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Multiple Weak Solutions of Biharmonic Systems

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We obtain sufficient conditions for the existence of at least three weak solutions of a (p, q)-biharmonic system. We also consider the applications of our main theorem to some special cases of the system. The proof of our main theorem utilizes the variational approaches and a recent theorem of Ricceri on the existence of two global minima of a functional.

Keywords: Biharmonic system, principal eigenvalues, variational methods, multiple weak solutions.

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1. Introduction

We assume that $\Omega \subseteq \mathbb{R}^N$, with $N > 2$, is a bounded domain with smooth boundary, $p, q > 1$ are constants satisfying $\max\{p, q\} < N/2$ and $p^*, q^* > 2$, where

$$p^* = \frac{Np}{N-2p} \quad \text{and} \quad q^* = \frac{Nq}{N-2q}.$$

Let $\mathbb{A}$ be the class of functions $H : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ which are measurable in Ω, continuously differentiable in $\mathbb{R}^2$, $H(\cdot, 0, 0) \in L^2(\Omega)$, and for which there exist constants $C_1 > 0$, $1 < \mu < p^*/2$, and $1 < \nu < q^*/2$ such that

$$|H_u(x, u, v)| \le C_1 \left(1 + |u|^{\mu-1} + |v|^{\frac{\nu(\mu-1)}{\mu}}\right)$$

and

$$|H_v(x, u, v)| \le C_1 \left(1 + |v|^{\nu-1} + |u|^{\frac{\mu(\nu-1)}{\nu}}\right)$$

for all $(x, u, v) \in \Omega \times \mathbb{R}^2$, where $H_u(x, u, v)$ and $H_v(x, u, v)$ denote the partial derivatives of $H(x, u, v)$ with respect to u and v, respectively.

In this work, we study the existence of multiple weak solutions to the (p,q)-biharmonic system

$$\begin{cases} \Delta\left(|\Delta u|^{p-2}\Delta u\right) = m_1(x)F_u(x, u, v) + m_2(x)G_u(x, u, v) + K_u(x, u, v) & \text{in } \Omega, \\ \Delta\left(|\Delta v|^{q-2}\Delta v\right) = m_1(x)F_v(x, u, v) + m_2(x)G_v(x, u, v) + K_v(x, u, v) & \text{in } \Omega, \\ u = \Delta u = 0, \ v = \Delta v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $F, G, K \in \mathbb{A}$ and $m_1, m_2 : \Omega \to \mathbb{R}$ are measurable functions. We first establish a theorem which ensures that system (1) has at least three weak solutions. Then,

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we derive two consequences of our theorem by applying it to some special cases of the system. See Theorem 3.2 and Corollaries 3.4 and 3.5 in Section 3 for details. This work is greatly motivated by a recent paper of Ricceri [13], where the author first proved a general theorem for a functional to have at least two global minima in a topological space, and then presented its applications to some second order elliptic systems. We explore further applications of Ricceri's theorem and obtain new existence results for biharmonic systems.

In recent years, biharmonic systems have been studied by many authors in the literature. We refer the reader to [1, 2, 3, 4, 6, 7, 8, 14] and the references therein for some recent work. For instance, the present authors [7] studied the (p,q)-biharmonic system

$$\begin{cases} \Delta\left(|\Delta u|^{p-2}\Delta u\right) = \lambda a(x)|u|^{p-2}u + \lambda c(x)|u|^{\alpha-1}|v|^{\beta+1}u & \text{in } \Omega, \\ \Delta\left(|\Delta v|^{q-2}\Delta v\right) = \lambda b(x)|v|^{q-2}v + \lambda c(x)|u|^{\alpha+1}|v|^{\beta-1}v & \text{in } \Omega, \\ u = \Delta u = 0,\ v = \Delta v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\lambda > 0$ is a parameter, and proved the existence, positivity, simplicity, uniqueness up to nonnegative eigenfunctions, and isolation of the principle eigenvalue of the system under the following assumptions:

(A1) α and β are nonnegative constants and $\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1$;

(A2) $a, b, c : \Omega \to \mathbb{R}$ are nonnegative measurable functions, $c \not\equiv 0$ in Ω, $a \in L^{\frac{p^*}{p^*-p}}(\Omega)$, $b \in L^{\frac{q^*}{q^*-q}}(\Omega)$, and $c \in L^{\rho}(\Omega)$, where $\rho = \left(1 - \frac{\alpha+1}{p^*} - \frac{\beta+1}{q^*}\right)^{-1}$.

