Minimax Theory and its Applications
Volume 1 (2016), No. 2, 245–256

New Hierarchical Minimax Inequalities for Non-Continuous Set-Valued Mappings

Yen-Cherng Lin
Department of Occupational Safety and Health, College of Public Health,
China Medical University, Taichung 40421, Taiwan
yclin@mail.cmu.edu.tw

Received: March 5, 2016
Accepted: July 8, 2016

We study minimax inequalities of set-valued mappings that possess hierarchical structures, and we propose two versions of minimax inequalities under topological vector space settings. We provide some examples to illustrate these theories. Our new results can be compared with existing results.

Keywords: Minimax inequalities, hierarchical minimax theorems, hierarchical structures.

2010 Mathematics Subject Classification: 49J35, 90C47

1. Introduction and Preliminaries

In many scalar and set situations, the validities that minimax inequalities hold have different conditions that involve cone convexities and semi-continuities. We refer to Zhang et al. [1, 2, 3]. Lin et al.[4] formulated minimax inequalities about noncontinuous mappings through the concept of a hierarchical structure. In this paper we prove that the minimax inequality is valid under several set-valued mappings that possess hierarchical structures. Here, the hierarchical structures of set-valued mappings mean that third condition in Theorem 1.2, or that in Theorem 2.1, or that in Theorem 2.4 holds.

We first describe the framework for the discussion of minimax inequalities under some set-valued mappings that possess hierarchical structures. Let X be a nonempty set in Hausdorff topological vector space; Z a Hausdorff topological vector space; $C \subset Z$ a closed convex and pointed cone with its apex at the origin; and $intC \neq \emptyset$. The scalar hierarchical minimax inequalities imply that, given mappings $F, S, T, H : X \times X \rightrightarrows \mathbb{R}$, under suitable conditions, the following inequality holds:

$$\min \bigcup_{x\in X} \max \bigcup_{y\in X} F(x,y) \leq \max \bigcup_{x\in X} H(x,x). \tag{s-Hi}$$

Given mappings $F, S, T, H : X \times X \rightrightarrows Z$, the first version of hierarchical minimax theorems implies that, under some suitable conditions, the following inequality holds:

$$Max \bigcup_{x\in X} H(x,x) \subset Min(co(\bigcup_{x\in X} Max_w \bigcup_{y\in X} F(x,y))) + C. \tag{Hi-1}$$

The second version of hierarchical minimax theorems implies that, under suitable conditions, the following inequality holds:

$$Max \bigcup_{x\in X} H(x,x) \subset Min \bigcup_{x\in X} Max_w \bigcup_{y\in X} F(x,y) + C. \tag{Hi-2}$$

These versions, (Hi-1) and (Hi-2), arise naturally from some minimax theorems in vector-valued or real-valued settings. We refer the reader to [4, 5, 6, 7, 8]. Note that the relation (Hi-1) used here and that used in [4] are different. Furthermore, if $Z = \mathbb{R}$ and $C = \mathbb{R}_+$, then both (Hi-1) and (Hi-2) can be reduced to (s-Hi).

The following notations will be used in the sequel. Let A be a nonempty subset of Z. A point $z \in A$ is called a *minimal point* [5, 7] (*weakly minimal point*, respectively) of A if $A \cap (z - C) = \{z\}$ ($A \cap (z - \text{int}C) = \emptyset$, respectively); $MinA$ (Min_wA, respectively) denotes the set of all minimal points(weakly minimal, respectively) of A; A point $z \in A$ is called a *maximal point* [5, 7] (*weakly maximal point*, respectively) of A if $A \cap (z + C) = \{z\}$ ($A \cap (z + \text{int}C) = \emptyset$, respectively); $MaxA$ (Max_wA, respectively) denotes the set of all maximal points (weakly maximal, respectively) of A. Note that, for a nonempty compact set A, both sets $MaxA$ and $MinA$ are nonempty. Furthermore, $MinA \subset \text{Min}_wA$, $MaxA \subset Max_wA$, $A \subset MinA + C$, and $A \subset MaxA - C$. According to [7], we denote both Max and Max_w by max (both Min and Min_w by min) in $\mathbb{R}$, because both Max and Max_w (both Min and Min_w) are the same in $\mathbb{R}$. Let U, V be Hausdorff topological spaces. A set-valued map $F : U \rightrightarrows V$ with nonempty values is said to be *upper semi-continuous on* U [9, 10] if for every $x_0 \in U$ and for every open set N containing $F(x_0)$, there exists a neighborhood M of x_0 such that $F(M) \subset N$; A set-valued map $F : U \rightrightarrows V$ with nonempty values is said to be *lower semi-continuous on* U if for every $x_0 \in U$ and for any sequence $\{x_n\} \subset U$ such that $x_n \to x_0$ and any $y_0 \in F(x_0)$, there exists a sequence $y_n \in F(x_n)$ such that $y_n \to y_0$; A set-valued map $F : U \rightrightarrows V$ with nonempty values is said to be *continuous on* U if F is upper semi-continuous as well as lower semi-continuous on U.

