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A Local Energy Estimate for a Class of Strongly Indefinite Functionals

Tian Xu
Center for Applied Mathematics, Tianjin University, Tianjin, P. R. China
xutian@amss.ac.cn

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Based on the generalized method of Nehari manifold, we present a unified approach for a refined local energy estimate for a class of strongly indefinite functionals. We give examples where this energy estimate is applied to the problems of describing qualitative properties for the linking levels.

Keywords: Strongly indefinite functional, Nehari manifold technique.

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1. Introduction

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. We consider in $\mathcal{H}$ the nonlinear equation of the form

$$Lz = N(z), \quad z \neq 0, \tag{1}$$

where $L : (\mathcal{H}, \langle\cdot,\cdot\rangle) \to (\mathcal{H}, \langle\cdot,\cdot\rangle)$ is an invertible continuous self-adjoint linear operator and N is a superlinear potential operator with potential Ψ, that is, $N = \nabla\Psi$. Let $\sigma(L)$ denote the spectrum of L. We set $\mathcal{H}^+$ and $\mathcal{H}^-$ the invariant subspaces of L corresponding to $\sigma(L) \cap (0, +\infty)$ and $\sigma(L) \cap (-\infty, 0)$, respectively, and assume that there exists $\delta > 0$ such that

$$\langle Lz, z\rangle \geq \delta\|z\|^2, \quad \forall z \in \mathcal{H}^+ \tag{2}$$

and

$$\langle Lz, z\rangle \leq -\delta\|z\|^2, \quad \forall z \in \mathcal{H}^-. \tag{3}$$

Here, we call L *strongly indefinite* if both the dimensions of $\mathcal{H}^\pm$ are infinite. Formally, (1) is the Euler-Lagrange equation of the functional

$$\Phi(z) = \frac{1}{2}\langle Lz, z\rangle - \Psi(z), \tag{4}$$

where z varies in $\mathcal{H}$. Then Φ is neither bounded from above nor below. Such problem arise for example in the study of periodic solutions of Hamiltonian systems of ODEs, in the study of stationary solutions of Schrödinger type equations, in the study of strongly coupled elliptic systems, in the existence theory for geometric equations involving Dirac-type operators, etc.

The basic common structure of these aforementioned problems has been previously emphasized, among others, by Amann [2] and Benci-Rabinowitz [5]. Since these

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problems have been the target of investigation for decades and it is not our intention to give a complete bibliography, we shall refer the interested reader to the basic references [11, 16, 17, 22].

Motivated by the study of a spinorial analogue of the Brezis-Nirenberg problem in [15], where a priori estimate of minimax levels of a strongly indefinite functional is involved, we are especially interested in the construction of a refined local energy estimate for the functional Φ in (4) near its lowest/smallest minimax level. More precisely, in contrast with the dual variational method in [15], we intend to establish a new kind of approximate calculation for the smallest minimax level of Φ so that one can insert test elements to obtain an accurate upper bound estimate. This is established in Section 2 (see Theorem 2.7) in an abstract setting. Then, in Section 3, we give applications of the abstract result to the following two representative nonlinear problems:

(A) Singularly perturbed Hamiltonian elliptic system on Riemannian manifolds

Let (M, g) be a smooth closed Riemannian m-manifold, $m \geq 3$. We consider the singularly perturbed nonlinear elliptic problem

$$\begin{cases} -\varepsilon^2 \Delta_g v + v = f_1(u), \\ -\varepsilon^2 \Delta_g u + u = f_2(v), \\ \quad u, v > 0, \end{cases} \qquad \text{on } M \tag{5}$$

where $\Delta_g = \mathrm{div}_g \nabla_g$ is the Laplace-Beltrami operator on (M, g) and f_1, f_2 fall into a typical class of superlinear functions. Beside the existence of a solution for a fixed value of ε, we are interested in the blow-up profile (the asymptotic behavior) of these solutions as ε goes to zero. Particularly, we are looking for a local geometric quantity of (M, g) which affects the location of a global maximum point of a solution of eq. (5) as $\varepsilon \searrow 0$.

(B) Spinorial Brezis-Nirenberg problem

Let (M, g, σ) be an m-dimensional closed spin manifold, $m \geq 2$, where g is a Riemannian metric and $\sigma : P_{Spin}(M) \to P_{SO}(M)$ is a spin structure on M. We denote $\mathbb{S}(M) = P_{Spin}(M) \times_\rho \mathbb{S}_m \to M$ the spinor bundle. It is a complex vector bundle associated to $P_{Spin}(M) \to M$ via the spin representation $\rho : Spin(m) \to \mathrm{Aut}(\mathbb{S}_m)$. The Dirac operator $D_g : C^\infty(M, \mathbb{S}(M)) \to C^\infty(M, \mathbb{S}(M))$ is defined by

$$\begin{aligned} D_g := \mathfrak{M} \circ \nabla : C^\infty(M, \mathbb{S}(M)) &\xrightarrow{\nabla} C^\infty(M, T^*M \otimes \mathbb{S}(M)) \\ &\cong C^\infty(M, TM \otimes \mathbb{S}(M)) \xrightarrow{\mathfrak{M}} C^\infty(M, \mathbb{S}(M)) \end{aligned}$$

where ∇ is the canonical lift of the Levi-Civita connection on $P_{SO}(M)$ via the double covering $P_{Spin}(M) \to P_{SO}(M)$ and $\mathfrak{M}$ is the Clifford multiplication.

We consider the spinorial Brezis-Nirenberg problem of the following form

$$D_g \psi = \lambda \psi + |\psi|^{2^*-2} \psi \quad \text{on } M, \tag{6}$$

where $2^* = \frac{2m}{m-1}$, $\lambda \in \mathbb{R}$ and $\psi \in C^1(M, \mathbb{S}(M))$ is a spinor.

The main characteristic of the problem is that the exponent 2^* is critical in the sense of the Sobolev embedding $H^{\frac{1}{2}}(M, \mathbb{S}(M)) \hookrightarrow L^q(M, \mathbb{S}(M))$ for $1 \leq q \leq 2^*$, which is compact precisely if $q < 2^*$.

Partial existence results for eq. (6) were previously established in [15] for some positive values of λ. And recently [4] considered a more generalized situation. However, the problem remains widely open.

Both of the just mentioned problems are of strongly indefinite type. And from the viewpoint of analysis, the main difficulties in them are to deal with blow-up profiles appearing in the variational framework.

For the first application, we mention that the study of existence of solutions to the Hamiltonian elliptic systems has been of great interest in recent years. For this type of result see, among others, the survey papers [6] and [12]. Under the variational framework in [6, 12], for a fixed $\varepsilon > 0$, it is not difficult to see the existence result for eq. (5) via standard variational argument (particularly for subcritical nonlinearities). When the parameter ε is varying, it is a classical problem to localize the peak of a solution as ε goes to 0. Generally speaking, as $\varepsilon \to 0$, the solution exhibits a spike layer and the blow-up profile localizes the asymptotic behavior of the spike layer. In order to characterize the specific asymptotic behavior one needs a delicate description of the min-max levels in the variational procedure in terms of local geometric quantities of the background Riemannian manifold (M, g). One of our main results is Theorem 3.7 which says that the ground-state solution of (5) blows up around a maximum point of the scalar curvature Scal_g on M as $\varepsilon \to 0$. This is a new result in the study of Hamiltonian elliptic systems.

Our second application in Section 3 concerns the eq. (6). Here we report one of the existence results obtained in [4] to show the broad applicability of our abstract result. Analogous to its counterpart of elliptic type, the criticality of the nonlinearity causes the so-called bubbling-phenomenon, that is, energy concentration associated to the appearance of blow-up profiles. In [15] Isobe characterized the behavior of a generic Palais-Smale sequence for (6). In general, when the bubbling-phenomenon occurs for a Palais-Smale sequence, the bubbles are in a finite number (since one can prove a uniform lower bound for their energies). This implies that we have compactness only in a suitable energy range and it gives a threshold value for the appearance of bubbles in the min-max methods. In our approach, we are able to complement the result in [15] where a same problem was studied. The result in [15] provides the existence of a solution to (6) for $m \geq 4$, $\lambda > 0$ and $\lambda \notin Spec(D_g)$ ($Spec(D_g)$ is the spectrum of D_g). Here, our main result is Theorem 3.11, which states that eq. (6) possesses at least one non-trivial solution for all $\lambda > 0$ and dimension $m \geq 2$.

From a variational point of view, the problem (A), as well as the problem (B), is typically solved by minimizing an associated constrained problem. This formulation was developed by Szulkin and Weth [23]. The advantage of this kind of approach is to give the most simplified characterization of the critical values for a indefinite functional. As the corresponding functionals of problems (A) and (B) are of strongly indefinite type, the direct method of the calculus of variation fails, but the reduction techniques and Nehari manifold techniques typically work. Hence these problems share many similarities.

However, only a few statements can be proven simultaneously for both of the two problems. First of all, let us mention that both problems (A) and (B) concern with a family of differential equations (parameterized by $\varepsilon > 0$ in (A) and by $\lambda \in \mathbb{R}$ in (B)). In our applications in Section 3, for a fixed $\varepsilon > 0$, the functional associated to eq. (5) satisfies the Palais-Smale condition. By employing the Nehari manifold technique in [23], one does not need to worry about the existence of a least energy solution. In this regard, the main target is to study the limiting behavior of the solutions and the corresponding energies as $\varepsilon \to 0$. Since the least energy solution of eq. (5) cannot be explicitly formulated, one possible way to characterize the qualitative properties of the limiting behavior is to compute approximations of those least energies by using some suitable test functions. These test functions are varying synchronously with ε. Being differently, the analysis in (B) is more involved due to the appearance of the critical nonlinearity in eq. (6) and the number one issue is to show the existence result. Standard methods developed in [11, 16, 17, 22, 23] etc. do not apply here due to the lack of compactness. To attack this problem, it is sufficient to deal with only one functional (i.e. λ is fixed) and, as mentioned before, the main ingredient here is to establish a priori energy estimate for the associated functional and to show the lowest minimax level stays below the threshold for noncompactness. As in (A), the energy estimate is based on a careful choice of some test functions. Since λ is fixed, the test functions should be constructed independently. Because of these differences, most analytical properties have to be proven for each problem separately. We consider it hence as remarkable that our abstract result on the energy estimate can be extended to all such problems.

2. The abstract results

We suppose that there is $\Psi \in C^2(\mathcal{H}, \mathbb{R})$ satisfying

(H1) $N(z) = \nabla\Psi(z)$ and there is a continuous map $\varrho : [0,\infty) \to [0,\infty)$ such that $|||\nabla N(z)||| \le \varrho(\|z\|)$, where $|||\cdot|||$ denotes the operator norm for continuous linear operators on $(\mathcal{H}, \langle\cdot,\cdot\rangle)$.

(H2) $\lim\limits_{\|z\|\to 0} \Psi(z)/\|z\| = 0$ and there exists $p > 2$ such that $\langle N(z), z\rangle \ge p\Psi(z)$ for all $z \in \mathcal{H}$.

(H3) $\Psi \not\equiv 0$ on $\mathcal{H}$ and is convex.

(H4) There exist $K > 0$ and $\mu \in (\frac{1}{2}, 1)$ such that $\|N(z)\| \le K \langle N(z), z\rangle^{\mu} + \mu\delta\|z\|$ for all $z \in \mathcal{H}$, where $\delta > 0$ is the constant in (2) and (3).

(H5) There exists $\kappa > 1$ such that, for arbitrary $z, w \in \mathcal{H}$,

$$\langle \nabla N(z)[z+w], z+w\rangle - 2\langle N(z), w\rangle \ge \kappa \langle N(z), z\rangle .$$

(H6) If there is a bounded sequence $\{z_j\} \subset \mathcal{H}$ such that

$$Lz_j - N(z_j) \to 0 \quad \text{and} \quad \liminf_{j\to\infty} \langle N(z_j), z_j\rangle > 0,$$

then there exists $z \neq 0$ with $Lz = N(z)$ in $\mathcal{H}$.

It can be seen from (H2) that $\Psi(0) = 0$ and the function $\Psi(tz)/t^p$, $t > 0$, is nondecreasing in t for all $z \in \mathcal{H}$. Combined with the convexity in (H3), it follows that $\Psi \ge 0$ (i.e. 0 is the global minimum of Ψ) and

$$\Psi(z) = 0 \Longrightarrow \Psi(tz) = 0 \text{ for all } t \in [0,1].$$

Furthermore, from (H4) and the obvious inequality

$$ab \le \mu a^{\frac{1}{\mu}} + (1-\mu)b^{\frac{1}{1-\mu}}, \quad \text{for } a, b \ge 0$$

we deduce that $\|N(z)\| \le \epsilon\mu\|N(z)\| + C_\epsilon(1-\mu)K^{\frac{1}{1-\mu}}\|z\|^{\frac{\mu}{1-\mu}} + \mu\delta\|z\|$ where $\epsilon > 0$ is arbitrary and $C_\epsilon > 0$ is a constant depending only on ϵ. If ϵ is fixed small, then we have the following boundedness of N:

$$\|N(z)\| \le \frac{C_\epsilon(1-\mu)K^{\frac{1}{1-\mu}}}{1-\epsilon\mu}\|z\|^{\frac{\mu}{1-\mu}} + \frac{\mu\delta}{1-\epsilon\mu}\|z\|, \quad \text{for all } z \in \mathcal{H}. \tag{7}$$

Beyond the positiveness and the convexity of Ψ, (H5) implies that $\nabla N(z) \neq 0$ when $\Psi(z) > 0$. And finally, it is clear that if $N : \mathcal{H} \to \mathcal{H}$ is compact then (H6) holds. We emphasize here that the sequence $\{z_j\}$ in (H6) may not be compact in general. Actually such sequence can even converge weakly to 0 in $\mathcal{H}$, and then a concentration-compactness type argument can be employed (see for instance [1, 7, 11, 12, 16] and references therein, where some semilinear problems on $\mathbb{R}^N$ are considered).

In what follows, we intend to characterize qualitative properties of critical values of the functional

$$\Phi(z) = \frac{1}{2}\langle Lz, z\rangle - \Psi(z), \quad z \in \mathcal{H}.$$

Under the hypotheses imposed above, the functional Φ is a well-defined C^2-functional on $\mathcal{H}$. Recall that 0 lies in a gap of the spectrum of $L : \mathcal{H} \to \mathcal{H}$, the spectral theory asserts that the L-invariant subspaces $\mathcal{H}^\pm$ associated to the positive and negative components of $\sigma(L)$ are orthogonal with respect to the scalar product $\langle\cdot,\cdot\rangle$. Particularly, $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ and each element $z \in \mathcal{H}$ possesses the representation $z = z^+ + z^-$, where $z^\pm \in \mathcal{H}^\pm$. And we easily see that, for $z, w, \varphi \in \mathcal{H}$,

$$\Phi(z) = \frac{1}{2}\left\langle Lz^+, z^+\right\rangle + \frac{1}{2}\left\langle Lz^-, z^-\right\rangle - \Psi(z), \quad \langle\nabla\Phi(z), w\rangle = \langle Lz - N(z), w\rangle,$$

and

$$\left\langle\nabla^2\Phi(z)[\varphi], w\right\rangle = \langle L\varphi - \nabla N(z)[\varphi], w\rangle$$

2.1. Variational formulation

The first step is a generalized version of the Lyapunov-Schmidt reduction procedure, developed for strongly indefinite functionals. This allows to study an equivalent reduced functional whose critical points are in one-to-one correspondence with those of Φ. This idea can be traced back to [2, 9], and was also used in [1, 7, 23]. Here, in the hypotheses (H1)-(H6) above, we allow the nonlinearity Ψ to vanish on certain part of $\mathcal{H}$, and this is not the case in [1, 2, 7, 9, 23].