In this paper, our goal is to further study the existence of multiple weak solutions of biharmonic systems. Our results enrich the existing literature on this subject.

The remainder of this paper is organized as follows. Section 2 presents some preliminaries and Section 3 contains the main theorem and its proof and corollaries.

2. Preliminary results

Throughout this paper, for any $r \in (1,\infty)$, we denote the norm of the space $L^r(\Omega)$ by $\|u\|_r = \left(\int_\Omega |u|^r\right)^{1/r}$, $u \in L^r(\Omega)$, and let the space X be defined by

$$X = \left(W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)\right) \times \left(W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)\right). \tag{3}$$

Then, X is a separable and reflexive Banach space equipped with the standard norm

$$\|(u,v)\|_X = \|\Delta u\|_p + \|\Delta v\|_q, \quad (u,v) \in X.$$

In the sequel, we assume that the constants α, β and the functions a, b, c satisfy the conditions (A1) and (A2), given in Section 1. Let the functionals $\Phi, \Psi : X \to \mathbb{R}$ be defined by

$$\Phi(u,v) = \frac{\alpha+1}{p}\int_\Omega |\Delta u|^p\,dx + \frac{\beta+1}{q}\int_\Omega |\Delta v|^q\,dx \tag{4}$$

and

$$\Psi(u,v)=\frac{\alpha+1}{p}\int_\Omega a(x)|u|^p\,dx+\frac{\beta+1}{q}\int_\Omega b(x)|v|^q\,dx+\int_\Omega c(x)|u|^{\alpha+1}|v|^{\beta+1}\,dx, \quad (5)$$

where $(u,v)\in X$. Then, from [7, Lemma 2.1], Φ and Ψ are well defined. Moreover, by [7, Theorem 2.3], we have the following lemma.

Lemma 2.1. *System* (2) *admits an eigenpair* $(\lambda_1,(u_1,v_1))$ *such that* $\lambda_1>0$ *is the principal eigenvalue,* u_1 *and* v_1 *are nonnegative in* Ω*, and*

$$\lambda_1=\Phi(u_1,v_1)=\inf_{\Psi(u,v)=1}\Phi(u,v). \quad (6)$$

From (6), it follows that

$$\Phi(u,v)\ge\lambda_1\Psi(u,v)\quad\text{for all } (u,v)\in X. \quad (7)$$

For any given $H\in\mathbb{A}$, define a functional $\widetilde{H}:X\to\mathbb{R}$ by

$$\widetilde{H}(u,v)=\int_\Omega H(x,u,v)\,dx,\quad (u,v)\in X.$$

Lemma 2.2. *For any* $H\in\mathbb{A}$*, we have*

(a) $\widetilde{H}$ *is well defined and* $H(\,\cdot\,,u(\cdot),v(\cdot))\in L^2(\Omega)$ *for any* $(u,v)\in X$*;*

(b) $\widetilde{H}\in C^1(X,\mathbb{R})$ *with its Gâteaux derivative given by*

$$\left\langle\widetilde{H}'(u,v),(\phi,\psi)\right\rangle=\int_\Omega H_u(x,u,v)\phi\,dx+\int_\Omega H_v(x,u,v)\psi\,dx \quad (8)$$

for any $(\phi,\psi)\in X$*;*

(c) $\widetilde{H}$ *is sequentially weakly continuous in* X*.*

Proof. For any $(x,u_0,v_0)\in\Omega\times\mathbb{R}^2$, in view of the fact that $H\in\mathbb{A}$, we have

