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Approximate Solutions to Nonsmooth Multiobjective Programming Problems

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We consider a multiobjective mathematical programming problem with inequality and equality constraints, where all functions are locally Lipschitz. An approximate strong Karush-Kuhn-Tucker (ASKKT for short) condition is defined and we show that every local efficient solution is an ASKKT point without any additional condition. Then a nonsmooth version of cone-continuity regularity is defined for this kind of problem. It is revealed that every ASKKT point under the cone-continuity regularity is a strong Karush-Kuhn-Tucker (SKKT for short) point. Correspondingly, the ASKKTs and the cone-continuity property are defined and the relations between them are investigated.

Keywords: Mathematical programming, optimality conditions, nonlinear programming, nonsmooth analysis and approximate conditions.

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1. Introductino

It is well-known that the Karush-Kuhn-Tucker (KKT for short) condition is one of the most important results in the nonlinear optimization, in both scalar and multiobjective optimization (see [4, 5, 14] and the references therein). True utilizing the KKT condition, in an optimization problem, the multiplier associated with the objective function will be nonzero, satisfying that the objective function plays an active role in the optimization. Many researches have been carried out in regard to the KKT condition in both smooth and nonsmooth cases. In the multiobjective programming problems, the KKT condition will be satisfied, provided that the corresponding multiplier of the objective function is nonzero. The condition, which implies that all the objective function components play a role in the optimization, is known as a strong KKT (SKKT for short) condition (see [12] and the references therein).

On the other hand, there are several approaches to solving multiobjective problems, for example, scalarization methods, descent methods and metaheuristics. In the most practical cases, we should use algorithms. Algorithms always produce approximate solutions. Hence, it is important to use an appropriate notion of approximate solutions. There are several notions for the approximate solution in the literature; for example, Loridan has defined six kinds of approximate solutions [13]. Andreani et al. [1] defined the notion of the *approximate KKT* (AKKT for short) condition.

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They proved that every local minimum point of the smooth constraint optimization satisfies the AKKT conditions. This notion has to do with practical algorithms applications, including the SQP method [16], the interior-point method [7], and the augmented Lagrangian algorithms [6].

We can derive the KKT condition from the AKKT condition, by using a constraint qualification (CQ for short). This kind of CQs is known as *strict constraint qualification* (SCQ for short). Andreani et al. [2] showed that the *cone-continuity property* (CCP for short) is the weakest SCQ under which the AKKT condition implies the KKT condition. They revealed that the CCP implies Abadie's CQ. In a general case, the SCQs are different from the CQs.

In recent years, Giorgi et al. [11] have extended the AKKT condition to the smooth multiobjective optimization problems. They proved that such conditions are necessary for a weakly efficient solution and under the *quasi-normality constraint qualification* (QNCQ for short), the KKT condition holds. Tuyen et al. [19] extended the results of Giorgi et al. [11] to the nonsmooth case, by using the limiting subdifferntial. They defined the nonsmooth version of the AKKT condition of Giorgi et al. [11]. They proved that every local weak efficient solution satisfies the AKKT condition and that the limit point of an AKKT sequence, under the QNCQ, is a KKT point. The approximate version of the SKKT condition was introduced by Feng et al. [9]. They showed that every local efficient point, for a smooth multiobjective mathematical problem with inequality constraints, satisfies such optimality conditions and that the limit point of an ASKKT sequence, under *cone-continuity regularity* (CCR for short), is an SKKT point.

In this paper, a multiobjective mathematical programming problem with inequality and equality constraints, where all functions are locally Lipschitz, has been taken into consideration. Besides, it has been revealed that every local efficient solution satisfies the ASKKT condition, without any additional condition. Also, a nonsmooth version of the CCR has been defined for such a multiobjective mathematical programming problem. Additionally, it has been shown that every limit point of an ASKKT, under the CCR, is an SKKT point. Moreover, the relation between the CCR and the CCP has been investigated and it has been verified that every limit point of an AKKT sequence, under the CCP, is a KKT point. Furthermore, in this investigation it has been demonstrated that CCP is the weakest condition that is necessary to derive the KKT condition from the AKKT condition.

This paper has been organized as follows. In Section 1, some preliminaries and notations, which are used in the sequel, have been introduced. Section 2 pertains to the AKKT condition and the ASKKT condition, as well as to their relations with the KKT condition and the SKKT condition.

2. Notations and preliminaries

In this paper, we have utilized the finite dimensional Euclidean space with the usual scalar product and Euclidean norm.

Let $x = (x_1, \dots, x_\ell)$ and $y = (y_1, \dots, y_\ell)$ be two vectors in $\mathbb{R}^\ell$. It is helpful to use the following notations:

$$\begin{aligned} x = y, &\quad \text{if } x_i = y_i, \quad \text{for all } i,\\ x \leqq y, &\quad \text{if } x_i \le y_i, \quad \text{for all } i,\\ x < y, &\quad \text{if } x_i < y_i, \quad \text{for all } i,\\ x \le y, &\quad \text{if } x \leqq y \text{ and } x \ne y. \end{aligned}$$

Let S be a subset of $\mathbb{R}^\ell$. Here and subsequently, $\operatorname{cl} S$ and $\operatorname{co} S$ denote the closure and the convex hull of S, respectively. We denote $a_+ := \max\{a, 0\}$ and $a_+^2 := (a_+)^2$ for $a \in \mathbb{R}$ and $\mathbb{R}^\ell_+$ and $\mathbb{R}^\ell_{++}$ are the nonnegative orthant and positive orthant in $\mathbb{R}^\ell$, respectively. From now onward, $x \xrightarrow{S} \bar{x}$ stands for $x \to \bar{x}$ and $x \in S$ and $B_\delta(\bar{x})$ is the ball in the center $\bar{x}$ with the radius δ.