Let $k \in \text{int}C$ and $v \in Z$. The *Gerstewitz function* [7, 11] $\xi_{kv} : Z \to \mathbb{R}$ is defined by $\xi_{kv}(u) = \min\{t \in \mathbb{R} : u \in v + tk - C\}$. Some fundamental properties for the Gerstewitz function can be found in [7, 11].

The following types of cone-convexities for set-valued mappings are generalized from single-valued functions cases.

Definition 1.1. [5] Let each of X, Y be a nonempty convex subset of a topological vector space. A set-valued mapping $F : X \times Y \rightrightarrows Z$ is said to be

(a) *above-C-convex* (respectively, *above-C-concave*) on X if for all $x_1, x_2 \in X$

and all $\lambda \in [0,1]$,

$$F(\lambda x_1 + (1-\lambda)x_2, y) \subset \lambda F(x_1, y) + (1-\lambda)F(x_2, y) - C$$

(resp., $\lambda F(x_1, y) + (1-\lambda)F(x_2, y) \subset F(\lambda x_1 + (1-\lambda)x_2, y) - C$), $\forall y \in Y$;

(b) *above-naturally C-quasi-convex* on X if for all $x_1, x_2 \in X$ and all $\lambda \in [0,1]$,

$$F(\lambda x_1 + (1-\lambda)x_2, y) \subset co\{F(x_1, y) \cup F(x_2, y)\} - C, \ \forall y \in Y$$

where coA denotes the convex hull of a set A; and

(c) *above-C-convex-like* (respectively, *above-C-concave-like*) on X (X is not necessary convex) if for all $x_1, x_2 \in X$ and all $\lambda \in [0,1]$, there exists an $x' \in X$ such that

$$F(x', y) \subset \lambda F(x_1, y) + (1-\lambda)F(x_2, y) - C.$$

$$\text{(respectively, } \lambda F(x_1, y) + (1-\lambda)F(x_2, y) \subset F(x', y) - C\text{)}, \ \forall y \in Y.$$

The following theory possesses the hierarchical structures of a scalar minimax theorem. The proof can be written with a method similar to that used in [5]; therefore, we omit its proof here and leave the reader to prove it.

Theorem 1.2. *Let X, Y be two nonempty compact (not necessarily convex) subsets of real Hausdorff topological vector spaces, respectively. Consider the set-valued mappings $F, S, T, H : X \times Y \rightrightarrows \mathbb{R}$ with nonempty compact values such that the sets $\bigcup_{y\in Y} F(x, y)$, $\bigcup_{x\in X} H(x, y)$ are compact for all $(x, y) \in X \times Y$, and satisfy the following conditions:*

(i) *the mapping $y \mapsto H(x, y)$ is continuous on Y for each $x \in X$, the mapping $x \mapsto F(x, y)$ is lower semi-continuous for each $x \in X$ and for each $y \in Y$;*

(ii) *the mapping $x \mapsto \max S(x, y)$ is convex-like on X for each $y \in Y$, and the mapping $y \mapsto \max T(x, y)$ is concave-like on Y for each $x \in X$;*

(iii) *for each $(x, y) \in X \times Y$,*

$$\max F(x, y) \le \max S(x, y) \le \max T(x, y) \le \max H(x, y); \ \textit{and}$$

(iv) *for each $y \in Y$, there exists $x_y \in X$, such that*

$$\max H(x_y, y) \le \max \bigcup_{y\in Y} \min \bigcup_{x\in X} H(x, y);$$

then the relation

$$\min \bigcup_{x\in X} \max \bigcup_{y\in Y} F(x, y) \le \max \bigcup_{y\in Y} \min \bigcup_{x\in X} H(x, y)$$

holds.

