Minimax Theory and its Applications
Volume 7 (2022), No. 1, 131–158

Sequential Pareto Subdifferential Multi-Composition Rule and Application to Multiobjective Minimax Location Problems

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Received: November 29, 2020
Accepted: June 28, 2021

In the absence of any qualification condition, we provide a sequential formula for the weak/proper Pareto subdifferential of multi-composed convex vector mappings. For illustrating this formula, we propose deriving sequential optimality conditions characterizing weakly/properly efficient solutions of multiobjective minimax location problems with infimal distances. We present an example of multiobjective bilevel programming problems with an extremal value function, where the standard Lagrange multipliers conditions can not be derived due to the lack of constraint qualification and the sequential conditions hold.

Keywords: Sequential Pareto subdifferential, multi-composed vector mappings, sequential weak efficiency, sequential proper efficiency, multiobjective minimax location problems, multiobjective bilevel programming problems.

2010 Mathematics Subject Classification: 49K35, 90C25, 90C29, 90C47, 90C48.

1. Introduction

Multi-composed scalar optimization problems were first introduced and examined by Wanka et al. [18, 8]. In fact, this new model can also be defined in the context of vector optimization because it is possible to write certain problems as multi-composed vector optimization problems (see the last section) such as, but not limited to, multiobjective minimax location problems with infimal distances [16], multiobjective bilevel programming problems with extremal-value function [15], etc.

In the literature, several works have investigated vector optimization problems from a duality point of view under certain qualification conditions like generalized Slater condition (see, for instance, the book [4]). However, in the optimization framework, it is well-known that vector as well as scalar optimization problems may not satisfy any qualification condition.

To avoid imposing a qualification condition, several authors [3, 10, 12, 13, 14] have developed an alternative approach called sequential approach that consists in deriving optimality conditions in terms of nets or sequences of subdifferentials at nearby points. To our knowledge, this approach was initiated by Thibault [13, 14]. The latter author has derived sequential optimality conditions for convex scalar optimization problems without any qualification condition through sequential calculus

ISSN 2199-1413 (printed), ISSN 2199-1421 (electronic) / $ 2.50

rules for the Brøndsted-Rockafellar subdifferential of the sum and the composition of convex and lower semicontinuous functions. In vector optimization, Laghdir et al. [10] have obtained sequential optimality conditions for weakly and properly efficient solutions of composed convex vector optimization problems via sequential formulae for the weak and proper Pareto subdifferential of composed convex vector mappings.

In order to deal with multi-composed vector optimization problems from a sequential point of view, the aim of this work is to extend the results of Laghdir et al. [10] by deriving a sequential formula more general for the weak/proper Pareto subdifferential of multi-composed convex vector mappings in the setting of Banach spaces and in the absence of any qualification condition. To do this, we will employ two interesting theorems established by El Maghri and Laghdir [7] and Laghdir et al. [9]. The first one characterizes scalarly the weak/proper Pareto subdifferential of convex vector mappings, while the second one gives a sequential formula for the Brøndsted-Rockafellar subdifferential of a finite sum of proper, convex and lower semicontinuous functions, without any qualification condition. As an application of our main result, we propose deriving, in the absence of any qualification condition, sequential optimality conditions for weak/proper efficiency to multiobjective minimax location problems with infimal distances and to multiobjective bilevel programming problems with an extremal value function.

The paper is organized as follows. In section 2, we recall some definitions and give some preliminary results. Section 3 is devoted to providing a sequential formula for the weak/proper Pareto subdifferential of multi-composed convex vector mappings in the absence of any qualification condition. In section 4, we derive sequential optimality conditions characterizing weakly/properly efficient solutions of multiobjective minimax location problems with infimal distances, without any qualification condition. Furthermore, we present another multiobjective optimization problem dealing with bilevel programming problems to illustrate our main result.

2. Preliminaries

In this section, we give some basic definitions and results. Let X, Y and Z be real Hausdorff topological vector spaces with duality pairing denoting by $\langle .,. \rangle$ and X^*, Y^* and Z^* their topological dual spaces, respectively. The topological dual spaces X^* and Y^* are endowed with the weak-star topology denoted by $w(X^*, X)$ and $w(Y^*, Y)$, respectively. Consider a nonempty convex cone $Q \subseteq Z$ with $\operatorname{int} Q \neq \emptyset$ (i.e. the topological interior of Q is nonempty). The cone Q is called *pointed* when its lineality $l(Q) := -Q \cap Q$ is null. On Z, we define the following ordering relations

$$\begin{aligned} z_1 &\leqq_Q z_2 \iff z_2 - z_1 \in Q, \\ z_1 &<_Q z_2 \iff z_2 - z_1 \in \operatorname{int} Q, \\ z_1 &\lneq_Q z_2 \iff z_2 - z_1 \in Q \backslash l(Q). \end{aligned}$$

