x|^{-\frac{N-1-\nu\sqrt{V_\infty}}{2}} e^{-\sqrt{V_\infty}|x|}, \quad \text{with } c > 0 \text{ and as } |x| \rightarrow \infty. \quad (10)$$

where  $\nu$  is a positive constant depending on the  $L^2(\mathbb{R}^N)$  norm of  $\omega$  (see (12) below).

The above result shows that the interaction of the Riesz potential and the nonlinearity affects in a substantial way the decay of the least action solutions. In our context we can see different perturbations on the decay of  $\omega$  depending on  $\alpha$ . In general we have

$$\omega(x) = (c + o(1)) \frac{e^{-\sqrt{V_\infty}Q(|x|)}}{|x|^{\frac{N-1}{2}}} \quad \text{as } |x| \rightarrow \infty, \quad (11)$$

where

$$Q(t) = \int_\nu^t \sqrt{1 - \frac{\nu^{N-\alpha}}{s^{N-\alpha}}} ds, \quad \nu^{N-\alpha} = \frac{1}{V_\infty} \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha} \int_{\mathbb{R}^N} |\omega|^2. \quad (12)$$

Notice that  $\nu$  does not actually depend on the choice of  $\omega$ , as  $\|\omega\|_2^2$  is invariant among least action solutions (see [21]). Nevertheless, when  $\alpha < N - 1$ , a careful analysis of the function  $Q$  shows that (9) holds, whereas if  $\alpha = N - 1$  (which includes the physical case  $N = 3, \alpha = 2$ ), a perturbation in the polynomial part occurs and one gets (10); if  $N - 1 < \alpha < N$ , then more involved perturbations appear, as a result of the Taylor expansion for the square root. However, in our Theorems 1.1 and 2.2 we will take into consideration the asymptotical decay given in (9) for the cases  $\alpha < N - 1$  and in (10) for  $\alpha = N - 1$ , which includes the physical case  $\alpha = 2$  and  $N = 3$ .

We will prove Theorem 1.1 as a consequence of the following result.

**Theorem 2.2.** *Let  $N \geq 2, \alpha \in ((N - 4)^+, N - 1]$ , and suppose that (2) holds. We also assume that the potential  $V(x)$  satisfies*

$$V(x) \leq V_\infty + A_0 |x|^\sigma e^{-\beta|x|}, \quad \text{with } A_0 > 0, \forall x \in \mathbb{R}^N, \quad (13)$$

and the exponent  $\sigma$  is such that

$$\begin{cases} \sigma \in \mathbb{R} & \text{if } \beta > 2\sqrt{V_\infty} \\ \sigma < \min \left\{ -1, -\frac{N-1}{2} \right\} & \text{if } \beta = 2\sqrt{V_\infty} \text{ and } \alpha < N - 1, \\ \sigma < \min \left\{ -1, -\frac{N-1}{2} + \nu\sqrt{V_\infty} \right\} & \text{if } \beta = 2\sqrt{V_\infty} \text{ and } \alpha = N - 1, \end{cases} \quad (14)$$

where  $\nu$  is introduced in (10). Then, if the limit Problem  $(P_\infty)$  has a unique positive solution, there exists a positive solution of  $(P_V)$ .

**Remark 2.3.** Let us observe that, for  $\alpha < N - 1$  the hypothesis (14) requires  $\sigma < -1$  when  $N = 2$  and  $\sigma < -(N - 1)/2$  when  $N \geq 3$ .

**Remark 2.4.** Let us note that as observed in [12] the hypothesis  $\alpha \in ((N-4)^+, N)$  is fundamental in order to have the convolution term well defined in  $H^1(\mathbb{R}^N)$  as a consequence of the Hardy-Littlewood-Sobolev inequality. Notice that when  $\alpha = 2$  this amounts to consider  $N \leq 5$ , so that this upper bound on the dimension is needed from the beginning, in order to have the convolution term well-defined.

**Remark 2.5.** Theorem 2.2 does not include the case  $\beta = 2\sqrt{V_\infty}$  and  $\alpha \in (N-1, N)$ . In this range the decay of  $\omega$  changes ([21], [20]). An analogous result can be obtained, also in the case  $p = 2, \alpha > N - 1$ . But, the principal tool in order to obtain the decay estimates (see Lemma 3.1) cannot be directly applied; the interested reader can see [19] where we prove an extension of Lemma 3.1 to handle the case  $p = 2, \alpha \in (N - 1, N - \frac{1}{2})$ .

Let us conclude this section by recalling the following decay information concerning the convolution term.

**Lemma 2.6.** *Let  $h \geq 0, h \in L^\infty$  such that*

$$\sup_{\mathbb{R}^N} h(x)(1 + |x|)^s < +\infty \tag{15}$$

*for some  $s > N$ . Then*

$$I_\alpha * h(x) = I_\alpha(x) \|h\|_1 (1 + o(1)), \quad \text{as } |x| \rightarrow \infty.$$

**Proof.** The conclusion follows immediately from Lemma 6.2 in [21]. □

As an immediate consequence of Lemma 2.6, we get the following asymptotical decay of the convolution term

$$I_\alpha * \omega^2(x) = I_\alpha(x) \|\omega\|_2 (1 + o(1)), \quad \text{as } |x| \rightarrow \infty. \tag{16}$$

Indeed, taking  $h = \omega^2$  one immediately has that (15) is satisfied for every  $s$ .

### 3. Asymptotic estimates

In this section we will obtain all the asymptotic estimates needed in proving our main results. We will first introduce the threshold that will guide our study and we will establish its decay. Then, the asymptotical decay of the integral term involving the potential will be studied, and at last we will deal with the nonlinearity term.

Let us precise that with the expression  $f \sim g$  as  $|x| \rightarrow \infty$  we mean that the quotient  $f/g \rightarrow l \in (0, +\infty)$  as  $|x| \rightarrow \infty$ . The following Lemma will be repeatedly exploited.