We begin with the following result concerning the boundedness of a Palais-Smale sequence of Φ.

Lemma 2.1. *Let $\{z_n\} \subset \mathcal{H}$ and $R > 0$ such that $\sup_n \Phi(z_n) \le R$ and $\nabla\Phi(z_n) \to 0$. Then, for large n, $\|z_n\|$ is bounded by a constant depending only on R.*

Proof. For sufficient large n, we have

$$\begin{aligned}\langle N(z_n), z_n\rangle &= \langle Lz_n, z_n\rangle - \langle \nabla\Phi(z_n), z_n\rangle = 2\Phi(z_n) + 2\Psi(z_n) + o_n(1)\|z_n\| \\ &\le 2R + \frac{2}{p}\langle N(z_n), z_n\rangle + o_n(1)\|z_n\|.\end{aligned}$$

Hence $\langle N(z_n), z_n\rangle \le \frac{p}{p-2}\big(2R + o_n(1)\|z_n\|\big)$ and so, by (H4),

$$\begin{aligned}\delta\|z_n\| \le \|Lz_n\| &\le \|\nabla\Phi(z_n)\| + \|N(z_n)\| \\ &\le o_n(1) + K\langle N(z_n), z_n\rangle^{\mu} + \mu\delta\|z_n\| \\ &\le o_n(1) + K\frac{p^{\mu}}{(p-2)^{\mu}}\big(2R + o_n(1)\|z_n\|\big)^{\mu} + \mu\delta\|z_n\|.\end{aligned}$$

Since $\mu < 1$, this implies the boundedness of $\|z_n\|$. □

By taking the advantage of the positiveness and convexity of Ψ, together with (3), we have a very good geometric behavior of the functional Φ in the following sense:

(i) Since Ψ is non-negative, for a given $\varphi \in \mathcal{H}^+$, the functional Φ^{φ} defined on $\mathcal{H}^-$ by
$$\Phi^{\varphi}(w) = \frac{1}{2}\langle L\varphi, \varphi\rangle + \frac{1}{2}\langle Lw, w\rangle - \Psi(\varphi + w), \quad w \in \mathcal{H}^-$$
is anti-coercive, i.e., $\Phi^{\varphi}(w) \to -\infty$ as $\|w\| \to \infty$.

(ii) Since Ψ is convex, Φ^{φ} is strictly concave, in other words, the quadratic form
$$\big\langle \nabla^2\Phi^{\varphi}(w)[\,\cdot\,], \cdot\,\big\rangle : \mathcal{H}^- \times \mathcal{H}^- \to \mathbb{R}$$
is negative definite.

Combining the above two properties, for each $\varphi \in \mathcal{H}^+ \setminus \{0\}$, we get a chance to maximize the functional Φ on $\mathbb{R}^+\varphi \oplus \mathcal{H}^-$, where $\mathbb{R}^+ = (0, \infty)$. Remark that φ is varying in the space $\mathcal{H}^+ \setminus \{0\}$. Therefore we can restrict ourselves to the choice $\varphi \in S^+ := \{\varphi \in \mathcal{H}^+ : \|\varphi\| = 1\}$ without changing the maxima of Φ on $\mathbb{R}^+\varphi \oplus \mathcal{H}^-$. By virtue of (7) in which we can fix $\epsilon \le 1 - \mu$, we deduce from the second part of (H2) that

$$\Psi(z) \le \frac{1}{p}\langle N(z), z\rangle \le \frac{\delta}{p}\|z\|^2 + C\|z\|^{\frac{1}{1-\mu}} \tag{8}$$

for all $z \in \mathcal{H}$, where $C > 0$ is a constant. Indeed, this is a direct consequence of $\frac{\mu}{1-\epsilon\mu} < \frac{\mu}{1-\epsilon} \le 1$. And so, by (2) and the fact $\frac{1}{1-\mu} > 2$, we have the existence of $\tau > 0$ such that

$$\max_{z \in \mathbb{R}^+\varphi \oplus \mathcal{H}^-} \Phi(z) \ge \max_{t>0} \Phi(t\varphi) \ge \max_{t>0}\Big(\frac{p-2}{2p}\delta t^2 - Ct^{\frac{1}{1-\mu}}\Big) \ge \tau \quad \text{for all } \varphi \in S^+. \tag{9}$$

Therefore, we have found a candidate min-max scheme for Φ: first maximize the functional Φ on $\mathbb{R}^+\varphi \oplus \mathcal{H}^-$ and then minimize with respect to $\varphi \in S^+$. Specifically, let $P : \mathcal{H} \to \mathcal{H}^+$ denote the orthogonal projection onto $\mathcal{H}^+$, we have the following variational formulation.

Proposition 2.2. (1) *There exists $\chi \in C^1(\mathcal{H}^+, \mathcal{H}^-)$ such that for $\varphi \in \mathcal{H}^+$*

$$w \in \mathcal{H}^-,\ w \neq \chi(\varphi) \Longrightarrow \Phi(\varphi + w) < \Phi(\varphi + \chi(\varphi)),$$

that is $\chi(\varphi)$ is the unique maximizer of Φ^φ on $\mathcal{H}^-$ and hence

$$L\big(\chi(\varphi)\big) = (I - P)N(\varphi + \chi(\varphi)); \tag{10}$$

furthermore, for $\varphi \in \mathcal{H}^+$,

we have $$\|\chi(\varphi)\|^2 \leq \frac{2}{\delta}\,\Psi(\varphi) \tag{11}$$

and $$|||\nabla\chi(\varphi)||| \leq \frac{1}{\delta}\,|||\nabla N(\varphi + \chi(\varphi))|||. \tag{12}$$

(2) *If $\{\varphi_n\} \subset \mathcal{H}^+$ is a Palais-Smale sequence for the reduced functional*

$$J : \mathcal{H}^+ \to \mathbb{R}, \quad J(\varphi) = \Phi(\varphi + \chi(\varphi)),$$

then $\{\varphi_n + \chi(\varphi_n)\}$ is a Palais-Smale sequence for Φ, and in particular we have $\|\nabla J(\varphi)\| = \|\nabla\Phi(\varphi + \chi(\varphi))\|$ for all $\varphi \in \mathcal{H}^+$.

Proof. For a given $\varphi \in \mathcal{H}^+$, due to the anti-coerciveness and the strictly concavity of Φ^φ, we have the existence of a unique maximizer $\chi(\varphi) \in \mathcal{H}^-$ for Φ^φ. Since $\chi(\varphi)$ is a critical point of Φ^φ, it satisfies $\langle\nabla\Phi^\varphi(\chi(\varphi)), w\rangle = 0$ for all $w \in \mathcal{H}^-$, that is, $\chi(\varphi)$ is the unique element in $\mathcal{H}^-$ such that

$$L\big(\chi(\varphi)\big) - (1 - P)N(\varphi + \chi(\varphi)) = 0.$$

By taking derivative of the left hand side with respect to $\chi(\varphi)$, we get a self-adjoint operator on $\mathcal{H}^-$:

$$w \mapsto Lw - (1 - P)\nabla N(\varphi + \chi(\varphi))[w], \quad \forall w \in \mathcal{H}^-. \tag{13}$$

Since

$$\langle Lw - (1 - P)\nabla N(\varphi + \chi(\varphi))[w], w\rangle = \langle Lw - \nabla N(\varphi + \chi(\varphi))[w], w\rangle$$
$$\leq \langle Lw, w\rangle \ \text{[by the convexity of } \Psi] \ \leq -\delta\|w\|^2 \ \text{[by (3)]}$$

for all $w \in \mathcal{H}^-$, we see that the operator in (13) is invertible and the operator norm of its inverse is bounded by $1/\delta$. Hence, the implicit function theorem shows $\chi \in C^1(\mathcal{H}^+, \mathcal{H}^-)$ and

$$|||\nabla\chi(\varphi)||| \leq \frac{1}{\delta}\,|||\nabla N(\varphi + \chi(\varphi))|||.$$

Meanwhile, since $\chi(\varphi)$ maximize Φ^φ,

$$0 \leq \Phi^\varphi(\chi(\varphi)) - \Phi^\varphi(0) = \frac{1}{2}\left\langle L\big(\chi(\varphi)\big), \chi(\varphi)\right\rangle - \Psi(\varphi + \chi(\varphi)) + \Psi(\varphi)$$
$$\leq \frac{1}{2}\left\langle L\big(\chi(\varphi)\big), \chi(\varphi)\right\rangle + \Psi(\varphi) \qquad \text{[by the non-negativeness of } \Psi]$$
$$\leq -\frac{\delta}{2}\|\chi(\varphi)\|^2 + \Psi(\varphi) \qquad \text{[by (3)]}$$

we conclude the boundedness of $\|\chi(\varphi)\|$.

Finally note that for $\phi \in \mathcal{H}$,

$$\begin{aligned}\langle \nabla\Phi(\varphi+\chi(\varphi)),\phi\rangle &= \langle \nabla\Phi(\varphi+\chi(\varphi)),P\phi\rangle + \langle \nabla\Phi(\varphi+\chi(\varphi)),(I-P)\phi\rangle \\ &= \langle \nabla J(\varphi),P\phi\rangle\end{aligned}$$

we have $\|\nabla J(\varphi)\| \geq \|\nabla\Phi(\varphi+\chi(\varphi))\|$. And on the other hand, we find for $w \in \mathcal{H}^+$

$$\begin{aligned}\langle \nabla J(\varphi),w\rangle &= \langle \nabla\Phi(\varphi+\chi(\varphi)),w\rangle + \langle \nabla\Phi(\varphi+\chi(\varphi)),\nabla\chi(\varphi)[w]\rangle \\ &= \langle \nabla\Phi(\varphi+\chi(\varphi)),w\rangle\end{aligned}$$

and so $\|\nabla J(\varphi)\| \leq \|\nabla\Phi(\varphi+\chi(\varphi))\|$. This completes the proof. □

2.2. Generalized Nehari manifold

By virtue of Proposition 2.2, we are reduced to consider the critical values of the functional J on $\mathcal{H}^+$. The essence of our analysis is to consider the minimax value

$$\gamma = \inf_{\varphi\in\mathcal{H}^+\setminus\{0\}} \max_{t>0} J(t\varphi). \tag{14}$$

Since $J(\varphi) \geq \Phi(\varphi)$ for all $\varphi \in \mathcal{H}^+$, together with (8) and (9), we have the existence of $r > 0$ such that

$$\|\varphi\| \leq r \Longrightarrow J(\varphi) \geq 0 \quad \text{and} \quad \|\varphi\| = r \Longrightarrow J(\varphi) \geq \tau.$$

It is clear that if there exists additionally $\phi \in \mathcal{H}^+$, $\|\phi\| > r$, such that $J(\phi) < 0$, then the reduced functional J possesses the mountain pass geometry, and hence $\gamma \geq \tau > 0$ is the lowest value of J at which there exist non-trivial critical points. This is the case when the nonlinearity $\Psi(\varphi)$ vanishes if and only if $\varphi = 0$, see for instance [1, 7, 23]. Unfortunately, since we allow Ψ to vanish on certain part of $\mathcal{H}$, we do not have an apparent linking structure for J. Here, our strategy consists of finding critical point of J on its Nehari-type manifold $\mathscr{N}$ defined as

$$\mathscr{N} = \big\{\varphi \in \mathcal{H}^+ \setminus \{0\} : \langle \nabla J(\varphi),\varphi\rangle = 0\big\}.$$

Different from the usual concept of Nehari manifold, we emphasize here that the set $\mathscr{N}$ above is not homeomorphic to the sphere S^+ in general. In fact, as we will see later in Section 3, we allow that there exists $\varphi \in \mathcal{H}^+\setminus\{0\}$ such that $\Psi(t\varphi) \equiv 0$ for all $t > 0$. Then, by Proposition 2.2, we have $\chi(t\varphi) \equiv 0$, and hence $J(t\varphi) = \frac{t^2}{2}\langle L\varphi,\varphi\rangle$, i.e. $J(t\varphi)$ is a quadratic function and has no critical points for $t \in [0,\infty)$ other than $t = 0$.

Lemma 2.3. *If $\varphi \in \mathscr{N}$, then $\Psi(\varphi) > 0$ and $\Psi(\varphi+\chi(\varphi)) > 0$.*

Proof. Assume contrarily that $\Psi(\varphi) = 0$, then it follows that $N(\varphi) = 0$ (since 0 is the global minimum of Ψ). And from (11), we have $\chi(\varphi) = 0$. Hence

$$\langle \nabla J(\varphi),\varphi\rangle = \langle L\varphi - PN(\varphi),\varphi\rangle = \langle L\varphi,\varphi\rangle > 0,$$

which is a contradiction.

If $\Psi(\varphi+\chi(\varphi)) = 0$, then it also follows that $N(\varphi+\chi(\varphi)) = 0$. By (10) and the invertibility of L on $\mathcal{H}^-$, we deduce that $\chi(\varphi) = 0$. Thus we get a similar contradiction as above. □

Let us introduce the functional $\mathcal{K} : \mathcal{H}^+ \to \mathbb{R}$ by

$$\mathcal{K}(\varphi) = \langle \nabla J(\varphi), \varphi \rangle . \tag{15}$$

We have $\mathcal{K} \in C^1(\mathcal{H}^+, \mathbb{R})$ and its derivative is given by

$$\langle \nabla \mathcal{K}(\varphi), w \rangle = \langle \nabla J(\varphi), w \rangle + \left\langle \nabla^2 J(\varphi)[w], \varphi \right\rangle$$

for all $\varphi, w \in \mathcal{H}^+$. Also, we have $\mathscr{N} = \mathcal{K}^{-1}(0) \setminus \{0\}$.

Lemma 2.4. *For $\varphi \in \mathcal{H}^+$,*

$$\langle \nabla \mathcal{K}(\varphi), \varphi \rangle \le 2\mathcal{K}(\varphi) - (\kappa - 1) \langle N(\varphi + \chi(\varphi)), \varphi + \chi(\varphi) \rangle ,$$

where $\kappa > 1$ is the constant in (H5).

