Minimax Theory and its Applications
Volume 7 (2022), No. 1, 57–78

Optimality and Duality for Weak Quasi Efficiency of Multiobjective Fractional Problems via Convexificators

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Received: November 4, 2020
Accepted: January 11, 2021

Fritz John and Kuhn-Tucker necessary conditions for weak quasi-efficiency of multiobjective fractional optimization problems with equality, inequality and set constraints are derived. Under asumptions on asymptotic pseudoinvexity of the objective and asymptotic quasiinvexity of constraint functions, sufficient conditions for weak quasi-efficiency are also given together with duality theorems of Wolfe and Mond-Weir types.

Keywords: Multiobjiective fractional problem, local weak quasi-efficient solution, Fritz John and Kuhn-Tucker efficiency conditions.

2010 Mathematics Subject Classification: 90C46, 90C29, 49J52.

1. Introduction

Optimality conditions and duality in multiobjective fractional programming have been of much interest in the recent past. There are a large number of articles on optimality conditions for fractional programs. Liang et al. [12] derived sufficient optimality conditions and duality for single fractional programming problems. Kuk et al. [11] established necessary and sufficient optimality conditions for nonsmooth multiobjective fractional programming problems with inequality constraints via the Clarke subdifferentiabls. Gadhi [4] derived optimality conditions for fractional multiobjective problems involving inequality constraints in terms of convexificators. Recently, Chuong et al. [2] established optimality conditions for nonsmooth fractional multiobjective optimization problems with inequality constraints, while Hong et al. [8] derived optimality conditions for nonsmooth fractional robust multiobjective optimization problems.

This paper derives necessary and sufficient conditions for weak quasi- efficiency in multiobjective fractional problems with equality, inequality and set constraints.

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ISSN 2199-1413 (printed), ISSN 2199-1421 (electronic) / $ 2.50

Studying the approximate solutions is useful, because in algorithms for solving optimization problems after a finite number of steps one gets only approximate solutions. The notion of approximate efficient solutions may be given flexible by making the error dependency on the decision variables. In [6], Golestani et al. established Kuhn-Tucker and strong Kuhn-Tucker necessary optimality conditions for weak quasi- efficiency via Clarke subdifferentials. Capătă [1] established necessary and sufficient conditions for a point to be an ϵ-quasi solution of a scalar optimization problem via ϵ-convexificators.

In recent years, many authors have studied optimality conditions for vector optimization problems and obtained Kuhn-Tucker optimality conditions via Lagrange multipliers rules (see, e.g., [1]–[5], [7], [11]–[19]). The notion of nonconvex closed convexificator by Jeyakumar-Luc [10] is a generalization of some notions of known subdifferentials such as the subdifferentials of Clarke [3], Michel-Penot [17], Mordukhovich [18]. It has provided good calculus rules for establishing necessary efficiency conditions in nonsmooth optimization. Necessary conditions for efficiency via convexificators may be sharper than those expressed in terms of known subdifferentials as the Clarke, Michel-Penot subdifferentials.

The paper is organized as follows. Section 2 gives some notions and preliminaries. Section 3 is devoted to derive Fritz John necessary optimality conditions for fractional multiobjective problems involving equality, inequality and set constraints via convexificators. Section 4 deals with Kuhn-Tucker necessary conditions for weak quasi-efficiency under a constraint qualification of Mangasarian-Fromovitz type. In Section 5, under assumptions on asymptotic pseudoconvexity of the objective and asymptotic quasiconvexity of constraint functions, sufficient optimality conditions are given as well. Section 6 gives theorems of weak duality, strong duality, and inverse duality of Wolfe and Mond-Weir types.

2. Preliminaries

Given a Banach space X, we denote by X^* its topological dual with the canonical dual pairing $\langle .,.\rangle$, and w^* denotes the weak* topology on the dual space X^*. For an extended-real-valued function $f : X \to \mathbb{R} \cup \{+\infty\}$, the expressions

$$f^-(x;v) = \liminf_{t\downarrow 0} \frac{f(x+tv)-f(x)}{t} \quad \text{and} \quad f^+(x;v) = \limsup_{t\downarrow 0} \frac{f(x+tv)-f(x)}{t}.$$

stand for the *lower* and *upper Dini directional derivatives* of f at x in the direction of v, respectively.

Consider the following multiobjective fractional optimization problem:

$$\text{(MFP)} \qquad \begin{cases} \min\Big(\dfrac{f_1(x)}{g_1(x)},...,\dfrac{f_p(x)}{g_p(x)}\Big), \quad \text{subject to} \\ h_j(x) \leqslant 0,\ j \in I := \{1,...,m\}, \\ \ell_k(x) = 0,\ k \in L := \{1,...r\},\ x \in D, \end{cases}$$

where D is a closed subset of X, f_i, g_i, h_j and ℓ_k are continous functions defined on X such that $g_i(x) > 0$ for all $i = 1, \dots, p$ and $x \in X$.

Denote by M the *feasible set* of (MFP)

$$M := \left\{ x \in D : \begin{array}{ll} h_j(x) \leq 0 & \text{for all } j = 1, ..., m, \text{ and} \\ \ell_k(x) = 0 & \text{for all } k = 1, \ldots, r \end{array} \right\},$$

and for $\overline{x} \in M$, $I(\overline{x}) := \{j \in I : h_j(\overline{x}) = 0\}$. We set

$$\phi(x) := \Big(\frac{f_1(x)}{g_1(x)}, \ \ldots \ , \frac{f_p(x)}{g_p(x)}\Big).$$

A point $\overline{x} \in M$ is said to be a *local efficient* (resp., *local weak efficient*) solution for (MFP) if there exists a neighborhood V of $\overline{x}$ such that there is no $x \in V \cap M$ with

$$\phi_i(x) \leqslant \phi_i(\overline{x}) \ (i = 1, \ldots, p),$$
$$\phi_s(x) < \phi_s(\overline{x}) \text{ at least one } s \in \{1, \ldots, p\},$$
$$(\text{resp., } \phi_i(x) < \phi_i(\overline{x}) \ (i = 1, \ldots, p)).$$

In case $V = X$ we obtain the notions of efficient and weak efficient solutions.

Recall [6] that a point $\overline{x} \in M$ is said to be a *local quasi-efficient* (resp., *local weak quasi efficient*) *solution* for (MFP) if there exist a number $\alpha := (\alpha_1, \ldots, \alpha_p) \in \mathbb{R}^p_{++}$ and a neighborhood V of $\overline{x}$ such that there is no $x \in V \cap M$ satisfying

$$\phi_i(x) \leqslant \phi_i(\overline{x}) - \alpha_i||x - \overline{x}|| \quad (i = 1, \ldots, p),$$
$$\phi_s(x) < \phi_s(\overline{x}) - \alpha_s||x - \overline{x}|| \text{ at least one } s \in \{1, \ldots, p\}$$
$$(\text{resp., } \phi_i(x) < \phi_i(\overline{x}) - \alpha_i||x - \overline{x}|| \quad (i = 1, \ldots, p)).$$

where $\mathbb{R}_{++} = \text{int}\mathbb{R}_+$. In case $V = X$ we obtain the notions of *efficient* and *weak quasi-efficient solutions.*

Note that a local efficient (local weak efficient) solution is also a local quasi-efficient (local weak quasi efficient) solution, but the converse relation is not true in general.

Recall [9] that a point $\overline{x}$ is said to be a *regular point* in the sense of Ioffe for ℓ relative to D if there exist numbers $K > 0$ and $\delta > 0$ such that for all $x \in D \cap B(\overline{x}; \delta)$,

$$d(x, Q) \leqslant K||\ell(x) - \ell(\overline{x})||,$$

where $Q = \{x \in D : \ell(x) = \ell(\overline{x})\}$, $d(x, Q)$ denotes the distance from x to Q, $B(\overline{x}; \delta)$ stands for the open ball of radius δ around $\overline{x}$ (see [16]). Note that for problem (MFP) without equality constraint, a local weak efficient solution of (MFP) is also a regular point in the sense of Ioffe.

Recall [10] that the function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to have a *convexificator* $\partial^* f(x)$ at x if $\partial^* f(x)$ ($\subset X^*$) is weak* closed, and for each $v \in X$,

$$f^-(x, v) \leqslant \sup_{x^* \in \partial^* f(x)} \langle x^*, v\rangle, \text{ and } f^+(x, v) \geqslant \inf_{x^* \in \partial^* f(x)} \langle x^*, v\rangle.$$

Note that convexificators are not necessarily weak* compact or convex. These relaxations allow applications to a large class of nonsmooth continuous functions.

For instance, the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sqrt{x}, & \text{if } x \geqslant 0, \\ -\sqrt{-x}, & \text{if } x < 0, \end{cases}$$

admits noncompact convexificators at 0 of the form $[\alpha, \infty)$ with $\alpha \in \mathbb{R}$.

Remark 2.1. The convex hull of a convexificator of a locally Lipschitz function may be strictly contained in both the Clarke and Michel-Penot subdifferentials (see [10]). Hence, necessary optimality conditions that are expressed in terms of $\partial^* f(x)$ may provide sharp conditions even for locally Lipschitz functions. For example,

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then $\partial^C f(0) = \partial^{MP} f(0) = [-1, 1]$ (Example 2.2.3 in [3]), where ∂^C and ∂^{MP} indicate the Clarke and Michel-Penot subdifferentials, respectively. The set $\{-1; 1\}$ is a convexificator of f at $\overline{x} = 0$. Then, $\{-1; 1\} \subsetneqq [-1, 1]$. □

The chain rule for composite functions in [10] is needed in the following.

Proposition 2.2. *Let $f = (f_1, ..., f_n)$ be a continuous function from X to $\mathbb{R}^n$, and let g be continuous function from $\mathbb{R}^n$ to $\mathbb{R}$. Suppose that, for each $i = 1, \ldots, n$, f_i admits a bounded convexificator $\partial^* f_i(\overline{x})$ at $\overline{x}$ and that g admits a bounded convexificator $\partial^* g(f(\overline{x}))$ at $f(\overline{x})$. For each $i = 1, ..., n$, if $\partial^* f_i$ is upper semicontinuous at $\overline{x}$ and $\partial^* g$ is upper semicontinuous at $f(\overline{x})$, then the set*

$$\partial^* g(f(\overline{x}))(\partial^* f_1(\overline{x}), ..., \partial^* f_n(\overline{x}))$$

is a convexificator of $g \circ f$ at $\overline{x}$.

From Proposition 2.2, we obtain the following

Proposition 2.3. *Let f, g be continuous functions defined on a set $D \subset X$, $g(x) > 0$ $(\forall x \in D)$. Assume that f, g admit bounded convexificators $\partial^* f(\overline{x})$, $\partial^* g(\overline{x})$ at $\overline{x} \in D$, respectively; $\partial^* f, \partial^* g$ are upper semicontinuous at $\overline{x}$. Then the set*

$$\frac{g(\overline{x})\partial^* f(\overline{x}) - f(\overline{x})\partial^* g(\overline{x})}{g^2(\overline{x})}$$

is a convexificator of f/g at $\overline{x}$.