**Lemma 3.1.** (Lemma 3.7 in [4]) *Let  $u, v : \mathbb{R}^N \rightarrow \mathbb{R}$  be two positive continuous radial functions such that*

$$u(x) \sim |x|^a e^{-b|x|}, \quad v(x) \sim |x|^{a'} e^{-b'|x|} \quad \text{as } |x| \rightarrow \infty,$$

*where  $a, a' \in \mathbb{R}$ , and  $b, b' > 0$ . Let  $\xi \in \mathbb{R}^N$  tend to infinity. Define  $u_\xi(x) := u(x - \xi)$ . Then the following asymptotic estimates hold:*

(i) *If  $b < b'$ ,* 
$$\int_{\mathbb{R}^N} u_\xi v \sim e^{-b|\xi|} |\xi|^a.$$

*A similar expression holds if  $b > b'$ , by replacing  $a$  and  $b$  with  $a'$  and  $b'$ .*

(ii) If  $b = b'$ , suppose that  $a \geq a'$ . Then

$$\int_{\mathbb{R}^N} u_\xi v \sim \begin{cases} e^{-b|\xi|} |\xi|^{a+a'+\frac{N+1}{2}} & \text{if } a' > -\frac{N+1}{2}, \\ e^{-b|\xi|} |\xi|^a \log \xi & \text{if } a' = -\frac{N+1}{2}, \\ e^{-b|\xi|} |\xi|^a & \text{if } a' < -\frac{N+1}{2}. \end{cases}$$

Let  $z_1 \in \mathbb{R}^N$  with  $|z_1| = 1$  and  $z_2 \in \partial B_2(z_1)$ , we denote with  $\omega_{i,R}(x)$  a positive solution of  $(P_\infty)$  achieving  $c_\infty$  (see (8)) of the limit problem translated in  $Rz_i$ , namely

$$\omega_{i,R}(x) = \omega(x - Rz_i). \tag{17}$$

Moreover, the threshold guiding all the asymptotic estimates is

$$\varepsilon_R = \int_{\mathbb{R}^N} (I_\alpha * \omega_{1,R}^2) \omega_{1,R} \omega_{2,R} = \int_{\mathbb{R}^N} (I_\alpha * \omega_{2,R}^2) \omega_{2,R} \omega_{1,R}. \tag{18}$$

The precise decay of  $\varepsilon_R$  is obtained in the following lemma.

**Lemma 3.2.** *Let  $\alpha \in ((N - 4)^+, N - 1]$ . Then, for  $R$  large enough, the following conclusions hold:*

$$\varepsilon_R \sim \begin{cases} e^{-2\sqrt{V_\infty}R} R^{-\frac{N-1}{2}}, & \text{if } \alpha < N - 1. \\ e^{-2\sqrt{V_\infty}R} R^{-\frac{N-1}{2} + \nu\sqrt{V_\infty}}, & \text{if } \alpha = N - 1. \end{cases} \tag{19}$$

where  $\nu$  is introduced in (12).

**Remark 3.3.** Notice that, for any  $\alpha \in (0, N)$ , it results

$$\varepsilon_R \geq CR^{-\frac{N-1}{2}} e^{-\sqrt{V_\infty}Q(2R)}$$

where  $Q$  is introduced in (12). Indeed, one has

$$\begin{aligned} \inf_{x \in B_1(0)} I_\alpha * \omega^2(x) &\geq \inf_{x \in B_1(0)} \frac{A_\alpha}{R_0^{N-\alpha}} \int_{B_{R_0}(x)} \omega^2(y) dy \\ &\geq A_\alpha \frac{|B_{R_0}(0)|}{R_0^{N-\alpha}} \min_{y \in B_{R_0+1}(0)} \omega^2(y) \geq C > 0. \end{aligned}$$

Therefore, denoting with  $C$  possibly different constants and taking into account the general decay given in (11) and recalling that  $Q$  is monotone increasing for  $t > \nu$ , one gets

$$\begin{aligned} \varepsilon_R &\geq \int_{B_1(Rz_1)} (I_\alpha * \omega_{i,R}^2) \omega_{i,R} \omega_{j,R} = \int_{B_1(0)} (I_\alpha * \omega^2(x)) \omega(x) \omega(x - R(z_2 - z_1)) dx \\ &\geq \inf_{x \in B_1(0)} (I_\alpha * \omega^2(x) \omega(x)) \int_{B_1(0)} \omega(x - R(z_2 - z_1)) dx \\ &\geq C \int_{B_1(0)} e^{-\sqrt{V_\infty}Q(|x - R(z_2 - z_1)|)} \frac{1}{(1 + |x - R(z_2 - z_1)|)^{\frac{N-1}{2}}} \\ &\geq CR^{-\frac{N-1}{2}} e^{-\sqrt{V_\infty}Q(1+2R)}, \quad \text{for } R \text{ sufficiently large.} \end{aligned}$$

Notice that

$$\begin{aligned} Q(1 + 2R) - Q(2R) &= \int_{\nu}^{1+2R} \sqrt{1 - \frac{\nu^{N-\alpha}}{s^{N-\alpha}}} ds - \int_{\nu}^{2R} \sqrt{1 - \frac{\nu^{N-\alpha}}{s^{N-\alpha}}} ds \\ &= \int_0^1 \sqrt{1 - \frac{\nu^{N-\alpha}}{(t + 2R)^{N-\alpha}}} dt \leq c, \quad \text{as } R \rightarrow \infty. \end{aligned}$$

In addition  $Q(2R) \leq 2R - \nu$  so that  $\varepsilon_R \geq C_0 R^{-\frac{N-1}{2}} e^{-2R\sqrt{V_\infty}}$ , which shows that (19) is optimal for  $\alpha < N - 1$ . On the other hand, when  $\alpha = N - 1$ , this estimate from below is consistent with estimates obtained in Lemma 3.2 and also with estimates in [11], however it is far from being sharp.