Proof. Since

$$\langle \nabla J(\varphi), \phi \rangle = \langle L\varphi - PN(\varphi + \chi(\varphi)), \phi \rangle = \langle L\varphi, \phi \rangle - \langle N(\varphi + \chi(\varphi)), \phi \rangle$$

for all $\varphi, \phi \in \mathcal{H}^+$ it follows directly that

$$\left\langle \nabla^2 J(\varphi)[\phi], \phi \right\rangle = \langle L\phi, \phi \rangle - \left\langle \nabla N(\varphi + \chi(\varphi)) \big[\phi + \nabla \chi(\varphi)[\phi]\big], \phi \right\rangle . \tag{16}$$

By taking derivative with respect to φ in the relation (10), we have

$$L\big(\nabla \chi(\varphi)[\varphi]\big) = (I - P) \nabla N(\varphi + \chi(\varphi)) \big[\varphi + \nabla \chi(\varphi)[\varphi]\big] \tag{17}$$

for $\varphi \in \mathcal{H}^+$. For ease of notation, we write $z_\varphi = \varphi + \chi(\varphi)$ and $w_\varphi = \nabla \chi(\varphi)[\varphi] - \chi(\varphi)$. Then we deduce from (16) and the relations (10) and (17) that

$$\begin{aligned}
\left\langle \nabla^2 J(\varphi)[\varphi], \varphi \right\rangle &= \langle L\varphi, \varphi \rangle - \langle \nabla N(z_\varphi)[z_\varphi + w_\varphi], \varphi \rangle \\
&= \langle \nabla J(\varphi), \varphi \rangle + \langle N(z_\varphi), \varphi \rangle - \langle \nabla N(z_\varphi)[z_\varphi + w_\varphi], \varphi \rangle \\
&= \langle \nabla J(\varphi), \varphi \rangle - \langle \nabla N(z_\varphi)[z_\varphi + w_\varphi], z_\varphi + w_\varphi \rangle + 2 \langle N(z_\varphi), w_\varphi \rangle \\
&\qquad + \langle N(z_\varphi), z_\varphi \rangle + \langle L w_\varphi, w_\varphi \rangle \\
&\le \langle \nabla J(\varphi), \varphi \rangle - (\kappa - 1) \langle N(z_\varphi), z_\varphi \rangle - \delta \|w_\varphi\|^2 \text{ [by (H5) and (3)]}
\end{aligned}$$

for $\varphi \in \mathcal{H}^+$. Hence the assertion follows. □

It follows from the above two lemmas and the second part of (H2) that, for $\varphi \in \mathscr{N}$, $\nabla \mathcal{K}(\varphi) \neq 0$. Hence $\mathscr{N}$ is locally a C^1-manifold. Moreover, if $\varphi \in \mathscr{N}$ is a constrained critical point of J, then φ is also a critical point of J on $\mathcal{H}^+$. Indeed, according to the Lagrange multiplier rule, we have the existence of $\lambda \in \mathbb{R}$ with $\nabla J(\varphi) = \lambda \nabla \mathcal{K}(\varphi)$. Note that $\varphi \in \mathscr{N}$, we find $\langle \nabla J(\varphi), \varphi \rangle = 0$. Together with Lemma 2.4, this implies $\lambda = 0$ and φ is a critical point of J on $\mathcal{H}^+$.

For later use, we also mention that if $\mathscr{N} \neq \emptyset$ then $\mathscr{N}$ is bounded away from 0, i.e., there exists $r_0 > 0$ such that $\|\varphi\| \ge r_0$ for all $\varphi \in \mathscr{N}$. In fact, for a given $\varphi \in S^+$, let us consider the function $t \mapsto J(t\varphi)$, $t \ge 0$. Then we have

$$\frac{d}{dt} J(t\varphi) = \langle \nabla J(t\varphi), \varphi \rangle = \frac{\mathcal{K}(t\varphi)}{t}$$

and

$$\frac{d^2}{dt^2} J(t\varphi) = \left\langle \nabla^2 J(t\varphi)[\varphi], \varphi \right\rangle = \frac{1}{t^2} \big(\langle \nabla \mathcal{K}(t\varphi), t\varphi \rangle - \mathcal{K}(t\varphi) \big).$$

It follows from Lemma 2.4 that if there is $t_\varphi > 0$ such that $\mathcal{K}(t_\varphi\varphi) = 0$, i.e., $t_\varphi\varphi \in \mathscr{N}$, then $\frac{d^2}{dt^2}J(t\varphi)\big|_{t=t_\varphi} < 0$. And hence t_φ is a strict local maximum point. We actually have more. In fact, by Lemma 2.3 and 2.4, we can see that $\mathcal{K}(t\varphi) < 0$ for all $t > t_\varphi$. Therefore, t_φ should be the unique critical point of $J(t\varphi)$, $t > 0$, and hence the global maximum point. Now, by (9), we find $\max_{t>0} J(t\varphi) \geq \tau$, and this implies the existence of $r_0 > 0$ such that $t_\varphi \geq r_0$.

We also note that the min-max value in (14) has the following characterization

$$\gamma = \inf_{\varphi\in\mathcal{H}^+\setminus\{0\}} \max_{t>0} J(t\varphi) = \inf_{\varphi\in\mathcal{H}^+\setminus\{0\}} \max_{z\in\mathbb{R}^+\varphi\oplus\mathcal{H}^-} \Phi(z) = \inf_{\varphi\in\mathscr{N}} J(\varphi). \tag{18}$$

Particularly, $\gamma < +\infty$ provided that $\mathscr{N} \neq \emptyset$.

2.3. A local energy estimate

Our next step is to establish an approximate calculation for the minimax level γ in (18). Particularly, this approximate calculation is based on a sequence of test elements in $\mathcal{H}$. Motivated by the applications mentioned in the Introduction, we need to cover the case that the test elements are varying synchronously with some parameter in the variational problem.

To this end, let us take $\mathcal{A} \subset \mathbb{R}$ to be an open interval. For $\alpha \in \mathcal{A}$, we equip $\mathcal{H}$ with a scalar product $\langle\cdot,\cdot\rangle_\alpha$ and the induced norm $\|\cdot\|_\alpha$. Then we consider in $\mathcal{H}$ the following family of equations

$$L_\alpha z = N_\alpha(z), \quad z \neq 0,\ \alpha \in \mathcal{A}, \tag{19}$$

where $L_\alpha : (\mathcal{H}, \langle\cdot,\cdot\rangle_\alpha) \to (\mathcal{H}, \langle\cdot,\cdot\rangle_\alpha)$ is of strongly indefinite type as in (2) and (3), N_α satisfies (H1)–(H5) with the function ϱ and the positive constants δ, p, K, μ and κ independent of $\alpha \in \mathcal{A}$. We shall denote $\mathcal{H}^\pm_\alpha$ the L_α-invariant subspaces associated to the positive and negative components of $\sigma(L_\alpha)$, which are orthogonal with respect to the scalar product $\langle\cdot,\cdot\rangle_\alpha$. Then we have $\mathcal{H} = \mathcal{H}^+_\alpha \oplus \mathcal{H}^-_\alpha$, and each element $z \in \mathcal{H}$ possesses the representation $z = z^+_\alpha + z^-_\alpha$ with $z^\pm_\alpha \in \mathcal{H}^\pm_\alpha$. The energy functional corresponds to eq. (19) is given by

$$\Phi_\alpha(z) = \frac{1}{2}\langle L_\alpha z, z\rangle_\alpha - \Psi_\alpha(z), \quad z \in \mathcal{H},\ \alpha \in \mathcal{A}.$$

It is clear that, for a fixed $\alpha \in \mathcal{A}$, we are dealing with only one equation and one functional, particularly, all the results set up previously apply to Φ_α. In what follows, let us consider $\{\alpha_n\} \subset \mathcal{A}$ and $\{z_n\} \subset \mathcal{H}$ such that

$$c_1 \leq \Phi_n(z_n) \leq c_2 \quad \text{and} \quad \|\nabla\Phi_n(z_n)\|_n \to 0 \quad \text{as } n \to \infty, \tag{20}$$

for some fixed constants $c_1, c_2 > 0$, where we write $\Phi_n = \Phi_{\alpha_n}$ and $\|\cdot\|_n = \|\cdot\|_{\alpha_n}$ for simplicity. In this setting, if $\alpha_n \equiv \alpha \in \mathcal{A}$ (that is, α_n equals to a fixed value), then $\{z_n\}$ is nothing but a generic Palais-Smale sequence for Φ_α. When α_n is varying for different n, the situation is more involved since we have a sequence of functionals Φ_n and $\{z_n\}$ is no longer a Palais-Smale sequence. Particularly, the boundedness of z_n needs to be measured by $\|\cdot\|_n$ correspondingly.

Lemma 2.5. *Under* (20), *we have:*

(1) $\|z_n\|_n$ *is uniformly bounded in* n;

(2) $\|(I-P_n)z_n-\chi_n(P_nz_n)\|_n = O(\|\nabla\Phi_n(z_n)\|_n)$ *as* $n\to\infty$, *where* $P_n=P_{\alpha_n}:\mathcal{H}\to\mathcal{H}^+_{\alpha_n}$ *is the orthogonal projection and* $\chi_n=\chi_{\alpha_n}\in C^1(\mathcal{H}^+_{\alpha_n},\mathcal{H}^-_{\alpha_n})$ *is the reduction map;*

(3) $\nabla J_n(P_nz_n)\to 0$ *as* $n\to\infty$, *where* $J_n=J_{\alpha_n}$.

Proof. By repeating the arguments in Lemma 2.1, we see the boundedness of $\|z_n\|_n$. To check (2), let us set $\phi_n = P_nz_n+\chi_n(P_nz_n)$ and $w_n = z_n-\phi_n$. Then, we have $w_n=(I-P_n)z_n-\chi_n(P_nz_n)\in\mathcal{H}^-_{\alpha_n}$. Thanks to the relation (10), we see that

$$0=\langle\nabla\Phi_n(\phi_n),w_n\rangle_n=\langle L_n\phi_n,w_n\rangle_n-\langle N_n(\phi_n),w_n\rangle_n$$

where, for simplicity, we write $L_n=L_{\alpha_n}$, $N_n=N_{\alpha_n}$ and $\langle\cdot,\cdot\rangle_n=\langle\cdot,\cdot\rangle_{\alpha_n}$.

Since $$\langle\nabla\Phi_n(z_n),w_n\rangle_n=\langle L_nz_n,w_n\rangle_n-\langle N_n(z_n),w_n\rangle_n\,,$$

we get $$\langle\nabla\Phi_n(z_n),w_n\rangle_n=\langle L_nw_n,w_n\rangle_n+\langle N_n(\phi_n),w_n\rangle_n-\langle N_n(z_n),w_n\rangle_n\,. \tag{21}$$

By the convexity in (H3), we see that

$$\langle N_n(z_n),w_n\rangle_n-\langle N_n(\phi_n),w_n\rangle_n=\langle\nabla N_n(\phi_n+\theta_nw_n)[w_n],w_n\rangle_n\geq 0$$

for some $\theta_n\in(0,1)$. Together with (3) and (21), this implies

$$\|w_n\|_n^2\leq\frac{1}{\delta}\|\nabla\Phi_n(z_n)\|_n\cdot\|w_n\|_n,$$

and hence $\|w_n\|_n=O(\|\nabla\Phi_n(z_n)\|_n)$ as $n\to\infty$.

To see (3), it follows from (2) and the C^2-smoothness of Φ_n that $\|\nabla\Phi_n(\phi_n)\|_n\to 0$. Finally, since we have $\|\nabla J_n(P_nz_n)\|_n=\|\nabla\Phi_n(\phi_n)\|_n$ we obtain $\|\nabla J_n(P_nz_n)\|_n\to 0$ as $n\to\infty$. □

Let $\mathcal{N}_n=\mathcal{N}_{\alpha_n}$ be the Nehari-type manifold for J_n, the next lemma guarantees that the infimum in (18) is well-defined, i.e., $\gamma_n=\gamma_{\alpha_n}<\infty$.

Lemma 2.6. *Under* (20), *we have* $\mathcal{N}_n\neq\emptyset$ *for* n *large. In particular, there eixsts* $t_n>0$ *such that* $t_nP_nz_n\in\mathcal{N}_n$ *and* $|t_n-1|=O(\|\nabla J_n(P_nz_n)\|_n)$ *as* $n\to\infty$.

Proof. We use the same notation $\phi_n=P_nz_n+\chi_n(P_nz_n)$ as in the previous lemma. Then, from (20) and Lemma 2.5 (3), it follows that

$$\liminf_{n\to\infty}\langle N_n(\phi_n),\phi_n\rangle_n\geq c_0 \tag{22}$$

for some constant $c_0>0$. Indeed, up to a subsequence, if $\langle N_n(\phi_n),\phi_n\rangle_n\to 0$ as $n\to\infty$ then

$$\begin{aligned}\delta\|\phi_n\|_n&\leq\|L_n\phi_n\|_n\leq\|\nabla\Phi_n(\phi_n)\|_n+\|N_n(\phi_n)\|_n\\&\leq\|\nabla\Phi_n(\phi_n)\|_n+K\,\langle N_n(\phi_n),\phi_n\rangle_n^{\mu}+\mu\delta\|\phi_n\|_n\quad[\text{by (H4)}]\\&=o_n(1)+\mu\delta\|\phi_n\|_n.\end{aligned}$$

Since $\mu<1$, we obtain $\|\phi_n\|_n\to 0$ as $n\to\infty$. This stands in contradiction to $\Phi_n(\phi_n)\geq\Phi_n(z_n)\geq c_1>0$ as assumed in (20).

Let us set $\eta_n : [0, \infty) \to \mathbb{R}$ by $\eta_n(t) = \mathcal{K}_n(tP_n z_n)$, where $\mathcal{K}_n = \mathcal{K}_{\alpha_n}$ is defined by (15). One easily checks that $\eta_n'(t) = \langle \nabla \mathcal{K}_n(tP_n z_n), P_n z_n \rangle_n$. And hence, by Lemma 2.4 and the Taylor's expansion of η_n around $t = 1$, we get

$$t\eta_n'(t) \le 2\eta_n(1) - (\kappa - 1)\langle N_n(\phi_n), \phi_n \rangle_n + O(|t-1|) \tag{23}$$

where the $O(|t-1|)$ term is independent of n due to the boundedness of $\eta_n'(t)$ (a direct consequence of Lemma 2.5 (1)).

Note that $\eta_n(1) = \langle \nabla J_n(P_n z_n), P_n z_n \rangle_n \to 0$ (by Lemma 2.5 (3)), we can conclude from (22) and (23) that there exists a small constant $\nu > 0$ such that

$$\eta_n'(t) \le -\nu \text{ for all } t \in [1-\nu, 1+\nu] \text{ and } n \text{ large enough.}$$

In particular, $\eta_n(1-\nu) \ge \eta_n(1) + \nu^2 > 0$ and $\eta_n(1+\nu) \le \eta_n(1) - \nu^2 < 0$ for n large. Then, by the Inverse Function Theorem, $t_n = \eta_n^{-1}(0)$ exists and $t_n P_n z_n \in \mathcal{N}_n$ for n large enough. Furthermore, since $|\eta_n'(t)^{-1}| \le \frac{1}{\nu}$ on $[1-\nu, 1+\nu]$, we consequently get

$$|t_n - 1| = \left|\eta_n^{-1}(0) - \eta_n^{-1}(\mathcal{K}_n(P_n z_n))\right| \le \frac{1}{\nu}|\mathcal{K}_n(P_n z_n)|.$$

Now the conclusion follows from $\mathcal{K}_n(P_n z_n) = O(\|\nabla J_n(P_n z_n)\|_n)$. □

Combining Lemma 2.1 to 2.6, we have the next general result for the abstract problem (19). Notice that the proofs of Lemma 2.1 to 2.6 and Proposition 2.2 do not need the hypothesis (H6), we state our result as follows.