In Theorem 1.2, the both compact sets $\bigcup_{y\in Y} F(x, y)$, $\bigcup_{x\in X} H(x, y)$ are more weaker than the continuity of the set-valued mapping. It is not difficult to find out some examples to show this fact. In the sequel, we also need the following two lemmas.

Lemma 1.3. [4] *Let* $F : X \rightrightarrows \mathbb{R}$ *be a set-valued mapping such that*

$$\max \bigcup_{x\in X} F(x),\ \max \bigcup_{x\in X} \max F(x) \text{ and } \max F(x)$$

exist for all $x \in X$*. Then* $\max \bigcup_{x\in X} F(x) = \max \bigcup_{x\in X} \max F(x)$.

We denote $Z^\star$ as the topological dual of Z and $C^\star = \{g \in Z^\star : g(c) \geq 0 \text{ for all } c \in C\}$.

Lemma 1.4. *Suppose that* $S : X \rightrightarrows \mathbb{R}$ *is a set-valued mapping and* $\max S(x)$ *exists for all* $x \in X$*.* (i) *If* S *is above-*$\mathbb{R}_+$*-convex-like, then the function* $x \mapsto \max S(x)$ *is convex-like.* (ii) *If* S *is above-*$\mathbb{R}_+$*-concave-like, then the function* $x \mapsto \max S(x)$ *is concave-like.*

Proof. Consider any $x_1, x_2 \in X$ and $\lambda \in [0,1]$. Because $\max S(x) \in S(x)$, $S(x) \subset \max S(x) - \mathbb{R}_+$, and the mapping is above-$\mathbb{R}_+$-convex-like, there exists an $x' \in X$ such that

$$\begin{array}{rcl} \max S(x') & \in & S(x') \\ & \subset & \lambda S(x_1) + (1-\lambda) S(x_2) - \mathbb{R}_+ \\ & \subset & \lambda \max S(x_1) + (1-\lambda) \max S(x_2) - \mathbb{R}_+. \end{array}$$

Hence, the function $x \mapsto \max S(x)$ is convex-like. This proves (i). The proof of (ii) is analogous to (i); therefore, we leave it to the reader. ☐

Proposition 1.5. *Let* X *be a nonempty set,* $k \in intC$ *and* $v \in Z$*. Suppose that the set-valued mappings* $F, G : X \rightrightarrows Z$ *with nonempty compact values and, for some* $x \in X$*,* $Max_w F(x) \subset Max_w G(x) - C$*. We note the following two results:*

(a) *for any* $\xi \in C^\star$*, the inequality* $\max \xi F(x) \leq \max \xi G(x)$ *holds;*

(b) *for the Gerstewitz function* $\xi_{kv} : Z \to \mathbb{R}$*, the inequality*

$$\max \xi_{kv} F(x) \leq \max \xi_{kv} G(x)$$

holds.

Proof. (a) Fix any $\xi \in C^\star$. Because $Max_w F(x) \subset Max_w G(x) - C$, for each $f \in Max_w F(x)$, there exists a $g \in Max_w G(x)$ such that $f \in g - C$. We have

$$\xi f \leq \xi g.$$

In particular,

$$\xi f = \max \xi F(x).$$

Hence,

$$\max \xi F(x) = \xi f \leq \xi g \leq \max \xi G(x).$$

The last inequality holds because $G(x)$ is nonempty compact, ξ is continuous, and $Max_w G(x) \subset G(x)$.

(b) For a similar argument as in (a), there exists $f \in Max_w F(x)$ and $g \in Max_w G(x)$ such that

$$\xi_{kv} f \leq \xi_{kv} g,$$

and

$$\xi_{kv} f = \max \xi_{kv} F(x).$$

The previous equality uses the continuity of ξ_{kv}, the compactness of $F(x)$, and Proposition 3.14 [5]. Hence, $\max \xi_{kv} F(x) \leq \max \xi_{kv} G(x)$, thus completing the proof. ☐

2. Hierarchical Minimax Inequalities

In this section, we describe two versions of hierarchical minimax inequalities. The first one is concerned about the relation (Hi-1) is as follows:

Theorem 2.1. *Let X be a nonempty compact (not necessarily convex) subset of a real Hausdorff topological space and Z be a complete locally convex Hausdorff topological vector space. Consider set-valued mappings $F, S, T, H : X \times X \rightrightarrows Z$ with nonempty compact values such that the sets $\bigcup_{y \in X} F(x,y)$, $\bigcup_{x \in X} H(x,y)$ and $\bigcup_{x \in X} H(x,x)$ are compact for all $(x,y) \in X \times X$ and satisfy the following conditions:*

- (i) *the mapping $y \mapsto H(x,y)$ is continuous on X for each $x \in X$, the mapping $x \mapsto F(x,y)$ is lower semi-continuous for each $y \in X$;*
- (ii) *the mapping $x \mapsto \xi S(x,y)$ is above-$\mathbb{R}_+$-convex-like on X for each $y \in X$, and the mapping $y \mapsto \xi T(x,y)$ is above-$\mathbb{R}_+$-concave-like on X for each $x \in X$;*
- (iii) *for each $(x,y) \in X \times X$, $Max_w F(x,y) \subset Max_w S(x,y) - C$, $Max_w S(x,y) \subset Max_w T(x,y) - C$, and $Max_w T(x,y) \subset Max_w H(x,y) - C$;*
- (iv) *for each $w \in X$, there is an $x_w \in X$ such that*

$$\max \xi H(x_w, w) \leq \max \bigcup_{y \in X} \min \bigcup_{x \in X} \xi H(x,y); \text{ and}$$

- (v) *for each $y \in X$, $Max \bigcup_{x \in X} H(x,x) \subset Min_w \bigcup_{x \in X} H(x,y) + C$,*

then the relation (Hi-1) *is valid.*

Proof. Let $\Lambda(x) = Max_w \bigcup_{y \in X} F(x,y)$, and k, ϵ, ξ be given in the same manner as in Theorem 4.5 [5] such that

$$\xi(v) \leq k - \epsilon < k \leq \xi(u + c) \tag{1}$$

holds for all $u \in co(\bigcup_{x \in X} \Lambda(x))$, $c \in C$, $\xi \in C^\star$ and

$$\xi(v) < \xi(u)$$

for all $u \in co(\bigcup_{x\in X} \Lambda(x))$. Through the use of a similar process as in Theorem 4.5 [5], we see that

$$\xi(v) < \min \bigcup_{x\in X} \max \bigcup_{y\in X} \xi F(x,y).$$

From condition (i), the mapping $y \mapsto \xi H(x,y)$ is continuous on Y for each $x \in X$, and the mapping $x \mapsto \xi F(x,y)$ is lower semi-continuous for each $x \in X$ and for each $y \in Y$. From condition (ii) and Lemma 1.4, the mapping $x \mapsto \max \xi S(x,y)$ is above-$\mathbb{R}_+$-convex-like for each $y \in Y$, and $y \mapsto \max \xi T(x,y)$ is above-$\mathbb{R}_+$-concave-like on Y for each $x \in X$. Combining Proposition 1.5 with (iii) and (iv), we determine that all conditions of Theorem 1.2 hold for the mappings $\xi F, \xi S, \xi T$ and ξH. Hence, according to Theorem 1.2, we determine that

$$\xi(v) < \max \bigcup_{y\in X} \min \bigcup_{x\in X} \xi H(x,y). \tag{2}$$

Because X is compact, there exists a $y' \in X$ such that

$$\xi(v) < \min \bigcup_{x\in X} \xi H(x,y').$$

Therefore,

$$v \notin \bigcup_{x\in X} H(x,y') + C,$$

and hence,

$$v \notin Min_w \bigcup_{x\in X} H(x,y') + C. \tag{3}$$

If $v \in Max \bigcup_{x\in X} H(x,x)$, then, according to (v), we have

$$v \in Min_w \bigcup_{x\in X} H(x,y') + C$$

which contradicts (3). Hence, the relation (Hi-1) is valid. □

Although the conclusions of Theorem 2.1 and Theorem 3.1 [4] are the same, they are used under highly different conditions. Furthermore, they involve the use of relatively different proof techniques.