Suppose that $\varphi : \mathbb{R}^\ell \to \bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is an extended real-valued function. The *epigraph* and the *domain* of φ are, respectively, defined by

$$\operatorname{epi}\varphi := \{(x,\alpha) \in \mathbb{R}^\ell \times \mathbb{R} \,:\, \alpha \geqslant \varphi(x)\},$$
$$\operatorname{dom}\varphi := \{x \in \mathbb{R}^\ell \,:\, |\varphi(x)| < +\infty\},$$

and φ will be called *proper* if $\operatorname{dom}\varphi \neq \emptyset$.

Suppose that $\Phi : \mathbb{R}^\ell \rightrightarrows \mathbb{R}^{\ell'}$ is a set-valued map. The *sequential Painlevé-Kuratowski outer limit* of Φ as $x \longrightarrow \bar{x}$ is defined by

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} \Phi(x) := \left\{ y \in \mathbb{R}^{\ell'} \,:\, \exists x^k \to \bar{x}, \exists y^k \to y \,\text{such that}\, y^k \in \Phi(x^k),\, \forall k \in \mathbb{N} \right\},$$

and the *sequential Painlevé-Kuratowski inner limit* by

$$\operatorname*{Lim\,inf}_{x\to\bar{x}} \Phi(x) := \left\{ y \in \mathbb{R}^{\ell'} \,:\, \forall x^k \to \bar{x}, \exists y^k \to y \,\text{such that}\, y^k \in \Phi(x^k),\, \forall k \in \mathbb{N} \right\}.$$

Φ will be called *outer semicontinuous* at $\bar{x}$ if $\operatorname*{Lim\,sup}_{x\to\bar{x}} \Phi(x) \subset \Phi(\bar{x})$, and *inner semicontinuous* at $\bar{x}$ if $\Phi(\bar{x}) \subset \operatorname*{Lim\,inf}_{x\to\bar{x}} \Phi(x)$.

If Φ is both outer semicontinuous and inner semicontinuous at $\bar{x}$, we will say that Φ is *continuous* at $\bar{x}$.

Definition 2.1. Suppose that S is a nonempty set of $\mathbb{R}^\ell$ and $\bar{x} \in S$.

(i) Let $\varepsilon \geqslant 0$, the *ε-normal cone* to S at $\bar{x}$ is defined by

$$\widehat{N}_\varepsilon(x; S) := \left\{ \xi \in \mathbb{R}^\ell \,:\, \limsup_{u \xrightarrow{S} x} \frac{\langle \xi, u - x\rangle}{\|u - x\|} \leqslant \varepsilon \right\}.$$

When $\varepsilon = 0$, the elements of the above relation are called the *Fréchet normals* and their collection, denoted by $\widehat{N}(x; S)$, is the *prenormal cone* to S at x. If $x \notin S$, we will put $\widehat{N}_\varepsilon(x; S) := \emptyset$ for all $\varepsilon \geqslant 0$.

(ii) The *basic/limiting normal* to S at $\bar{x}$, is denoted by $N(\bar{x}; S)$ and defined by

$$N(\bar{x}; S) := \operatorname*{Lim\,sup}_{\substack{x\to\bar{x}\\ \varepsilon\downarrow 0}} \widehat{N}_\varepsilon(x; S).$$

Let $\varphi : \mathbb{R}^\ell \longrightarrow \bar{\mathbb{R}}$ be a proper function and $\bar{x} \in \operatorname{dom}\varphi$. The *Mordukhovich/limiting subdifferential* of φ at $\bar{x}$ has been denoted by $\partial f(\bar{x})$ and defined by

$$\partial\varphi(\bar{x}) := \{\xi \in \mathbb{R}^\ell \,:\, (\xi, -1) \in N((\bar{x}, \varphi(\bar{x})); \operatorname{epi}\varphi)\}.$$

Proposition 2.2. *Let $\varphi, \psi : \mathbb{R}^\ell \longrightarrow \mathbb{R}$ be locally Lipschitz functions near $\bar{x}$. Then, the following statements will be true. The limiting subdifferential has the following properties:*

(1) *$\partial\varphi(\bar{x})$ is a nonempty and compact subset of $\mathbb{R}^\ell$.*

(2) *Let φ be Lipschitz of the rank κ. Then $\|\xi\| \leqslant \kappa$, for all $\xi \in \partial\varphi(\bar{x})$.*

(3) *Let φ be strictly differentiable at $\bar{x}$. Then $\partial f(\bar{x}) = \{\nabla\varphi(\bar{x})\}$.*

(4) *For any scalar $\lambda > 0$, $\partial\lambda\varphi(\bar{x}) = \lambda\partial\varphi(\bar{x})$.*