To Z, we attach an abstract maximal element with respect to " $\leqq_Q$ ", denoted by $+\infty_Z$. Besides, on $Z \cup \{+\infty_Z\}$ we consider the following operations and conventions:

$$z \leqq_Q +\infty_Z,\ z + (+\infty_Z) := (+\infty_Z) + z := +\infty_Z,\ \forall z \in Z \cup \{+\infty_Z\},$$

$$\alpha \cdot (+\infty_Z) := +\infty_Z,\ \forall \alpha \geq 0.$$

The *dual cone* Q^* and the *strict polar cone* $(Q^*)^\circ$ of Q are defined respectively as

$$Q^* := \{z^* \in Z^* : z^*(Q) \subseteq \mathbb{R}_+\}$$

and
$$(Q^*)^\circ := \{z^* \in Z^* : z^*(Q \setminus l(Q)) \subseteq \mathbb{R}_+ \setminus \{0\}\}.$$

By convention, $\langle z^*, +\infty_Z\rangle := +\infty$ for all $z^* \in Q^*$.

Definition 2.1. Let $f : X \to Z \cup \{+\infty_Z\}$ be a vector mapping. Then, f is called

- *proper* if its effective domain
$$\text{dom} f := \{x \in X : f(x) \in Z\} \neq \emptyset,$$
- Q-*convex* if for all $\lambda \in [0,1]$ and all $x_1, x_2 \in X$
$$f(\lambda x_1 + (1-\lambda)x_2) \leqq_Q \lambda f(x_1) + (1-\lambda) f(x_2),$$
- Q-*epi closed* if its epigraph
$$\text{epi} f := \{(x,z) \in X \times Z : f(x) \leqq_Q z\} \text{ is closed.}$$

Let $\leqq_K$ be a partial order on Y induced by a nonempty convex cone $K \subseteq Y$.

Definition 2.2. Let $g : Y \to Z \cup \{+\infty_Z\}$ be a vector mapping. Then, we say that g is (K,Q)-*nondecreasing* on $M \subseteq Y$ if for all $y_1, y_2 \in M$, we have

$$y_1 \leqq_K y_2 \implies g(y_1) \leqq_Q g(y_2).$$

When $Z = \mathbb{R}$ and $Q = \mathbb{R}_+$, we call it K-nondecreasing on M.

Definition 2.3. Let $h : X \to Y \cup \{+\infty_Y\}$ and $g : Y \to Z \cup \{+\infty_Z\}$ be two vector mappings, then the composed vector mapping $g \circ h : X \to Z \cup \{+\infty_Z\}$ is defined by

$$(g \circ h)(x) := \begin{cases} g(h(x)), & \text{if } x \in \text{dom} h, \\ +\infty_Z, & \text{otherwise.} \end{cases}$$

Proposition 2.4. *Let $h : X \to Y \cup \{+\infty_Y\}$ and $g : Y \to Z \cup \{+\infty_Z\}$ be two vector mappings. If g is (K,Q)-nondecreasing on* dom$g$ *and Q-convex, and h is proper and K-convex with $h(\text{dom} h) \subseteq \text{dom} g$, then $g \circ h$ is Q-convex.*

Proof. The proof is straightforward. □

The following definition of the lower semicontinuity of a vector mapping can be found for instance in [11].

Definition 2.5. Let $f : X \to Z \cup \{+\infty_Z\}$ be a vector mapping. Then, we say that f is *lower semicontinuous* at $\overline{x} \in \text{dom} f$ if for any neighborhood V of $f(\bar{x})$ in Z, there exists a neighborhood U of $\overline{x}$ such that

$$f(U) \subseteq (V + Q) \cup \{+\infty_Z\}. \tag{1}$$

If $f(\bar{x}) = +\infty_Z$, then f is called lower semicontinuous at $\overline{x}$ if for any $z \in Z$, any neighborhood V of z, there exists a neighborhood U of $\overline{x}$ such that (1) is satisfied. When f is lower semicontinuous at every point of X, then it is called lower semicontinuous.