Proof. We define the functions $F : X \to \mathbb{R}^2, G : \mathbb{R}^2 \to \mathbb{R}$ as follows:

$$G(u_1, u_2) = \frac{u_1}{u_2}, \quad F(x) = (f(x), g(x)).$$

Then $f/g = G \circ F$. Taking account of Proposition 2.2, we deduce that the set

$$\frac{g(\overline{x})\partial^* f(\overline{x}) - f(\overline{x})\partial^* g(\overline{x})}{g^2(\overline{x})}$$

is a convexificator of f/g at $\overline{x}$. □

For a set $D \subset X$, the *Clarke tangent cone* to D at $\overline{x}$ is defined as

$$T(D;\overline{x}) = \left\{ v \in X : \begin{array}{l} \forall x_n \in D,\ x_n \to \overline{x},\ \forall t_n \downarrow 0,\ \exists v_n \to v \\ \text{such that } x_n + t_n v_n \in D,\ \forall n \end{array} \right\}.$$

The *Clarke normal cone* to D at $\overline{x}$ is

$$N(D;\overline{x}) = \big\{ \xi \in X^* : \langle \xi, v \rangle \leqslant 0, \forall v \in T(D;\overline{x}) \big\}.$$

3. Fritz John necessary efficient conditions

To derive Kuhn-Tucker necessary optimality conditions in terms of convexificators for (MFP), we first begin with establishing a Fritz John necessary efficiency condition for (MFP).

Theorem 3.1. *Let $\overline{x} \in M$ be a local weak quasi-efficient solution with respect to $\alpha := (\alpha_1, \ldots, \alpha_p) \in \mathbb{R}_{++}$ of* (MFP)*. Suppose that $\overline{x}$ is a regular point in the sense of Ioffe for ℓ relative to D. Assume that f_i, g_i, h_j, ℓ_k are continuous and admit convexificators $\partial^* f_i(x)$, $\partial^* g_i(x)$, $\partial^* h_j(x)$, $\partial^* \ell_k(x)$ at x near $\overline{x}$, respectively. They admit bounded convexificators $\partial^* f_i(\overline{x})$, $\partial^* g_i(\overline{x})$, $\partial^* h_j(\overline{x})$, $\partial^* \ell_k(\overline{x})$ at $\overline{x}$, where the $|\ell_k|$ $(k \in L)$ are regular in the sense of Clarke. Moreover, $\partial^* f_i$, $\partial^* g_i$,$\partial^* h_j$, $\partial^* \ell_k$ are upper semicontinuous at $\overline{x}$. Then there exists vector $(\theta_1^*, ..., \theta_p^*) \in \mathbb{R}^p_+ \setminus \{0\}$, $(\mu_0^*, \mu_1^*, ..., \mu_m^*) \in \mathbb{R}^{m+1}_+$, and $(\nu_1^*, ..., \nu_r^*) \in \mathbb{R}^r$ such that*

$$0 \in cl\Big(\mu_0^* \sum_{i=1}^{p} \theta_i^* \left(conv \partial^* f_i(\overline{x}) - \phi_i(\overline{x}) conv \partial^* g_i(\overline{x}) + \alpha_i g_i(\overline{x}) \mathbb{B} \right)$$
$$+ \sum_{j=1}^{m} \mu_j^* conv\ \partial^* h_j(\overline{x}) + \sum_{k=1}^{r} \nu_k^* conv\ \partial^* \ell_k(\overline{x}) + N(D;\overline{x}) \Big), \tag{1}$$

$$\mu_0^* + \sum_{j=1}^{m} \mu_j^* = 1, \quad \mu_j^* h_j(\overline{x}) = 0, \quad j = 1, ..., m, \tag{2}$$

where $\mathbb{B}$ is closed unit ball in X, cl indicates the weak closure, conv stands for the convex hull.*

Proof. Since $\overline{x} \in M$ is a local weak quasi-efficient solution of (MFP), there exists a neighborhood U of $\overline{x}$ such that

$$\phi(x) - \phi(\overline{x}) + \alpha ||x - \overline{x}|| \in \mathbb{R}^p \setminus -\mathbb{R}^p_{++}, \tag{3}$$

for all $x \in U \cap M$. Then $\overline{x}$ is a local weak minimal solution of the following problem:

$$(\mathrm{P}_1) \qquad \begin{cases} \min\ (\widetilde{\phi}_1(x), ..., \widetilde{\phi}_p(x)|, \\ \text{subject to } \ x \in M, \end{cases} \tag{4}$$

where $\widetilde{\phi}_i(\overline{x}) = \frac{f_i(\overline{x})}{g_i(\overline{x})} + \alpha_i ||x - \overline{x}||$ $(i = 1, \ldots, p)$.

Applying the scalarization theorem by Gong ([7], Theorem 3.1) to Problem (P_1) yields the existence of a continuous positively homogeneous subadditive function Λ on X satisfying

$$y_2 - y_1 \in \mathbb{R}^p_{++} \Longrightarrow \Lambda(y_1) < \Lambda(y_2),$$

and
$$\Lambda(\widetilde{\phi}(x) - \widetilde{\phi}(\overline{x})) \geqslant 0 \ (\forall x \in U \cap M),$$

where $\widetilde{\phi}(x) := (\widetilde{\phi}_1(x), \ldots, \widetilde{\phi}_p(x))$. Hence,

$$\Lambda(\widetilde{\phi}(x) - \widetilde{\phi}(\overline{x})) \geqslant 0 = \Lambda(\widetilde{\phi}(\overline{x}) - \widetilde{\phi}(\overline{x})) \ \ (\forall x \in U \cap M). \tag{5}$$

It follows from (5) that $\overline{x}$ is a local minimal solution of the following scalar problem:

$$(\mathrm{P}_2) \quad \begin{cases} \text{minimize } \Lambda(\phi_1(x) - \phi_1(\overline{x}) + \alpha_1||x - \overline{x}||, ..., \phi_p(x) - \phi_p(\overline{x}) + \alpha_p||x - \overline{x}||), \\ \text{subject to } x \in M. \end{cases}$$

By assumption, all hypotheses of Theorem 6.4 [16] hold. We invoke Theorem 6.4 [16] to deduce the existence of $\mu_0^* \geqslant 0$, $\mu_j^* \geqslant 0$, $j = 1, ..., m$, $\nu_k^* \in \mathbb{R}$, $k = 1, ..., r$ with

$$\mu_0^* + \sum_{i=1}^{m} \mu_j^* = 1, \quad \text{and}$$

$$\begin{aligned} 0 \in \mathrm{cl}\Big(&\mu_0^* \mathrm{conv}\, \partial^* \Lambda(\phi_1() - \phi_1(\overline{x}) + \alpha_1||. - \overline{x}||, ..., \phi_p(.) - \phi_p(\overline{x}) + \alpha_p||. - \overline{x}||)(\overline{x}) \\ &+ \sum_{j \in I(\overline{x})} \mu_j^* \mathrm{conv}\ \partial^* h_j(\overline{x}) + \sum_{k=1}^{r} \nu_k^* \mathrm{conv}\ \partial^* \ell_k(\overline{x}) + N(D; \overline{x})\Big), \end{aligned}$$

which is equivalent to

$$\begin{aligned} 0 \in \mathrm{cl}\Big(&\mu_0^* \mathrm{conv}\, \partial^* \Lambda(\phi_1(.) - \phi_1(\overline{x}) + \alpha_1||. - \overline{x}||, ..., \phi_p(.) - \phi_p(\overline{x}) + \alpha_p||. - \overline{x}||)(\overline{x}) \\ &+ \sum_{j=1}^{m} \mu_j^* \mathrm{conv}\ \partial^* h_j(\overline{x}) + \sum_{k=1}^{r} \nu_k^* \mathrm{conv}\ \partial^* \ell_k(\overline{x}) + N(D; \overline{x})\Big), \ \mu_j^* h_j(\overline{x}) = 0, \end{aligned} \tag{6}$$

where $j = 1, \ldots, m$. Note that Λ is a continuous convex function on $\mathbb{R}^p$, and so, it is locally Lipschitz on $\mathbb{R}^p$. Therefore, we can choose the convex subdifferential $\partial\Lambda(0)$ as a convexificator of Λ at

$$(\phi_1(\overline{x}) - \phi_1(\overline{x}) + \alpha_1||\overline{x} - \overline{x}||, \ldots, \varphi_p(\overline{x} - \overline{x}) + \alpha_p||\overline{x} - \overline{x}||) = (0, \ldots, 0).$$

Then $\partial\Lambda(0)$ is a bounded convexificator of Λ at $(0, \ldots, 0)$. Moreover, the mapping $\partial\Lambda$ is upper semicontinuous at $(0, \ldots, 0)$, since Λ is locally Lipschitz on $\mathbb{R}^p$. Observe that the function $\alpha_i||. - \overline{x}||$ is Lipschitz, and its Clarke subdifferential at $\overline{x}$ is $\alpha_i\mathbb{B}$, and thus a its convexificator can be taken as $\alpha_i\mathbb{B}$. Taking account of Proposition 2.2 to the composite function

$$\Lambda(\phi_1(.) - \phi_1(\overline{x}) + \alpha_1||. - \overline{x}||, \ldots, \phi_p(.) - \phi_p(\overline{x}) + \alpha_p||. - \overline{x}||)(.),$$

one gets that $\partial\Lambda(0)(\partial^*\phi_1(\overline{x}) + \alpha_1\mathbb{B}, ..., \partial^*\phi_p(\overline{x}) + \alpha_p\mathbb{B})$ is a convexificator of

$$\Lambda(\phi_1(.) - \phi_1(\overline{x}) + \alpha_1||. - \overline{x}||, \ldots, \phi_p(.) - \phi_p(\overline{x}) + \alpha_p||. - \overline{x}||)(.)$$

at $\overline{x}$, where $\partial\Lambda(0)$ is the convex subdifferential of Λ at 0.