(5) *φ and ψ are two locally Lipschitz functions, $\partial(\varphi + \psi)(x) \subset \partial\varphi(x) + \partial\psi(x)$ and equality when functions are convex.*

(6) *The set-valued function $x \rightrightarrows \partial\varphi(x)$ is closed.*

Proposition 2.3. *Let $\varphi : \mathbb{R}^\ell \longrightarrow \bar{\mathbb{R}}$ be a proper function. If φ has a local minimum at $\bar{x}$, then $0 \in \partial\varphi(\bar{x})$.*

Proposition 2.4. *Let $\psi : \mathbb{R}^\ell \longrightarrow \mathbb{R}^{\ell'}$ and $\varphi : \mathbb{R}^{\ell'} \longrightarrow \mathbb{R}$ be locally Lipschitz. Then*

$$\partial(\varphi \circ \psi)(\bar{x}) \subset \bigcup_{\xi \in \partial(\varphi(\psi(\bar{x})))} \partial\langle \xi, \psi\rangle(\bar{x}).$$

Proposition 2.5. *Assume that $\varphi_1, \varphi_2 : \mathbb{R}^\ell \longrightarrow \bar{\mathbb{R}}$ are Lipschitz at $\bar{x}$. We define $\varphi(x) := \max\{\varphi_1(x), \varphi_2(x)\}$.*

- *If $\varphi(\bar{x}) = \varphi_1(\bar{x}) = \varphi_2(\bar{x})$, then $\partial\varphi(\bar{x}) \subset \bigcup_{0 \leqslant \lambda \leqslant 1} \{\lambda\partial\varphi_1(\bar{x}) + (1 - \lambda)\partial\varphi_2(\bar{x})\}$.*
- *If $\varphi(\bar{x}) = \varphi_1(\bar{x})$, then $\partial\varphi(\bar{x}) = \partial\varphi_1(\bar{x})$.*

Example 2.6. Let $\varphi(x) = |x|$. Then $\partial\varphi(0) = [-1, 1]$.

Example 2.7. Let $\varphi(x) = -|x|$. Then $\partial\varphi(0) = \{-1, 1\}$.

For more information about nonsmooth analysis (see [3, 8, 15, 17, 18]).

In this paper we discuss the following multiobjective programming problem:

$$\text{(MOP)} \qquad \begin{array}{ll} \min & f(x) = (f_1(x), \dots, f_m(x)), \\ \text{s.t.} & g(x) = (g_1(x), \dots, g_n(x)) \leqq 0, \\ & h(x) = (h_1(x), \dots, h_p(x)) = 0, \end{array}$$

where f_i, $i \in I = \{1, \dots, m\}$, g_j, $j \in J = \{1, \dots, n\}$ and h_k, $k \in K = \{1, \dots, p\}$, are the real-valued locally Lipschitz functions defined on $\mathbb{R}^\ell$.

S denotes the feasible region of (MOP), namely,

$$S := \Big\{x \in \mathbb{R}^\ell \,:\, g(x) \leqq 0, h(x) = 0\Big\},$$

and $J(x)$ represents the active index set of the inequality constraints at x.

In vector optimization, objectives tend to often conflict with each other. In this regard the concepts of efficient and weak efficient solutions are widely used.

Definition 2.8. The feasible point $\bar{x}$ is said to be

(1) a *local efficient (or efficient) solution* if there exists a neighbourhood U of $\bar{x}$ such that for any $x \in U \cap S$ (or $x \in S$) the following inequality does not hold:

$$f(x) \leq f(\bar{x}).$$

(2) a *local weak efficient (or weak efficient) solution* if there exists a neighbourhood U of $\bar{x}$ such that for any $x \in U \cap S$ (or $x \in S$) the following inequality does not hold:

$$f(x) < f(\bar{x}).$$

Definition 2.9. We say that $\bar{x} \in S$ satisfies the *strong Karush-Kuhn-Tucker* (SKKT for short) condition if there exists $(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}) \in \mathbb{R}^m_{++} \times \mathbb{R}^n_{+} \times \mathbb{R}^p$ such that

$$0 \in \sum_{i=1}^{m} \lambda_i \partial f_i(\bar{x}) + \sum_{i=1}^{n} \mu_i \partial g_i(\bar{x}) + \sum_{i=1}^{p} \nu_i [\partial(-h_i)(\bar{x}) \cup \partial h_i(\bar{x})] \tag{1}$$

$$\sum_{i=1}^{m} \lambda_i = 1, \qquad \mu_i g_i(\bar{x}) = 0, \quad \forall i \in J. \tag{2}$$

Remark 2.10. In Definition 2.9 interchanging $\boldsymbol{\lambda} \in \mathbb{R}^m_{++}$ and $\boldsymbol{\lambda} \in \mathbb{R}^m_{+}$, we arrive at the *Karush-Kuhn-Tucker* (KKT for short) condition. It is obvious that every SKKT point, yet is also a KKT point but the converse is not true in the general.

Remark 2.11. In Definition 2.9, if we replace the limiting subdifferential with the Clark subdifferential in view of Proposition 2.2, we will arrive at the SKKT condition, which has been mentioned in [12].