$$\begin{aligned}
|H(x,u_0,v_0)|&=\left|H(x,0,0)+\int_0^1\frac{\partial H(x,su_0,sv_0)}{\partial s}\,ds\right|\\
&\le|H(x,0,0)|+\int_0^1\big(|u_0|\,|H_u(x,su_0,sv_0)|+|v_0|\,|H_v(x,su_0,sv_0)|\big)\,ds\\
&\le|H(x,0,0)|\\
&\quad+C_1\int_0^1\Big(|u_0|+|u_0|^\mu s^{\mu-1}+|sv_0|^{\frac{\nu(\mu-1)}{\mu}}|u_0|+|v_0|+|v_0|^\nu s^{\nu-1}+|su_0|^{\frac{\mu(\nu-1)}{\nu}}|v_0|\Big)\,ds\\
&\le|H(x,0,0)|+C_1\left(|u_0|+|v_0|+|u_0|^\mu+|v_0|^\nu+|v_0|^{\frac{\nu(\mu-1)}{\mu}}|u_0|+|u_0|^{\frac{\mu(\nu-1)}{\nu}}|v_0|\right).
\end{aligned}$$

This, together with Young's inequality, implies that

$$|H(x,u_0,v_0)|\le C_2\left(|H(x,0,0)|+|u_0|+|v_0|+|u_0|^\mu+|v_0|^\nu\right), \quad (9)$$

where C_2 is a positive constant independent of $(u_0,v_0)\in\mathbb{R}^2$.

Recall that $H(\,\cdot\,,0,0) \in L^2(\Omega)$, $1 < \mu < p^*/2$, and $1 < \nu < q^*/2$. Then, for any $(u,v) \in X$, from the Sobolev embedding theorem, we see that

$$\big|\widetilde{H}(u,v)\big| \le C_2 \int_\Omega \big(|H(x,0,0)| + |u| + |v| + |u|^\mu + |v|^\nu\big)\, dx < \infty.$$

Hence, $\widetilde{H}$ is well defined. In view of the fact that $H(\,\cdot\,,0,0) \in L^2(\Omega)$, $2 < 2\mu < p^*$, and $2 < 2\nu < q^*$, from (9), Hölder's inequality, and the Sobolev embedding theorem, it is a easy to check that $H(\cdot, u(\cdot), v(\cdot)) \in L^2(\Omega)$ for any $(u,v) \in X$. Thus, we have proved part (a).

Moreover, by a standard argument, we can show that $\widetilde{H} \in C^1(X,\mathbb{R})$ and (8) holds. Hence, part (b) holds.

Below, we show part (c). Let (u_n, v_n) be a bounded sequence in X. Then, passing to a subsequence if necessary, we may assume that (u_n, v_n) converges weakly to $(\hat{u},\hat{v}) \in X$. Thus, by the Sobolev embedding theorem, we have $u_n \to \hat{u}$ in $L^{2\mu}(\Omega)$ and $v_n \to \hat{v}$ in $L^{2\nu}(\Omega)$. Since $H \in \mathbb{A}$, there exists a constant $C_3 > 0$ such that

$$|H_u(x,u,v)| \le C_3 \left(1 + |u|^{\frac{2\mu}{2}} + |v|^{\frac{2\nu}{2}}\right)$$

and

$$|H_v(x,u,v)| \le C_3 \left(1 + |u|^{\frac{2\mu}{2}} + |v|^{\frac{2\nu}{2}}\right)$$

for all $(x,u,v) \in \Omega \times \mathbb{R}^2$. Hence, $H_u(\,\cdot\,, u(\cdot), v(\cdot))$ and $H_v(\,\cdot\,, u(\cdot), v(\cdot))$ are continuous in $L^2(\Omega)$ with respect to $(u,v) \in L^{2\mu}(\Omega) \times L^{2\nu}(\Omega)$ (see, for example, [5, Theorem 1.1]). Thus, from (8) and Hölder's inequality,