We show that the set

$$\frac{g_i(\overline{x})\partial^* f_i(\overline{x}) - \phi_i(\overline{x})\partial^* g_i(\overline{x})}{g_i^2(\overline{x})} + \alpha_i \mathbb{B}$$

is a convexificator of the function $\frac{f_i(.)}{g_i(.)} + \alpha_1 ||. - \overline{x}||$ at $\overline{x}$ $(i = 1, \ldots, p)$. In fact, for $i = 1, \ldots, p$, we set

$$G(u_i, v_i, w_i) := \frac{u_i}{v_i} + \alpha_i w_i, \quad \text{and} \quad F(x) := \big(f_i(x), g_i(x), ||x - \overline{x}||\big).$$

Then $\frac{f_i(x)}{g_i(x)} = (G \circ F)(x)$. Taking account of Proposition 2.3, it follows that the following set is a convexificator of $G \circ F$ at $\overline{x}$:

$$\frac{g_i(\overline{x})\partial^* f_i(\overline{x}) - \phi_i(\overline{x})\partial^* g_i(\overline{x}))}{g_i^2(\overline{x})} + \alpha_i \mathbb{B}\,.$$

In view of (6), one gets

$$\begin{aligned}
0 \in \mathrm{cl}\,\Big(&\mu_0^* \,\partial\Lambda(0)(\frac{1}{g_1(\overline{x})} \text{conv}\ \partial^* f_1(\overline{x}) - \frac{\phi_1(\overline{x})}{g_1(\overline{x})} \text{conv}\partial^* g_1(\overline{x}) + \alpha_1 \mathbb{B}, \ldots\,, \\
&\frac{1}{g_p(\overline{x})} \text{conv}\partial^* f_p(\overline{x}) - \frac{\phi_p(\overline{x})}{g_p(\overline{x})} \text{conv}\partial^* g_p(\overline{x}) + \alpha_p \mathbb{B}) + \sum_{j=1}^m \mu_j^* \,\text{conv}\ \partial^* h_j(\overline{x}) \\
&+ \sum_{k=1}^r \nu_k^* \,\text{conv}\ \partial^* \ell_k(\overline{x}) + N(D; \overline{x})\Big),
\end{aligned}$$

which implies that there exists a sequence

$$\begin{aligned}
\theta_n \in\ &\mu_0^* \,\partial\Lambda(0)(\frac{1}{g_1(\overline{x})}\text{conv}\ \partial^* f_1(\overline{x}) - \frac{\phi_1(\overline{x})}{g_1(\overline{x})}\text{conv}\partial^* g_1(\overline{x}) + \alpha_1 \mathbb{B}, \ldots, \\
&\frac{1}{g_p(\overline{x})}\text{conv}\,\partial^* f_p(\overline{x}) - \frac{\phi_p(\overline{x})}{g_p(\overline{x})}\text{conv}\partial^* g_p(\overline{x}) + \alpha_p \mathbb{B}) + \sum_{j=1}^m \mu_j^*\text{conv}\ \partial^* h_j(\overline{x}) \\
&+ \sum_{k=1}^r \nu_k^*\text{conv}\ \partial^* \ell_k(\overline{x}) + N(D; \overline{x}).
\end{aligned}$$

such that $\lim\limits_{n\to\infty} \theta_n = 0$. The latter implies that there exists a sequence $\{\lambda_n\} \subset \partial\Lambda(0)$ such that

$$\begin{aligned}
\theta_n \in\ &\mu_0^* \lambda_n(\frac{1}{g_1(\overline{x})}\text{conv}\partial^* f_1(\overline{x}) - \frac{\phi_1(\overline{x})}{g_1(\overline{x})}\text{conv}\partial^* g_1(\overline{x}) + \alpha_1 \mathbb{B}, \ldots, \\
&\frac{1}{g_p(\overline{x})}\text{conv}\partial^* f_p(\overline{x}) - \frac{\phi_p(\overline{x})}{g_p(\overline{x})}\text{conv}\partial^* g_p(\overline{x}) + \alpha_p \mathbb{B}) + \sum_{j=1}^m \mu_j^*\text{conv}\ \partial^* h_j(\overline{x}) \\
&+ \sum_{k=1}^r \nu_k^*\text{conv}\ \partial^* \ell_k(\overline{x}) + N(D; \overline{x}).
\end{aligned}$$

Since $\partial\Lambda(0)$ is a compact set in $\mathbb{R}^p$, without loss of generality, we can suppose that $\lambda_n \to \lambda^* \in \partial\Lambda(0)$.

By letting $n \to \infty$, one gets

$$0 \in \mathrm{cl}\Big(\mu_0^* \lambda^* (\frac{1}{g_1(\overline{x})} \mathrm{conv}\partial^* f_1(\overline{x}) - \frac{\phi_1(\overline{x})}{g_1(\overline{x})} \mathrm{conv}\partial^* g_1(\overline{x}) + \alpha_1 \mathbb{B}, ..., \frac{1}{g_p(\overline{x})} \mathrm{conv}\partial^* f_p(\overline{x})$$
$$- \frac{\phi_p(\overline{x})}{g_p(\overline{x})} \mathrm{conv}\partial^* g_p(\overline{x}) + \alpha_p \mathbb{B})) + \sum_{j=1}^{m} \mu_j^* \, \mathrm{conv}\ \partial^* h_j(\overline{x})$$
$$+ \sum_{k=1}^{r} \nu_k^* \, \mathrm{conv}\ \partial^* \ell_k(\overline{x}) + N(D; \overline{x})\Big).$$

Let us see that $\lambda^* \in \mathbb{R}^p_+ \setminus \{0\}$. In fact, for $y \in \mathbb{R}^p_{++}$, we can write $0 - (-y) \in \mathbb{R}^p_{++}$. Therefore,

$$\langle \lambda^*, -y \rangle = \langle \lambda^*, -y - 0 \rangle \leqslant \Lambda(-y) - \Lambda(0) = \Lambda(-y) < \Lambda(0) = 0.$$

Consequently, $\lambda^* \in \mathbb{R}^p_+ \setminus \{0\}$. We have $\lambda^* = (\lambda_1^*, ..., \lambda_p^*) \in \mathbb{R}^p_+ \setminus \{0\}$, and

$$0 \in \mathrm{cl}\Big(\mu_0^* (\lambda_1^*, \ldots, \lambda_p^*) (\frac{1}{g_1 \overline{x})} (\mathrm{conv}\partial^* f_1(\overline{x}) - \frac{\phi_1(\overline{x})}{g_1(\overline{x})} \mathrm{conv}\partial^* g_1(\overline{x}) + \alpha_1 \mathbb{B}, ...,$$
$$\frac{1}{g_p(\overline{x})} \mathrm{conv}\partial^* f_p(\overline{x}) - \frac{\phi_p(\overline{x})}{g_p(\overline{x})} \mathrm{conv}\partial^* g_p(\overline{x}) + \alpha_p \mathbb{B})) + \sum_{j=1}^{m} \mu_j^* \mathrm{conv}\ \partial^* h_j(\overline{x})$$
$$+ \sum_{k=1}^{r} \nu_k^* \mathrm{conv}\ \partial^* \ell_k(\overline{x}) + N(D; \overline{x})\Big).$$

Putting $\theta_i^* = \lambda_i^* / g_i(\overline{x})$ $(i = 1, \ldots p)$, we get

$$0 \in \mathrm{cl}\Big(\mu_0^* \sum_{i=1}^{p} \theta_i^* (\mathrm{conv}\partial^* f_i(\overline{x}) - \phi_i(\overline{x}) \mathrm{conv}\partial^* g_i(\overline{x}) + \alpha_i g_i(\overline{x}) \mathbb{B})$$
$$+ \sum_{j=1}^{m} \mu_j^* \mathrm{conv}\ \partial^* h_j(\overline{x}) + \sum_{k=1}^{r} \nu_k^* \mathrm{conv}\ \partial^* \ell_k(\overline{x}) + N(D; \overline{x})\Big).$$

The proof is complete. □

4. Kuhn-Tucker necessary efficiency conditions

For the purpose to derive Kuhn-Tucker necessary conditions we introduce the following Mangasarian-Fromovitz type constraint qualification (CQ) for multiobjective fractional programming problem (MFP): There exist $v_0 \in T(D; \overline{x})$ and numbers $a_i > 0$ $(i \in I(\overline{x}))$ such that

(i) $\langle \xi, v_0 \rangle \leqslant -a_i$ $(\forall \xi \in \mathrm{conv}\partial^* h_i(\overline{x}),\ \forall i \in I(\overline{x}))$;

(ii) $\langle \eta_j, v_0 \rangle = 0$ $(\forall \eta_j \in \mathrm{conv}\partial^* \ell(\overline{x}), \forall j \in \{1, \ldots, n\})$.

A Kuhn-Tucker necessary efficiency condition for (MFP) can be stated as follows.

Theorem 4.1. *Let $\overline{x} \in M$ be a local weak quasi-efficient solution of* (MFP), *and let all hypotheses of Theorem* 3.1 *be fulfilled. Assume furthermore the constraint qualification* (CQ) *holds. Then, there exist vectors $(\tau_1^*, ..., \tau_p^*) \in \mathbb{R}_+^p \setminus \{0\}$, $(\mu_1^*, ..., \mu_m^*) \in \mathbb{R}_+^m$, and $(\nu_1^*, ..., \nu_r^*) \in \mathbb{R}^r$ such that*

$$\begin{aligned} 0 \in cl\Big(&\sum_{i=1}^{p} \tau_i^* \left(conv\, \partial^* f_i(\overline{x}) - \phi_i(\overline{x}) conv\, \partial^* g_i(\overline{x}) + \alpha_i g_i(\overline{x})\mathbb{B}\right) \\ &+ \sum_{j=1}^{m} \mu_j^* conv\ \partial^* h_j(\overline{x}) + \sum_{k=1}^{r} \nu_k^* conv\ \partial^* \ell_k(\overline{x}) + N(D;\overline{x})\Big), \\ &\mu_j^* h_j(\overline{x}) = 0, \quad j = 1, ..., m. \end{aligned} \tag{7}$$

Proof. Under Assumptions of Theorem 3.1, we claim that there exists vector $(\theta_1^*, ..., \theta_p^*) \in \mathbb{R}_+^p \setminus \{0\}$, $(\mu_0^*, \mu_1^*, ..., \mu_m^*) \in \mathbb{R}_+^{m+1}$, and $(\nu_1^*, ..., \nu_r^*) \in \mathbb{R}^r$ such that

$$\begin{aligned} 0 \in \mathrm{cl}\Big(&\mu_0^* \sum_{i=1}^{p} \theta_i^* \left(\mathrm{conv}\, \partial^* f_i(\overline{x}) - \phi_i(\overline{x}) \mathrm{conv}\, \partial^* g_i(\overline{x}) + \alpha_i g_i(\overline{x})\mathbb{B}\right) \\ &+ \sum_{j=1}^{m} \mu_j^* \mathrm{conv}\ \partial^* h_j(\overline{x}) + \sum_{k=1}^{r} \nu_k^* \mathrm{conv}\ \partial^* \ell_k(\overline{x}) + N(D;\overline{x})\Big), \end{aligned} \tag{8}$$

$$\mu_0^* + \sum_{j=1}^{m} \mu_j^* = 1, \quad \mu_j^* h_j(\overline{x}) = 0, \quad j = 1, \ldots, m. \tag{9}$$

If $\mu_0^* = 0$, then $\sum_{j=1}^m \mu_j^* = 1$. By virtue of (8) there exist $\xi_j^{(n)} \in \mathrm{conv}\, \partial^* h_j(\overline{x})$, $\eta_k^{(n)} \in \mathrm{conv}\, \partial^* \ell_k(\overline{x})$ and $\zeta^{(n)} \in N(C;\overline{x})$ such that

$$\lim_{n\to\infty} \Big[\sum_{j \in I(\overline{x})} \mu_j^* \xi_j^{(n)} + \sum_{k \in L} \nu_k^* \eta_k^{(n)} + \zeta^{(n)} \Big] = 0. \tag{10}$$