Theorem 2.7. *Under the hypotheses* (H1) *to* (H5)*, if there exist* $\{\alpha_n\} \subset \mathcal{A}$ *and* $\{z_n\} \subset \mathcal{H}$ *verifying* (20)*, then, for* n *large, the min-max value* $\gamma_n = \gamma_{\alpha_n} > 0$ *defined in* (18) *satisfies*

$$\gamma_n \le \Phi_n(z_n) + O(\|\nabla \Phi_n(z_n)\|_n^2) \quad \text{as } n \to \infty.$$

Furthermore, if the hypothesis (H6) *is satisfied, then* γ_n *is a critical value of* Φ_n *for* n *large.*

Remark 2.8. Clearly, under the hypotheses of Theorem 2.7, there is a natural upper bound of γ_n, that is the constant c_2 in the assumption (20). However, this is not enough since sometimes it is crucial to have a more explicit asymptotic characterization for the energies (see for instance the resolution of the classical Yamabe problem and its variations, the elliptic Brezis-Nirenberg problem, etc.). As we will see later in the applications, the sequence $\{z_n\}$ in Theorem 2.7 plays the role of certain test elements for the functionals Φ_n. Our result shows a refined upper bound estimate for the min-max value γ_n, and makes it computable whenever $\{z_n\}$ is constructible or is known beforehand.

Proof of Theorem 2.7. We set $\phi_n = P_n z_n + \chi_n(P_n z_n)$ as before and $t_n > 0$ as was in Lemma 2.6, we also let $\psi_n = t_n P_n z_n + \chi_n(t_n P_n z_n)$. It follows from Lemma 2.5 and 2.6 that

$$\begin{aligned} \|z_n - \psi_n\|_n &\le \|z_n - \phi_n\|_n + |t_n - 1|\|P_n z_n\|_n + \|\chi_n(P_n z_n) - \chi_n(t_n P_n z_n)\|_n \\ &= O(\|\nabla \Phi_n(z_n)\|_n) + O(\|\nabla J_n(P_n z_n)\|_n) \end{aligned} \tag{24}$$

where we have used the inequality

$$\|\chi_n(P_n z_n) - \chi_n(t_n P_n z_n)\|_n \le \|\|\nabla \chi_n(\theta_n P_n z_n)\|\|_n \cdot \|P_n z_n\|_n \cdot |t_n - 1|$$

for some θ_n between t_n and 1.

Since we have $\|\nabla J_n(P_n z_n)\|_n = \|\nabla \Phi_n(\phi_n)\|_n$ (by Proposition 2.2 (2)), we can see from Lemma 2.5 that

$$\|\nabla J_n(P_n z_n)\|_n \le \|\nabla \Phi_n(z_n)\|_n + O(\|\phi_n - z_n\|_n) = O(\|\nabla \Phi_n(z_n)\|_n).$$

Together with (24), this implies $\|z_n - \psi_n\|_n = O(\|\nabla \Phi_n(z_n)\|_n)$.

Now, using Taylor's expansion, we obtain

$$\begin{aligned}\Phi_n(z_n) &= \Phi_n(\psi_n) + \langle \nabla \Phi_n(\psi_n), z_n - \psi_n \rangle_n + O(\|z_n - \psi_n\|_n^2) \\ &= \Phi_n(\psi_n) + \langle \nabla \Phi_n(\psi_n), z_n - \psi_n \rangle_n + O(\|\nabla \Phi_n(z_n)\|_n^2).\end{aligned}$$

Since $\psi_n = t_n P_n z_n + \chi_n(t_n P_n z_n)$ and $t_n P_n z_n \in \mathcal{N}_n$, we have (by (10))

$$\begin{aligned}\langle \nabla \Phi_n(\psi_n), z_n - \psi_n \rangle_n &= \langle \nabla \Phi_n(\psi_n), P_n(z_n - \psi_n) \rangle_n \\ &= (1 - t_n) \langle \nabla J_n(t_n P_n z_n), P_n z_n \rangle_n = 0.\end{aligned}$$

Therefore we have, as desired,

$$\gamma_n = \inf_{\varphi \in \mathcal{N}_n} J_n(\varphi) \le J_n(t_n P_n z_n) = \Phi_n(\psi_n) = \Phi_n(z_n) + O(\|\nabla \Phi_n(z_n)\|_n^2).$$

Finally, for a fixed n large, let us consider a minimizing sequence $\{\varphi_j\} \subset \mathcal{N}_n$ for J_n. By Lemma 2.1 and Proposition 2.2, we see that $\{z_j := \varphi_j + \chi_n(\varphi_j)\}$ is a bounded Palais-Smale sequence for Φ_n and $\liminf_{j\to\infty} \langle N_n(z_j), z_j \rangle_n > 0$ (similar to (22)). Thus the conclusion follows from (H6). □

3. Applications

We shall now apply Theorem 2.7 to the problems (A) and (B) mentioned before. And in order to describe our results, it is useful to recall some notations and definitions. For a closed Riemannian m-manifold (M, g), let $\exp : TM \to M$ be the exponential map defined on the tangent bundle TM of M. Since we assume M to be closed, there exists $r > 0$ such that $\exp_\xi : B_r(0) \supset \mathbb{R}^m \cong T_\xi M \to B_r(\xi) \subset M$ is a diffeomorphism for any $\xi \in M$. In what follows, $B_r(0)$ will denote the ball in $\mathbb{R}^m$ centered at 0 with radius r and, for $\xi \in M$, $B_r(\xi)$ will denote the ball in M centered at ξ with respect to the metric g. For vector fields X, Y, W, Z on M, the Riemannian curvature tensor $\mathcal{R}$ is given by

$$\mathcal{R}(X, Y, W, Z) = g(\nabla_Y \nabla_X W, Z) - g(\nabla_X \nabla_Y W, Z) - g(\nabla_{[Y,X]} W, Z) \tag{25}$$

where ∇ is the Riemannian connection. The Ricci tensor $\mathrm{Ric} : TM \times TM \to \mathbb{R}$ is defined by the trace of $\mathcal{R}$, that is,

$$\mathrm{Ric}_\xi(X, Y) = \sum_{j=1}^{m} \mathcal{R}_\xi(X, e_j, Y, e_j) \tag{26}$$

where $\{e_1, \ldots, e_m\}$ is a basis of the (Riemann) normal coordinates in $T_\xi M$. The scalar curvature will be denoted by Scal_g.

(A) Singularly perturbed Hamiltonian elliptic system on Riemannian manifolds

Let (M, g) be a closed Riemannian m-manifold, $m \geq 3$, we set

$$H_g^1(M) = \Big\{ u : M \to \mathbb{R} : \int_M |\nabla_g u|^2 + |u|^2 d\mathrm{vol}_g < +\infty \Big\}$$

where $d\mathrm{vol}_g$ is the volume form on M associated to the metric g. We also set $L_g^p(M)$, $p \geq 1$, for the space of functions on (M, g) for which the p-th power of their absolute value is integrable, and is equipped with its canonical norm. It is clear that $L_g^2(M)$ is a Hilbert space.

Let us take the index $\alpha = \varepsilon \in (0, +\infty)$, and introduce the self-adjoint operator

$$S_\varepsilon : \mathcal{D}(S_\varepsilon) \subset L_g^2(M) \times L_g^2(M) \to L_g^2(M) \times L_g^2(M)$$

defined by

$$S_\varepsilon \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\varepsilon^2 \Delta_g v + v \\ -\varepsilon^2 \Delta_g u + u \end{pmatrix}.$$

Then eq. (5) can be equivalently rewritten as

$$S_\varepsilon z = \mathcal{F}(z) \tag{27}$$

where the nonlinear part is defined by

$$\mathcal{F}(z) = \begin{pmatrix} f_1(u) \\ f_2(v) \end{pmatrix} \quad \text{for } z = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}(S_\varepsilon).$$

Associated with eq. (27), we choose the Hilbert space $\mathcal{H} = H_g^1(M) \times H_g^1(M)$, and equip it with a family of inner products

$$\langle z_1, z_2 \rangle_\varepsilon = \frac{1}{\varepsilon^m} \int_M \big(\varepsilon^2 \nabla_g u_1 \cdot \nabla_g u_2 + u_1 u_2 \big) d\mathrm{vol}_g + \frac{1}{\varepsilon^m} \int_M \big(\varepsilon^2 \nabla_g v_1 \cdot \nabla_g v_2 + v_1 v_2 \big) d\mathrm{vol}_g$$

for $z_i = \begin{pmatrix} u_i \\ v_i \end{pmatrix} \in \mathcal{H}$, $i = 1, 2$, and the induced norms $\| \cdot \|_\varepsilon$.

We now take the operator $L_\varepsilon : (\mathcal{H}, \langle \cdot, \cdot \rangle_\varepsilon) \to (\mathcal{H}, \langle \cdot, \cdot \rangle_\varepsilon)$ by

$$L_\varepsilon \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ u \end{pmatrix}.$$

Then we have L_ε is self-adjoint, continuous, invertible and $\sigma(L_\varepsilon) = \{1, -1\}$. In particular we can split the space $\mathcal{H}$ with respect to L_ε as

$$\mathcal{H} = \mathcal{H}_\varepsilon^+ \oplus \mathcal{H}_\varepsilon^-, \quad \text{where } \mathcal{H}_\varepsilon^\pm \equiv \big\{ (u, \pm u)^T : u \in H_g^1(M) \big\}.$$

It is clear that (2) and (3) are satisfied on $\mathcal{H}_\varepsilon^+$ and $\mathcal{H}_\varepsilon^-$, respectively, with the constant $\delta = 1$. Moreover, eq. (27) (or equivalently eq. (5)) reads as $L_\varepsilon z = N_\varepsilon(z)$ for $z \in \mathcal{H}$, where the relation between S_ε and L_ε is given by

$$L_\varepsilon = T_\varepsilon^{-1} \circ S_\varepsilon \quad \text{with } T_\varepsilon = -\varepsilon^2 \Delta_g + 1 : \mathcal{H} \to \mathcal{H}^*$$

and, similarly, $N_\varepsilon = T_\varepsilon^{-1} \circ \mathcal{F}$.

In order to study this nonlinear problem, we impose the following conditions on the nonlinear functions f_i, $i = 1, 2$, involved in eq. (5):

(F) $f_i \in C^1(\mathbb{R})$, $f_i(s) = 0$ for $s \le 0$, and there exist $q_1, q_2 > 2$ such that $\frac{1}{q_1} + \frac{1}{q_2} > \frac{m-2}{m}$ and

$$\lim_{s\to+\infty} \frac{f_1'(s)}{s^{q_1-2}} = \lim_{s\to+\infty} \frac{f_2'(s)}{s^{q_2-2}} = 0. \tag{28}$$

Moreover, for some $\theta > 1$ and every $s > 0$,

$$0 < \theta f_i(s) \le f_i'(s)s \quad i = 1, 2. \tag{29}$$

As was explained in [18, 21], we can assume without loss of generality that the numbers q_1, q_2 in (F) are such that $2 < q_1 = q_2 < \frac{2m}{m-2}$. In fact, since the solutions we work with are of ground-state type, they are a priori bounded in L^∞-norm and this bound keeps invariant even if we truncate f_1 and f_2 at infinity so that (F) is satisfied for the modified functions with $q_1 = q_2$ in (28).

Now, we denote $F_i(s) = \int_0^s f_i(t)dt$, $i = 1, 2$, the primitive functions of f_i, and $\Psi_\varepsilon : (\mathcal{H}, \langle\cdot,\cdot\rangle_\varepsilon) \to \mathbb{R}$ with

$$\Psi_\varepsilon(z) = \frac{1}{\varepsilon^m}\int_M F_1(u)d\mathrm{vol}_g + \frac{1}{\varepsilon^m}\int_M F_2(v)d\mathrm{vol}_g \quad \text{for } z = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{H}.$$

We also equip $L^p_g(M) \times L^p_g(M)$, $p \ge 1$, with the norm

$$|z|_{p,\varepsilon} = \Big(\frac{1}{\varepsilon^m}\int_M |z|^p d\mathrm{vol}_g\Big)^{\frac{1}{p}} \quad \text{for } z = \begin{pmatrix} u \\ v \end{pmatrix} \in L^p_g(M) \times L^p_g(M).$$

By Sobolev's embedding theorem, we have that there exists $c_p > 0$, independent of ε, such that $|z|_{p,\varepsilon} \le c_p\|z\|_\varepsilon$ for each $p \in [2, \frac{2m}{m-2}]$. Then we verify the hypotheses (H1) to (H6) as follows.

Lemma 3.1. *Under condition* (F)*, we have* $\Psi_\varepsilon \in C^2(\mathcal{H}, \mathbb{R})$ *satisfies* (H1)–(H6) *for every* $\varepsilon > 0$.

Proof. Since the proof is quite standard, we will be sketchy.
The C^2-smoothness follows from (28) and our choice of $q := q_1 = q_2 \in (2, \frac{2m}{m-2})$. Hence (H1) is verified with $N_\varepsilon = \nabla\Psi_\varepsilon$. To proceed, we mention that (29) implies

$$\theta F_i(s) = \theta\int_0^s f_i(t)dt \le \int_0^s f_i'(t)tdt = f_i(t)t\Big|_{t=0}^{t=s} - \int_0^s f_i(t)dt = f_i(s)s - F_i(s).$$

Thus $(\theta + 1)F_i(s) \le f_i(s)s$, $i = 1, 2$, and this gives (H2). Using the positiveness of the $f_i'(s)$ in (29), the convexity follows immediately from the computation

$$\langle\nabla N_\varepsilon(z)[w], w\rangle_\varepsilon = \frac{1}{\varepsilon^m}\int_M f_1'(u)\tilde{u}^2 d\mathrm{vol}_g + \frac{1}{\varepsilon^m}\int_M f_2'(v)\tilde{v}^2 d\mathrm{vol}_g \ge 0,$$

where $z = \begin{pmatrix} u \\ v \end{pmatrix}$ and $w = \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \in \mathcal{H}$. Moreover, since we have $f_i(0) = f_i'(0) = 0$, we can conclude from eq. (28) that for arbitrary $\sigma > 0$ there exists $c_\sigma > 0$ such that $f_i(s) \le \sigma s + c_\sigma\big(f_i(s)s\big)^{\frac{q-1}{q}}$ for all $s \ge 0$.