$$\begin{aligned}
&\big|\big\langle \widetilde{H}'(u_n,v_n) - \widetilde{H}'(\hat{u},\hat{v}), (\phi,\psi)\big\rangle\big| \\
&\quad \le \int_\Omega |H_u(x,u_n,v_n) - H_u(x,\hat{u},\hat{v})|\,|\phi|\, dx + \int_\Omega |H_v(x,u_n,v_n) - H_v(x,\hat{u},\hat{v})|\,|\psi|\, dx \\
&\quad \le \left(\int_\Omega |H_u(x,u_n,v_n) - H_u(x,\hat{u},\hat{v})|^2\, dx\right)^{\frac{1}{2}} \|\phi\|_2 \\
&\qquad + \left(\int_\Omega |H_v(x,u_n,v_n) - H_v(x,\hat{u},\hat{v})|^2\, dx\right)^{\frac{1}{2}} \|\psi\|_2 \quad \to 0 \quad \text{for any } (\phi,\psi) \in X.
\end{aligned}$$

Hence, $\widetilde{H}'(u_n,v_n) \to \widetilde{H}'(\hat{u},\hat{v})$. This shows that $\widetilde{H}'$ is compact. Thus, by [15, Corollary 4.19], $\widetilde{H}$ is sequentially weakly continuous in X; that is, part(c) holds. This completes the proof of the lemma. □

Let S be a topological space. Recall that a function $g : S \to \mathbb{R}$ is said to be *inf-compact* if the set $g^{-1}((-\infty, r]) := \{u \in S : g(u) \le r\}$ is compact for each $r \in \mathbb{R}$.

The following lemma plays a key role in the proof of our main theorem and is taken from [13, Theorem 2.1]. For some related results, see [10, 11, 12] and the references therein.

Lemma 2.3. *Let X be a topological space, $(Y, \langle\cdot,\cdot\rangle)$ a real Hilbert space, $T \subseteq Y$ a convex set dense in Y, and $I : X \to \mathbb{R}$, $\varphi : X \to Y$ two functions such that, for each $y \in T$, the function $x \to I(x) + \langle\varphi(x), y\rangle$ is lower semicontinuous and inf-compact. Moreover, assume that there exists a point $x_0 \in X$, with $\varphi(x_0) \neq 0$, such that*

(a) *x_0 is a global minimum of both functions I and $\|\varphi(\cdot)\|$;*

(b) $\inf_{x\in X}\langle\varphi(x), \varphi(x_0)\rangle < \|\varphi(x_0)\|^2$.

Then, for each convex set $S \subseteq T$ dense in Y, there exists $\tilde{y} \in S$ such that the functional $x \to I(x) + \langle\varphi(x), \tilde{y}\rangle$ has at least two global minima in X.

3. Main results

Recall that $F, G, K \in \mathbb{A}$ are the functions appearing in system (1) and that the constants α, β and the functions a, b, c satisfy the conditions (A1) and (A2). We further make the following assumptions:

(H1) meas$(A) > 0$, meas$(B) > 0$, and

$$|F(x,0,0)|^2+|G(x,0,0)|^2 \le |F(x,s,t)|^2+|G(x,s,t)|^2 \text{ for all } x\in\Omega \text{ and } s,t\in\mathbb{R},$$

where $$A = \big\{x \in \Omega \,:\, |F(x,0,0)|^2 + |G(x,0,0)|^2 > 0\big\}$$

and $$B = \Big\{x \in \Omega \,:\, \inf_{(s,t)\in\mathbb{R}^2} P(x,s,t) < 0\Big\} \quad \text{with}$$

$$P(x,s,t) = F(x,0,0)F(x,s,t)+G(x,0,0)G(x,s,t)-|F(x,0,0)|^2-|G(x,0,0)|^2;$$

(H2) $K(x,0,0) = 0$ for all $x \in \Omega$ and there exists a constant $\kappa \in (0,1)$ such that

$$K(x,s,t) \le \kappa\theta_2\lambda_1\big[\theta_1\big(a(x)|s|^p + b(x)|t|^q\big) + c(x)|s|^{\alpha+1}|t|^{\beta+1}\big]$$

for all $x \in \Omega$ and $s,t \in \mathbb{R}$, where λ_1 is given by (6),

$$\theta_1 = \min\left\{\frac{\alpha+1}{p}, \frac{\beta+1}{q}\right\}, \quad \text{and} \quad \theta_2 = \frac{1}{\max\{\alpha+1,\, \beta+1\}};$$