**Proof of Lemma 3.2.** Let us first observe that, performing a change of variable

$$\varepsilon_R = \int_{\mathbb{R}^N} (I_\alpha * \omega^2)(x) \omega(x) \omega(x - R(z_1 - z_2)) dx. \tag{20}$$

We are going to apply Lemma 3.1 with

$$v = I_\alpha * \omega^2 \omega, \quad u = \omega, \quad \xi = R(z_1 - z_2), \quad \text{and } |\xi| = 2R, \tag{21}$$

with the exponents

$$b = b' = \sqrt{V_\infty}, \quad a = -\frac{N-1}{2}, \quad a' = a - N + \alpha < a$$

and  $a' < -(N + 1)/2$  iff  $\alpha < N - 1$ , so that in this case we get the first information in (19), while when  $\alpha = N - 1$  we have to consider (10)

$$b = b' = \sqrt{V_\infty} \quad a = -\frac{N-1}{2} + \frac{\nu\sqrt{V_\infty}}{2}, \quad a' = a - N + \alpha = a - 1 < a$$

and now  $a' > -(N + 1)/2$  as  $a > -(N - 1)/2$  so that the second information in (19) follows. □

Let us now prove the asymptotic estimates on the term with  $V(x)$  that will be used in the following.

**Lemma 3.4.** *Let  $N \geq 2$ , and  $\alpha \in ((N - 4)^+, N - 1]$  and assume (2), (13), (14). Then, for  $R$  large enough, it results*

$$\mathcal{A}_V := \int_{\mathbb{R}^N} (V(x) - V_\infty) (\omega_{i,R})^2 \leq o(\varepsilon_R), \quad \text{for } i = 1, 2.$$

**Remark 3.5.** In the proof of this Lemma we will exploit Lemma 3.1. Notice that, in order to do this, we will first make use of (13) which gives an upper bound on  $V$  by a positive radial function. As a consequence, we will get the conclusion, even if  $V(x) - V_\infty$  is not radial, nor positive.

**Proof.** We want now to apply Lemma 3.1. However, this only applies to radial functions, hence we preliminary notice that

$$\int_{\mathbb{R}^N} (V(x) - V_\infty) \omega_{i,R}^2 \leq C \int_{\mathbb{R}^N} (|x| + 1)^\sigma e^{-\beta|x|} \omega_{i,R}^2.$$

Take  $u = \omega^2$ ,  $v = (|x| + 1)^\sigma e^{-\beta|x|}$ , and  $\xi = Rz_i$ .

Notice that

$$\begin{aligned} u &\leq C e^{-2\sqrt{V_\infty}|x|} |x|^{-N+1}, & \text{for } \alpha < N - 1 \\ u &\leq C e^{-2\sqrt{V_\infty}|x|} |x|^{-N+1+\nu\sqrt{V_\infty}}, & \text{for } \alpha = N - 1, \end{aligned}$$

with  $C$  a suitable positive constant. Then, in the case  $\beta > 2\sqrt{V_\infty}$  Lemma 3.1 immediately implies the conclusion. Let us now deal with the case  $\beta = 2\sqrt{V_\infty}$  considering first  $N = 2$  and  $\alpha < N - 1$ , i.e.  $\alpha \in (0, 1)$ . we again exploit Lemma 3.1 with (recalling (14))

$$b = b' = 2\sqrt{V_\infty} \quad a = -1, \quad a' = \sigma < -1;$$

then one has  $a > a'$ , and we can suppose without loss of generality that  $a' > -3/2$ , so that Lemma 3.1 yields the conclusion as  $\sigma < -1$ . If  $N = 2$ ,  $\alpha = 1$ , we have

$$b = b' = 2\sqrt{V_\infty} \quad a = -1 + \nu\sqrt{V_\infty}, \quad a' = \sigma < -1;$$

so that  $a > a'$  and again we can suppose that  $a' > -3/2$ , and we reach the conclusion taking into account (19). When  $N \geq 3$  and  $\alpha < N - 1$  we can use Lemma 3.1 with

$$b = b' = 2\sqrt{V_\infty}, \quad a = 1 - N, \quad a' = \sigma < -\frac{N - 1}{2}$$

and  $a' > a$  and  $a < -(N + 1)/2$  for every  $N > 3$  and  $a = -(N + 1)/2$  if  $N = 3$ . In both cases, we get  $a'$  as the exponent of the polynomial part and the conclusion follows as  $a' = \sigma < -\frac{N-1}{2}$ . As a last case, let us consider  $N \geq 3$  and  $\alpha = N - 1$ . Take

$$b = b' = 2\sqrt{V_\infty}, \quad a = 1 - N + \nu\sqrt{V_\infty}, \quad a' = \sigma < \min \left\{ -1, -\frac{N-1}{2} + \nu\sqrt{V_\infty} \right\}.$$

If  $\nu\sqrt{V_\infty} \leq \frac{N-3}{2}$  we have  $a \leq a'$  and  $a \leq -(N + 1)/2$  so that the exponent in the polynomial term will be  $a' = \sigma$  yielding the conclusion as  $\sigma < -\frac{N-1}{2} + \nu\sqrt{V_\infty}$ . In the case in which  $\frac{N-3}{2} < \nu\sqrt{V_\infty} \leq N - 2$  one has  $a \leq a'$  and  $a > -(N + 1)/2$  so that the exponent in the polynomial term will be

$$a' + a + \frac{N + 1}{2} = \sigma - \frac{N - 3}{2} + \nu\sqrt{V_\infty}$$

implying again the conclusion thanks to (14). Finally, when  $\nu\sqrt{V_\infty} > N - 2$  it results  $a > a'$ , and as  $-(N + 1)/2 < -(N - 1)/2 + \nu\sqrt{V_\infty}$  for every  $N$  we can suppose w.l.o.g. that  $\sigma > -(N + 1)/2$  so that the exponent in the polynomial term will be  $a + \sigma + \frac{N+1}{2}$  which again gives a decay faster than the one of  $\varepsilon_R$ .  $\square$

We conclude this section by studying the nonlinearity term.