Lemma 2.12. *In Definition* 2.9 *we can say that* $\bar{x} \in S$ *will satisfy the SKKT condition if and only if there exists* $(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}) \in \mathbb{R}^m_{++} \times \mathbb{R}^n_{+} \times \mathbb{R}^p$ *such that*

$$0 \in \sum_{i=1}^{m} \lambda_i \partial f_i(\bar{x}) + \sum_{i=1}^{n} \mu_i \partial g_i(\bar{x}) + \sum_{i=1}^{p} \nu_i [\partial(-h_i)(\bar{x}) \cup \partial h_i(\bar{x})] \tag{3}$$

$$\lambda_i \geq 1, \forall i \in I, \qquad \mu_i g_i(\bar{x}) = 0, \forall i \in J. \tag{4}$$

Proof. Suppose that the conditions of Definition 2.9 are satisfied. We put

$$e = \max \left\{ \frac{1}{\lambda_i} \, : \, i \in I \right\},$$

and define $(\bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\nu}}) = e(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu})$, thus completing the first part of the proof.

To prove the opposite implication, it is sufficient to suppose that $e = \sum_{i=1}^{m} \lambda_i$ and to put $(\bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\nu}}) = \frac{1}{e}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu})$. □

3. Necessary condition

Definition 3.1. We say that the *approximate strong Karush-Kuhn-Tucker* (ASKKT for short) condition is satisfied at the feasible point $\bar{x}$ if there exist sequences $\{x^k\} \subset \mathbb{R}^\ell$ (which is called an ASKKT sequence) and $\{(\boldsymbol{\lambda}^k, \boldsymbol{\mu}^k, \boldsymbol{\nu}^k)\} \subset \mathbb{R}^m_+ \times \mathbb{R}^n_+ \times \mathbb{R}^p$ such that

(i) $\lim_{k\to\infty} x^k = \bar{x}$,

(ii)) $\sum_{i=1}^m \lambda_i^k \xi_i^k + \sum_{i=1}^n \mu_i^k \eta_i^k + \sum_{i=1}^p \nu_i^k \zeta_i^k \longrightarrow 0$, for some $\xi_i^k \in \partial f_i(x^k)$, $i \in I$, $\eta_i^k \in \partial g_i(x^k)$, $i \in J$, $\zeta_i^k \in [\partial(-h_i)(x^k) \cup \partial h_i(x^k)]$, $i \in K$ for each $k \in \mathbb{N}$,

(iii) $\lambda_i^k \geqslant 1$ for all $i \in I$ and for each $k \in \mathbb{N}$,

(iv) if for some $i \in J$ we have $g_i(\bar{x}) < 0$, then for k large enough we will have $\mu_i^k = 0$.

Remark 3.2. If in Definition 3.1 we replace $\lambda_i^k \geqslant 1$ for all $i \in I$ with $\sum_{i=1}^m \lambda_i^k = 1$, we will arrive at the *approximate KKT* (AKKT for short) condition.

Remark 3.3. In Definition 3.1, if we put

$$\{(\bar{\boldsymbol{\lambda}}^k, \bar{\boldsymbol{\mu}}^k, \bar{\boldsymbol{\nu}}^k)\} := \frac{1}{\sum_{i=1}^m \lambda_i^k}\{(\boldsymbol{\lambda}^k, \boldsymbol{\mu}^k, \boldsymbol{\nu}^k)\},$$

we can see that if the ASKKT condition is satisfied at $\bar{x}$, the AKKT condition will also be satisfied at this point. Yet even in the differentiable case, the AKKT condition does not imply the ASKKT condition (for a differentable case example, see [9, Example 3.2]).

Theorem 3.4. *If $\bar{x} \in S$ is a local efficient solution for* (MOP)*, there will be sequences $\{x_n\}$ and $\{(\boldsymbol{\lambda}^k, \boldsymbol{\mu}^k, \boldsymbol{\nu}^k)\} \in \mathbb{R}^m_+ \times \mathbb{R}^n_+ \times \mathbb{R}^p$, satisfying the ASKKT condition in Definition* 3.1 *at $\bar{x}$.*

Proof. Let $\bar{x}$ be a local efficient solution for (MOP). By [20, Theorem 1], $\bar{x}$ is a local solution to the following scalar problem

$$\min \sum_{i=1}^m f_i(x), \quad \text{s.t. } f_i(x) \leqslant f_i(\bar{x}), \quad i \in I, \quad x \in S.$$

Hence, there will exist $\delta > 0$ such that

$$\sum_{i=1}^m f_i(\bar{x}) \leqslant \sum_{i=1}^m f_i(x), \qquad x \in \operatorname{cl} B_\delta(\bar{x}) \cap S.$$

Therefore, $\bar{x}$ is the unique global solution to the following problem

$$\min \sum_{i=1}^m f_i(x) + \frac{1}{2}\|x-\bar{x}\|^2, \quad f_i(x) \leqslant f_i(\bar{x}), \quad i = 1, \dots, p, \quad x \in S, \quad x \in \operatorname{cl} B_\delta(\bar{x}) \cap S.$$