Corollary 2.2. *The relation* (Hi-1) *remains valid if we replace a sub-condition corresponding to S or T in Theorem* 2.1(ii) *with any of the following sub-conditions:*

(i) *the mapping $x \mapsto S(x,y)$ is above-C-convex on X for each $y \in X$, where X is convex;*

(ii) *the mapping $x \mapsto S(x,y)$ is above-C-convex-like on X for each $y \in X$;*

(iii) *the mapping $x \mapsto \xi S(x,y)$ is above-$\mathbb{R}_+$-convex on X for each $y \in X$, where X is convex;*

(iv) *the mapping $y \mapsto T(x,y)$ is above-C-concave on X for each $x \in X$, where X is convex; or*

(v) *the mapping* $y \mapsto \xi T(x,y)$ *is above-*$\mathbb{R}_+$*-concave on* X *for each* $x \in X$*, where* X *is convex.*

Proof. In order to make this proof complete, we must evaluate the following steps: Step 1. By definition, if we choose $x' = \lambda x_1 + (1-\lambda)x_2$, we can determine that (i) $\Rightarrow$ (ii) holds. Sept 2. According to Proposition 3.9 [5], (i) $\Rightarrow$ (iii) is valid. Step 3. According to Proposition 3.8 [5], (ii) implies that condition (ii) of Theorem 2.1 for the mapping $\xi S(x,y)$ is true. Step 4. The claim is trivially valid for (iii) $\Rightarrow$ condition (ii) of Theorem 2.1 for the mapping $\xi S(x,y)$. Step 5. According to Proposition 3.9 [5], (iv) $\Rightarrow$ (v) is valid. Step 6. The claim is obviously valid for (v) $\Rightarrow$ condition (ii) of Theorem 2.1 for the mapping $\xi T(x,y)$. Hence, we complete the proof. ☐

The following example illustrates that Theorem 2.1 is true.

Example 2.3. Let $X = [0,1]$, $C = \mathbb{R}^2_+$ and $\varphi : X \mapsto \mathbb{R}$ be defined by

$$\varphi(x) = \begin{cases} [-1,0], & x \neq 0, \\ \{0\}, & x = 0. \end{cases}$$

Define $F, S, T, H : X \times X \mapsto \mathbb{R}^2$ by

$$F(x,y) = \varphi(x) \times \{-\tan(y\pi/4)\},$$

$$S(x,y) = [-1/2, 1/2 + x^2(1-y^2)] \times \{1 - \tan(y\pi/4)\},$$

$$T(x,y) = [0, 1 + x^2(1-y^2)] \times \{2 - \tan(y\pi/4)\},$$

$$H(x,y) = [1, 1+x^2] \times [2, 3 - \tan(y\pi/4)].$$

for all $(x,y) \in X \times X$.

Obviously, conditions (i) and (iii) of Theorem 2.1 are valid. We can easily observe that the mapping $x \mapsto S(x,y)$ is above-C-convex on X for each $y \in X$ and that the mapping $y \mapsto T(x,y)$ is above-C-concave on X for each $x \in X$; therefore, by Corollary 2.2, condition (ii) holds. We now claim that condition (iv) holds. Indeed, for each $\xi = (\xi_1, \xi_2) \in C^\star$, because

$$\xi H(x,y) = \{\xi_1 s + \xi_2(3-t) : s \in [1, 1+x^2], t \in [2, 3 - \tan(y\pi/4)]\},$$

$$\max \bigcup_{y \in X} \min \bigcup_{x \in X} \xi H(x,y) = \xi_1 + 3\xi_2.$$

For each $w \in X$, we choose $x_w \in X$ through the following strategy:

$$x_w = \begin{cases} \varsigma \text{ is an arbitrary number in } X, & \text{if } \xi_1 = 0, \\ \varsigma \text{ is an arbitrary number in } [0, \sqrt{\xi_2/\xi_1 \tan(w\pi/4)}] \cap X, & \text{if } \xi_1 > 0. \end{cases}$$

Then

$$\max \xi H(x_w, w) \leq \max \bigcup_{y\in X} \min \bigcup_{x\in X} \xi H(x,y),$$

and hence, condition (iv) is valid. Finally, because

$$Max \bigcup_{x\in X} H(x,x) = \{(1+t^2, 3-\tan(t\pi/4) : t\in[0,1]\}$$

and

$$Max_w \bigcup_{x\in X} H(x,y) = (\{1\}\times[2, 3-\tan(y\pi/4)]) \bigcup ([1,2]\times\{2\}),$$

we see that condition (v) holds. Therefore, according to Theorem 2.1, the relation (Hi-1) is valid. Indeed, we can deduce that

$$\bigcup_{x\in X} Max_w \bigcup_{y\in X} F(x,y) = ([-1,0]\times\{0\}) \bigcup (\{0\}\times[-1,0]),$$

hence

$$co(\bigcup_{x\in X} Max_w \bigcup_{y\in X} F(x,y)) = \{(s,t) : -1-s \leq t \leq 0, -1\leq s\leq 0\}.$$