**Proposition 3.6.** *Given  $s, t \in (0, +\infty)$ , it results*

$$\int_{\mathbb{R}^N} (I_\alpha * (s\omega_{1,R} + t\omega_{2,R})^2) (s\omega_{1,R} + t\omega_{2,R})^2 - s^4 \int_{\mathbb{R}^N} (I_\alpha * \omega_{1,R}^2) \omega_{1,R}^2 - t^4 \int_{\mathbb{R}^N} (I_\alpha * \omega_{2,R}^2) \omega_{2,R}^2 \geq 4st(s^2 + t^2)\varepsilon_R.$$

where  $\varepsilon_R$  is defined in (18).

**Proof.** Direct computations show

$$\int_{\mathbb{R}^N} (I_\alpha * (s\omega_{1,R} + t\omega_{2,R})^2) (s\omega_{1,R} + t\omega_{2,R})^2 \geq s^4 \int_{\mathbb{R}^N} (I_\alpha * \omega_{1,R}^2) \omega_{1,R}^2 + t^4 \int_{\mathbb{R}^N} (I_\alpha * \omega_{2,R}^2) \omega_{2,R}^2 + 4st(s^2 + t^2)\varepsilon_R$$

where we have used that

$$\int_{\mathbb{R}^N} (I_\alpha * \omega_{i,R}^2) \omega_{j,R}^2 \geq 0 \quad \text{for every } i, j = 1, 2 \text{ with } i \neq j$$

and that

$$\begin{aligned} \int_{\mathbb{R}^N} (I_\alpha * \omega_{i,R} \omega_{j,R}) \omega_{i,R}^2 &= \int_{\mathbb{R}^N} \omega^2(\theta - Rz_i) d\theta \int_{\mathbb{R}^N} \frac{\omega(x - Rz_i) \omega(x - Rz_j)}{|x - \theta|^{N-\alpha}} dx \\ &= \int_{\mathbb{R}^N} \omega(x - Rz_i) \omega(x - Rz_j) dx \int_{\mathbb{R}^N} \frac{\omega^2(\theta - Rz_i)}{|x - \theta|^{N-\alpha}} d\theta \\ &= \int_{\mathbb{R}^N} (I_\alpha * \omega_{i,R}^2) \omega_{j,R} \omega_{i,R} = \varepsilon_R. \end{aligned} \quad \square$$

#### 4. Proof of Theorems 2.2 and 1.1

Let us start this section by proving some results concerning the compactness properties of the functional  $\mathcal{I}_V$ . These are nowadays quite classical in this context, we will follow arguments in [18, 9].

**Lemma 4.1.** *Let  $\mathcal{N}_V$  be defined in (6). Any sequence  $(u_k)$  such that*

$$u_k \in \mathcal{N}_V \quad \text{and} \quad \mathcal{I}_V(u_k) \rightarrow d, \quad \nabla_{\mathcal{N}} \mathcal{I}_V(u_k) \rightarrow 0,$$

*satisfies  $\nabla \mathcal{I}_V(u_k) \rightarrow 0$  in  $H^{-1}(\mathbb{R}^N)$  and  $(u_k)$  has a subsequence which is bounded in  $H^1(\mathbb{R}^N)$ .*

**Proof.** In order to show that  $(u_k)$  is a free Palais-Smale sequence, it is possible to perform the same argument as in the local case thanks to the homogeneity power of the Choquard nonlinearity. Indeed, the proof of Corollary 3.2 in [10] can be adapted in a straightforward way to the framework without symmetries.  $\square$

**Lemma 4.2.** (Splitting Lemma) *Let  $(u_k)$  be a bounded  $(PS)_d$  sequence for  $\mathcal{I}_V$ . Up to a subsequence, there exists a solution  $u_0$  of problem  $(P_V)$ , a number  $m \in \mathbb{N} \cup \{0\}$ ,  $m$  non trivial solutions  $\omega^1, \omega^2, \dots, \omega^m$  to the limit problem  $(P_\infty)$ , and  $m$  sequences of points  $(y_k^j) \in \mathbb{R}^N$ ,  $1 \leq j \leq m$ , satisfying*

- (i)  $|y_k^j| \rightarrow \infty$  and  $|y_k^j - y_k^i| \rightarrow \infty$  if  $i \neq j$ ,
- (ii)  $u_k - \sum_{i=1}^m \omega^i(\cdot - y_k^i) \rightarrow u_0$  in  $H^{-1}(\mathbb{R}^N)$
- (iii)  $d = \mathcal{I}_V(u_0) + \sum_{i=1}^m \mathcal{I}_\infty(\omega^i)$ .

**Proof.** Since  $(u_k)$  is bounded, there exists  $u_0$  such that up to a subsequence  $u_k \rightharpoonup u_0$ . Then  $\mathcal{I}'_V(u_0) = 0$  in  $H^{-1}(\mathbb{R}^N)$ . Let  $u_k^1 = u_k - u_0$ . By standard arguments and exploiting (2), one has

$$\int_{\mathbb{R}^N} |\nabla u_k^1|^2 + V_\infty \int_{\mathbb{R}^N} |u_k^1|^2 = \|u_k\|_V^2 - \|u_0\|_V^2 + o_k(1).$$