For each $k \in \mathbb{N}$, we have defined

$$\varphi_k(x) := \sum_{i=1}^m f_i(x) + \frac{1}{2}\|x-\bar{x}\|^2 + \frac{k}{2}\left(\sum_{i=1}^m (f_i(x) - f_i(\bar{x}))_+^2 + \sum_{i=1}^n (g_i(x))_+^2 + \sum_{i=1}^p |h_i(x)|^2\right).$$

Consider the following problem:

$$\min \varphi_k(x), \quad \text{s.t.}\, x \in \operatorname{cl} B_\delta(\bar{x}).$$

In view of the fact that for each $k \in \mathbb{N}$ the $\varphi_k(\cdot)$ is a continuous function and that $\operatorname{cl} B_\delta(\bar{x})$ is a compact set, by the Weierstrass the $\varphi_k(\cdot)$ will have a solution called x^k. Accordingly, by Proposition 2.3: $0 \in \partial\varphi_k(x^k)$.

On the other hand, by Propositions 2.2 and 2.4, we have

$$\begin{aligned}
\partial\varphi_k(x^k) \subset & \sum_{i=1}^{m} \partial f_i(x^k) + (x^k - \bar{x}) \\
& + \frac{k}{2}\left(\sum_{i=1}^{m} \partial\left(f_i(x^k) - f_i(\bar{x})\right)_+^2 + \sum_{i=1}^{n} \partial g_i(x^k)_+^2 + \sum_{i=1}^{p} \partial h_i^2(x^k)\right) \\
& \subset \sum_{i=1}^{m} \partial f_i(x^k) + (x^k - \bar{x}) + \frac{k}{2}\left(\sum_{i=1}^{m} \partial[2(f_i(x^k) - f_i(\bar{x}))_+(f_i(\cdot) - f_i(\bar{x}))](x^k)\right. \\
& \left. + \sum_{i=1}^{n} \partial[2g_i(x^k)_+ g_i(\cdot)](x^k) + \sum_{i=1}^{p} \partial[2|h_i(x^k)|h_i(\cdot)](x^k)\right) \\
& = \sum_{i=1}^{m} [1 + k(f_i(x^k) - f_i(\bar{x}))_+]\partial f_i(x^k) + (x^k - \bar{x}) \\
& + k\left(\sum_{i=1}^{n} g_i(x^k)_+ \partial g_i(x^k) + \sum_{i=1}^{p} |h_i(x^k)|[\partial(-h_i)(x^k) \cup \partial h_i(x^k)]\right)
\end{aligned}$$

For each $k \in \mathbb{N}$, by setting

$$\lambda_i^k = [1 + k(f_i(x^k) - f_i(\bar{x}))_+], \qquad i \in I, \tag{5}$$

$$\mu_i^k = k g_i(x^k)_+, \qquad i \in J, \tag{6}$$

$$\nu_i^k = |h_i(x^k)|, \qquad i \in K,$$

we have

$$0 \in \sum_{i=1}^{m} \lambda_i^k \partial f_i(x^k) + (x^k - \bar{x}) + \left(\sum_{i=1}^{n} \mu_i^k \partial g_i(x^k) + \sum_{i=1}^{p} \nu_i^k [\partial(-h_i)(x^k) \cup \partial h_i(x^k)]\right).$$

For each $k \in \mathbb{N}$, suppose $\xi_i^k \in \partial f_i(x^k)$ for $i \in I$, $\eta_i^k \in \partial g_i(x^k)$ for $i \in J$, and $\zeta_i^k \in \partial(-h_i)(x^k) \cup \partial h_i(x^k)$ for $i \in K$, such that

$$\sum_{i=1}^{m} \lambda_i^k \xi_i^k + (x^k - \bar{x}) + \sum_{i=1}^{n} \mu_i^k \eta_i^k + \sum_{i=1}^{p} \nu_i^k \zeta_i^k = 0.$$

Therefore,

$$\sum_{i=1}^{m} \lambda_i^k \xi_i^k + \sum_{i=1}^{n} \mu_i^k \eta_i^k + \sum_{i=1}^{p} \nu_i^k \zeta_i^k = -(x^k - \bar{x}). \tag{7}$$

By [10, Theorem 9], we have $x^k \longrightarrow \bar{x}$. Therefore, the right side of (7) will tend toward zero. Hence, the first and second parts of Definition 3.1 will be satisfied.

As can be seen, by (5), the third part of Definition 3.1 is also true.

To see the last part of Definition 3.1, suppose that $g_i(\bar{x}) < 0$, for some $i \in J$. By using the continuity of g_i for k to be sufficiently large, we have $g_i(x^k) < 0$. Thus, $(g_i(x^k))_+ = 0$ for k to be sufficiently large, and (6) implies $\mu_i^k = 0$, for k to be sufficiently large. □

Corollary 3.5. *If $\bar{x}$ is a local efficient solution for* (MOP)*, the AKKT condition is satisfied at $\bar{x}$.*

Proof. Using Theorem 3.4 and Remark 3.3 the statement follows directly. □

Remark 3.6. In the above theorem, without particular changes in the procedure of the proof, we can replace k with ρ^k where $\lim_{k\to\infty} \rho_k = +\infty$. This signifies that we have an extended version of [9, Theorem 3.1].