(H3) F and G satisfy the following growth condition

$$\lim_{|s|+|t|\to\infty} \sup_{x\in\Omega} \frac{|F(x,s,t)| + |G(x,s,t)|}{a(x)|s|^p + b(x)|t|^q + c(x)|s|^{\alpha+1}|t|^{\beta+1}} = 0.$$

Definition 3.1. We say that $(u,v) \in X$ is a *weak solution* of system (1) if

$$\int_\Omega |\Delta u|^{p-2}\Delta u\Delta\phi\,dx = \int_\Omega \big(m_1(x)F_u(x,u,v) + m_2(x)G_u(x,u,v) + K_u(x,u,v)\big)\phi\,dx$$

and

$$\int_\Omega |\Delta v|^{q-2}\Delta v\Delta\psi\,dx + \int_\Omega \big(m_1(x)F_v(x,u,v) + m_2(x)G_v(x,u,v) + K_v(x,u,v)\big)\phi\,dx$$

for all $(\phi,\psi) \in X$.

We now state our main theorem.

Theorem 3.2. *Assume that* (A1), (A2), *and* (H1)–(H3) *hold. Then, for each convex set* $S \subseteq L^\infty(\Omega) \times L^\infty(\Omega)$ *dense in* $L^2(\Omega) \times L^2(\Omega)$*, there exists* $(m_1, m_2) \in S$ *such that system* (1) *has at least three weak solutions, two of which are the global minima in* X *of the functional* J *defined by*

$$\begin{aligned} J(u,v) &= \frac{1}{p}\int_\Omega |\Delta u|^p\,dx + \frac{1}{q}\int_\Omega |\Delta v|^q\,dx \\ &\quad - \int_\Omega \big(m_1(x)F(x,u,v) + m_2(x)G(x,u,v) + K(x,u,v)\big)\,dx. \end{aligned} \tag{10}$$

Remark 3.3. From Definition 3.1 and (10), the pair $(u,v) \in X$ is a weak solution of system (1) if and only if it is a critical point of J; that is, $\langle J'(u,v), (\phi,\psi)\rangle = 0$ for all $(\phi,\psi) \in X$.

Proof. Our goal is to apply Lemma 2.3 to system (1). To this end, let X be defined by (3) endowed with the weak topology and Y be the Hilbert space $L^2(\Omega) \times L^2(\Omega)$ equipped with the inner product

$$\langle (f,g),(h,k)\rangle_Y = \int_\Omega (f(x)h(x) + g(x)k(x))\,dx,$$

which further induces the norm

$$\|(f,g)\|_Y = \left(\int_\Omega \left(|f(x)|^2 + |g(x)|^2\right)\,dx\right)^{\frac{1}{2}}.$$

Let T be the space $L^\infty(\Omega) \times L^\infty(\Omega)$. Then, T is a convex subset of Y and dense in Y. Define the functional $I : X \to \mathbb{R}$ by

$$I(u,v) = \frac{1}{p}\int_\Omega |\Delta u|^p\,dx + \frac{1}{q}\int_\Omega |\Delta v|^q\,dx - \int_\Omega K(x,u,v)\,dx \tag{11}$$

and the function $\varphi : X \to Y$ by

$$\varphi(u,v) = \big(-F(\,\cdot\,,u(\cdot),v(\cdot)), -G(\,\cdot\,,u(\cdot),v(\cdot))\big),$$

and let $x_0 = (0,0) \in X$. In view of Lemma 2.2 (a) and (b), I and φ are well defined. We now verify that all the conditions of Lemma 2.3 are satisfied. Using the conditions in (H1), we have

$$\|\varphi(0,0)\|_Y^2 = \int_\Omega (|F(x,0,0)|^2 + |G(x,0,0)|^2)\,dx > 0$$

and

$$\|\varphi(0,0)\|_Y^2 \le \|\varphi(u,v)\|_Y^2 \quad \text{for any } (u,v) \in X.$$