Moreover, by a Brezis-Lieb type lemma (see Lemma 2.4 in [21]), one has

$$\begin{aligned} \mathcal{I}_\infty(u_k^1) - \mathcal{I}_V(u_k) + \mathcal{I}_V(u_0) &= \int_{\mathbb{R}^N} (I_\alpha * u_k^2) u_k^2 - \\ &- \int_{\mathbb{R}^N} (I_\alpha * (u_k - u_0)^2) (u_k - u_0)^2 - \int_{\mathbb{R}^N} (I_\alpha * u_0^2) u_0^2 + o_k(1) = o_k(1), \end{aligned}$$

where  $o_k(1) \rightarrow 0$  as  $k \rightarrow +\infty$ . Moreover, as  $(u_k)$  is a Palais-Smale of  $\mathcal{I}_V$  at level  $d$ , one has

$$\mathcal{I}_\infty(u_k^1) \rightarrow d - \mathcal{I}_V(u_0).$$

Now, exploiting [1, Lemma 3.4], and recalling that  $\mathcal{I}'_V(u_0) = 0$  in  $H^{-1}(\mathbb{R}^N)$ , it results, for any  $\varphi \in H^1$

$$\begin{aligned} o_k(1) \|\varphi\| &\geq |\langle \mathcal{I}'_V(u_k), \varphi \rangle| \geq |\langle \mathcal{I}'_V(u_0), \varphi \rangle + \langle \mathcal{I}'_\infty(u_k^1), \varphi \rangle| - \\ &- \left| \int_{\mathbb{R}^N} \left( (I_\alpha * |u_k^1 - u_0|^2) |u_k^1 - u_0| - (I_\alpha * |u_k^1|^2) |u_k^1| - (I_\alpha * |u_0|^2) |u_0| \right) \varphi \right| \\ &= |\langle \mathcal{I}'_\infty(u_k^1), \varphi \rangle| - \\ &- \left| \int_{\mathbb{R}^N} \left( (I_\alpha * |u_k^1 - u_0|^2) |u_k^1 - u_0| - (I_\alpha * |u_k^1|^2) |u_k^1| - (I_\alpha * |u_0|^2) |u_0| \right) \varphi \right| \\ &\geq |\langle \mathcal{I}'_\infty(u_k^1), \varphi \rangle| - o_k(1) \|\varphi\|, \end{aligned}$$

Hence  $\mathcal{I}'_\infty(u_k^1) \rightarrow 0$  in  $H^{-1}(\mathbb{R}^N)$ .

If  $u_k^1 \rightarrow 0$  strongly, then we are done, choosing  $m = 0$ . Otherwise, by applying Lions lemma [15], there exists  $\delta > 0$  and a sequence  $y_k^1$  such that

$$\int_{B_1(y_k^1)} (u_k^1)^2 > \delta.$$

We define  $v_k^1(x) = u_k^1(x + y_k^1)$ . By the boundedness of  $(u_k^1)$ , there exists  $\omega^1$  such that  $v_k^1 \rightharpoonup \omega^1$  and  $v_k^1 \rightarrow \omega^1$  a.e. Therefore,  $\omega^1 \neq 0$ , and  $|y_k^1| \rightarrow \infty$ . We now show that  $\omega^1$  is a solution to  $(P_\infty)$ . Indeed, let  $\varphi \in C_c^\infty(\mathbb{R}^N)$  and set  $\varphi_k^1(x) = \varphi(x - y_k^1)$ .

One has  $\mathcal{I}'_\infty(\omega^1)\varphi + o_k(1) = \mathcal{I}'_\infty(v_k^1)\varphi = \mathcal{I}'_\infty(u_k^1)\varphi_k^1 = o_k(1)$ ,

thus  $\mathcal{I}'_\infty(\omega^1) = 0$  and  $\omega^1$  is a solution to  $(P_\infty)$ . Moreover, by [21, Lemma 2.4] and [1, Lemma 3.4],

$$\mathcal{I}_\infty(v_k^1) = \mathcal{I}_\infty(v_k^1 - \omega^1) + \mathcal{I}_\infty(\omega^1) + o(1),$$

so that  $o(1) = \mathcal{I}'_\infty(v_k^1) = \mathcal{I}'_\infty(v_k^1 - \omega^1) + o(1)$ .

Now, we iterate the argument and define  $u_k^2(x) = u_k^1(x) - \omega^1(x - y_k^1)$ . Then, we have

$$\begin{aligned} \mathcal{I}_\infty(u_k^2) &= \mathcal{I}_\infty(u_k^1) - \mathcal{I}_\infty(\omega^1) + o(1) = \mathcal{I}_V(u_k) - \mathcal{I}_V(u_0) - \mathcal{I}_\infty(\omega^1) + o(1) \\ &= d - \mathcal{I}_V(u_0) - \mathcal{I}_\infty(\omega^1) + o(1) \end{aligned}$$

and  $\mathcal{I}'_\infty(u_k^2) = \mathcal{I}'_\infty(u_k^1) + o_k(1) = o_k(1)$ .

If  $u_k^2 \rightarrow 0$  strongly, then we are done, choosing  $m = 1$ . If not, we repeat the argument. After a finite number of steps (as  $d$  is finite), we will arrive to a sequence  $u_k^{m+1}$  which converges strongly to 0, and the proof is completed.  $\square$

**Corollary 4.3.** *Suppose that problem  $(P_\infty)$  has a unique positive solution. If  $c_V$  is not attained, then  $c_V \geq c_\infty$ . Moreover,  $\mathcal{I}_V$  satisfies the Palais-Smale condition at every level  $d \in (c_\infty, 2c_\infty)$ .*

**Proof.** First of all let us observe that if  $d = c_V < c_\infty$  then, conclusion (iii) of Lemma 4.2 and (8) imply

$$d = \mathcal{I}_V(u_0) + \sum_{i=1}^m \mathcal{I}_\infty(\omega^i) > \mathcal{I}_V(u_0) + md \geq md$$

which immediately gives  $m = 0$ , namely  $(u_k)$  strongly converges to  $u_0$  so that  $c_V$  is attained.