Remark 3.7. As mentioned in [9, Example 3.4], we cannot deduce that the ASKKT condition is satisfied at a weak efficient solution, which is not an efficient solution.

Remark 3.8. If we use the Clarke subdifferential instead of the limiting subdifferential, in Definition 3.1, we can replace $\zeta_i^k \in [\partial(-h_i)(x^k) \cup \partial h_i(x^k)]$ by $\zeta_i^k \in \partial h_i(x^k)$ and we get Theorem 3.4 without any change.

To investigate the opposite implication of Theorem 3.4, we need a strict constraint qualification (SCQ for short). For this purpose, we must find an equivalent form of the SKKT condition and for the simplicity of notation, we will utilize the following notation. Suppose $x \in S$, we will define

$$\mathcal{D}(x) := \left\{ d \ : \begin{array}{l} d \in \displaystyle\sum_{i=1}^{m} \lambda_i \partial f_i(x) + \sum_{i\in J(x)} \mu_i \partial g_i(x) + \sum_{j=1}^{p} \nu_i [\partial(-h_i)(x) \cup \partial h_i(x)], \\ \lambda_i \in \mathbb{R}_+,\ \forall i \in I,\ \mu_i \in \mathbb{R}_+,\ \forall i \in J(x),\ \nu_i \in \mathbb{R},\ i \in J \end{array} \right\}.$$

Lemma 3.9. *Let $\bar{x} \in S$. Then $\bar{x}$ satisfies the* SKKT *condition for* (MOP) *if and only if for each $i_0 \in I$ there exists $\xi_{i_0} \in \partial f_{i_0}(\bar{x})$ such that $-\xi_{i_0} \in \mathcal{D}(\bar{x})$.*

Proof. Suppose that $\bar{x}$ satisfies the SKKT condition and $i_0 \in I$. By Definition 2.9, there exist $(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}) \in \mathbb{R}^m_{++} \times \mathbb{R}^n_+ \times \mathbb{R}^p$, $\xi_i \in \partial f_i(\bar{x})$, $i \in I$, $\eta_i \in \partial g_i(\bar{x})$, $i \in J$, and $\zeta_i \in [\partial(-h_i)(\bar{x}) \cup \partial h_i(\bar{x})]$, $i \in K$, such that

$$\sum_{i=1}^{m} \lambda_i \xi_i + \sum_{i=1}^{n} \mu_i \eta_i + \sum_{i=1}^{p} \nu_i \zeta_i = 0$$

Therefore,

$$
\begin{aligned}
-\xi_{i_0} &= \sum_{i\in I\setminus\{i_0\}} \frac{\lambda_i}{\lambda_{i_0}}\xi_i + \sum_{i=1}^{n} \frac{\mu_i}{\lambda_{i_0}}\eta_i + \sum_{i=1}^{p} \frac{\nu_i}{\lambda_{i_0}}\zeta_i \\
&\in \sum_{i\in I\setminus\{i_0\}} \frac{\lambda_i}{\lambda_{i_0}}\partial f_i(\bar{x}) + \sum_{i=1}^{n} \frac{\mu_i}{\lambda_{i_0}}\partial g_i(\bar{x}) + \sum_{i=1}^{p} \frac{\nu_i}{\lambda_{i_0}}[\partial(-h_i)(\bar{x}) \cup \partial h_i(\bar{x})] \\
&\in \mathcal{D}(\bar{x}).
\end{aligned}
$$

For the converse, suppose that for each $i_0 \in I$, there exists $\xi_{i_0} \in \partial f_{i_0}(\bar{x})$ such that $-\xi_{i_0} \in \mathcal{D}(\bar{x})$. Thus, there exists $(\boldsymbol{\lambda}^{i_0}, \boldsymbol{\mu}^{i_0}, \boldsymbol{\nu}^{i_0}) \in \mathbb{R}_+^m \times \mathbb{R}_+^n \times \mathbb{R}^p$ such that

$$
-\xi_{i_0} \in \sum_{i=1}^{m} \lambda_i^{i_0}\partial f_i(\bar{x}) + \sum_{i\in J(\bar{x})} \mu_i^{i_0}\partial g_i(\bar{x}) + \sum_{j=1}^{p} \nu_i^{i_0}[\partial(-h_i)(\bar{x}) \cup \partial h_i(\bar{x})],
$$

Therefore,

$$
0 \in (1+\lambda_{i_0}^{i_0})\partial f_{i_0}(\bar{x}) + \sum_{i\in I\setminus\{i_0\}} \lambda_i^{i_0}\partial f_i(x) + \sum_{i\in J(\bar{x})} \mu_i^{i_0}\partial g_i(\bar{x}) + \sum_{j=1}^{p} \nu_i^{i_0}[\partial(-h_i)(\bar{x}) \cup \partial h_i(\bar{x})].
$$

Since $i_0 \in I$ is arbitrary, we have

$$
0 \in \sum_{i=1}^{m} \bar{\lambda}_i\partial f_i(\bar{x}) + \sum_{i\in J(\bar{x})} \bar{\mu}_i\partial g_i(\bar{x}) + \sum_{j=1}^{p} \bar{\nu}_i[\partial(-h_i)(\bar{x}) \cup \partial h_i(\bar{x})],
$$

where $\bar{\lambda}_i = \sum_{i_0=1}^{m}(1+\lambda_i^{i_0})$, $i \in I$, $\bar{\mu}_i = \sum_{i_0=1}^{m}\mu_i^{i_0}$, $i \in J(\bar{x})$ and $\bar{\nu}_i = \sum_{i_0=1}^{m}\nu_i^{i_0}$, $i \in K$. By using Lemma 2.12 $\bar{x}$ satisfies the SKKT condition. ☐

In the following, we will define the nonsmooth form of an SCQ, which is required to prove the converse of Theorem 3.4.