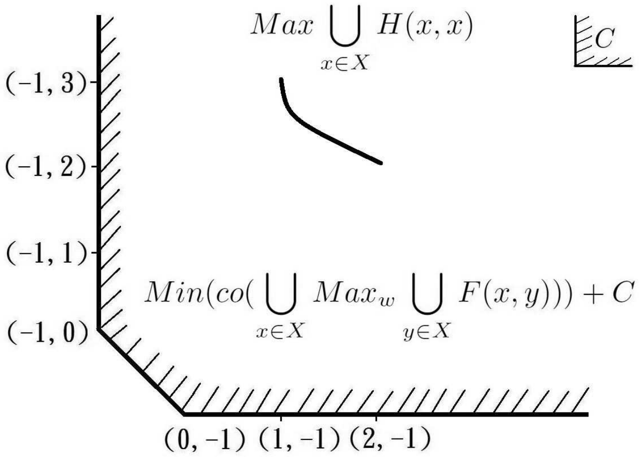

Figure 1: $Max \bigcup_{x\in X} H(x,x) \subset Min(co(\bigcup_{x\in X} Max_w \bigcup_{y\in X} F(x,y))) + C$

Thus,

$$Min(co(\bigcup_{x\in X} Max_w \bigcup_{y\in X} F(x,y))) + C = \{(x,y)\in\mathbb{R}^2 : x\geq -1, x+y\geq -1, y\geq -1\}.$$

Therefore, as shown in Figure 1, the relation (Hi-1) is valid. □

Theorem 2.4. *Let X be a nonempty compact (not necessarily convex) subset of a real Hausdorff topological space and, Z be a Hausdorff topological vector space. Consider set-valued mappings $F, S, T, H : X\times X \rightrightarrows Z$ with nonempty compact values such that the sets $\bigcup_{y\in X} F(x,y)$, $\bigcup_{x\in X} H(x,y)$ and $\bigcup_{x\in X} H(x,x)$ are compact for all $(x,y)\in X\times X$, and satisfy the following conditions:*

(i) *the mapping* $y \mapsto H(x,y)$ *is continuous on* X *for each* $x \in X$, *and the mapping* $x \mapsto F(x,y)$ *is lower semi-continuous for each* $y \in X$;

(ii) *given any Gerstewitz function* ξ_{kv} *with* $v \notin \bigcup_{x\in X} Max_w \bigcup_{y\in X} F(x,y) + C$ *satisfies the following conditions:*

  (ii$_a$) *the mapping* $x \mapsto \xi_{kv}S(x,y)$ *is above-*$\mathbb{R}_+$*-convex-like on* X *for each* $y \in X$, *and the mapping* $y \mapsto \xi_{kv}T(x,y)$ *is above-*$\mathbb{R}_+$*-concave-like on* X *for each* $x \in X$;

  (ii$_b$) *for each* $w \in X$, *there exists an* $x_w \in X$ *such that*

$$\max \xi_{kv}H(x_w,w) \le \max \bigcup_{y\in X} \min \bigcup_{x\in X} \xi_{kv}H(x,y); \text{ and}$$

(iii) *for each* $(x,y) \in X \times X$, $Max_w F(x,y) \subset Max_w S(x,y) - C$, $Max_w S(x,y) \subset Max_w T(x,y) - C$, *and* $Max_w T(x,y) \subset Max_w H(x,y) - C$;