On the other hand, if  $d \in (c_\infty, 2c_\infty)$ , one again exploits conclusion (iii) of Lemma 4.2, recalling that  $u_0$  is a solution (so that  $\mathcal{I}_V(u_0) = 0$  if  $u_0 = 0$ , otherwise  $\mathcal{I}_V(u_0) \geq c_V$ ), and takes into account (6) to obtain

$$2c_\infty > d = \mathcal{I}_V(u_0) + \sum_{i=1}^m \mathcal{I}_\infty(\omega^i) \geq mc_\infty$$

because for every  $\omega^i$ ,  $\mathcal{I}_\infty(\omega^i) \geq c_\infty$ . Then  $m \leq 1$  and it is only left to rule out the case  $m = 1$ . In this case, if  $u_0 \not\equiv 0$

$$2c_\infty > d = \mathcal{I}_V(u_0) + \mathcal{I}_\infty(\omega^1) \geq c_V + \mathcal{I}_\infty(\omega^1) \geq c_\infty + \mathcal{I}_\infty(\omega^1) \geq 2c_\infty \tag{22}$$

which is an evident contradiction, so that  $u_0 \equiv 0$ . Reading again (22), we get  $\mathcal{I}_\infty(\omega^1) \in (c_\infty, 2c_\infty)$ . On the other hand  $(\omega^1)^\pm$  both belong to  $\mathcal{N}_\infty$ , so that

$$2c_\infty > \mathcal{I}_\infty(\omega^1) = \mathcal{I}_\infty(\omega^1)^+ + \mathcal{I}_\infty(\omega^1)^- \geq 2c_\infty,$$

as a consequence  $\omega^1$  does not change sign and it is, up to a translation, the unique positive solution of problem  $(P_\infty)$  and  $\mathcal{I}_\infty(\omega^1) = c_\infty$ .

Finally, writing again (iii) of Lemma 4.2 with  $u_0 \equiv 0$  and  $m = 1$  gives

$$d = \mathcal{I}_\infty(\omega^1) = c_\infty$$

which is again a contradiction, yielding  $m = 0$ . □

Recall (18) and define for every  $\lambda \in [0, 1]$

$$\chi_{\lambda,R} = \lambda\omega_{1,R} + (1 - \lambda)\omega_{2,R}, \quad \text{where } \omega_{i,R} \text{ is defined in (17).} \tag{23}$$

**Lemma 4.4.** *For every  $u \in H^1(\mathbb{R}^N) \setminus \{0\}$  the number*

$$\bar{T} := \left[ \frac{\|u\|_V^2}{\int_{\mathbb{R}^N} (I_\alpha * u^2) u^2} \right]^{1/2} \tag{24}$$

*is the unique positive one such that  $\bar{T}u \in \mathcal{N}_V$  and the map  $T : H^1(\mathbb{R}^N) \setminus \{0\} \mapsto \mathbb{R}^+$  is continuous. In addition, having defined  $T_{\lambda,R} := \bar{T}(\chi_{\lambda,R})$ , there exists  $R_0 > 0$  and  $T_0 > 0$  such that*

$$T_{\lambda,R} \leq T_0, \quad \text{for every } R \geq R_0, \text{ and } \lambda \in [0, 1]. \tag{25}$$

**Proof.** The fact that  $\bar{T}$  given in (24) is the unique positive number such that  $\bar{T}u \in \mathcal{N}_V$  comes from the characterization

$$\mathcal{I}_V(\bar{T}u) = \max_{t>0} \mathcal{I}_V(tu),$$

and the continuity is a direct consequence of (24). In order to prove (25) let us note that Lemma 3.4 and Theorem 2.1 imply

$$\begin{aligned} \|\chi_{\lambda,R}\|_V^2 &= (\lambda^2 + (1 - \lambda)^2) \|\nabla\omega\|_2^2 + 2\lambda(1 - \lambda) \int_{\mathbb{R}^N} \nabla\omega_{1,R} \nabla\omega_{2,R} + \\ &+ \lambda^2 \int_{\mathbb{R}^N} V(x + Rz_1)\omega^2 + (1 - \lambda)^2 \int_{\mathbb{R}^N} V(x + Rz_2)\omega^2 + 2\lambda(1 - \lambda) \int_{\mathbb{R}^N} V(x)\omega_{1,R}\omega_{2,R} \\ &= [\lambda^2 + (1 - \lambda)^2] \left[ \|\nabla\omega\|_2^2 + \int_{\mathbb{R}^N} V_\infty\omega^2 \right] + o_R(1), \end{aligned}$$

where  $o_R(1)$  is a quantity tending to zero as  $R \rightarrow +\infty$ . In addition, one has

$$\int_{\mathbb{R}^N} (I_\alpha * \chi_{\lambda,R}^2) \chi_{\lambda,R}^2 \geq [\lambda^4 + (1 - \lambda)^4] \int_{\mathbb{R}^N} (I_\alpha * \omega^2) \omega^2$$

so that

$$T_{\lambda,R}^2 = \frac{\lambda^2 + (1 - \lambda)^2 \|\nabla\omega\|_2^2 + \int_{\mathbb{R}^N} V_\infty\omega^2 + o_R(1)}{\lambda^4 + (1 - \lambda)^4 \int_{\mathbb{R}^N} (I_\alpha * \omega^2) \omega^2} \leq 4 + o_R(1). \tag{26} \quad \square$$

**Proposition 4.5.** *Assume (2), (13), (14). Then, there exists  $R_1 > 0$  and for each  $R > R_1$  there exists  $\eta_R > 0$  such that*

$$\mathcal{I}_V(T_{\lambda,R}\chi_{\lambda,R}) \leq 2c_\infty - \eta_R \tag{26}$$

for all  $\lambda \in [0, 1]$  and all  $z \in \partial B_2(z_1)$ . Moreover, for any  $\delta > 0$  there exists  $R_2 > 0$  such that

$$\mathcal{I}_V(T_{0,R}\chi_{0,R}) = \mathcal{I}_V(T_{0,R}\omega_{2,R}) < c_\infty + \delta \tag{27}$$

for every  $z \in \partial B_2(z_0)$  and  $R > R_2$ . In particular,  $c_V \leq c_\infty$ .