For any $(u, v) \in X$, from (4), (5), (7), and (H2), it follows that

$$\begin{aligned}
\int_\Omega K(x,u,v)dx &\le \kappa\theta_2\lambda_1 \left(\theta_1 \int_\Omega \left(a(x)|u|^p + b(x)|v|^q\right) dx + \int_\Omega c(x)|u|^{\alpha+1}|v|^{\beta+1}\, dx\right) \\
&\le \kappa\theta_2\lambda_1 \left(\frac{\alpha+1}{p}\int_\Omega a(x)|u|^p\, dx + \frac{\beta+1}{q}\int_\Omega b(x)|v|^q\, dx + \int_\Omega c(x)|u|^{\alpha+1}|v|^{\beta+1}\, dx\right) \\
&\le \kappa\theta_2 \left(\frac{\alpha+1}{p}\int_\Omega |\Delta u|^p\, dx + \frac{\beta+1}{q}\int_\Omega |\Delta v|^q\, dx\right) \\
&\le \kappa \left(\frac{1}{p}\int_\Omega |\Delta u|^p\, dx + \frac{1}{q}\int_\Omega |\Delta v|^q\, dx\right) \\
&\le \frac{1}{p}\int_\Omega |\Delta u|^p\, dx + \frac{1}{q}\int_\Omega |\Delta v|^q\, dx. \qquad (12)
\end{aligned}$$

Note from (H2) that $K(x, 0, 0) = 0$ in Ω. Then, by (11) and (12), we see that $(0, 0)$ is a global minimum of I in X. Hence, condition (a) of Lemma 2.3 is satisfied.

Now, let the set B be defined as in (H1), Then, meas$(B) > 0$ by (H1). Then, using the Scorza-Dragoni theorem and arguing exactly as in the proof of [13, Theorem 2.2], we can verify condition (b) of Lemma 2.3.

Next, let $(m_1, m_2) \in T$ be fixed and the functional J be defined by (10). Then, we have

$$H(x,s,t) := m_1(x)F(x,s,t) + m_2(x)G(x,s,t) + K(x,s,t) \in \mathbb{A}$$

and
$$J(u,v) = I(u,v) + \langle \varphi(u,v), (m_1,m_2)\rangle_Y. \qquad (13)$$

By Lemma 2.3 (c), J is sequentially weakly lower semicontinuous in X. Below, we show that J is coercive. Let

$$\delta = \max\{\|m_1\|_\infty,\ \|m_2\|_\infty,\ 1\} > 0$$

and choose $\epsilon > 0$ small enough so that

$$\epsilon < \frac{1}{\delta}\theta_1\lambda_1(1-\kappa)\min\left\{\frac{1}{p}, \frac{1}{q}\right\}. \qquad (14)$$

From (H3), there exists a constant $C_\epsilon > 0$ such that

$$|F(x,s,t)| + |G(x,s,t)| \le \epsilon\left(a(x)|s|^p + b(x)|t|^q + c(x)|s|^{\alpha+1}|t|^{\beta+1}\right) + C_\epsilon$$

for all $(x, s, t) \in \Omega \times \mathbb{R}^2$. Then, for any $(u, v) \in X$, from (12), we obtain that

$$\begin{aligned}
J(u,v) &= I(u,v) + \langle \varphi(u,v), (\alpha,\beta)\rangle_Y \qquad (15) \\
&\ge (1-\kappa)\left(\frac{1}{p}\int_\Omega |\Delta u|^p\, dx + \frac{1}{q}\int_\Omega |\Delta v|^q\, dx\right) \\
&\quad - \int_\Omega \left(|m_1(x)|\,|F(x,u,v)| + |m_2(x)|\,|G(x,u,v)|\right) dx
\end{aligned}$$