Hence,

$$\begin{aligned}
&\langle N_\varepsilon(z), w\rangle_\varepsilon = \frac{1}{\varepsilon^m}\int_M f_1(u)\tilde{u}d\mathrm{vol}_g + \frac{1}{\varepsilon^m}\int_M f_2(v)\tilde{v}d\mathrm{vol}_g \\
&\leq \frac{\sigma}{\varepsilon^m}\int_M \big(|u||\tilde{u}| + |v||\tilde{v}|\big)d\mathrm{vol}_g + \frac{c_\sigma}{\varepsilon^m}\int_M \Big(\big(f_1(u)u\big)^{\frac{q-1}{q}}|\tilde{u}| + \big(f_2(v)v\big)^{\frac{q-1}{q}}|\tilde{v}|\Big)d\mathrm{vol}_g \\
&\leq \sigma|z|_{2,\varepsilon}|w|_{2,\varepsilon} + c_\sigma\,\langle N_\varepsilon(z), z\rangle_\varepsilon^{\frac{q-1}{q}}\,|w|_{q,\varepsilon} \quad \text{[by the Hölder inequality]}
\end{aligned}$$

which verifies (H4) by choosing σ small. To see (H5), we first remark that, for $s > 0$ and $t \in \mathbb{R}$,

$$\begin{aligned}
f_i'(s)(s+t)^2 - 2f_i(s)t &= f_i'(s)\bigg(s^2 + 2\Big(1 - \frac{f_i(s)}{f_i'(s)s}\Big)st + t^2\bigg) \\
&= f_i'(s)\bigg(\Big(1 - \frac{f_i(s)}{f_i'(s)s}\Big)s + t\bigg)^2 + f_i'(s)\bigg(1 - \Big(1 - \frac{f_i(s)}{f_i'(s)s}\Big)^2\bigg)s^2 \\
&\geq \Big(2 - \frac{f_i(s)}{f_i'(s)s}\Big)f_i(s)s \quad \text{[by the positiveness of the} f_i'(s)\text{]}.
\end{aligned}$$

And since, by (29), we have $0 < \frac{f_i(s)}{f_i'(s)s} \leq \frac{1}{\theta} < 1$. This, together with above estimate, implies (H5). Finally, the hypothesis (H6) follows from the compact embedding

$$(\mathcal{H}, \|\cdot\|_\varepsilon) \hookrightarrow \big(L_g^q(M)\times L_g^q(M), |\cdot|_{q,\varepsilon}\big), \quad \text{for } \ q \in (2, \frac{2m}{m-2}). \qquad \square$$

By virtue of the above lemma, we see that solutions of eq. (5) correspond to critical points of the functional

$$\Phi_\varepsilon(z) = \frac{1}{2}\langle L_\varepsilon z, z\rangle_\varepsilon - \Psi_\varepsilon(z) \quad \text{for } z \in \mathcal{H} \text{ and } \varepsilon \in (0, +\infty).$$

And, in order to apply Theorem 2.7 (with the index $\alpha = \varepsilon$), one needs to find a sequence of test functions satisfy (20) for the functionals Φ_ε. In the sequel, the following problem on the Euclidean space $\mathbb{R}^m$ will play a key role:

$$\begin{cases} -\Delta v + v = f_1(u), \\ -\Delta u + u = f_2(v), \\ \ u, v > 0 \end{cases} \qquad \text{on } \mathbb{R}^m \tag{30}$$

whose energy functional is given by

$$\Phi_e(z) = \int_{\mathbb{R}^m}\big(\nabla u\cdot\nabla v + uv - F_1(u) - F_2(v)\big)dx, \quad \text{for } z = \begin{pmatrix} u \\ v \end{pmatrix} \in H^1(\mathbb{R}^m)\times H^1(\mathbb{R}^m).$$

Let us denote γ_e the ground-state critical value of Φ_e, that is,

$$\gamma_e = \inf\big\{\Phi_e(z) :\ \nabla\Phi_e(z) = 0,\ z \neq 0\big\}.$$

Under the same condition (F), it follows from [19, 20] that $\gamma_e > 0$ is attained at some positive functions U, V which satisfy the system (30). Moreover, by elliptic estimates, U and V decay exponentially and they are radially symmetric with respect

to a common point of $\mathbb{R}^m$ (see for instance [8]). For later use, we denote by $\mathcal{L}_e$ the set of all ground-state solutions (U,V) of eq. (30) satisfying $U(0)=\max_{x\in\mathbb{R}^m}U(x)$ and $V(0)=\max_{x\in\mathbb{R}^m}V(x)$. Then it follows that any $(U,V)\in\mathcal{L}_e$ is radially symmetric with respect to the origin and $\frac{\partial U}{\partial\rho}<0$, $\frac{\partial V}{\partial\rho}<0$ for $\rho=|x|>0$ (see [8, 13]).

Lemma 3.2. *Let $(U,V)\in\mathcal{L}_e$. Then, for any $k\geq 0$ we have*

$$\int_{\mathbb{R}^m}\Big(\nabla U\cdot\nabla V+UV-F_1(U)-F_2(V)\Big)|x|^k dx=\frac{2(k+1)}{m+k}\int_{\mathbb{R}^m}\big(\nabla U\cdot\nabla V\big)|x|^k dx \quad (31)$$

Proof. Since U, V are decreasing exponentially at infinity, multiplying both sides of system (30) by the couple $(|x|^k x\cdot\nabla U,|x|^k x\cdot\nabla V)$, integrating by parts on $B_R(0)$ and taking $R\to\infty$, we get the identity. □

Next, we define a map $\mathcal{S}: M\times\mathcal{L}_e\to\mathbb{R}$ by

$$\begin{aligned}\mathcal{S}(\xi,Z):=&\sum_{i,j=1}^m\int_{\mathbb{R}^m}\mathcal{R}_\xi(X,e_i,X,e_j)\partial_i U\partial_j V dx-\frac{1}{2}\int_{\mathbb{R}^m}\mathrm{Ric}_\xi(X,X)\nabla U\cdot\nabla V dx\\&-\frac{1}{2}\int_{\mathbb{R}^m}\mathrm{Ric}_\xi(X,X)UV dx+\frac{1}{2}\int_{\mathbb{R}^m}\mathrm{Ric}_\xi(X,X)\big(F_1(U)+F_2(V)\big)dx\end{aligned} \quad (32)$$

where $X\in T_\xi M$, $X=\sum_{i=1}^m x_i e_i$. Then we have the following

Proposition 3.3. *Let $Z=(U,V)\in\mathcal{L}_e$. Then we have*

$$\mathcal{S}(\xi,Z)=-\frac{3\,\mathrm{Scal}_g(\xi)}{m(m+2)}\int_{\mathbb{R}^m}\nabla U\cdot\nabla V|x|^2dx,$$

where $\mathrm{Scal}_g(\xi)$ is the scalar curvature of M at ξ.

Proof. Note that $X=\sum_{i=1}^m x_i e_i$ and $Z=(U,V)$ is radially symmetric, it follows from (25) and (26) that

$$\sum_{i,j=1}^m\int_{\mathbb{R}^m}\mathcal{R}_\xi(X,e_i,X,e_j)\partial_i U\partial_j V dx=\sum_{i,j=1}^m\sum_{k,l=1}^m\int_{\mathbb{R}^m}\mathcal{R}_\xi(e_k,e_i,e_l,e_j)\frac{x_k x_l x_i x_j}{|x|^2}\frac{\partial U}{\partial\rho}\frac{\partial V}{\partial\rho}dx=0$$

and
$$\begin{aligned}\int_{\mathbb{R}^m}\mathrm{Ric}_\xi(X,X)\nabla U\cdot\nabla V dx&=\sum_{i,j=1}^m\int_{\mathbb{R}^m}\mathrm{Ric}_\xi(e_i,e_j)x_i x_j\nabla U\cdot\nabla V dx\\&=\sum_{i=1}^m\mathrm{Ric}_\xi(e_i,e_i)\int_{\mathbb{R}^m}(x_i)^2\nabla U\cdot\nabla V dx=\frac{\mathrm{Scal}_g(\xi)}{m}\int_{\mathbb{R}^m}\nabla U\cdot\nabla V|x|^2dx.\end{aligned}$$

In the last equality we used $\mathrm{Scal}_g(\xi)=\sum_{i=1}^m\mathrm{Ric}_\xi(e_i,e_i)$. Analogously, we have

$$\int_{\mathbb{R}^m}\mathrm{Ric}_\xi(X,X)UV dx=\frac{\mathrm{Scal}_g(\xi)}{m}\int_{\mathbb{R}^m}UV|x|^2dx,\quad\text{and}$$

$$\int_{\mathbb{R}^m}\mathrm{Ric}_\xi(X,X)\big(F_1(U)+F_2(V)\big)dx=\frac{\mathrm{Scal}_g(\xi)}{m}\int_{\mathbb{R}^m}\big(F_1(U)+F_2(V)\big)|x|^2dx.$$

Finally, the assertion can be straightforwardly verified via Lemma 3.2. □

For the given Riemannian metric g, we write it locally as $g = \sum_{i,j=1}^m g_{ij} dx_i \otimes dx_j$ on $B_r(\xi)$, $\xi \in M$. We also let $G(\xi) = (g_{ij}(\xi))_{ij}$ and $G^{-1}(\xi) = (g^{ij}(\xi))_{ij}$. Henceforth, det stands for the determinant. We now consider the Riemann normal coordinates on $B_r(\xi)$. For ease of notation, we identify $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ with $X = \sum_i x_i e_i \in T_\xi M$, then it is well known (see [10]) that

$$g_{ij}(\exp_\xi x) = \delta_{ij} - \frac{1}{3}\mathcal{R}_\xi(x, e_i, x, e_j) + O(|x|^3), \tag{33}$$

$$g^{ij}(\exp_\xi x) = \delta_{ij} + \frac{1}{3}\mathcal{R}_\xi(x, e_i, x, e_j) + O(|x|^3) \tag{34}$$

and

$$\sqrt{\det G}(\exp_\xi x) = 1 - \frac{1}{6}\operatorname{Ric}_\xi(x, x) + O(|x|^3), \tag{35}$$

where δ_{ij} is the Kronecker delta.

Let $0 < r < inj(M)/2$ where $inj(M) > 0$ is the injectivity radius of M, and let $\eta \in C_c^\infty(\mathbb{R}^m, [0,1])$ be such that $|\nabla\eta| \le 2/r$, $\eta(x) = 1$ for $|x| \le r$ and $\eta(x) = 0$ for $|x| \ge 2r$. For a ground-state solution $Z = (U, V) \in \mathcal{L}_e$, we set $U_\varepsilon(x) = U(\frac{x}{\varepsilon})$ and $V_\varepsilon(x) = V(\frac{x}{\varepsilon})$. Define $W_\varepsilon = (\eta U_\varepsilon, \eta V_\varepsilon)$ and $z_\varepsilon = (u_\varepsilon, v_\varepsilon) = W_\varepsilon \circ \exp_\xi^{-1}$.

Then, via the expression $\Delta_g u = \dfrac{1}{\sqrt{\det G}} \sum_{i,j} \partial_i\big(g^{ij}\sqrt{\det G}\,\partial_j u\big)$, we have

Lemma 3.4. *As $\varepsilon \to 0$, $\Phi_\varepsilon(z_\varepsilon) = \gamma_e + \dfrac{\varepsilon^2}{3}\mathcal{S}(\xi, Z) + O(\varepsilon^3)$ and $\|\nabla\Phi_\varepsilon(z_\varepsilon)\|_\varepsilon = O(\varepsilon^2)$, i.e. $\{z_\varepsilon\}$ verifies the condition* (20).

Proof. Since $Z = (U, V) \in \mathcal{L}_e$ decays exponentially at infinity, by the change of variables and (33)-(35), we see immediately that

$$\begin{aligned}
\Phi_\varepsilon(z_\varepsilon) &= \frac{1}{\varepsilon^m}\int_M \Big(\varepsilon^2 \sum_{i,j=1}^m g^{ij}\partial_i u_\varepsilon \partial_j v_\varepsilon + u_\varepsilon v_\varepsilon\Big) d\mathrm{vol}_g - \frac{1}{\varepsilon^m}\int_M \big(F_1(u_\varepsilon) + F_2(v_\varepsilon)\big) d\mathrm{vol}_g \\
&= \frac{1}{\varepsilon^m}\int_{B_{2r}(0)} \Big(\varepsilon^2 \sum_{i,j=1}^m g^{ij}(\exp_\xi x)\partial_i(\eta U_\varepsilon)\partial_j(\eta V_\varepsilon) + \eta^2 U_\varepsilon V_\varepsilon\Big)\sqrt{\det G}(\exp_\xi x)dx \\
&\quad - \frac{1}{\varepsilon^m}\int_{B_{2r}(0)} \big(F_1(\eta U_\varepsilon) + F_2(\eta V_\varepsilon)\big)\sqrt{\det G}(\exp_\xi x)dx \\
&= \Phi_e(Z) + \frac{\varepsilon^2}{3}\mathcal{S}(\xi, Z) + O(\varepsilon^3) \quad \text{[by using the formulation of } \mathcal{S} \text{ in (32)]}
\end{aligned}$$

and analogously, for arbitrary $z = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{H}$,

$$\begin{aligned}
\langle \nabla\Phi_\varepsilon(z_\varepsilon), z\rangle_\varepsilon &= \frac{1}{\varepsilon^m}\int_{B_{2r}(0)} \Big(\varepsilon^2 \sum_{i,j=1}^m g^{ij}(\exp_\xi x)\partial_i u_\varepsilon \partial_j v + u_\varepsilon v\Big)\sqrt{\det G}(\exp_\xi x)dx \\
&\quad - \frac{1}{\varepsilon^m}\int_{B_{2r}(0)} f_2(\eta V_\varepsilon) v\sqrt{\det G}(\exp_\xi x)dx \\
&\quad + \frac{1}{\varepsilon^m}\int_{B_{2r}(0)} \Big(\varepsilon^2 \sum_{i,j=1}^m g^{ij}(\exp_\xi x)\partial_i u \partial_j v_\varepsilon + u v_\varepsilon\Big)\sqrt{\det G}(\exp_\xi x)dx
\end{aligned}$$

$$-\frac{1}{\varepsilon^m}\int_{B_{2r}(0)} f_1(\eta U_\varepsilon)u\sqrt{\det G}(\exp_\xi x)dx$$
$$= O(\varepsilon^2)\|z\|_\varepsilon, \quad \text{[by using the exponential decay of } U \text{ and } V\text{]}$$

hence the lemma is proved. □

Since we have checked that all the hypotheses in Theorem 2.7 hold true, we are now ready to state

Proposition 3.5. *For all small numbers $\varepsilon > 0$, there exists a ground-state solution $\bar{z}_\varepsilon = (\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ of (5) with energy evaluation*

$$\gamma_\varepsilon := \Phi_\varepsilon(\bar{z}_\varepsilon) \le \gamma_e + \frac{\varepsilon^2}{3}\mathcal{S}(\xi, Z) + O(\varepsilon^3), \tag{36}$$

where $\xi \in M$ and $Z \in \mathcal{L}_e$ are given arbitrarily. Moreover, for each ε, there exists a unique maximum point $\xi_\varepsilon \in M$ of $\bar{u}_\varepsilon + \bar{v}_\varepsilon$ such that for some constants $C, c > 0$,

$$\bar{u}_\varepsilon(\xi) + \bar{v}_\varepsilon(\xi) + |\nabla\bar{u}_\varepsilon(\xi)| + |\nabla\bar{v}_\varepsilon(\xi)| \le Ce^{-\frac{c}{\varepsilon}\operatorname{dist}_g(\xi,\xi_\varepsilon)}, \quad \xi \in M$$

where dist_g is the distance induced by the metric g. And, up to a subsequence, the transformed functions $\bar{U}_\varepsilon(x) = \bar{u}_\varepsilon \circ \exp_{\xi_\varepsilon}(\varepsilon x)$ and $\bar{V}_\varepsilon(x) = \bar{v}_\varepsilon \circ \exp_{\xi_\varepsilon}(\varepsilon x)$ converge to a ground-state solution $\bar{Z} = (\bar{U}, \bar{V}) \in \mathcal{L}_e$ of (30) in C^2_{loc}-sense.