(iv) *for each* $y \in X$,

$$Max \bigcup_{x\in X} H(x,x) \subset Min_w \bigcup_{x\in X} H(x,y) + C,$$

then the relation (Hi-2) *is valid.*

Proof. Let $\Lambda(x)$ be defined in the same manner as in Theorem 2.1 for all $x \in X$. From the process involved in the proof of Theorem 2.1, we determine that the set $\bigcup_{x\in X} \Lambda(x)$ is nonempty compact. Assume that $v \notin \bigcup_{x\in X} \Lambda(x) + C$. For any $k \in intC$, there exists a Gerstewitz function $\xi_{kv} : Z \mapsto \mathbb{R}$ such that

$$\xi_{kv}(u) > 0 \tag{4}$$

for all $u \in \bigcup_{x\in X} \Lambda(x)$. Then, for each $x \in X$, there exists $y_x^\star \in X$ and $f(x,y_x^\star) \in F(x,y_x^\star)$ with $f(x,y_x^\star) \in Max_w \bigcup_{y\in X} F(x,y)$ such that

$$\xi_{kv}(f(x,y_x^\star)) = \max \bigcup_{y\in X} \xi_{kv}F(x,y).$$

Choosing $u = f(x,y_x^\star)$ in equation (4), we have

$$\max \bigcup_{y\in X} \xi_{kv}F(x,y) > 0$$

for all $x \in X$. Therefore,

$$\min \bigcup_{x\in X} \max \bigcup_{y\in X} \xi_{kv}F(x,y) > 0.$$

According to conditions (i)-(iii), we determine that all conditions of Theorem 1.2 hold for the mappings $\xi_{kv}F(x,y), \xi_{kv}S(x,y), \xi_{kv}T(x,y)$ and $\xi_{kv}H(x,y)$, and hence,

$$\max \bigcup_{y\in X} \min \bigcup_{x\in X} \xi_{kv}H(x,y) > 0.$$

Because X is compact, there exists a $y' \in X$ such that

$$\min \bigcup_{x\in X} \xi_{kv} H(x,y') > 0.$$

Thus,

$$v \notin \bigcup_{x\in X} H(x,y') + C,$$

and hence,

$$v \notin Min_w \bigcup_{x\in X} H(x,y') + C. \tag{5}$$

If $v \in Max \bigcup_{x\in X} H(x,x)$, then, according to (iv), we have

$$v \in Min_w \bigcup_{x\in X} H(x,y') + C$$

which contradicts (5). Therefore, we can deduce that the relation (Hi-2) is valid.

□

The following example illustrates that Theorem 2.4 is true.

Example 2.5. Let X, C, φ, F be defined in the same manner as in Example 2.3. Define $S, T, H : X \times X \mapsto \mathbb{R}^2$ by

$$S(x,y) = [3/2, 2 + x^2(1-y^2)] \times \{1 - \tan(y\pi/4)\},$$

$$T(x,y) = [5/2, 5/2 + x^2(1-y^2)] \times \{2 - \tan(y\pi/4)\},$$

$$H(x,y) = [7/2, 7/2 + x^2] \times [2, 3 - \tan(y\pi/4)].$$

for all $(x,y) \in X \times X$.

Obviously, conditions (i) and (iii) of Theorem 2.4 are valid. Let $v = (-1/2, -1) \notin \bigcup_{x\in X} Max_w \bigcup_{y\in X} F(x,y) + C$ and $k = (1,1) \in intC$. Using a similar calculation to the calculation in [12], we have $\xi_{kv} S(x,y) = [2, 5/2 + x^2(1-y^2)]$, and $\xi_{kv} T(x,y) = [3, 3 + x^2(1-y^2)]$ for $(x,y) \in X \times X$. Therefore, the mapping $x \mapsto \xi_{kv} S(x,y)$ is above-$\mathbb{R}_+$-convex on X for each $y \in X$ and the mapping $y \mapsto \xi_{kv} T(x,y)$ is above-$\mathbb{R}_+$-concave on X for each $x \in X$. According to Corollary 2.2, condition (ii$_a$) holds. We now claim that condition (ii$_b$) holds. Indeed, for each $(x,y) \in X \times X$, $\xi_{kv} H(x,y) = [4, 4+x^2]$, $\max \bigcup_{y\in X} \min \bigcup_{x\in X} \xi_{kv} H(x,y) = 4$. For each $w \in X$, we choose $x_w = 0$; then, condition (ii$_b$) of Theorem 2.4 is valid. Finally, because

$$Max \bigcup_{x\in X} H(x,x) = \{(7/2 + t^2, 3 - \tan(t\pi/4) : t \in [0,1]\}$$

and

$$Max_w \bigcup_{x\in X} H(x,y) = (\{7/2\} \times [2, 3 - \tan(y\pi/4)]) \bigcup ([7/2, 9/2] \times \{2\}),$$

we determine that condition (vi) holds. Therefore, according to Theorem 2.4, the relation (Hi-2) is valid. Indeed, from Example 2.3,

$$\bigcup_{x\in X} Max_w \bigcup_{y\in X} F(x,y) = ([-1,0]\times\{0\})\bigcup(\{0\}\times[-1,0]),$$

and

$$Min \bigcup_{x\in X} Max_w \bigcup_{y\in X} F(x,y) + C = \{(-1,0),(0,-1)\} + C.$$