Definition 3.10. We say that $\bar{x} \in S$ satisfies the *cone-continuity regularity* (CCR for short) if

$$
\operatorname*{Lim\,sup}_{x\longrightarrow\bar{x}} \mathcal{D}(x) \subset \mathcal{D}(\bar{x}).
$$

The following theorem provides us the sufficient conditions for an ASKKT sequence to converge to an SKKT point.

Theorem 3.11. *Let $\bar{x}$ be a limit point of an ASKKT sequence $\{x^k\}$ of (MOP) and the CCR is satisfied at $\bar{x}$; then $\bar{x}$ is an SKKT point.*

Proof. In view of Lemma 3.9, it is sufficient to show that for each $i_0 \in I$, there exists $\xi_{i_0} \in \partial f_{i_0}(\bar{x})$ such that $-\xi_{i_0} \in \mathcal{D}(\bar{x})$. Accordingly, suppose that $i_0 \in I$ is arbitrary. By Definition 3.1, for each $k \in \mathbb{N}$, there exist $\{(\boldsymbol{\lambda}^k, \boldsymbol{\mu}^k, \boldsymbol{\nu}^k)\} \subset \mathbb{R}_+^m \times \mathbb{R}_+^n \times \mathbb{R}^p$, $\xi_i^k \in \partial f_i(x^k)$, $\lambda_i^k \geqslant 1$, $i \in I$, $\eta_i^k \in \partial g_i(x^k)$, $i \in J$, and $\zeta_i^k \in [\partial(-h_i)(x^k) \cup \partial h_i(x^k)]$, $i \in K$ such that $x^k \longrightarrow \bar{x}$ and

$$
d^k := \sum_{i=1}^{m} \lambda_i^k\xi_i^k + \sum_{i=1}^{n} \mu_i^k\eta_i^k + \sum_{i=1}^{p} \nu_i^k\zeta_i^k \longrightarrow 0. \tag{8}
$$

Since $\lambda_{i_0}^k \geqslant 1$

$$d^k - \xi_{i_0}^k = (\lambda_{i_0}^k - 1)\xi_{i_0}^k + \sum_{i\in I\setminus\{i_0\}} \lambda_i^k \xi_i^k + \sum_{i=1}^{n} \mu_i^k \eta_i^k + \sum_{i=1}^{p} \nu_i^k \zeta_i^k \in \mathcal{D}(x^k).$$

On the other hand, f_{i_0} is locally Lipschitz. Hence, for $k \in \mathbb{N}$ sufficiently large, $\partial f_{i_0}(x^k)$ is a compact set. Thus, by using the Bolzano-Weierstrass theorem, we can consider the subsequence, namely $\{x^k\}$, convergence to $\bar{x}$, $\xi_{i_0}^k \in \partial f_{i_0}(x^k)$ and ξ_{i_0} such that $\xi_{i_0}^k \to \xi_{i_0}$. By the upper semicontinuity of subdifferentials, Proposition 2.2, we have $\xi_{i_0} \in \partial f_i(\bar{x})$. By using this fact, (8), and the CCR we will have

$$-\xi_{i_0} = \lim_{k\to\infty} (d^k - \xi_{i_0}^k) \in \operatorname*{Lim\,sup}_{k\longrightarrow\infty} \mathcal{D}(x^k) \subset \operatorname*{Lim\,sup}_{x\longrightarrow\bar{x}} \mathcal{D}(x) \subset \mathcal{D}(\bar{x}).$$

In view of Lemma 3.9 the proof will be completed. □

We can state and prove analogous forms of the above theorem for the AKKT and the KKT point. First of all, let us consider the nonsmooth version of the cone-continuity property. For this purpose, we have used the following notation. Suppose $x \in S$, we will define

$$\mathcal{K}(x) := \left\{ d \ : \begin{array}{l} d \in \displaystyle\sum_{i\in J(x)} \mu_i \partial g_i(x) + \sum_{i=1}^{p} \nu_i [\partial(-h_i)(x) \cup \partial h_i(x)], \\ \mu_i \in \mathbb{R}_+, \ \forall i \in I(x), \ \nu_i \in \mathbb{R}, \ i \in K \end{array} \right\}.$$

Definition 3.12. We say that $\bar{x} \in S$ satisfies the *cone-continuity property* (CCP for short) if

$$\operatorname*{Lim\,sup}_{x\longrightarrow\bar{x}} \mathcal{K}(x) \subset \mathcal{K}(\bar{x}).$$

Remark 3.13. In relation to the CCP some remarks are adequate:

- The CCP and the CCR do not imply each other [9, Examples 4.2 and 4.3];
- Under the CCP, the ASKKT condition does not imply the SKKT condition [9, Example 4.1]. Hence, to derive the ASKKT condition, using the CCP is not useful.