Proof. Since the existence and the energy evaluation are direct consequences of Lemma 3.1, 3.4 and Theorem 2.7, we only need to describe the asymptotic behavior of the solutions $\bar{z}_\varepsilon = (\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ as $\varepsilon \to 0$.

By the Sobolev embedding and the arguments in Lemma 2.1, we deduce that $\|\bar{z}_\varepsilon\|_\varepsilon$ is bounded. And by bootstrap arguments, we see also that $\|\bar{u}_\varepsilon\|_\infty$ and $\|\bar{v}_\varepsilon\|_\infty$ are bounded. Furthermore, via the maximum principle, we have $\|\bar{u}_\varepsilon + \bar{v}_\varepsilon\|_\infty$ is bounded away from 0.

Let $\xi_\varepsilon \in M$ be such that $\liminf_{\varepsilon\to 0} \bar{u}_\varepsilon(\xi_\varepsilon) + \bar{v}_\varepsilon(\xi_\varepsilon) > 0$. Consider the Riemann normal coordinates on $B_r(\xi_\varepsilon) \subset M$, for $r > 0$ fixed small. Denoted by $\bar{U}_\varepsilon(x) = \bar{u}_\varepsilon \circ \exp_{\xi_\varepsilon}(\varepsilon x)$ and $\bar{V}_\varepsilon(x) = \bar{v}_\varepsilon \circ \exp_{\xi_\varepsilon}(\varepsilon x)$, we find

$$\begin{cases} -\dfrac{1}{\sqrt{\det G}(\exp_{\xi_\varepsilon}(\varepsilon x))}\displaystyle\sum_{i,j}\partial_i\big(g^{ij}(\exp_{\xi_\varepsilon}(\varepsilon x))\sqrt{\det G}(\exp_{\xi_\varepsilon}(\varepsilon x))\partial_j\bar{U}_\varepsilon\big) + \bar{U}_\varepsilon = f_2(\bar{V}_\varepsilon), \\ -\dfrac{1}{\sqrt{\det G}(\exp_{\xi_\varepsilon}(\varepsilon x))}\displaystyle\sum_{i,j}\partial_i\big(g^{ij}(\exp_{\xi_\varepsilon}(\varepsilon x))\sqrt{\det G}(\exp_{\xi_\varepsilon}(\varepsilon x))\partial_j\bar{V}_\varepsilon\big) + \bar{V}_\varepsilon = f_1(\bar{U}_\varepsilon), \end{cases}$$

on $B_{r/\varepsilon}(0)$. Then, from elliptic estimates, we deduce that $\bar{U}_\varepsilon$ and $\bar{V}_\varepsilon$ converge locally uniformly in C^2-topology to a finite energy solution $\bar{Z} = (\bar{U}, \bar{V})$ of (30). Note that $\Phi_e(\bar{Z}) \ge \gamma_e$, we can conclude that there exist a non-negative integer l and $\xi^i_\varepsilon \in M$ for $1 \le i \le l$ satisfying

$$\lim_{\varepsilon\to 0}\frac{\min_{i\neq j}\operatorname{dist}(\xi^i_\varepsilon, \xi^j_\varepsilon)}{\varepsilon} = \infty$$

and $\liminf_{\varepsilon\to 0} \bar{u}_\varepsilon(\xi^i_\varepsilon) + \bar{v}_\varepsilon(\xi^i_\varepsilon) > 0$, $i = 1, \dots, l$. It also follows from the standard comparison principle that for some constants $C, c > 0$

$$\bar{u}_\varepsilon(\xi) + \bar{v}_\varepsilon(\xi) + |\nabla\bar{u}_\varepsilon(\xi)| + |\nabla\bar{v}_\varepsilon(\xi)| \le C\exp\Big(-\frac{c}{\varepsilon}\operatorname{dist}_g\big(\xi, \{\xi^1_\varepsilon, \dots, \xi^l_\varepsilon\}\big)\Big), \quad \xi \in M.$$

This implies that $\liminf_{\varepsilon\to 0} \Phi_\varepsilon(\bar z_\varepsilon) \geq l\gamma_e$.

Since we have already known from Theorem 2.7 the upper energy estimate (36), we see that $l = 1$. And thus, for a unique maximum point $\xi_\varepsilon \in M$ of $\bar u_\varepsilon + \bar v_\varepsilon$, we have

$$\bar u_\varepsilon(\xi) + \bar v_\varepsilon(\xi) + |\nabla \bar u_\varepsilon(\xi)| + |\nabla \bar v_\varepsilon(\xi)| \leq C \exp\Big(-\frac{c}{\varepsilon}\,\mathrm{dist}_g(\xi,\, \xi_\varepsilon)\Big), \quad \xi \in M.$$

and, in this way, the transformed functions $\bar U_\varepsilon$ and $\bar V_\varepsilon$ converge locally in C^2-topology to a ground-state solution $\bar Z = (\bar U, \bar V) \in \mathcal{L}_e$ as $\varepsilon \to 0$. □

The next result gives a refined lower energy estimate.

Lemma 3.6. *Let $\bar z_\varepsilon = (\bar u_\varepsilon, \bar v_\varepsilon)$, $\bar Z = (\bar U, \bar V) \in \mathcal{L}_e$ and $\xi_\varepsilon \in M$ be as in Proposition 3.5. Suppose that, up to a subsequence, $\xi_\varepsilon \to \xi$ in M (with respect to the Riemannian metric g) as $\varepsilon \to 0$, then*

$$\gamma_\varepsilon \geq \gamma_e + \frac{\varepsilon^2}{3}\mathcal{S}(\xi, \bar Z) + o(\varepsilon^2).$$

Proof. For $\varepsilon > 0$ small, let us denote by $\bar z_\varepsilon = (\bar u_\varepsilon, \bar v_\varepsilon)$ the ground-state solution of (5) as in the previous proposition and $\xi_\varepsilon \in M$ be the maximum point of $\bar u_\varepsilon + \bar v_\varepsilon$.

Take $\beta_\varepsilon \in C^\infty(M, [0,1])$ in such a way that $|\nabla_g \beta_\varepsilon| \leq 4/r$, $\beta_\varepsilon \equiv 1$ on $B_r(\xi_\varepsilon)$ and $\mathrm{supp}\,\beta_\varepsilon \subset B_{2r}(\xi_\varepsilon)$, for some $r < inj(M)/2$. Since $\bar u_\varepsilon$ and $\bar v_\varepsilon$ solves eq. (5), we have

$$\Phi_\varepsilon(\bar z_\varepsilon) = \frac{1}{\varepsilon^m}\int_M \Big(\frac{1}{2} f_1(\bar u_\varepsilon)\bar u_\varepsilon - F_1(\bar u_\varepsilon)\Big)d\mathrm{vol}_g + \frac{1}{\varepsilon^m}\int_M \Big(\frac{1}{2} f_2(\bar v_\varepsilon)\bar v_\varepsilon - F_2(\bar v_\varepsilon)\Big)d\mathrm{vol}_g$$

and, multiplying both sides of (5) by $(\beta_\varepsilon^2 \bar u_\varepsilon, \beta_\varepsilon^2 \bar v_\varepsilon)$ and integrating by parts, we find

$$\begin{aligned}\Phi_\varepsilon(\beta_\varepsilon \bar z_\varepsilon) = {} & \frac{1}{\varepsilon^m}\int_M \Big(\frac{\beta_\varepsilon^2}{2} f_1(\bar u_\varepsilon)\bar u_\varepsilon - F_1(\beta_\varepsilon \bar u_\varepsilon)\Big)d\mathrm{vol}_g \\ & + \frac{1}{\varepsilon^m}\int_M \Big(\frac{\beta_\varepsilon^2}{2} f_2(\bar v_\varepsilon)\bar v_\varepsilon - F_2(\beta_\varepsilon \bar v_\varepsilon)\Big)d\mathrm{vol}_g + \frac{1}{\varepsilon^m}\int_M \varepsilon^2 \bar u_\varepsilon \bar v_\varepsilon |\nabla_g \beta_\varepsilon|^2 d\mathrm{vol}_g\,.\end{aligned}$$

For $i = 1, 2$ and $s > 0$, let us consider the function $h_i(t) = \frac{t^2}{2} f_i(s)s - F_i(ts)$. It is all clear that $h_i(0) = 0$ and $h_i'(0) = h_i'(1) = 0$. Assume $t_0 > 0$ satisfies $h_i'(t_0) = 0$, then it follows from $0 = h_i'(t_0) = t_0 f_i(s)s - f_i(t_0 s)s$ and (29) that

$$\begin{aligned} h_i''(t_0) &= f_i(s)s - f_i'(t_0 s)s^2 = \frac{f_i(t_0 s)}{t_0}s - f_i'(t_0 s)s^2 \\ &\leq \frac{1}{\theta} f_i'(t_0 s)s^2 - f_i'(t_0 s)s^2 < 0 \qquad [\text{by (29) and } \theta > 1]\end{aligned}$$

i.e., $t_0 > 0$ is a strict maximum point of h_i. Hence, we must have $t_0 = 1$, $h_i(t)$ is increasing for $t \in [0, 1]$ and $h_i(1)$ is the unique global maximum. Together with the exponential decay of $\bar u_\varepsilon$ and $\bar v_\varepsilon$ on M, this implies

$$\Phi_\varepsilon(\beta_\varepsilon \bar z_\varepsilon) \leq \Phi_\varepsilon(\bar z_\varepsilon) + \frac{1}{\varepsilon^m}\int_M \varepsilon^2 \bar u_\varepsilon \bar v_\varepsilon |\nabla_g \beta_\varepsilon|^2 d\mathrm{vol}_g = \gamma_\varepsilon + o(\varepsilon^2). \tag{37}$$

On the other hand, use the local coordinates around ξ_ε and argue similarly as in Lemma 3.4, we can get

$$\begin{aligned}\Phi_\varepsilon(\beta_\varepsilon\bar{z}_\varepsilon) &= \int\limits_{B_{2r/\varepsilon}(0)} \Big(\sum_{i,j=1}^m g^{ij}(\exp_{\xi_\varepsilon}\varepsilon x)\partial_i(\bar\beta_\varepsilon\bar U_\varepsilon)\partial_j(\bar\beta_\varepsilon\bar V_\varepsilon) + \bar\beta_\varepsilon^2\bar U_\varepsilon\bar V_\varepsilon\Big)\sqrt{\det G}(\exp_{\xi_\varepsilon}\varepsilon x)dx \\ &\quad - \int\limits_{B_{2r/\varepsilon}(0)} \big(F_1(\bar\beta_\varepsilon\bar U_\varepsilon) + F_2(\bar\beta_\varepsilon\bar V_\varepsilon)\big)\sqrt{\det G}(\exp_{\xi_\varepsilon}\varepsilon x)dx \\ &= \Phi_e(\bar W_\varepsilon) + \frac{\varepsilon^2}{3}\mathcal{S}(\xi_\varepsilon,\bar W_\varepsilon) + o(\varepsilon^2) \text{ [use the exponential deacy of } \bar U_\varepsilon \text{ and } \bar V_\varepsilon]\end{aligned}$$

where, for simplicity, we denote $\bar U_\varepsilon(x) = \bar u_\varepsilon\circ\exp_{\xi_\varepsilon}(\varepsilon x)$, $\bar V_\varepsilon(x) = \bar v_\varepsilon\circ\exp_{\xi_\varepsilon}(\varepsilon x)$, $\bar\beta_\varepsilon(x) = \beta_\varepsilon\circ\exp_{\xi_\varepsilon}(\varepsilon x)$ and $\bar W_\varepsilon = (\bar\beta_\varepsilon\bar U_\varepsilon, \bar\beta_\varepsilon\bar V_\varepsilon)$. Note that, by Proposition 3.5, $\bar W_\varepsilon$ converges in C^2_{loc}-topology to $\bar Z\in\mathcal{L}_e$. Hence, together with $\xi_\varepsilon\to\xi$ in M, we have

$$\mathcal{S}(\xi_\varepsilon,\bar W_\varepsilon) = \mathcal{S}(\xi,\bar Z) + o_\varepsilon(1).$$

where $o_\varepsilon(1)\to 0$ as $\varepsilon\to 0$. Note that the energy estimate in Theorem 2.7 also applies to the functional Φ_e (since the condition (F) guarantees the validity of (H1) to (H5)), we have

$$\Phi_e(\bar Z) = \gamma_e \le \Phi_e(\bar W_\varepsilon) + O(\|\nabla\Phi_e(\bar W_\varepsilon)\|^2). \tag{38}$$

where $\|\cdot\|$ is the usual norm on $H^1(\mathbb{R}^m)\times H^1(\mathbb{R}^m)$. Since, for arbitrary $W=(U,V)$,

$$\begin{aligned}\big\langle\nabla\Phi_e(\bar W_\varepsilon), W\big\rangle &= \int_{\mathbb{R}^m}\big(\nabla(\bar\beta_\varepsilon\bar U_\varepsilon)\cdot\nabla V + \bar\beta_\varepsilon\bar U_\varepsilon V - f_2(\bar\beta_\varepsilon\bar V_\varepsilon)V\big)dx \\ &\quad + \int_{\mathbb{R}^m}\big(\nabla U\cdot\nabla(\bar\beta_\varepsilon\bar V_\varepsilon) + \bar\beta_\varepsilon U\bar V_\varepsilon - f_1(\bar\beta_\varepsilon\bar U_\varepsilon)U\big)dx \\ &= \int_{B_{2r/\varepsilon}(0)}\Big(\sum_{i,j=1}^m g^{ij}(\exp_{\xi_\varepsilon}\varepsilon x)\partial_i(\bar\beta_\varepsilon\bar U_\varepsilon)\partial_j V + \bar\beta_\varepsilon\bar U_\varepsilon V\Big)\sqrt{\det G}(\exp_{\xi_\varepsilon}\varepsilon x)dx \\ &\quad - \int_{B_{2r/\varepsilon}(0)} f_2(\bar\beta_\varepsilon\bar V_\varepsilon)V\sqrt{\det G}(\exp_{\xi_\varepsilon}\varepsilon x)dx \\ &\quad + \int_{B_{2r/\varepsilon}(0)}\Big(\sum_{i,j=1}^m g^{ij}(\exp_{\xi_\varepsilon}\varepsilon x)\partial_i U\partial_j(\bar\beta_\varepsilon\bar V_\varepsilon) + \bar\beta_\varepsilon U\bar V_\varepsilon\Big)\sqrt{\det G}(\exp_{\xi_\varepsilon}\varepsilon x)dx \\ &\quad - \int_{B_{2r/\varepsilon}(0)} f_1(\bar\beta_\varepsilon\bar U_\varepsilon)U\sqrt{\det G}(\exp_{\xi_\varepsilon}\varepsilon x)dx + O(\varepsilon^2)\|W\| \\ &= \big\langle\nabla\Phi_\varepsilon(\beta_\varepsilon\bar z_\varepsilon), w_\varepsilon\big\rangle_\varepsilon + O(\varepsilon^2)\|W\| \qquad \text{with } w_\varepsilon(\xi) = W\Big(\frac{\exp_{\xi_\varepsilon}^{-1}\xi}{\varepsilon}\Big) \\ &= \big\langle\nabla\Phi_\varepsilon(\bar z_\varepsilon), w_\varepsilon\big\rangle_\varepsilon + o(\varepsilon^2)\|w_\varepsilon\|_\varepsilon + O(\varepsilon^2)\|W\|,\end{aligned}$$

and $\|w_\varepsilon\|_\varepsilon = O(\|W\|)$, we have $\|\nabla\Phi_e(\bar W_\varepsilon)\| = O(\varepsilon^2)$. Therefore, by (37) and (38), we obtain, as desired,

$$\gamma_\varepsilon \ge \Phi_e(\bar W_\varepsilon) + \frac{\varepsilon^2}{3}\mathcal{S}(\xi_\varepsilon,\bar W_\varepsilon) + o(\varepsilon^2) \ge \gamma_e + \frac{\varepsilon^2}{3}\mathcal{S}(\xi,\bar Z) + o(\varepsilon^2). \qquad \square$$