$$\begin{aligned}
&\geq (1-\kappa)\min\left\{\frac{1}{p},\,\frac{1}{q}\right\}\left(\int_\Omega |\Delta u|^p\,dx + \int_\Omega |\Delta v|^q\,dx\right)\\
&\qquad - \delta\int_\Omega \big(|F(x,u,v)| + |G(x,u,v)|\big)\,dx\\
&\geq (1-\kappa)\min\left\{\frac{1}{p},\,\frac{1}{q}\right\}\left(\int_\Omega |\Delta u|^p\,dx + \int_\Omega |\Delta v|^q\,dx\right)\\
&\qquad - \delta\epsilon\int_\Omega \left(a(x)|u|^p + b(x)|v|^q + c(x)|u|^{\alpha+1}|v|^{\beta+1}\right)\,dx - \delta C_\epsilon\,\mathrm{meas}(\Omega).
\end{aligned}$$

For θ_1 defined in (H2), we have

$$\frac{1}{\theta_1} = \max\left\{\frac{p}{\alpha+1},\,\frac{q}{\beta+1}\right\} > 1.$$

From (4), (5), and (7), we derive that

$$\begin{aligned}
&\int_\Omega \left(a(x)|u|^p + b(x)|v|^q + c(x)|u|^{\alpha+1}|v|^{\beta+1}\right)\,dx\\
&\quad \leq \frac{1}{\theta_1}\int_\Omega \left(\frac{\alpha+1}{p}a(x)|u|^p + \frac{\beta+1}{q}b(x)|v|^q + c(x)|u|^{\alpha+1}|v|^{\beta+1}\right)\,dx\\
&\quad \leq \frac{1}{\theta_1\lambda_1}\left(\frac{\alpha+1}{p}\int_\Omega |\Delta u|^p\,dx + \frac{\beta+1}{q}\int_\Omega |\Delta v|^q\,dx\right)\\
&\quad \leq \frac{1}{\theta_1\lambda_1}\left(\int_\Omega |\Delta u|^p\,dx + \int_\Omega |\Delta v|^q\,dx\right).
\end{aligned}$$

This, together with (15), yields that

$$J(u,v) \geq \left((1-\kappa)\min\left\{\frac{1}{p},\,\frac{1}{q}\right\} - \frac{\delta\epsilon}{\theta_1\lambda_1}\right)\left(\int_\Omega |\Delta u|^p\,dx + \int_\Omega |\Delta v|^q\,dx\right) - \delta C_\epsilon\,\mathrm{meas}(\Omega).$$

By virtue of (14), we have $(1-\kappa)\min\left\{\frac{1}{p},\,\frac{1}{q}\right\} - \frac{\delta\epsilon}{\theta_1\lambda_1} > 0$. Then,

$$\lim_{\|(u,v)\|_X\to\infty} J(u,v) = \infty,$$

i.e., J is coercive. Thus, $J^{-1}((0,r])$ is bounded for any $r\in\mathbb{R}$. So J is weakly lower semicontinuous and inf-compact in view of the Eberlein-Smulyan theorem. Hence, all the conditions of Lemma 2.3 are satisfied. Therefore, by Lemma 2.3 and in view of (13), for each convex set $S \subseteq L^\infty(\Omega)\times L^\infty(\Omega)$ dense in $L^2(\Omega)\times L^2(\Omega)$, there exists $(m_1,m_2)\in S$ such that the functional J has at least two global minima in X. Moreover, from [15, Example 38.25], J satisfies the Palais-Smale condition. Thus, using [9, Corollary 1], we conclude that J has at least three critical points, which are weak solutions of system (1) by Remark 3.3. This completes the proof of the theorem. □

Finally, we present two simple applications of Theorem 3.2. The first one considers some special functions F and G.