Hence,
$$\lim_{n\to\infty} \Big[\sum_{j \in I(\overline{x})} \mu_j^* \langle \xi_j^{(n)}, v_0 \rangle + \sum_{k \in L} \nu_k^* \langle \eta_k^{(n)}, v_0 \rangle + \langle \zeta^{(n)}, v_0 \rangle \Big] = 0, \tag{11}$$

where $v_0 \in T(C; x_0)$ is available in the constraint qualification (CQ).
On the other hand, since $\mu_i^* \geqslant 0$, $i = 1, \ldots, m$, $\sum_{j=1}^m \mu_j^* = 1$, we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} \Big[\sum_{j \in I(\overline{x})} \mu_j^* \langle \xi_j^{(n)}, v_0 \rangle + \sum_{k \in L} \nu_k^* \langle \eta_k^{(n)}, v_0 \rangle + \langle \zeta^{(n)}, v_0 \rangle \Big] \\ &\leqslant \lim_{n\to\infty} \Big[\sum_{j \in I(\overline{x})} \mu_j^* \langle \xi_j^{(n)}, v_0 \rangle + \sum_{k \in L} \nu_k^* \langle \eta_k^{(n)}, v_0 \rangle \Big] \leqslant - \sum_{i \in I(\overline{x})} \mu_j^* a_j < 0, \end{aligned}$$

a contradiction(!). Hence, $\mu_0 > 0$. Putting $\tau_i^* = \mu_0^* \theta_i^*$ $(i = 1, \ldots, p)$, we obtain $(\tau_1^*, \ldots, \tau_p^*) \in \mathbb{R}_+^m \setminus \{0\}$, and

$$\begin{aligned} 0 \in \mathrm{cl}\Big(&\sum_{i=1}^{p} \tau_i^* \left(\mathrm{conv}\, \partial^* f_i(\overline{x}) - \phi_i(\overline{x}) \mathrm{conv}\, \partial^* g_i(\overline{x}) + \alpha_i g_i(\overline{x})\mathbb{B}(\overline{x})\right) \\ &+ \sum_{j=1}^{p} \mu_j^* \mathrm{conv}\ \partial^* h_j(\overline{x}) + \sum_{k=1}^{r} \nu_k^* \mathrm{conv}\ \partial^* \ell_k(\overline{x}) + N(D;\overline{x})\Big). \end{aligned}$$

This completes the proof. □

Remark 4.2. When $D = X$, problem (MFP) reduces to problem (UP) which has been examined in [8] by a different approach using the Mordukhovich subdifferential. Since problem (MFP) is more general than the one in [8], and the Mordukhovich subdifferential is a convexificator also, it follows that Theorem 4.1 recovers Theorem 3.2 in [8].

Theorem 4.1 is illustrated by the following example

Example 4.3. Let $D = [0, 1]$, $\alpha = (1, 1)$, and $\overline{x} = 0$. We consider the following functions:

$$f_1(x) := \begin{cases} x^2|\cos\frac{\pi}{x}|, & \text{if} \quad x \neq 0, \\ 0, & \text{if} \quad x = 0 \end{cases}, \quad f_2(x) := \begin{cases} \frac{1}{1+e^{\frac{1}{x}}}, & \text{if} \quad x \neq 0, \\ 0, & \text{if} \quad x = 0. \end{cases},$$

$$\ell(x) := \begin{cases} \frac{\sqrt{x^2+x^3}}{x}, & \text{if} \quad x < 0, \\ 0, & \text{if} \quad x > 0. \end{cases}$$

$$g_1(x) = |x| + 1, \quad g_2(x) = x^2 + 2, \quad h(x) := x^2 - x.$$

Consider the following multiobjective fractional programming problem:

$$(\mathrm{P}_3) \qquad \begin{cases} \min\Big(\dfrac{f_1(x)}{g_1(x)}, \dfrac{f_2(x)}{g_2(x)}\Big), \\ \text{subject to} \quad h(x) \leqslant 0, \quad \ell(x) = 0, \ x \in D. \end{cases}$$

The feasible set of (P_3) is $M = \{x \in D : h(x) \leqslant 0, \ \ell(x) = 0\} = [0, 1]$.
It can be seen that $\phi_1(\overline{x}) = 0$, $\phi_2(\overline{x}) = 0$, and for all $x \in (0, 1]$,

$$\phi_1(x) \not< \phi(0) - x \quad \text{and} \quad \phi_2(x) \not< \phi(0) - x.$$

Hence, $\overline{x} = 0$ is a quasi efficient solution of (P_3). It can be seen that

$$\partial^* f_1(0) = \{-\pi; \pi\}, \quad \partial^* g_1(0) = \{-1; 1\}, \quad \partial^* f_2(0) = \{0\},$$
$$\partial^* g_2(0) = \{0\}, \quad \partial^* h(0) = \{-1\}, \quad \partial^* \ell(0) = \{(-1/2, 1/2)\}.$$

One has $T(D; \overline{x}) = \mathbb{R}_+$, $N(D; \overline{x}) = \mathbb{R}_-$. The constraint qualification (CQ) is satisfied with $v_0 = 1 \in \mathbb{R} = T(D; \overline{x})$. Thus all hypotheses of Theorem 4.1 are fulfilled, and the Kuhn-Tucker necessary condition (7) holds with $\tau_1^* = \tau_2^* = 1$, $\alpha_1^* = 1$, $\alpha_2^* = 1/2$, $\mu^* = 1$, $\nu^* = 1$.

5. Sufficient optimality conditions

To derive sufficient optimality conditions, we need some assumptions on the generalized convexity. Adapting to the definitions of asymptotic pseudoconvex and quasiconvex functions in [16], we introduce the following definitions.

Definition 5.1. Let $f : X \to \mathbb{R}$ be a function which admits a convexificator $\partial^* f(x) \subset X^*$.

(1) We say that f is *asymptotic pseudoconvex* at $\overline{x}$ on D if for every $x \in D$, for some $x_n^* \in \operatorname{conv} \partial^* f(\overline{x})$, $\lim_{n\to\infty}\langle x_n^*, x - \overline{x})\rangle \geqslant 0 \implies f(x) \geqslant f(\overline{x})$.

(2) We say that f is *asymptotic strict pseudoconvex* at $\overline{x}$ if for every $x \in D$, $x \neq \overline{x}$, for some $x_n^* \in \operatorname{conv} \partial^* f(\overline{x})$, $\lim_{n\to\infty}\langle x_n^*, x - \overline{x})\rangle \geqslant 0 \implies f(x) > f(\overline{x})$.

(3) The function f is called *asymptotic quasiconvex* at $\overline{x}$ on D if for all $x \in D$, $f(x) \leqslant f(\overline{x}) \implies$ for any $x_n^* \in \mathrm{conv}\,\partial^* f(\overline{x})$, $\lim_{n\to\infty}\langle x_n^*, x - \overline{x}\rangle \leqslant 0$.

Remark 5.2. If we assume that $\partial^*(f_1 + f_2)(\overline{x}) = \partial^* f_1(\overline{x}) + \partial^* f_2(\overline{x})$, and f_1, f_2 are asymptotic pseudoconvex (asymptotic quasiconvex, asymptotic strict pseudoconvex) at $\overline{x}$, then $f_1 + f_2$ is also asymptotic pseudoconvex (resp. asymptotic quasiconvex, asymptotic strict pseudoconvex) at $\overline{x}$.

We now strengthen necessary condition (7) and get the following sufficient condition for weak efficiency of (MFP).

Theorem 5.3. *Let $\overline{x} \in E$. Assume that for $i = 1, \ldots, p$, f_i, g_i admits bounded convexificators $\partial^* f_i(\overline{x}), \partial^* g_i(\overline{x})$ at $\overline{x}$, the mappings $\partial^* f_i$, $\partial^* g_i$ are upper semicontinuous at $\overline{x}$; $|\ell_k|(k \in L)$ are regular in the sense of Clarke. Suppose also that*

(i) *There exist $(\tau_1^*, ..., \tau_p^*) \in \mathbb{R}^p_+ \setminus \{0\}$, $(\mu_0^*, \mu_1^*, ..., \mu_m^*) \in \mathbb{R}^{m+1}_+$, and $(\nu_1^*, ..., \nu_r^*) \in \mathbb{R}^r$ such that*

$$0 \in cl\Big(\sum_{i=1}^{p} \tau_i^* \left(conv\,\partial^* f_i(\overline{x}) - \phi_i(\overline{x}) conv\,\partial^* g_i(\overline{x}) + \alpha_i g_i^* \mathbb{B}\right),$$

$$+ \sum_{j=1}^{m} \mu_j^* conv\ \partial^* h_j(\overline{x}) + \sum_{k=1}^{r} \nu_k^* conv\ \partial^* \ell_k(\overline{x}) + N(D; \overline{x})\Big), \qquad (12)$$

where $g_i^* = \inf_{x\in D} g_i(x)$ $(i = 1, \ldots, p)$.

(ii) *The function $\sum_{i=1}^p \alpha_i^*(f_i(x) - \phi_i(\overline{x})g_i(x))$ is asymptotic pseudoconvex at $\overline{x}$ on D; the function h_j are asymptotic quasiconvex at $\overline{x}$ on D for all $j = 1, ..., m$; the functions $\pm\ell_k$ are asymptotic quasiconvex at $\overline{x}$on D for all $k = 1, ..., r$.*
Then $\overline{x}$ is a weakly quasi-efficient solution of (MFP).