Combining Proposition 3.3, 3.5 and Lemma 3.6, we are now ready to state

Theorem 3.7. *Under the condition* (F)*, for small* $\varepsilon > 0$*, there exists a ground-state solution* $\bar{z}_\varepsilon = (\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ *of eq.* (5) *and a unique maximum point* $\xi_\varepsilon \in M$ *of* $\bar{u}_\varepsilon + \bar{v}_\varepsilon$ *such that*

(1) *for some constants* $C, c > 0$: $\bar{u}_\varepsilon(\xi) + \bar{v}_\varepsilon(\xi) \le C \exp\Big(- \dfrac{c}{\varepsilon} \operatorname{dist}_g(\xi, \xi_\varepsilon)\Big)$;

(2) $\lim\limits_{\varepsilon\to 0} \operatorname{Scal}_g(\xi_\varepsilon) = \max\limits_{\xi\in M} \operatorname{Scal}_g(\xi)$;

(3) *up to a subsequence, the transformed functions* $\bar{U}_\varepsilon(x) = \bar{u}_\varepsilon \circ \exp_{\xi_\varepsilon}(\varepsilon x)$ *and* $\bar{V}_\varepsilon(x) = \bar{v}_\varepsilon \circ \exp_{\xi_\varepsilon}(\varepsilon x)$ *converge to a ground-state solution* $\bar{Z} = (\bar{U}, \bar{V})$ *of eq.* (30)*, where* $\bar{Z}$ *is selected by*

$$\int_{\mathbb{R}^m} \nabla\bar{U}\cdot\nabla\bar{V}|x|^2 dx = \begin{cases} \max\limits_{Z=(U,V)\in\mathcal{L}_e} \displaystyle\int_{\mathbb{R}^m} \nabla U\cdot\nabla V|x|^2 dx, & \text{if } \max\limits_{\xi\in M} \operatorname{Scal}_g(\xi) > 0, \\ \min\limits_{Z=(U,V)\in\mathcal{L}_e} \displaystyle\int_{\mathbb{R}^m} \nabla U\cdot\nabla V|x|^2 dx, & \text{if } \max\limits_{\xi\in M} \operatorname{Scal}_g(\xi) < 0. \end{cases}$$

(B) Spinorial Brezis-Nirenberg problem

Let (M, g, σ) be an m-dimensional closed spin manifold, $m \ge 2$, where g is a Riemannian metric and $\sigma : P_{Spin}(M) \to P_{SO}(M)$ is a spin structure on M. Let $\rho : Spin(m) \to \operatorname{Aut}(\mathbb{S}_m)$ be the spin representation. Then, on the spinor bundle $\mathbb{S}(M) : P_{Spin}(M) \times_\rho \mathbb{S}_m \to M$, we shall consider the nonlinear equation

$$D\psi = \lambda\psi + |\psi|^{\frac{2}{m-1}}\psi, \quad \psi : M \to \mathbb{S}(M) \tag{39}$$

where D is the Dirac operator and $\lambda \in \mathbb{R}$.

Since the Dirac operator is self-adjoint, solutions of eq. (39) are the critical points of the energy functional

$$\Phi_\lambda(\psi) = \frac{1}{2}\int_M (D\psi, \psi) d\mathrm{vol}_g - \frac{\lambda}{2}\int_M |\psi|^2 d\mathrm{vol}_g - \frac{m-1}{2m}\int_M |\psi|^{\frac{2m}{m-1}} d\mathrm{vol}_g$$

where $(\cdot,\cdot) : \mathbb{S}(M)\times\mathbb{S}(M) \to \mathbb{C}$ stands for the hermitian inner product on the spinor bundle.

Let us introduce the functional settings of this problem: First of all, the natural function space is the $H^{1/2}$-spinors on M denoted by $H^{1/2}(M, \mathbb{S}(M))$, that is,

$$H^{1/2}(M, \mathbb{S}(M)) = \big\{\psi \in L^2(M, \mathbb{S}(M)) : |D|^{1/2}\psi \in L^2(M, \mathbb{S}(M))\big\}.$$

Evidently, $H^{1/2}(M, \mathbb{S}(M))$ is a Hilbert space with the following inner product

$$\langle\psi, \varphi\rangle_{H^{1/2}} = \operatorname{Re}\big(|D|^{1/2}\psi, |D|^{1/2}\varphi\big)_{L^2} + \operatorname{Re}(\psi, \varphi)_{L^2}.$$

We also have the Sobolev embedding $H^{1/2}(M, \mathbb{S}(M)) \hookrightarrow L^q(M, \mathbb{S}(M))$ for $q \in [1, \frac{2m}{m-1}]$. The embedding is compact for $1 \le q < \frac{2m}{m-1}$, but not for $q = \frac{2m}{m-1}$. Hence $2^* = \frac{2m}{m-1}$ is called the critical exponent. Due to the lack of compactness of the embedding $H^{1/2}(M, \mathbb{S}(M)) \hookrightarrow L^{2^*}(M, \mathbb{S}(M))$, the functional Φ_λ does not satisfy the Palais-Smale condition in the energy range $(-\infty, +\infty)$. The following lemma shows that Φ_λ satisfies a local Palais-Smale condition (its proof can be found in [15, Section 5]).

Lemma 3.8. *Let $\gamma_{crit} = \frac{1}{2m}\left(\frac{m}{2}\right)^m \omega_m$, where ω_m stands for the volume of the standard sphere S^m. Then, for any $\lambda \in \mathbb{R}$, the functional Φ_λ satisfies the Palais-Smale condition in the energy range $(-\infty, \gamma_{crit})$ in the following sense:*

If $c < \gamma_{crit}$ and $\{\psi_n\}$ is a sequence in $H^{1/2}(M, \mathbb{S}(M))$ such that $\Phi_\lambda(\psi_n) \to c$ and $\nabla\Phi_\lambda(\psi_n) \to 0$, as $n \to \infty$, then $\{\psi_n\}$ is compact in $H^{1/2}(M, \mathbb{S}(M))$.

Motivated by this lemma, our next step is to use the first part of Theorem 2.7 to find a suitable energy estimate (which is going to stay below the threshold γ_{crit}) for the functional Φ_λ.

In the sequel, we denote λ_k, $k \in \mathbb{Z}$, the eigenvalues of D and $E(\lambda_k)$ the corresponding eigenspaces. Since $\{\lambda_k\} \subset \mathbb{R}$ consists of a two-sided unbounded discrete sequence of eigenvalues with finite multiplicities (see for instance [14]), we may order the set as

$$\cdots < \lambda_{-1} < \lambda_0 \leq 0 < \lambda_1 < \lambda_2 < \cdots$$

with $|\lambda_k| \to \infty$ as $|k| \to \infty$. For $\lambda \in \mathbb{R}$, we denote $P^0_\lambda : L^2(M, \mathbb{S}(M)) \to E(\lambda)$ the orthogonal projection, where $E(\lambda) = E(\lambda_k)$ if $\lambda = \lambda_k$ and $E(\lambda) = \{0\}$ if $\lambda \notin \{\lambda_k\}$. Then we can equip $H^{1/2}(M, \mathbb{S}(M))$ with a family of inner products

$$\langle \psi, \varphi \rangle_\lambda = \mathrm{Re}\left(|D - \lambda|^{1/2}\psi, |D - \lambda|^{1/2}\varphi\right)_{L^2} + \mathrm{Re}\left(P^0_\lambda \psi, P^0_\lambda \varphi\right)_{L^2}$$

and the induced norms $\|\cdot\|_\lambda = \langle \cdot, \cdot \rangle^{1/2}_\lambda$, for $\lambda \in \mathbb{R}$. Clearly, for a fixed λ, the norm $\|\cdot\|_\lambda$ is equivalent to the usual $H^{1/2}$-norm. Given $\lambda \in \mathbb{R}$, we set

$$\mathcal{H}^+_\lambda = \overline{\bigoplus_{\lambda_j > \lambda} E(\lambda_j)} \quad \text{and} \quad \mathcal{H}^-_\lambda = \overline{\bigoplus_{\lambda_j < \lambda} E(\lambda_j)}, \tag{40}$$

where the closure is taken in $H^{1/2}(M, \mathbb{S}(M))$. Then we have the following orthogonal decomposition with respect to $\langle \cdot, \cdot \rangle_\lambda$:

$$H^{1/2}(M, \mathbb{S}(M)) = \mathcal{H}^+_\lambda \oplus E(\lambda) \oplus \mathcal{H}^-_\lambda. \tag{41}$$

To proceed, we introduce

$$T_\lambda : L^{2^*}(M, \mathbb{S}(M)) \to E(\lambda), \quad T_\lambda(\psi) = \arg\min_{\phi \in E(\lambda)} \|\psi - \phi\|^{2^*}_{L^{2^*}}$$

i.e. $T_\lambda(\psi)$ is the best approximation of $\psi \in L^{2^*}(M, \mathbb{S}(M))$ in $E(\lambda)$. This element exists and is unique because the L^{2^*}-norm is uniformly convex. Generally, T_λ is not linear, and $T_\lambda(\psi) \equiv 0$ when $E(\lambda) = \{0\}$. Since $\|\psi\|_{L^{2^*}} \geq \|\psi - T_\lambda(\psi)\|_{L^{2^*}}$ for all $\psi \in L^{2^*}(M, \mathbb{S}(M))$, we have

$$\Phi_\lambda(\psi) \leq \widetilde{\Phi}_\lambda(\psi) := \frac{1}{2}\int_M (D\psi, \psi) d\mathrm{vol}_g - \frac{\lambda}{2}\int_M |\psi|^2 d\mathrm{vol}_g - \frac{1}{2^*}\int_M |\psi - T_\lambda(\psi)|^{2^*} d\mathrm{vol}_g$$

for all $\psi \in H^{1/2}(M, \mathbb{S}(M))$. Set

$$\Psi_\lambda(\psi) = \frac{1}{2^*}\int_M |\psi - T_\lambda(\psi)|^{2^*} d\mathrm{vol}_g, \quad \text{for } \psi \in H^{1/2}(M, \mathbb{S}(M))$$

Under the decomposition (41) and the Sobolev embedding, the following lemma holds:

Lemma 3.9. *In the case $E(\lambda) \neq \{0\}$, we have*

(1) *For $\psi \in \mathcal{H}_\lambda^+ \oplus \mathcal{H}_\lambda^-$, $\phi \in E(\lambda)$ and $t \in \mathbb{R}$, we have $T_\lambda(t\psi) = tT_\lambda(\psi)$, $T_\lambda(\psi+\phi) = T_\lambda(\psi) + \phi$ and $\Psi_\lambda(\psi+\phi) = \Psi_\lambda(\psi)$.*

(2) *T_λ is C^1 on $\mathcal{H}_\lambda^+ \oplus \mathcal{H}_\lambda^- \setminus \{0\}$ and $\nabla T_\lambda(\psi)[\psi] = T_\lambda(\psi)$ for $\psi \neq 0$.*

(3) *Ψ_λ is C^2 on $\mathcal{H}_\lambda^+ \oplus \mathcal{H}_\lambda^-$ and is convex, moreover, $P_\lambda^0 \nabla \Psi_\lambda(\psi) = 0$ for all $\psi \in \mathcal{H}_\lambda^+ \oplus \mathcal{H}_\lambda^-$.*

(4) *For any $\psi, \varphi \in \mathcal{H}_\lambda^+ \oplus \mathcal{H}_\lambda^-$, we have*

$$\left\langle \nabla^2 \Psi_\lambda(\psi)[\psi+\varphi], \psi+\varphi \right\rangle_\lambda - 2\left\langle \nabla\Psi_\lambda(\psi), \varphi\right\rangle_\lambda \geq \frac{m+3}{m+1}\left\langle \nabla\Psi_\lambda(\psi), \psi\right\rangle_\lambda .$$

(5) *Critical points of $\widetilde{\Phi}_\lambda$ on $\mathcal{H}_\lambda^+ \oplus \mathcal{H}_\lambda^-$ are in one-to-one correspondence with critical points of Φ_λ on $H^{1/2}(M, \mathbb{S}(M))$ via the injective map $\psi \mapsto \psi - T_\lambda(\psi)$ from $\mathcal{H}_\lambda^+ \oplus \mathcal{H}_\lambda^-$ into $H^{1/2}(M, \mathbb{S}(M))$.*

Proof. The proof of this lemma is contained in [4], moreover (1) can be straightforwardly verified. For the sake of completeness, we sketch the argument as follows.

To see (2), let us fix a $\psi \in \mathcal{H}_\lambda^+ \oplus \mathcal{H}_\lambda^- \setminus \{0\}$ and consider the functional Ψ_λ^ψ defined on $E(\lambda)$ by

$$\Psi_\lambda^\psi(\phi) = \frac{1}{2^*}\int_M |\psi - \phi|^{2^*} d\mathrm{vol}_g, \quad \phi \in E(\lambda).$$

Particularly, we only focus on the case where $E(\lambda)$ is non-trivial. Then we have Ψ_λ^ψ is C^2 on $E(\lambda)$ and $\Psi_\lambda^\psi(\phi) \geq \frac{1}{2^*}(\|\phi\|_{L^{2^*}} - \|\psi\|_{L^{2^*}})^{2^*}$. Moreover, for $\phi, \varphi \in E(\lambda)$, we get

$$\left\langle \nabla\Psi_\lambda^\psi(\phi), \varphi \right\rangle_\lambda = -\operatorname{Re}\int_M |\psi-\phi|^{2^*-2}(\psi-\phi,\varphi) d\mathrm{vol}_g, \quad \text{and}$$

$$\begin{aligned}\left\langle \nabla^2\Psi_\lambda^\psi(\phi)[\varphi], \varphi \right\rangle_\lambda &= \int_M \Big[|\psi-\phi|^{2^*-2}|\varphi|^2 + (2^*-2)|\psi-\phi|^{2^*-4}|\operatorname{Re}(\psi-\phi,\varphi)|^2\Big] d\mathrm{vol}_g \\ &\geq \int_M |\psi-\phi|^{2^*-2}|\varphi|^2 d\mathrm{vol}_g .\end{aligned}$$

Hence Ψ_λ^ψ is convex. Since $E(\lambda)$ is a finite dimensional eigenspace of the Dirac operator, by the convexity and coerciveness of Ψ_λ^ψ, we have $T_\lambda(\psi)$ is the unique strict minimum point for Ψ_λ^ψ on $E(\lambda)$, which is also the only critical point of Ψ_λ^ψ. By virtue of the above observation and Bär's theorem [3] on nodal sets of the eigenspinors, we can see that the self-adjoint operator

$$\varphi \mapsto \nabla^2\Psi_\lambda^\psi(T_\lambda(\psi))[\varphi], \quad \forall \varphi \in E(\lambda)$$

is invertible. Hence, the implicit function theorem implies $T_\lambda : \mathcal{H}_\lambda^+ \oplus \mathcal{H}_\lambda^- \setminus \{0\} \to E(\lambda)$ is of C^1.