**Proof.** Let us first note that (24) yields

$$\mathcal{I}_V(T_{\lambda,R}\chi_{\lambda,R}) = \frac{1}{4}T_{\lambda,R}^2\|\chi_{\lambda,R}\|_V^2. \tag{28}$$

Repeating the argument in the proof of Lemma 4.4, taking into account Lemma 3.4, (18) and the fact that  $\omega$  is a solution of  $(P_\infty)$  we get

$$\|\chi_{\lambda,R}\|_V^2 = (s^2 + t^2)\|\omega\|^2 + 2st\varepsilon_R + o(\varepsilon_R), \quad \text{where } s = \lambda, t = (1 - \lambda), \tag{29}$$

On the other hand, Proposition 3.6 and Lemma 3.4 yield

$$\begin{aligned} \int_{\mathbb{R}^N} (I_\alpha * \chi_{\lambda,R}^2)\chi_{\lambda,R}^2 &\geq (s^4 + t^4) \int_{\mathbb{R}^N} (I_\alpha * \omega^2)\omega^2 + 4st(s^2 + t^2)\varepsilon_R \\ &= (s^4 + t^4)\|\omega\|^2 + 4st(s^2 + t^2)\varepsilon_R. \end{aligned}$$

Using the information in (24) and taking the expansion

$$(a + bt)^{-1} = \frac{1}{a} - \frac{b}{a^2}t + o(t)$$

into account one gets

$$\begin{aligned} T^2 &\leq \left\{ (s^2 + t^2)\|\omega\|^2 + 2st\varepsilon_R + o(\varepsilon_R) \right\} \left\{ \frac{1}{(s^4 + t^4)\|\omega\|^2} - \frac{4st(s^2 + t^2)}{(s^4 + t^4)^2\|\omega\|_V^4} \varepsilon_R + o(\varepsilon_R) \right\} \\ &= \frac{s^2 + t^2}{(s^4 + t^4)} + \frac{2st}{(s^4 + t^4)\|\omega\|^2} \varepsilon_R \left\{ 1 - 2\frac{(s^2 + t^2)^2}{s^4 + t^4} \right\} + o(\varepsilon_R). \end{aligned}$$

When using this inequality in (28) one obtains, thanks to (29),

$$\begin{aligned} \mathcal{I}_V(T_{\lambda,R}\chi_{\lambda,R}) &\leq \frac{1}{4}\|\omega\|_V^2 \frac{(s^2 + t^2)^2}{s^4 + t^4} + \frac{st(s^2 + t^2)}{(s^4 + t^4)} \varepsilon_R \left\{ 1 - \frac{(s^2 + t^2)^2}{(s^4 + t^4)} \right\} \\ &= c_\infty \frac{(s^2 + t^2)^2}{s^4 + t^4} - \frac{2s^3t^3(s^2 + t^2)}{(s^4 + t^4)^2} \varepsilon_R + o(\varepsilon_R) \end{aligned} \tag{30}$$

Let now  $\delta$  be a positive constant less than  $1/2$ . If  $\lambda \in [\frac{1}{2} - \delta, \frac{1}{2} + \delta]$  one observes that

$$\frac{2s^3t^3(s^2 + t^2)}{(s^4 + t^4)^2} \geq \mu_\delta > 0, \quad \frac{(s^2 + t^2)^2}{s^4 + t^4} \leq 2,$$

so that (30) becomes  $\mathcal{I}_V(T_{\lambda,R}\chi_{\lambda,R}) \leq 2c_\infty - \mu_\delta\varepsilon_R + o(\varepsilon_R) \leq 2c_\infty - \eta_R$ .

In the case  $|\lambda - \frac{1}{2}| > \delta$ , there exists  $\sigma_\delta$  such that that

$$\frac{(s^2 + t^2)^2}{s^4 + t^4} \leq 2 - \sigma_\delta < 2$$

so that (30) becomes

$$\mathcal{I}_V(T_{\lambda,R}\chi_{\lambda,R}) \leq (2 - \sigma_\delta)c_\infty - \frac{2s^3t^3(s^2 + t^2)}{(s^4 + t^4)^2}\varepsilon_R + o(\varepsilon_R) < 2c_\infty.$$

Then, also in this case we get (26). In order to show (27) we repeat the same argument with  $\lambda = 0$  arriving at (30) which now reads as follows (recalling that  $s = 0, t = 1$ )

$$\mathcal{I}_V(\overline{T}(\omega_{2,R})\omega_{2,R}) \leq c_\infty + o(\varepsilon_R).$$

which immediately yields (27). □

From now on we will make use of a barycenter map, whose definition and properties we we briefly recall for the sake of completeness. For more details see [3], [5], [7], [18]. For every  $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ , the following maps are well defined

$$\mu(u)(x) := \frac{1}{|B_1(x)|} \int_{B_1(x)} |u(y)| dy, \quad \mu(u) \in L^\infty \cap C^0(0, +\infty),$$

$$\hat{u}(x) := \left[ \mu(u)(x) - \frac{\|\mu(u)\|_\infty}{2} \right]^+, \quad \hat{u} \in C_0(\mathbb{R}^N).$$

Then, the barycenter of a function  $u \in H^1(\mathbb{R}^N) \setminus \{0\}$  defined by

$$\beta(u) = \frac{1}{\|\hat{u}\|_1} \int_{\mathbb{R}^N} x \hat{u}(x) dx$$

is a continuous function enjoying the following properties

$$\beta(u(\cdot - y)) = \beta(u) + y \quad \forall y \in \mathbb{R}^N, \tag{31}$$

$$\beta(Tu) = \beta(u) \quad \forall T > 0. \tag{32}$$

Note that  $\beta(u) = 0$  if  $u$  is radial.