Proof. By contradiction, suppose that $\overline{x}$ is not a weakly quasi-efficient solution of (MFP). By an argument analogous to that used for the proof of Theorem 3.1, we deduce that $\overline{x}$ is not a weakly efficient solution of (P_1). Then, there exists a sequence $x_n \to \overline{x}$ such that

$$\begin{cases} (\tau_1^*(f_1(x_n) - \phi_1(\overline{x})g_1(x_n) + \alpha_1 g_1^*||x_n - \overline{x}||, ..., \tau_p^*(f_p(x_n) - \phi_p(\overline{x}))g_p(x_n) \\ \qquad\qquad + \alpha_p g_p^*||x - \overline{x}|| \in -\mathrm{int}\,\mathbb{R}^p_+, \\ h_j(x_n) \leqslant 0 \text{ for all } j = 1, ..., m,\ \ell_k(x_n) = 0 \text{ for all } k = 1, ..., r,\ x_n \in D. \end{cases}$$

Let us see that $\sum_{i=1}^p \tau_i^*(\partial^* f_i(\overline{x}) - \phi_i(\overline{x})\partial^* g_i(\overline{x}) + \alpha_i g_i^* \mathbb{B})$ is a convexificator of the function $\sum_{i=1}^p \tau_i^*(f_i - \phi_i(\overline{x})g_i + \alpha_i g_i^*||x - \overline{x}||)$ at $\overline{x}$. In fact, we set

$$\begin{aligned} &G((u_1, v_1, w_1), \ldots, (u_p, v_p, w_p)) \\ &\qquad\qquad := \big(\tau_1^*(u_1 - \phi_1(\overline{x})v_1 + \alpha_1 g_1^* w_1), \ldots, \tau_p^*(u_p - \phi_p(\overline{x})v_p + \alpha_p g_p^* w_p)\big), \\ &F(x) := \big((f_1(x), g_1(x), ||x - \overline{x}||), \ldots, (f_p(x), g_p(x), ||x - \overline{x}||)\big). \end{aligned}$$

Then, G is Fréchet differentiable at $\overline{x}$, and

$$\sum_{i=1}^{p} \tau_i^*(f_i(x) - \phi_i(\overline{x})g_i(x) + \alpha_i g_i^*||x - \overline{x}||) = (G \circ F)(x).$$

Moreover,

$$\partial^* G(F(\overline{x})) = \big\{\nabla G(F(\overline{x}))\big\} = \big\{\big(\tau_1^*(1, -\phi_1(\overline{x}), \alpha_1 g_1^*), \ldots, \tau_p^*(1, -\phi_p(\overline{x}), \alpha_p g_p^*)\big)\big\}.$$

Taking account of Proposition 2.2, we conclude that the following set is a convexificator of $G \circ F$ at $\overline{x}$

$$\big((\tau_1^*(1, -\phi_1(\overline{x}), \alpha_1 g_1^*), \ldots, \tau_p^*(1, -\phi_p(\overline{x}), \alpha_p g_p^*)\big)\big((\partial^* f_1(\overline{x}), \partial^* g_1(\overline{x}), \mathbb{B}), \ldots,$$
$$(\partial^* f_p(\overline{x}), \partial^* g_p(\overline{x}), \mathbb{B})\big) = \sum_{i=1}^{p} \tau_i^*(\partial^* f_i(\overline{x}) - \phi_i(\overline{x})\partial^* g_i(\overline{x}) + \alpha_i g_i^* \mathbb{B}).$$

It is a consequence of the Kuhn-Tucker conditon (12) that there exist $a_i^{(n)} \in \partial^* f_i(\overline{x})$, $b_j^{(n)} \in \partial^* g_i(\overline{x})$, $c_i^{(n)} \in \mathbb{B}$, $d_j^{(n)} \in \partial^* h_j(\overline{x})$, $e_k^{(n)} \in \partial^* \ell_k(\overline{x})$, $\sigma^{(n)} \in N(D; \overline{x})$ such that

$$\lim_{n\to\infty} \Big(\sum_{i=1}^{p} \tau_i^*(a_i^{(n)} + \phi_i(\overline{x}) b_i^{(n)} + \alpha_i g_i^* c_i^{(n)}) + \sum_{j\in I(\overline{x})} \mu_j^* d_j^{(n)} + \sum_{k=1}^{r} \nu_k^* e_k^{(n)} + \sigma^{(n)}\Big) = 0.$$

Hence,

$$\lim_{n\to\infty} \Big(\sum_{i=1}^{p} \tau_i^* \langle a_i^{(n)} + \phi_i(\overline{x}) b_i^{(n)} + \alpha_i g_i^* c_i^{(n)}, x_n - \overline{x})\rangle + \sum_{j\in I(\overline{x})} \mu_j^* \langle d_j^{(n)}, x_n - \overline{x})\rangle$$
$$+ \sum_{k=1}^{r} \nu_k^* \langle e_k^{(n)}, x_n - \overline{x})\rangle + \langle \sigma^{(n)}, x_n - \overline{x})\rangle\Big) = 0. \tag{13}$$

For $x_n \in M$, one has $h_j(x_n) \leqslant 0 = h_j(\overline{x})\,(\forall j \in I(\overline{x}))$. By virtue of the asymptotic quasiconvexity of h_j, it follows that

$$\lim_{n\to\infty} \langle d_j^{(n)}, x_n - \overline{x})\rangle \leqslant 0. \tag{14}$$

For each $k = 1, \ldots, r$, we have $\ell_k(x_n) = 0 = \ell_k(\overline{x})\,(\forall k \in L)$. In view of the asymptotic quasiconvexity of $\pm\ell_k$, it results that

$$\lim_{n\to\infty} \langle e_k^{(n)}, x_n - \overline{x})\rangle = 0. \tag{15}$$

Since $x_n \in D$ and $d^{(n)} \in N(D; \overline{x})$ we have $\lim\limits_{n\to\infty} \langle d^{(n)}, x_n - \overline{x})\rangle \leqslant 0.$ (16)

Combining (13)–(16) yields

$$\lim_{n\to\infty} \langle \sum_{i=1}^{p} \tau_i^*(a_i^{(n)} + \phi_i(\overline{x}) b_i^{(n)} + \alpha_i g_i^* c_i^{(n)}), x_n - \overline{x})\rangle \geqslant 0. \tag{17}$$

By virtue of Remark 5.2, since the functions $\sum_{i=1}^p \tau_i^*(f_i - \phi_i(\overline{x}) g_i$ and $\alpha_i g_i^* ||x - \overline{x}||$ are asymptotic pseudoconvex at $\overline{x}$, so is the function $\sum_{i=1}^p \tau_i^*(f_i - \phi_i(\overline{x}) g_i + \alpha_i g_i^* ||x - \overline{x}||)$. By the asymptotic pseudoconvexity of $\sum_{i=1}^p \tau_i^*(f_i - \phi_i(\overline{x}) g_i + \alpha_i g_i^* ||x - \overline{x}||)$, it follows that for all $x \in M$,

$$\sum_{i=1}^{p} \tau_i^*(f_i(x) - \phi_i(\overline{x}) g_i(x)) + \alpha_i g_i^* ||x - \overline{x}||)$$
$$\geqslant \sum_{i=1}^{p} \tau_i^*(f_i(\overline{x}) - \phi_i(\overline{x}) g_i(\overline{x}) + \alpha_i ||\overline{x} - \overline{x}||) = 0. \tag{18}$$

Let us see that $\overline{x}$ is a weakly quasi-efficient solution of (MFP). In fact, if it is were not so there would exist $x_1 \in M$ such that

$$\frac{f_i(x_1)}{g_i(x_1)} - \frac{f_i(\overline{x})}{g_i(\overline{x})} + \alpha_i ||x_1 - \overline{x}|| < 0, \quad i = 1, \ldots, p.$$

whence, $\quad \dfrac{1}{g_i(x_1)}(f_i(x_1) - \phi_i(\overline{x})g_i(x_1) + \alpha_i g_i(x_1)||x_1 - \overline{x}|| < 0, \quad i = 1, \ldots, p.$

Therefore,
$$\begin{aligned} & f_i(x_1) - \phi_i(\overline{x})g_i(x_1) + \alpha_i g_i^* ||x_1 - \overline{x}|| \\ & \leqslant f_i(x_1) - \phi_i(\overline{x})g_i(x_1) + \alpha_i g_i(x_1)||x_1 - \overline{x}|| < 0, \quad i = 1, \ldots, p. \end{aligned}$$

Since $(\tau_1^*, \ldots, \tau_p^*) \neq 0$ we conclude that

$$\sum_{i=1}^{p} \tau_i^* (f_i(x_1) - \phi_i(\overline{x})g_i(x_1) + \alpha_i g_i^* ||x_1 - \overline{x}||) < 0.$$

But this is in conflict with (18). Hence, $\overline{x}$ is a quasi efficient solution of (P_1), and the proof is complete. □

Remark 5.4. Golestani et al. [6] established sufficient conditions for quasi-efficient under assumptions on some generalized convexity but different here. They study multiobjective optimization problems which are KT- approximate pseudoconvex-affine at the considering point.

6. Duality

We now consider the following dual problem of Mond-Weir type for problem (MFP):

$$(\text{MWDP}) \begin{cases} \max \phi(u) := \Big(\dfrac{f_1(u)}{g_1(u)}, ..., \dfrac{f_p(u)}{g_p(u)}\Big), \\ \text{s.t. } \; 0 \in \text{cl}\Big(\displaystyle\sum_{i=1}^{p} \tau_i \left(\text{conv}\, \partial^* f_i(u) - \phi_i(u) \text{conv}\, \partial^* g_i(u) + \alpha_i g_i^* \mathbb{B}\right) \\ \qquad\qquad + \displaystyle\sum_{j=1}^{m} \mu_j \text{conv } \partial^* h_j(u) + \sum_{k=1}^{r} \nu_k \text{conv } \partial^* \ell_k(u) + N(D; u)\Big), \\ \mu_j h_j(u) \geqslant 0 \ (i \in I(u)), \nu_k \ell_k(u) = 0 \ (k = 1, \ldots, L), u \in D, \\ (\tau_1, \ldots, \tau_p) \in \mathbb{R}^p_+ \setminus \{0\}, \mu_i \geqslant 0 \ (i \in I(u)), \mu_r = 0 \ (r \notin I(u)), \\ \nu_k \in \mathbb{R} \ (k \in L). \end{cases}$$

Denote by M_1 the feasible set of problem (MWDP). We define $\tau := (\tau_1, \ldots, \tau_p)$, $\mu := (\mu_i)_{i \in I}$, $\nu := (\nu_k)_{k \in L}$. Recall that $\overline{x}$ is called a *local weak efficient maximum* of (MWDP) if there exists a number $\delta > 0$ such that for all $x \in M_1 \cap B(\overline{x}, \delta)$,

$$\phi(x) - \phi(\overline{x}) - \alpha ||x - \overline{x}|| \notin \mathbb{R}_{++}.$$

A weak duality theorem for the primal problem (FMP) and the dual problem (MWDP) can be stated as follows.