Now we are ready to derive the converse of Corollary 3.5.

Theorem 3.14. *Let $\bar{x}$ be a limit point of an AKKT sequence $\{x^k\}$ of* (MOP) *and assume that the* CCP *is satisfied at $\bar{x}$; then $\bar{x}$ is a KKT point.*

Proof. Suppose that $\bar{x}$ is a limit point of an AKKT sequence $\{x^k\}$ of (MOP). By Remark 3.2, for each $k \in \mathbb{N}$, there exist $\{(\boldsymbol{\lambda}^k, \boldsymbol{\mu}^k, \boldsymbol{\nu}^k)\} \subset \mathbb{R}_+^m \times \mathbb{R}_+^n \times \mathbb{R}^p$, $\xi_i^k \in \partial f_i(x^k)$, $i \in I$, $\eta_i^k \in \partial g_i(x^k)$, $i \in J$, and $\zeta_i^k \in [\partial(-h_i)(x^k) \cup \partial h_i(x^k)]$, $i \in K$ such that $\sum_{i=1} \lambda_i^k = 1$, $x^k \longrightarrow \bar{x}$ and

$$d^k := \sum_{i=1}^{m} \lambda_i^k \xi_i^k + \sum_{i=1}^{n} \mu_i^k \eta_i^k + \sum_{i=1}^{p} \nu_i^k \zeta_i^k \longrightarrow 0. \tag{9}$$

Therefore, $$d^k - \sum_{i=1}^{m} \lambda_i^k \xi_i^k = \sum_{i=1}^{n} \mu_i^k \eta_i^k + \sum_{i=1}^{p} \nu_i^k \zeta_i^k \in \mathcal{K}(x^k).$$

Since $\sum_{i=1}^{m} \lambda_i^k = 1$, there exists a subsequence, that is, $\{\boldsymbol{\lambda}^k\}$ and convergences to $\bar{\boldsymbol{\lambda}} \in \mathbb{R}_+^m$ such that $\sum_{i=1}^{m} \bar{\lambda}_i = 1$.

On the other hand, f_i, for each $i \in I$, f_i is locally Lipschitz; thus, for $k \in \mathbb{N}$ sufficiently large, $\partial f_i(x^k)$, $i \in I$, is a compact set. Therefore, by using the Bolzano-Weierstrass theorem, we can consider a subsequence, namely $\{x^k\}$, which converges to $\bar{x}$, $\xi_i^k \in \partial f_i(x^k)$ and ξ_i, such that $\xi_i^k \to \xi_i$, $i \in I$. By the upper semicontinuity of subdifferentials, Proposition 2.2, we have $\xi_i \in \partial f_i(\bar{x})$. By using this fact, (9) and the CCP, we obtain

$$-\sum_{i=1}^{m} \bar{\lambda}_i \xi_i = \lim_{k \to \infty} \left(d^k - \sum_{i=1}^{m} \lambda_i^k \xi_i^k \right) \in \operatorname*{Lim\,sup}_{k \longrightarrow \infty} \mathcal{K}(x^k) \subset \operatorname*{Lim\,sup}_{x \longrightarrow \bar{x}} \mathcal{K}(x) \subset \mathcal{K}(\bar{x}).$$

This relation reveals that $\bar{x}$ is a KKT point. □

In the following theorem we have shown that the CCP is the weakest SCQ under which the AKKT condition implies the KKT condition.

Theorem 3.15. *Suppose that for every objective function the AKKT condition implies the KKT condition at $\bar{x}$. Then the CCP holds at $\bar{x}$.*

Proof. Suppose $d \in \operatorname{Lim\,sup}_{x \to \bar{x}} \mathcal{K}(x)$. By the definition of the outer limit, there exist $\{x^k\}$ and $\{d^k\}$ such that $x^k \longrightarrow \bar{x}$ and $d^k \longrightarrow d$, where $d^k \in \mathcal{K}(x^k)$. We define $\varphi(x) := -\langle d, x \rangle$ and obtain $\partial \varphi(x^k) = \{-d\}$. Since $d^k \in \mathcal{K}(x^k)$, there exist, by Remark 3.2, for each $k \in \mathbb{N}$, $\{(\boldsymbol{\mu}^k, \boldsymbol{\nu}^k)\} \in \mathbb{R}_+^n \times \mathbb{R}^p$, $\eta_i^k \in \partial g_i(x^k)$, $i \in J(x^k)$, and $\zeta_i^k \in [\partial(-h_i)(x^k) \cup \partial h_i(x^k)]$, $i \in K$, such that

$$d^k = \sum_{i \in J(x^)} \mu_i^k \eta_i^k + \sum_{i=1}^{p} \nu_i^k \zeta_i^k.$$

Since $-d + d^k \longrightarrow 0$, $\{x^k\}$ is an AKKT sequence for φ at $\bar{x}$. By hypothesis, $\bar{x}$ is a KKT point, which means $-d \in \mathcal{K}(\bar{x})$. □

Remark 3.16. Theorem 3.14 and Theorem 3.15 mean that like the differentiable case, the CCP is the weakest condition and the AKKT condition implies the KKT condition.

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