Although (1) implies that T_λ cannot be differentiable at 0, we shall show that Ψ_λ is of class C^2. In fact, by $\Psi_\lambda(0) = 0$, we find that

$$\Psi_\lambda(t\psi) - \Psi_\lambda(0) = \frac{1}{2^*}\int_M |t\psi - T_\lambda(t\psi)|^{2^*} d\mathrm{vol}_g = \frac{t^{2^*}}{2^*}\int_M |\psi - T_\lambda(\psi)|^{2^*} d\mathrm{vol}_g$$

for $\psi \neq 0$. Hence, combined with the fact that $2^* > 2$ we see that $\nabla\Psi_\lambda(0) = 0$.

For $\psi \in \mathcal{H}_\lambda^+ \oplus \mathcal{H}_\lambda^- \setminus \{0\}$, it follows directly that $P_\lambda^0 \nabla \Psi_\lambda(\psi) = 0$ and that

$$\begin{aligned}\langle \nabla \Psi_\lambda(\psi), \varphi \rangle_\lambda &= \mathrm{Re} \int_M |\psi - T_\lambda(\psi)|^{2^*-2} (\psi - T_\lambda(\psi), \varphi - \nabla T_\lambda(\psi)[\varphi]) d\mathrm{vol}_g \\ &= \mathrm{Re} \int_M |\psi - T_\lambda(\psi)|^{2^*-2} (\psi - T_\lambda(\psi), \varphi - T_\lambda(\varphi)) d\mathrm{vol}_g \\ &= \mathrm{Re} \int_M |\psi - T_\lambda(\psi)|^{2^*-2} (\psi - T_\lambda(\psi), \varphi) d\mathrm{vol}_g\end{aligned}$$

for all $\varphi \in \mathcal{H}_\lambda^+ \oplus \mathcal{H}_\lambda^-$. Then we have the derivative of Ψ_λ is continuous in ψ, and hence Ψ_λ is of class C^1 on $\mathcal{H}_\lambda^+ \oplus \mathcal{H}_\lambda^-$.

For the C^2-smoothness, we only need to show that the second derivative of Ψ_λ is continuous at 0. To this end, let us take again $\psi \in \mathcal{H}_\lambda^+ \oplus \mathcal{H}_\lambda^- \setminus \{0\}$ and $\varphi \in \mathcal{H}_\lambda^+ \oplus \mathcal{H}_\lambda^-$. For the second derivative at 0, we first note that

$$\langle \nabla \Psi_\lambda(t\psi), \varphi \rangle_\lambda = t^{2^*-1} \,\mathrm{Re} \int_M |\psi - T_\lambda(\psi)|^{2^*-2} (\psi - T_\lambda(\psi), \varphi) d\mathrm{vol}_g \,.$$

Since $2^* > 2$ and $\nabla \Psi_\lambda(0) = 0$, this implies that Ψ_λ is twice differentiable at 0 and $\nabla^2 \Psi_\lambda(0) = 0$. For $\psi \neq 0$, by taking the derivative in the direction of φ at both sides of $P_\lambda^0 \nabla \Psi_\lambda(\psi) = 0$, we have $P_\lambda^0 \nabla^2 \Psi(\psi)[\varphi] = 0$. Hence, we see that

$$\begin{aligned}\left\langle \nabla^2 \Psi_\lambda(\psi)[\phi], \varphi \right\rangle_\lambda = \frac{2}{m-1} \int_M |\psi - T_\lambda(\psi)|^{2^*-4} \,\mathrm{Re}(\psi - T_\lambda(\psi), \phi) \,\mathrm{Re}(\psi - T_\lambda(\psi), \varphi) d\mathrm{vol}_g \\ + \mathrm{Re} \int_M |\psi - T_\lambda(\psi)|^{2^*-2} (\phi, \varphi) d\mathrm{vol}_g\end{aligned}$$

for $\phi, \varphi \in \mathcal{H}_\lambda^+ \oplus \mathcal{H}_\lambda^-$. It follows that $\nabla^2 \Psi_\lambda$ is continuous, and hence Ψ_λ is of class C^2. The convexity of Ψ_λ is an immediate consequence of the above calculation and, furthermore,

$$\begin{aligned}&\left\langle \nabla^2 \Psi_\lambda(\psi)[\psi + \varphi], \psi + \varphi \right\rangle_\lambda - 2 \left\langle \nabla \Psi_\lambda(\psi), \varphi \right\rangle_\lambda \\ &\quad = \int_M \Big[|\psi - T_\lambda(\psi)|^{2^*-2} \big| \psi + \varphi - \nabla T_\lambda(\psi)[\psi + \varphi] \big|^2 \\ &\qquad + (2^* - 2) |\psi - T_\lambda(\psi)|^{2^*-4} \big| \,\mathrm{Re}\left(\psi - T_\lambda(\psi), \psi + \varphi - \nabla T_\lambda(\psi)[\psi + \varphi] \right) \big|^2 \Big] d\mathrm{vol}_g \\ &\qquad - 2 \,\mathrm{Re} \int_M |\psi - T_\lambda(\psi)|^{2^*-2} (\psi - T_\lambda(\psi), \varphi - \nabla T_\lambda(\psi)[\varphi]) d\mathrm{vol}_g \,.\end{aligned}$$

For simplicity, we denote $z = \psi - T_\lambda(\psi)$ and $w = \varphi - \nabla T_\lambda(\psi)[\varphi]$. Then, using the fact $\nabla T_\lambda(\psi)[\psi] = T_\lambda(\psi)$, we can see from the above equality that

$$\begin{aligned}&\left\langle \nabla^2 \Psi_\lambda(\psi)[\psi + \varphi], \psi + \varphi \right\rangle_\lambda - 2 \left\langle \nabla \Psi_\lambda(\psi), \varphi \right\rangle_\lambda \\ &\quad = \int_M \Big[|z|^{2^*-2} |z + w|^2 + (2^* - 2) |z|^{2^*-4} |\,\mathrm{Re}(z, z + w)|^2 - 2 |z|^{2^*-2} \,\mathrm{Re}(z, w) \Big] d\mathrm{vol}_g \\ &\quad = \int_M |z|^{2^*-2} \Big[|w|^2 + (2^* - 1) |z|^2 + 2(2^* - 2) \,\mathrm{Re}(z, w) + (2^* - 2) \frac{|\,\mathrm{Re}(z, w)|^2}{|z|^2} \Big] d\mathrm{vol}_g\end{aligned}$$

$$\geq \int_M |z|^{2^*-2}\Big[(2^*-1)|z|^2 + 2(2^*-2)\operatorname{Re}(z,w) + (2^*-1)\frac{|\operatorname{Re}(z,w)|^2}{|z|^2}\Big]d\mathrm{vol}_g$$
$$\geq \Big((2^*-1) - \frac{(2^*-2)^2}{2^*-1}\Big)\int_M |z|^{2^*}d\mathrm{vol}_g = \frac{m+3}{m+1}\int_M |\psi - T_\lambda(\psi)|^{2^*}d\mathrm{vol}_g$$

where we used the fundamental inequalities

$$|w|^2 \geq \frac{|\operatorname{Re}(z,w)|^2}{|z|^2} \quad \text{and} \quad 2|\operatorname{Re}(z,w)| \leq \frac{2^*-2}{2^*-1}|z|^2 + \frac{2^*-1}{2^*-2}\frac{|\operatorname{Re}(z,w)|^2}{|z|^2}$$

for $|z| \neq 0$. This proves (4).

Finally, to see (5), we first mention that if $\varphi \in H^{1/2}(M, \mathbb{S}(M))$ is a critical point of Φ_λ then $\operatorname{Re}\int_M |\varphi|^{2^*-2}(\varphi,\phi)d\mathrm{vol}_g = 0$ for all $\phi \in E(\lambda)$. This implies $P^0_\lambda\varphi = -T_\lambda(\varphi)$ and $\varphi - P^0_\lambda\varphi \in \mathcal{H}^+_\lambda \oplus \mathcal{H}^-_\lambda$ is a critical point of $\widetilde{\Phi}_\lambda$. On the other hand, if $\psi \in \mathcal{H}^+_\lambda \oplus \mathcal{H}^-_\lambda$ is a (constrained) critical point of $\widetilde{\Phi}_\lambda$, then $\psi - T_\lambda(\psi)$ is an unconstrained critical point of $\widetilde{\Phi}_\lambda$ on $H^{1/2}(M, \mathbb{S}(M))$ and hence is a critical point of Φ_λ. □

Thanks to Lemma 3.9, we can reduce the problem (39) to the search of critical points of $\widetilde{\Phi}_\lambda$ on the subspace $\mathcal{H}_\lambda := \mathcal{H}^+_\lambda \oplus \mathcal{H}^-_\lambda \subset H^{1/2}(M, \mathbb{S}(M))$ with the inner product $\langle\cdot,\cdot\rangle_\lambda$. Now, for a fixed $\lambda \in \mathbb{R}$, we associate with $D - \lambda$ the operator L_λ defined on $\mathcal{H}_\lambda$ by

$$\langle L_\lambda\psi, \varphi\rangle_\lambda = \operatorname{Re}\int_M (D\psi, \varphi)d\mathrm{vol}_g - \lambda \operatorname{Re}\int_M (\psi,\varphi)d\mathrm{vol}_g, \quad \forall \psi, \varphi \in \mathcal{H}_\lambda.$$

Then $L_\lambda : \mathcal{H}_\lambda \to \mathcal{H}_\lambda$ is self-adjoint, continuous and invertible. Moreover, due to the definition in (40), we have (2) and (3) are satisfied (by taking the index $\alpha \equiv \lambda$). Hence we can rewrite eq. (39) in a form of (1) on $(\mathcal{H}_\lambda, \langle\cdot,\cdot\rangle_\lambda)$.

Our next step is based on a suitable choice of test spinor fields $\{\psi_\varepsilon\}$ to estimate $\widetilde{\Phi}_\lambda$. To begin with, we fix a constant spinor $\psi_0 \in \mathbb{S}_m$ with $|\psi_0| = m^{\frac{m-1}{2}}$. Let $\mu(x) = \frac{1}{1+|x|^2}$ for $x \in \mathbb{R}^m$, we consider the spinor

$$\varphi(x) = \mu(x)^{\frac{m}{2}}(1-x)\cdot\psi_0$$

where " $\cdot$ " stands for the action of Clifford multiplication on spinors. One may compute that

$$D\varphi = m\mu\varphi \quad \text{and} \quad |\varphi| = m^{\frac{m-1}{2}}\mu^{\frac{m-1}{2}} = \Big(\frac{m}{1+|x|^2}\Big)^{\frac{m-1}{2}}.$$

We choose $0 < r < inj(M)/2$ and let $\eta \in C^\infty(\mathbb{R}^m, [0,1])$ be a smooth cut-off function satisfying $\eta(x) = 1$ if $|x| \leq r$ and $\eta(x) = 0$ if $|x| \geq 2r$. Then we consider the spinors defined by

$$\varphi_\varepsilon(x) = \varepsilon^{-\frac{m-1}{2}}\eta(x)\varphi\Big(\frac{x}{\varepsilon}\Big), \quad \text{for } x \in \mathbb{R}^m \text{ and } \varepsilon > 0.$$

In order to transplant the above spinors onto M, we consider the Riemann normal coordinates on a neighborhood U of a given point $\xi_0 \in M$. On U, we adopt the well-known Bourguignon-Gauduchon trivialization which preserves orientation and spin structure.

This trivialization gives us a fiberwise isomorphism

$$\begin{array}{ccc} \mathbb{S}(\mathbb{R}^m) \supset \mathbb{S}(B_{2r}(0)) & \xrightarrow{f} & \mathbb{S}(U) \subset \mathbb{S}(M) \\ \big\downarrow & & \big\downarrow \\ \mathbb{R}^m \supset B_{2r}(0) & \xrightarrow{\exp_{\xi_0}} & U \subset M \end{array}$$

and, particularly, one may define on M the test spinors

$$\psi_\varepsilon(\xi) = f \circ \varphi_\varepsilon \circ \exp_{\xi_0}(\xi), \quad \text{for } \xi \in U \subset M. \tag{42}$$

The following evaluation is contained in [4], which is based on elementary calculations.

Lemma 3.10. *If ψ_ε is defined as in* (42)*, then as $\varepsilon \to 0$*

$$\widetilde{\Phi}_\lambda(\psi_\varepsilon) \le \frac{1}{2m}\big(\frac{m}{2}\big)^m \omega_m + \begin{cases} -\lambda C\varepsilon|\ln\varepsilon| + O(\varepsilon) & m = 2, \\ -\lambda C\varepsilon + O(\varepsilon^2|\ln\varepsilon|) & m = 3, \\ -\lambda C\varepsilon + O(\varepsilon^2) & m \ge 4, \end{cases}$$

and

$$\|\nabla\widetilde{\Phi}_\lambda(\psi_\varepsilon)\|_\lambda = \begin{cases} O(\varepsilon^{\frac{1}{2}}) & m = 2, \\ O(\varepsilon|\ln\varepsilon|^{\frac{2}{3}}) & m = 3, \\ O(\varepsilon) & m \ge 4. \end{cases}$$

As an immediate consequence of Lemma 3.10 and Theorem 2.7 (by taking the index $\alpha \equiv \lambda$), we have

$$\gamma_\lambda := \inf_{\phi\in\mathcal{H}_\lambda^+\setminus\{0\}} \sup_{\psi\in\mathbb{R}\phi\oplus\mathcal{H}_\lambda^-} \widetilde{\Phi}_\lambda(\psi) \le \frac{1}{2m}\big(\frac{m}{2}\big)^m \omega_m + \begin{cases} -\lambda C\varepsilon|\ln\varepsilon| + O(\varepsilon) & m = 2, \\ -\lambda C\varepsilon + O(\varepsilon^2|\ln\varepsilon|^{\frac{4}{3}}) & m = 3, \\ -\lambda C\varepsilon + O(\varepsilon^2) & m \ge 4. \end{cases}$$

Now, combining Lemma 3.8, we can state our crucial result:

Theorem 3.11. *For $\lambda > 0$, the spinorial Brezis-Nirenberg problem* (39) *has a nontrivial solution.*

For more details on the spinorial Brezis-Nirenberg equation and some related problems, we refer to [4].

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