Hence, as shown in Figure 2, relation (Hi-2) is valid. ☐

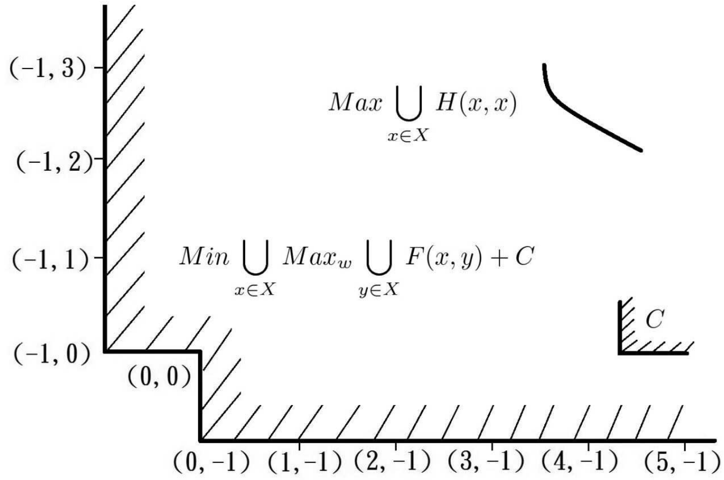

Figure 2: $Max \bigcup_{x\in X} H(x,x) \subset Min \bigcup_{x\in X} Max_w \bigcup_{y\in X} F(x,y) + C$

Acknowledgements The support of this work by Grant No. MOST104-2115-M-039-001- of the Ministry of Science and Technology of Taiwan (Republic of China) is gratefully acknowledged.

References

[1] Y. Zhang, S. J. Li: *Minimax theorems for scalar set-valued mappings with non-convex domains and applications*, J. Global Optimization 57 (2013) 1359–1373; doi:10.1007/s10898-012-9992-2.

[2] Y. Zhang, S. J. Li: *Ky Fan minimax inequalities for set-valued mappings*, Fixed Point Theory Appl. 2012 (2012): doi:10.1186/1687-1812-2012-64.

[3] Y. Zhang, S. J. Li, M. H. Li: *Minimax inequalities for set-valued mappings*, Positivity 16 (2012) 751–770; doi:10.1007/s11117-011-0144-6.

[4] Y. C. Lin, C.-T. Pang, C.-T: *The hierarchical minimax inequalities for set-valued mappings*, Abstract and Appl. Analysis (2014); doi:10.1155/2014/190821.

[5] Y. C. Lin, Q. H. Ansari, H. C. Lai: *Minimax theorems for set-valued mappings under cone-convexities*, Abstract and Appl. Analysis (2012); doi:10.1155/2012/310818.

[6] L. C. Zeng, S. Y. Wu, J.-C. Yao: *Generalized KKM theorem with applications to generalized minimax inequalities and generalized equilibrium problems*, Taiwanese J. Math. 10 (2006) 1497–1514.

[7] S. J. Li, G. Y. Chen, K. L. Teo, X. Q. Yang: *Generalized minimax inequalities for set-valued mappings*, Journal of Math. Analysis and Appl. 281 (2003) 707–723.

[8] F. Ferro: *Optimization and stability results through cone lower semi-continuity*, Set-Valued Analysis 5 (1997) 365-375.

[9] C. Berge: *Topological spaces*, Macmillan, New York (1963).

[10] J. P. Aubin, A. Cellina: *Differential inclusions*, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo (1984).

[11] C. Gerth, P. Weidner: *Nonconvex separation theorems and some applications in vector optimization*, Journal of Optimization Theory Appl. 67 (1990) 297-320.

[12] Y. C. Lin: *Bilevel minimax theorems for non-continuous set-valued mappings*, Journal of Inequalities and Appl. 2014:182 (2014).