Theorem 6.1. (Weak duality) *Let x and (u, τ, μ, ν) be the feasible points of* (MFP) *and* (MWDP)*, respectively. Assume that*

(i) *f_i, g_i, ℓ_k are continuous and admit convexificators $\partial^* f_i(x), \partial^* g_i(x)$, at x near $\overline{x}$, respectively. They admit bounded convexificators $\partial^* f_i(\overline{x})$, $\partial^* g_i(\overline{x})$ at $\overline{x}$. Moreover, $\partial^* f_i, \partial^* g_i$ are upper semicontinuous at $\overline{x}$.*

(ii) *The function $\sum_{i=1}^p \alpha_i^*(f_i - \phi_i(u)g_i$ is asymptotic pseudoconvex at u; the function $\mu_j h_j$ are asymptotic quasiconvex at u for all $j = 1, \dots, m$; the function $\pm\nu_k \ell_k$ are asymptotic quasiconvex at u for all $k = 1, \dots, r$.*

Then
$$\phi(x) + \alpha||x - u|| \not< \phi(u). \tag{19}$$

Proof. Due to $(u, \lambda, \mu, \gamma) \in M_1$, there exist $\xi_i^{(n)} \in \text{conv } \partial^* f_i(u)$ $(i = 1, \dots, p)$, $\eta_i^{(n)} \in \text{conv } \partial^* g_i(u)$ $(i = 1, \dots, p)$, $\theta_i^{(n)} \in \alpha_i g_i^* \mathbb{B}$, $\zeta_j^{(n)} \in \text{conv } \partial^*(\mu_j h_j)(u)$ $(j \in I)$, $\chi_k^{(n)} \in \text{conv}\partial^*(\nu_k \ell_k)(u)$ $(k \in L)$, $\sigma^{(n)} \in N(D, u)$ such that

$$0 = \lim_{n\to\infty} \Big[\sum_{i=1}^p \tau_i(\xi_i^{(n)} - \phi_i(u)\eta_i^{(n)} + \theta_i^{(n)}) + \sum_{j\in I(u)} \zeta_j^{(n)} + \sum_{k\in L} \chi_k^{(n)} + \sigma^{(n)} \Big].$$

Hence, for $x \in M$, we have

$$\lim_{n\to\infty} \langle \sum_{i=1}^p \tau_i(\xi_i^{(n)} - \phi_i(u)\eta_i^{(n)} + \theta_i^{(n)}), x - u\rangle + \lim_{n\to\infty} \sum_{j\in I(u)} \langle\, \zeta_j^{(n)}, x - u\rangle$$
$$+ \sum_{k\in L} \lim_{n\to\infty} \langle \chi_k^{(n)}, x - u\rangle + \lim_{n\to\infty} \langle \sigma^{(n)}, x - u\rangle = 0. \tag{20}$$

Observe that for $x \in M$, $\mu_j h_j(x) \leqslant 0\, \mu_i \leqslant \mu_j h_j(u)$ $(\forall j \in I(u))$. In view of the asymptotic quasiconvexity of $\mu_j h_j$ at u $(i \in I(u))$, we obtain

$$\lim_{n\to\infty} \langle \zeta_j^{(n)}, x - u\rangle \leqslant 0. \tag{21}$$

Since $\ell_k(x) = 0 = \ell_k(u)$, in view of the asymptotic quasilinearity of $\nu_k h_k$ $(\forall k \in L)$, for $x \in M$, we have

$$\lim_{n\to\infty} \langle \chi_k^{(n)}, x - u\rangle = 0. \tag{22}$$

The convexity of D implies that for $x \in M$,

$$\lim_{n\to\infty} \langle \sigma^{(n)}, x - u\rangle \leqslant 0. \tag{23}$$

Let us show that $\partial^* f_i(u) - \phi_i(u)\partial^* g_i(u) + \alpha_i g_i^* \mathbb{B}$ is a convexificator of the function $f_i(x) - \phi_i(u)g_i(x) + \alpha_i g_i^* ||x - u||$ at u $(i = 1, \dots, p)$. In fact, for $i = 1, \dots, p$, we set

$$G(v, w, z) := v - \phi_i(u)w + \alpha_i g_i^* z, \quad \text{and} \quad F(x) := \big(f_i(x), g_i(x), ||x - u||\big)||.$$

Then, G is Fréchet differentiable at $\overline{x}$, and

$$\partial^* G(.)) = \big\{\nabla G(.)\big\} = \big\{\big(1, -\phi_i(u), \alpha_i g_i^*\big)\big\}.$$

Taking account of Proposition 2.2, we conclude that the following set is a convexificator of $\varphi_i := G \circ F$ at u:

$$\big(1, -\phi_i(u), \alpha_i g_i^*)\big)\big((\partial^* f_i(u), \partial^* g_i(u), \mathbb{B})\big) = \partial^* f_i(u) - \phi_i(u)\partial^* g_i(u) + \alpha_i g_i^* \mathbb{B}.$$

Combining (20)–(23) yields

$$\lim_{n\to\infty}\sum_{i=1}^{p}\langle \tau_i(\xi_i^{(n)} - \phi_i(u)\eta_i^{(n)} + \theta_i^{(n)}), x-u\rangle \geqslant 0.$$

We observe that $\partial^* f_i(u) - \phi_i(u)\partial^* g_i(u) + \alpha_i g_i^* \mathbb{B}$ is a convexificator of the function $f_i(x) - \phi_i(u)g_i(x) + \alpha_i||x-u||$ at u. The function $\sum_{i=1}^{p}\tau^*(f_i(x) - \phi_i(u)g_i(x))$ is asymptotic pseudoconvex at u, just like the map $\tau_i^*(f_(x) - \phi_i(u)g_i(x) + \alpha_i g_i^*||x-u||)$. Hence we have for every $x \in M$,

$$\sum_{i=1}^{p}\tau_i^*(f_i(x) - \phi_i(u)g_i(x) + \alpha_i g_i^*||x-u||)$$
$$\geqslant \sum_{i=1}^{p}\tau_i^*(f_i(u) - \phi_i(u)g_i(u) + \alpha_i g_i^*||u-u||) = 0. \tag{24}$$

Hence,
$$\phi(x) + \alpha||x-u|| \nless \phi(u). \tag{25}$$

In fact, if it were not so, there would be exists $x_1 \in M$ such that

$$\phi_i(x_1) + \alpha_i||x_1 - u|| < \phi_i(u) \ (i = 1, \ldots, p),$$

which implies that

$$f_i(x_1) + \alpha_i g_i^*||x_1 - u|| - \phi_i(u)g_i(x_1)$$
$$\leqslant f_i(x_1) + \alpha_i g_i(x_1)||x_1 - u|| - \phi_i(u)g_i(x_1) < 0, \ (i = 1, \ldots, p).$$

Since $(\tau_1^*, \ldots, \tau_p^*) \in \mathbb{R}_+^p \setminus \{0\}$, which yields

$$\sum_{i=1}^{p}\tau_i^*(f_i(x_1) + \alpha_i g_i^*||x_1 - u|| - \phi_i(u)g_i(x_1)) < 0.$$

Thus we arrive at a contradiction to (24). Therefore (25) holds and the proof is complete. □

In what follows, we give a strong duality theorem for (FMP) and (MWDP).

Theorem 6.2. (Strong duality) *Let $\overline{x}$ be a local weak quasi-efficient efficient solution of* (MFP). *Assume that all hypotheses of Theorem* 3.2 *are fulfilled. Then there exist $\overline{\tau}_i \geqslant 0$ $(i = 1, \ldots, p)$, not all zero, $\overline{\mu}_j \geqslant 0$ $(\forall j \in I(\overline{x}))$, $\overline{\nu}_k \in \mathbb{R}$ $(\forall k \in L)$ such that $(\overline{x}, \overline{\tau}, \overline{\mu}, \overline{\nu})$ is a feasible point of* (MWDP)*, and the value of the objective functions of* (MFP) *and* (MWDP) *at $\overline{x}$ and $(\overline{x}, \overline{\tau}, \overline{\mu}, \overline{\gamma})$, respectively, are equal. Moreover, if Assumption* (ii) *of Theorem* 6.1 *holds, then $(\overline{x}, \overline{\tau}, \overline{\mu}, \overline{\nu})$ is a weakly efficient solution of* (MWDP).

Proof. Since $\overline{x}$ is a local weak quasi-efficient solution of (MFP), we can invoke Theorem 4.1 to deduce that there exist $\overline{\tau}_i \geqslant 0$ $(i = 1, , \ldots, p)$, not all zero, $\overline{\mu}_j \geqslant 0$ $(\forall i \in I(\overline{x}))$, $\overline{\nu}_k \in \mathbb{R}$ $(\forall k \in L)$ such that

$$0 \in \mathrm{cl}\Big(\sum_{i=1}^{p}\overline{\tau}_i(\mathrm{conv}\ \partial^* f(\overline{x}) - \phi_i(\overline{x})\partial^* g_i(\overline{x}) + \alpha_i g_i^* \mathbb{B})$$
$$+ \sum_{i\in I(\overline{x})}\overline{\mu}_j \mathrm{conv}\ \partial^* hj(\overline{x}) + \sum_{k\in L}\overline{\nu}_k \mathrm{conv}\ \partial^* \ell_k(\overline{x}) + N(D;\overline{x})\Big).$$

If $i \notin I(\overline{x})$, we take $\overline{\mu}_i = 0$. Then, $(\overline{x}, \overline{\tau}, \overline{\mu}, \overline{\gamma})$ is a feasible point of (MWDP), and the value of the objective functions of (MFP) and (MWDP) at $\overline{x}$ and $(\overline{x}, \overline{\tau}, \overline{\mu}, \overline{\gamma})$, respectively, are equal.

If Assumption (ii) of Theorem 6.1 holds, then the hypotheses of Theorem 6.1 on weak duality are fulfilled. By this theorem one gets that for every feasible points (u, τ, μ, ν) of (MWDP),

$$F(\overline{x}) + \alpha||u - \overline{x}|| \nleq F(u).$$

Hence, $(\overline{x}, \overline{\tau}, \overline{\mu}, \overline{\gamma})$ is a weakly quasi-efficient solution of (MWDP). □

Theorem 6.3. (Converse duality) *Let $\overline{x}$ and $(\overline{u}, \overline{\tau}, \overline{\mu}, \overline{\nu})$ be the feasible points of* (MFP) *and* (MWDP)*, respectively. Assume that*

(i) *f_i, g_i, ℓ_k are continuous and admit convexificators $\partial^* f_i(u), \partial^* g_i(u)$ at u near $\overline{u}$, respectively. They admit bounded convexificators $\partial^* f_i(\overline{u}), \partial^* g_i(\overline{u})$ at $\overline{u}$. Moreover, $\partial^* f_i, \partial^* g_i$, $\partial^* h_j$, and $\partial^* \ell_k$ are upper semicontinuous at $\overline{u}$.*

(ii) *The function $\sum_{i=1}^p \tau_i^*(f_i(x) - \phi_i(\overline{u})g_i(x))$ is asymptotic strict pseudoconvex at $\overline{u}$; D is convex, the function $\overline{\mu}_j h_j$ are asymptotic quasiconvex at $\overline{u}$ for all $j = 1, \dots, m$; the function $\pm\overline{\nu}_k \ell_k$ are asymptotic quasiconvex at $\overline{u}$ for all $k = 1, \dots, r$. Suppose, furthermore, that*

$$\sum_{i=1}^p \overline{\tau}_i(f_i(\overline{x}) - \phi_i(\overline{u})g_i(\overline{x}) + \alpha_i g_i^*||\overline{x} - \overline{u}||) \leqslant \sum_{i=1}^p \overline{\alpha}_i(f_i(\overline{u}) - \phi_i(\overline{u})g_i(\overline{u})). \tag{26}$$

Then $\overline{u} = \overline{x}$.