Corollary 3.4. *Assume that* (A1) *and* (A2) *hold and* $K \in \mathbb{A}$ *satisfies* (H2). *Let* $\Theta : \mathbb{R}^2 \to \mathbb{R}$ *be a non-constant* C^1 *function in* $\mathbb{A}$ *satisfying* $\Theta(0,0) = 0$. *Then, for each convex set* $S \subseteq L^\infty(\Omega) \times L^\infty(\Omega)$ *dense in* $L^2(\Omega) \times L^2(\Omega)$, *there exists* $(m_1, m_2) \in S$ *such that the system*

$$\begin{cases} \Delta\left(|\Delta u|^{p-2}\Delta u\right) = \left[m_1(x)\cos(\Theta(u,v)) - m_2(x)\sin(\Theta(u,v))\right]\Theta_u(u,v) + K_u(x,u,v) \\ \hfill \text{in } \Omega, \\ \Delta\left(|\Delta v|^{q-2}\Delta v\right) = \left[m_1(x)\cos(\Theta(u,v)) - m_2(x)\sin(\Theta(u,v))\right]\Theta_v(u,v) + K_v(x,u,v) \\ \hfill \text{in } \Omega, \\ u = \Delta u = 0, \;\; v = \Delta v = 0 \hfill \text{on } \partial\Omega, \end{cases}$$

has at least three weak solutions, two of which are the global minima in X *of the functional* J_1 *defined by*

$$\begin{aligned} J_1(u,v) \;\; &= \;\; \frac{1}{p}\int_\Omega |\Delta u|^p dx + \frac{1}{q}\int_\Omega |\Delta v|^q dx \\ &\quad - \int_\Omega \big(m_1(x)\sin(\Theta(u,v)) + m_2(x)\cos(\Theta(u,v)) + K(x,u,v)\big)dx. \end{aligned}$$

Proof. Let $F, G : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F(s,t) = \sin(\Theta(s,t)) \;\; \text{and} \;\; G(s,t) = \cos(\Theta(s,t)) \quad \text{for all } (s,t) \in \mathbb{R}^2.$$

It is easy to verify that $F, G \in \mathbb{A}$ and satisfy (H1) and (H3). The conclusion then follows from Theorem 3.2. □

When $F(x,s,t) \equiv F(s)$, $G(x,s,t) \equiv G(s)$, and $K(x,s,t) \equiv 0$ in system (1), applying Theorem 3.2, we obtain the following existence result for a scalar problem.

Corollary 3.5. *Assume that* $F, G : \mathbb{R} \to \mathbb{R}$ *are functions in* $\mathbb{A}$, *twice differentiable at* 0, *and that*

$$0 < |F(0)|^2 + |G(0)|^2 = \inf_{s\in\mathbb{R}}(|F(s)|^2 + |G(s)|^2), \tag{16}$$

$$F''(0)F(0) + G''(0)G(0) < 0, \tag{17}$$

and

$$\lim_{|s|\to\infty} \frac{|F(s)| + |G(s)|}{|s|^p} = 0. \tag{18}$$

Then, for each convex set $S \subseteq L^\infty(\Omega) \times L^\infty(\Omega)$ *dense in* $L^2(\Omega) \times L^2(\Omega)$, *there exists* $(m_1, m_2) \in S$ *such that the problem*

$$\begin{cases} \Delta\left(|\Delta u|^{p-2}\Delta u\right) = m_1(x)F'(u) + m_2(x)G'(u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

has at least three weak solutions, two of which are the global minima in the space $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ *of the functional* J_2 *defined by*

$$J_2(u) = \frac{1}{p}\int_\Omega |\Delta u|^p dx - \int_\Omega \big(m_1(x)F(u) + m_2(x)G(u)\big)dx.$$

Proof. Let $F(x,s,t) \equiv F(s)$, $G(x,s,t) \equiv G(s)$, $K(x,s,t) = 0$, $q = \frac{p}{p-1}$, $\alpha = \beta = 0$, and $a(x) = b(x) = c(x) = 1$ in Ω. Then, (A1), (A2), and (H2) hold. It is obvious that (18) implies (H3). Note from (16) that 0 is a global minimum of the function $|F(s)|^2 + |G(s)|^2$. Then, we have

$$F'(0)F(0) + G'(0)H(0) = 0.$$

Thus, from (17), it follows immediately that 0 is a strict local maximum of the function $F(s)F(0) + G(x)G(0)$. Hence, (H1) holds. The conclusion then follows from Theorem 3.2. □

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