**Lemma 4.6.** *If  $c_V$  is not attained then  $c_V = c_\infty$  and there exists  $\delta > 0$  such that*

$$\beta(u) \neq z_1 \quad \forall u \in \mathcal{N}_V \cap \mathcal{I}_V^{c_\infty + \delta}$$

where  $\mathcal{I}_V^c = \{u : \mathcal{I}_V(u) \leq c\}$ .

**Proof.** By Corollary 4.3 and Proposition 4.5 if  $c_V$  is not attained, then  $c_V = c_\infty$ . Let us assume by contradiction that for each  $k \in \mathbb{N}$  there exists  $u_k \in \mathcal{N}_V$  such that

$$\mathcal{I}_V(u_k) < c_V + \frac{1}{k}, \quad \text{and} \quad \beta(u_k) = z_1.$$

By Ekeland’s variational principle (see [26]) there exists a constrained Palais-Smale sequence, called  $(v_k)$ , at level  $c_V$  for  $\mathcal{I}_V$  on  $\mathcal{N}_V$  and such that  $\|v_k - u_k\|_V \rightarrow 0$ , so that  $\beta(v_k) \rightarrow z_1$ . By Lemma 4.1  $(v_k)$  is (up to a subsequence) a bounded Palais Smale sequence for  $\mathcal{I}_V$  at level  $c_V$  in  $H^1(\mathbb{R}^N)$ . Since  $c_V$  is not attained, we conclude by Lemma 4.2 that there exists a sequence  $(z_k)$  such that  $|z_k| \rightarrow \infty$  such that  $\|v_k - \omega(\cdot - z_k)\| \rightarrow 0$ . We set  $w_k(x) = v_k(x + z_k)$  and from (31), it follows that

$$z_1 - z_k = \beta(v_k) - z_k + o_k(1) = \beta(w_k) + o_k(1) \rightarrow \beta(\omega) = 0,$$

which is a contradiction. □

We are finally in the position to prove our main results.

**Proof of Theorem 2.2.** If  $c_V$  is attained at some  $u \in \mathcal{N}_V$ , taking into account that  $\mathcal{N}_V$  is a natural constraint for  $\mathcal{I}_V$  it turns out that  $u$  is a nontrivial solution of  $(P_V)$ . Let us assume that  $c_V$  is not attained. Then by Lemma 4.6  $c_V = c_\infty$ . Then, we are going to show that  $\mathcal{I}_V$  has a critical value in  $(c_\infty, 2c_\infty)$ . By Lemma 4.6 we may find a  $\delta > 0$  sufficiently small such that

$$\beta(u) \neq z_1 \quad \forall u \in \mathcal{N}_V \cap \mathcal{I}_V^{c_\infty + \delta},$$

where  $\mathcal{I}_V^b := \{u \in H^1(\mathbb{R}^N) : \mathcal{I}_V(u) \leq b\}$ .

Moreover, thanks to Proposition 4.5 we can choose  $\eta > 0$  sufficiently small and  $R > 0$  such that

$$\mathcal{I}_V(T_{\lambda,R}\chi_{\lambda,R}) \leq \begin{cases} 2c_\infty - \eta & \text{for all } \lambda \in [0, 1] \text{ and all } z_2 \in \partial B_2(z_1) \\ c_\infty + \delta & \text{for } \lambda = 0 \text{ and all } z_2 \in \partial B_2(z_1). \end{cases}$$

Let us define  $\psi : \overline{B_2(z_1)} \rightarrow \mathcal{N}_V \cap \mathcal{I}_V^{2c_\infty - \eta}$  by

$$\psi(\lambda z_1 + (1 - \lambda)z_2) = T_{\lambda,R}\chi_{\lambda,R}, \text{ with } \lambda \in [0, 1], z_2 \in \partial B_2(z_1).$$

Let us assume by contradiction that  $\mathcal{I}_V$  does not have a critical value in  $(c_\infty, 2c_\infty)$ . Thus, one can define a continuous deformation (see Lemma 5.15 in [26])

$$\rho : \mathcal{N}_V \cap \mathcal{I}_V^{2c_\infty - \eta} \mapsto \mathcal{N}_V \cap \mathcal{I}_V^{c_\infty + \delta}$$

such that  $\rho(u) = u$  for all  $u \in \mathcal{N}_V \cap \mathcal{I}_V^{c_\infty + \delta}$ . Then the function  $h : \overline{B_2(z_1)} \rightarrow \partial B_2(z_1)$  given by

$$h(x) = 2 \left( \frac{(\beta \circ \rho \circ \psi)(x) - z_1}{|(\beta \circ \rho \circ \psi)(x) - z_1|} \right) + z_1$$

is well defined and continuous. Moreover, if  $z_2 \in \partial B_2(z_1)$ , then

$$\psi(z_2) = T_{0,R}\chi_{0,R} = T_{0,R}\omega_{2,R} \in \mathcal{N}_V \cap \mathcal{I}_V^{c_\infty + \delta},$$

and  $(\beta \circ \rho \circ \psi)(z_2) = \beta(T_{0,R}\omega_{2,R}) = z_2$ . Therefore,  $h(z_2) = z_2$  for every  $z_2 \in \partial B_2(z_1)$ . Since such a map does not exists,  $\mathcal{I}_V$  must have a critical point  $u$ . Noting that  $u^\pm$  both belong to  $\mathcal{N}_V$  so that  $\mathcal{I}_V(u^\pm) \geq \mathcal{I}_V(u)$  and at the same time

$$\mathcal{I}_V(u^+) + \mathcal{I}_V(u^-) = \mathcal{I}_V(u) = c_V \in (c_\infty, 2c_\infty)$$

allows us to conclude that  $u$  can be chosen nonnegative and by the Maximum Principle  $u$  is positive. □

**Proof of Theorem 1.1.** Theorem 1.1 immediately follows once one notices that hypothesis (14) reduces to (5) when  $\alpha = 2$  and  $N = 3, 4, 5$ .  $\square$

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