Proof. Assume to the contrary that $\overline{u} \neq \overline{x}$. Since $(\overline{u}, \overline{\tau}, \overline{\mu}, \overline{\nu})$ are the feasible points of (MWDP), we get in consequence that there exist $\xi_i^{(n)} \in \mathrm{conv}\ \partial^* f_i(\overline{u})$ $(i = 1, \dots, p)$, $\eta_i^{(n)} \in \mathrm{conv}\ \partial^* g_i(\overline{u})$, $\theta_i \in \alpha_i g_i^* \mathbb{B}$ $(i = 1, \dots, p)$, $\zeta_j^{(n)} \in \mathrm{conv}\ \partial^*(\overline{\mu}_j h_j)(\overline{u})$ $(j \in I)$, $\chi_k^{(n)} \in \partial^*(\overline{\nu}_k \ell_k)(\overline{u})$ $(k \in L)$, $\sigma^{(n)} \in N(D, \overline{u})$ such that

$$0 = \lim_{n\to\infty} \Big[\sum_{i=1}^p \overline{\tau}_i(\xi_i^{(n)} - \phi_i(u)\eta_i^{(n)} + \theta^{(n)}) + \sum_{j\in I(\overline{u})} \zeta_j^{(n)} + \sum_{k\in L} \chi_k^{(n)} + \sigma^{(n)} \Big]. \tag{27}$$

Observe that for $x \in M$, $\overline{\mu}_j h_j(x) \leqslant 0 \ \leqslant \overline{\mu}_j h_j(u)$ $(\forall j \in I(\overline{u}))$. In view of the asymptotic quasiconvexity of $\mu_j h_j$ at $\overline{u}$ $(i \in I(\overline{u}))$ we obtain

$$\lim_{n\to\infty} \langle \zeta_j^{(n)}, x - \overline{u} \rangle \leqslant 0. \tag{28}$$

Since $\ell_k(x) = 0 = \ell_k(u)$, in view of the asymptotic quasiconnvex of $\pm\nu_k h_k$ $(\forall k \in L)$, for $x \in M$, we have

$$\lim_{n\to\infty} \langle \chi_k^{(n)}, x - \overline{u} \rangle = 0. \tag{29}$$

The convexity of D implies that for $x \in M$,

$$\lim_{n\to\infty} \langle \sigma^{(n)}, x - \overline{u} \rangle \leqslant 0. \tag{30}$$

By an argument analogous to that used for the proof of Theorem 6.1, it follows that $\partial^* f_i(\overline{x}) - \phi_i(\overline{u})\partial^* g_i(\overline{u}) + \alpha_i g_i^* \mathbb{B}$ is a convexificator of $f_i(x) - \phi_i(\overline{u})g_i(x) + \alpha_i g_i^*||x - \overline{u}||$ at $\overline{u}$ $(i = 1, \dots, p)$.

Combining (27)–(30) yields

$$\lim_{n\to\infty}\sum_{i=1}^{p}\langle\overline{\tau}_i(\xi_i^{(n)}-\phi_i(\overline{u})\eta_i^{(n)}+\theta_i),x-\overline{u}\rangle\geqslant 0.$$

Since the function $\sum_{i=1}^{p}\overline{\tau}_i(f_i-\phi_i(\overline{u})g_i))$ is asymptotic strict pseudoconvex at $\overline{u}$, so is the function $\sum_{i=1}^{p}\overline{\tau}_i(f_i(x)-\phi_i(\overline{u})g_i(x)+\alpha_i||x-\overline{u}||)$. Hence,

$$\sum_{i=1}^{p}\overline{\tau}_i(f_i\overline{x})-\phi_i(\overline{u})g_i(\overline{x})+\alpha_i||\overline{x}-\overline{u}||)>\sum_{i=1}^{p}\overline{\tau}_i(f_i(\overline{u})-\phi_i(\overline{u})g_i(\overline{u})).$$

But this contradicts (24). Hence, $\overline{u}=\overline{x}$. . □

Now we introduce the following dual problem of Wolfe type for (FMP):

$$(\text{WDP})\quad\begin{cases}\max\phi(x)+\sum\limits_{j\in I(u)}\mu_jh_j(u)+\sum\limits_{k\in L}\nu_k\ell_k(u),\\ \text{s.t. }\ 0\in\text{cl}\Big(\sum\limits_{i=1}^{p}\tau_i(\text{conv }\partial^*f_i(u)-\phi_i(u)\text{conv }\partial^*g_i(u)+\alpha_ig_i^*\mathbb{B})\\ \qquad+\sum\limits_{j\in I(u)}\mu_i\text{conv }\partial^*h_j(u)+\sum\limits_{k\in L}\nu_k\text{conv }\partial^*\ell_k(u)+N(D,u)\Big),\\ u\in D,\tau\in\mathbb{R}^m_+\ \setminus\{0\},\ \ \sum\limits_{i=1}^{m}\tau_i=1,\\ \sum\limits_{j\in I}\mu_jh_j(u)+\sum\limits_{k\in L}\nu_k\ell_k(u)=0,\ \nu_k\in\mathbb{R}\ (k\in L).\end{cases}$$

Denote by M_2 the feasible set of (WDP).
Note that we have $\phi(u)+b=(\phi_1(u)+b,\dots,\phi_p(u)+b)$ for $b\in\mathbb{R}$.

A weak duality theorem for (MFP) and (WDP) can be stated as follows.

Theorem 6.4. (Weak duality) *Let x and (u,τ,μ,ν) be the feasible points of* (MFP) *and* (WDP), *respectively. Assume that*

(i) *$f_i,g_i,h_j,\ \ell_k$ are continuous and admit convexificators $\partial^*f_i(v)$, $\partial^*g_i(v)$, $\partial^*h_j(v)$, $\partial^*\ell_k(v)$ at v near u, respectively. They admit bounded convexificators $\partial^*f_i(u)$, $\partial^*g_i(u)$, $\partial^*h_j(u)$, $\partial^*\ell_k(u)$ at u. Moreover, ∂^*f_i, ∂^*g_i, ∂^*h_j, and $\partial^*\ell_k$ are upper semicontinuous at u.*

(ii) *The map $\sum_{i=1}^{p}\tau_i^*(f_i(y)-\phi_i(u)g_i(y)+\alpha_ig_i^*||y-u||)+\sum_{j\in I(u)}\mu_jh_j(y)+\sum_{k\in L}\nu_k\ell_k(y)$ is asymptotic pseudoconvex at u.*

Then
$$\phi(x)+\alpha||x-u||\not<\phi(u).$$

Proof. Since $(u,\lambda,\mu,\gamma)\in M_2$, there exist $\xi_i^{(n)}\in$ conv $\partial^*f_i(u)$ $(k=1,\dots,p)$, $\eta_i^{(n)}\in\text{conv}\,\partial^*g_i(u)$ $(i\in I(u))$, $\theta_i^{(n)}\in\alpha_ig_i^*\mathbb{B}$, $\zeta_j^{(n)}\in\partial^*h_j(u)$, $\chi_k^{(n)}\in\partial^*\ell_k(u)$, and $\sigma^{(n)}\in N(D,u)$ such that

$$0=\lim_{n\to\infty}\Big[\tau_i(\xi_i^{(n)}-\phi_i\eta_i^{(n)}+\theta_i^{(n)})+\sum_{i\in I(u)}\mu_i\zeta_j^{(n)}+\sum_{j\in L}\nu_k\chi_k^{(n)}+\sigma^{(n)}\Big].$$

Hence, for $x \in M$, we have

$$\lim_{n\to\infty}\Big[\langle \tau_i(\xi_i^{(n)} - \phi_i(u)\eta_i^{(n)} + \theta_i^{(n)}), x-u\rangle + \sum_{i\in I(u)} \mu_i\langle \zeta_j^{(n)}, x-u\rangle$$

$$+\sum_{j\in L}\nu_k\langle \chi_k^{(n)}, x-u\rangle\Big] + \lim_{n\to\infty}\langle \sigma^{(n)}, x-u\rangle = 0. \tag{31}$$

Since D is convex we have for $x \in M$, $\lim_{n\to\infty}\langle \sigma^{(n)}, x-u\rangle \leqslant 0.$ (32)

Combining (31) and (32) yields

$$\lim_{n\to\infty}\Big[\langle \tau_i(\xi_i^{(n)} - \phi_i(u)\eta_i^{(n)}), x-u\rangle + \sum_{i\in I(u)} \mu_i\langle \zeta_j^{(n)}, x-u\rangle + \sum_{j\in L}\nu_k\langle \chi_k^{(n)}, x-u\rangle\Big] \geqslant 0. \tag{33}$$

By an argument analogous to that used in the proof of Theorem 6.1 we deduce that the set

$$\mathrm{cl}\Big(\sum_{i=1}^p \tau_i(\partial^* f_i(\overline{u}) - \phi_i(\overline{u})\partial^* g_i(\overline{u}) + \alpha_i g_i^*\mathbb{B}) + \sum_{j\in I(\overline{u})}\mu_j\partial^* h_j(\overline{u}) + \sum_{k\in L}\nu_k\partial^*\ell_k(\overline{u})\Big)$$

is a convexificator of the following function at u:

$$\sum_{i=1}^p \tau_i(f_i(y) - \phi_i(u)g_i(y) + \alpha_i g_i^*||y-u||) + \sum_{j\in I(u)}\mu_j h_j(y) + \sum_{k\in L}\nu_k\ell_k(y).$$

Since $\sum_{i=1}^p \tau_i(f_i - \phi_i(u)g_i) + \sum_{j\in I(\overline{u})}\mu_j h_j + \sum_{k\in L}\nu_k\ell_k$ is asymptotic pseudoconvex, so is the function

$$\sum_{i=1}^p \tau_i(f_i(y) - \phi_i(\overline{x})g_i(y) + \alpha_i g_i^*||y-u||) + \sum_{j\in I(\overline{u})}\mu_j h_j(y) + \sum_{k\in L}\nu_k\ell_k(y).$$

Therefore,

$$\sum_{i=1}^p \tau_i(f_i(x) - \phi_i(u)g_i(x) + \alpha_i g_i^*||x-u||) + \sum_{j\in I(u)}\mu_j h_j(x) + \sum_{k\in L}\nu_k\ell_k(x)$$

$$\geqslant \sum_{i=1}^p \tau_i(f_i(u) - \phi_i(u)g_i(u)) + \sum_{j\in I(u)}\mu_j h_j(u) + \sum_{k\in L}\nu_k\ell_k(u),$$

which implies that

$$\sum_{i=1}^p \tau_i(f_i(x) - \phi_i(u)g_i(x) + \alpha_i g_i^*||x-u||)$$

$$\geqslant \sum_{i=1}^p \tau_i(f_i(u) - \phi_i(u)g_i(u)) + \sum_{j\in I(u)}\mu_j h_j(u) + \sum_{k\in L}\nu_k\ell_k(u)$$

$$= \sum_{i=1}^p \tau_i(f_i(u) - \phi_i(u)g_i(u)) = 0. \tag{34}$$

Then, $$\phi(x) + \alpha||x - u||) \not< \phi(u). \tag{35}$$

Indeed, if it were not so, there would exist some $x_1 \in M$ such that

$$\phi_i(x_1) + \alpha_i||x_1 - u||) < \phi_i(u) \quad (\text{for } i = 1, \ldots, p).$$

Then $$\frac{f_i(x_1)}{g_i(x_1)} + \alpha_i||x_1 - u|| - \phi_i(u) < 0$$

Hence, $$f_i(x_1) + \alpha_i g(x_1)||x_1 - u|| - \phi_i(u)g_i(x_1) < 0$$

Since $\tau_i \geqslant 0, \sum_{i=1}^p \tau_i = 1$, which yields that

$$\begin{aligned}
&\sum_{i=1}^p \tau_i(f_i(x_1) - \phi_i(u)g_i(x_1) + \alpha_i g_i^*||x_1 - u||) \\
&\quad \leqslant \sum_{i=1}^p \tau_i(f_i(x_1) + \alpha_i g(x_1)||x_1 - u|| - \phi_i(u)g_i(x_1)) \\
&\quad < 0 = \sum_{i=1}^p \tau_i(f_i(u) - \phi_i(u)g_i(u)).
\end{aligned}$$

But this is in conflict to (34). Consequently, (35) holds. This completes the proof. □

A strong duality theorem for (FMP) and (WDP) can be stated as follows.

Theorem 6.5. (Strong duality) *Let $\overline{x} \in M$ be a local weak efficient solution of* (MFP)*, and let all hypotheses of Theorem* 4.1 *be fulfilled. Then, there exist vectors* $(\overline{\tau}_1, ..., \overline{\tau}_p) \in \mathbb{R}^p_+ \setminus \{0\}$, $(\overline{\mu}_1, ..., \overline{\mu}_m) \in \mathbb{R}^{m+1}_+$*, and* $(\overline{\nu}_1, ..., \overline{\nu}_r) \in \mathbb{R}^r$ *such that* $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\nu})$ *is a feasible point of* (WDP)*, and the value of the objective functions of* (FMP) *and* (WDP) *at* $\overline{x}$ *and* $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\nu})$*, respectively, are equal. Moreover, if assumptions of Theorem* 6.4 *hold, then* $(\overline{x}, \overline{\tau}, \overline{\mu}, \overline{\nu})$ *is a weak efficient solution of* (WDP).

Proof. As in the proof of Theorem 6.2, since $\overline{x}$ is a local weak efficient solution of (MFP), we can invoke Theorem 4.1 to deduce that there exist $(\overline{\tau}_1, ..., \overline{\tau}_p) \in \mathbb{R}^p_+ \setminus \{0\}$, $(\overline{\mu}_1, ..., \overline{\mu}_m) \in \mathbb{R}^{m+1}_+$, and $(\overline{\nu}_1, ..., \overline{\nu}_r) \in \mathbb{R}^r$ such that

$$\begin{aligned}
0 \in \mathrm{cl}\Big(&\sum_{i=1}^p \overline{\tau}_i(\mathrm{conv}\ \partial^* f_i(u) - \phi_i(u)\mathrm{conv}\ \partial^* g_i(u) + \alpha_i g_i^* \mathbb{B}) \\
&+ \sum_{j \in I(u)} \overline{\mu}_j \mathrm{conv}\ \partial^* h_j(u) + \sum_{k \in L} \overline{\nu}_k \mathrm{conv}\ \partial^* \ell_k(u) + N(D, u)\Big),
\end{aligned}$$

Taking $\overline{\mu}_i = 0$ if $i \notin I(\overline{x})$, we obtain $(\overline{x}, \overline{\tau}, \overline{\mu}, \overline{\nu})$ is a feasible point of (WDP), and the value of the objective functions at $\overline{x}$ and $(\overline{x}, \overline{\tau}, \overline{\mu}, \overline{\nu})$ of (MFP) and (WDP), respectively, are equal.

If all the assumptions of Theorem 6.4 are fulfilled, since $\overline{x} \in M$, we deduce that for every feasible points (u, τ, μ, γ) of (WDP),

$$\phi(x) + \alpha||x - u|| \not< \phi(u).$$

Hence,
$$f_i(x) + \alpha_i||x - \overline{x}|| + \sum_{i=1}^{m} \overline{\mu}_j h_j(x) + \sum_{j=1}^{\ell} \overline{\nu}_k \ell_k(x)$$
$$\not\leqslant f_i(u) + \sum_{i=1}^{m} \overline{\mu}_j h_j(u) + \sum_{j=1}^{\ell} \overline{\nu}_k \ell_k(u).$$

Consequently, $(\overline{x}, \overline{\tau}, \overline{\mu}, \overline{\gamma})$ is a weak efficient solution of (WDP). □

Theorem 6.6. (Converse duality) *Let $\overline{x}$ and $(\overline{u}, \overline{\tau}, \overline{\mu}, \overline{\nu})$ be the feasible points of* (MFP) *and* (WDP), *respectively. Assume that*

(i) *f_i, g_i,h_j ℓ_k are continuous and admit convexificators $\partial^* f_i(u)$, $\partial^* g_i(u)$, $\partial^* h_j(u)$, $\partial^* \ell_k(u)$ at u near $\overline{u}$, respectively. They admit bounded convexificators $\partial^* f_i(\overline{u})$, $\partial^* g_i(\overline{u})$, $\partial^* h_j(\overline{u})$, $\partial^* \ell_k(\overline{u})$ at $\overline{u}$. Moreover, $\partial^* f_i, \partial^* g_i$, $\partial^* h_j$, and $\partial^* \ell_k$ are upper semicontinuous at $\overline{u}$.*

(ii) *The function $\sum_{i=1}^{p} \overline{\tau}_i(f_i(x) - \phi_i(u) g_i(x)) + \sum_{j \in I(\overline{u})} \overline{\mu}_j h_j + \sum_{k \in L} \overline{\nu}_k \ell_k$ is strict asymptotic pseudoinvex at u; C is convex. Suppose, furthermore, that*

$$\sum_{i=1}^{p} \overline{\tau}_i(f_i(\overline{x}) - \phi_i(\overline{u}) g_i(\overline{x}) + \alpha_i g_i^* ||\overline{x} - \overline{u}||) + \sum_{j \in I(\overline{u})} \overline{\mu}_j h_j(\overline{x}) + \sum_{k \in L} \overline{\nu}_k \ell_k(\overline{x})$$
$$\leqslant \sum_{i=1}^{p} \overline{\tau}_i(f_i(\overline{u}) - \phi_i(\overline{u}) g_i(\overline{u})) + \sum_{j \in I(\overline{u})} \overline{\mu}_j h_j(\overline{u}) + \sum_{k \in L} \overline{\nu}_k \ell_k(\overline{u}). \tag{36}$$

Then $\overline{u} = \overline{x}$.

Proof. Assume the contrary, namely that $\overline{u} \neq \overline{x}$. Since $(\overline{u}, \overline{\tau}, \overline{\mu}, \overline{\nu})$ be the feasible points of (WDP), it follows that there exist $\xi_i^{(n)} \in \text{conv}\, \partial^* f_i(\overline{u})$ $(i = 1, \ldots, p)$, $\eta_i^{(n)} \in \text{conv}\, \partial^* g_i(\overline{u})$, $\theta_i^{(n)} \in \alpha_i g_i^* \mathbb{B}$ $(i = 1, \ldots, p)$, $\zeta_j^{(n)} \in \text{conv}\, \partial^*(\overline{\mu}_j h_j)(\overline{u})$ $(j \in L)$, $\chi_k^{(n)} \in \text{conv}\, \partial^*(\overline{\nu}_k \ell_k)(\overline{u})$ $(k \in L)$, $\sigma^{(n)} \in N(D, \overline{u})$ such that

$$0 = \lim_{n \to \infty} \Big[\sum_{i=1}^{p} \overline{\tau}_i(\xi_i^{(n)} - \phi_i(u) \eta_i^{(n)} + \theta_i^{(n)}) + \sum_{j \in I(u)} \zeta_j^{(n)} + \sum_{k \in L} \chi_k^{(n)} + \sigma^{(n)} \Big]. \tag{37}$$

The convexity of D implies that for $x \in M$,

$$\lim_{n \to \infty} \langle \sigma^{(n)}, x - \overline{u} \rangle \leqslant 0. \tag{38}$$

By an argument of the proof of Theorem 5.1 we deduce that the set

$$\sum_{i=1}^{p} \overline{\tau}_i(\partial^* f_i(\overline{u}) - \phi_i(\overline{u}) \partial^* g_i(\overline{u}) + \alpha_i g_i^* \mathbb{B}) + \sum_{j \in I(\overline{u})} \overline{\mu}_j \partial^* h_j(\overline{u}) + \sum_{k \in L} \overline{\nu}_k \partial^* \ell_k(\overline{u}) \tag{39}$$

is a convexificator of the the following function at $\overline{u}$:

$$\sum_{i=1}^{p} \overline{\tau}_i(f_i(u) - \phi_i(\overline{u}) g_i(u) + \alpha_i g_i^* ||u - \overline{u}||) + \sum_{j \in I(\overline{u})} \overline{\mu}_j h_j(u) + \sum_{k \in L} \overline{\nu}_k \ell_k(u).$$

Combining (37), (38) yields that

$$\lim_{n\to\infty}\sum_{i=1}^{p}\langle\overline{\tau}_i(\xi_i^{(n)}-\phi_i(\overline{u})\eta_i^{(n)}+\theta_i^{(n)}),x-\overline{u}\rangle\geqslant 0.$$

Since the function $\sum_{i=1}^{p}\overline{\tau}_i(f_i(x)-\phi_i(u)g_i)(x))+\sum_{i\in I(\overline{u}}\overline{\mu}_j h_j(x)+\sum_{k\in L}\overline{\nu}_k\ell_k(x)$ is strict asymptotic pseudoconvex at $\overline{u}$, so is the function

$$\sum_{i=1}^{p}\overline{\tau}_i(f_i(x)-\phi_i(u)g_i)(x)+\alpha_i||x-\overline{u}||)+\sum_{i\in I(\overline{u}}\overline{\mu}_j h_j(x)+\sum_{k\in L}\overline{\nu}_k\ell_k(x).$$

Consequently,

$$\sum_{i=1}^{p}\overline{\tau}_i(f_i(\overline{x})-\phi_i(\overline{u})g_i(\overline{x})+\alpha_i g_i^*||\overline{x}-\overline{u}||)+\sum_{j\in I(u)}\overline{\mu}_j h_j(\overline{x})+\sum_{k\in L}\overline{\nu}_k\ell_k(\overline{x})$$
$$>\sum_{i=1}^{p}\overline{\tau}_i(f_i(\overline{u})-\phi_i(\overline{u})g_i(\overline{u}))+\sum_{j\in I(\overline{u})}\overline{\mu}_j h_j(\overline{u})+\sum_{k\in L}\overline{\nu}_k\ell_k(\overline{u}).$$

But this contradicts (36). Hence, $\overline{u}=\overline{x}$. □

7. Conclusions

Using results in [16] we derive necessary and sufficient efficiency conditions via convexificators for multiobjective fractional problems involving equality, inequality and set constraints which are assumed to be continuous but not necessarily Lipschitz. Under assumptions on the asymptotic pseudoconvexity for the objective function and the asymptotic quasiconvexity for the constraint functions, sufficient condition for weak efficiency are also given together with theorems of weak duality, strong duality, inverse duality of Wolfe and Mond-Weir types. The results obtained here include as special cases some results of [4], [8], [11], [12].

It should be noted that the necessary optimality conditions that are expressed in terms of convexificators can be sharper than those expressed by Clarke and Michel-Penot.

Acknowledgements. This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.01-2017.301.

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