Proposition 2.6. *Let $f : X \to Z \cup \{+\infty_Z\}$ be a proper , Q-convex and lower semicontinuous vector mapping. Then for all $z^* \in Q^* \setminus \{0\}$, the function $z^* \circ f$ is proper, convex and lower semicontinuous.*

Proof. Let $z^* \in Q^* \setminus \{0\}$. It is easy to prove that $z^* \circ f$ is proper and convex. For the lower semicontinuity of $z^* \circ f$, it suffices to apply Proposition 1.1 in [11]. □

Now, we consider the following vector optimization problem

$$(\mathcal{P}) \qquad \text{v-}\min_{x \in X} f(x)$$

where $f : X \to Z \cup \{+\infty_Z\}$ is a vector mapping.

Definition 2.7. Let $\overline{x} \in \operatorname{dom} f$, then $\overline{x}$ is called

- *weakly efficient* (*w-efficient* for short) solution of $(\mathcal{P})$ if $\nexists x \in X,\ f(x) <_Q f(\overline{x})$,
- *properly efficient* (*p-efficient* for short) solution of $(\mathcal{P})$ if $\exists \hat{Q} \subsetneq Z$ convex cone with $Q \setminus l(Q) \subseteq \operatorname{int} \hat{Q}$ such that $\nexists x \in X,\ f(x) \lneq_{\hat{Q}} f(\overline{x})$.

These latter definitions enable us to define two kinds of Pareto subdifferentials of a given vector mapping $f : X \to Z \cup \{+\infty_Z\}$ at $\bar{x} \in \operatorname{dom} f$ (see [7])

- weak Pareto subdifferential of f at $\bar{x}$
$$\partial^w f(\overline{x}) := \{A \in \mathrm{L}(X, Z) : \nexists x \in X,\ f(x) - f(\overline{x}) <_Q A(x - \overline{x})\},$$
- proper Pareto subdifferential of f at $\bar{x}$
$$\partial^p f(\overline{x}) := \left\{ A \in \mathrm{L}(X, Z) : \begin{array}{l} \exists \hat{Q} \subsetneq Z \text{ convex cone with } Q \setminus l(Q) \subseteq \operatorname{int} \hat{Q} \\ \text{such that } \nexists x \in X,\ f(x) - f(\overline{x}) \lneq_{\hat{Q}} A(x - \overline{x}) \end{array} \right\},$$

where $\mathrm{L}(X, Z)$ denotes the space of linear continuous operators from X to Z. When f is a real valued function ($Z = \mathbb{R}$ and $Q = \mathbb{R}_+$), the latter sets coincide with the subdifferential of convex analysis denoted by ∂f and defined as follows

$$\begin{aligned} \partial f(\overline{x}) : &= \{x^* \in X^* : f(x) \geq f(\overline{x}) + \langle x^*, x - \bar{x} \rangle,\ \forall x \in X\} \\ &= \{x^* \in X^* : f^*(x^*) + f(\overline{x}) = \langle x^*, \overline{x} \rangle\}, \end{aligned}$$

where the function $f^* : X^* \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ is defined by

$$f^*(x^*) = \sup_{x \in X} \{\langle x^*, x \rangle - f(x)\}$$

and called the *conjugate function* of f. For simplicity, we unify in one notation σ-efficient solution and $\partial^\sigma f$ for $\sigma \in \{w, p\}$ and put

$$Q^\sigma := \begin{cases} Q^* \setminus \{0\}, & \text{if } \sigma = w, \\ (Q^*)^\circ, & \text{if } \sigma = p. \end{cases}$$

Let $C \subseteq X$, then we denote by $\delta_C^v : X \to Z \cup \{+\infty_Z\}$ the vector indicator mapping of C defined by

$$\delta_C^v(x) := \begin{cases} 0, & \text{if } x \in C, \\ +\infty_Z, & \text{otherwise,} \end{cases}$$

and by $$N_C(\overline{x}) := \{x^* \in X^* : \langle x^*, x - \overline{x}\rangle \leq 0,\ \forall x \in C\},$$

the normal cone of C at $\overline{x} \in C$. When $Z = \mathbb{R}$ and $Q = \mathbb{R}_+$, the vector indicator mapping δ_C^v is nothing else than the well-known scalar indicator function which will be denoted by δ_C. It is worth noting that the vector indicator mapping δ_C^v possesses properties like the scalar one δ_C (see [7]). Moreover, we point out the relation between δ_C^v and δ_C

$$z^* \circ \delta_C^v = \delta_C,\ \forall z^* \in Q^*.$$

In case $(X, \|.\|_X)$ is a Banach space and $(X^*, w(X^*, X))$ its topological dual space, the notation $x_j \xrightarrow[j\in J]{\|\cdot\|_X} x$ (resp. $x_j^* \xrightarrow[j\in J]{w(X^*,X)} x^*$) means that the net $\{x_j\}_{j\in J}$ converges to $x \in X$ in $(X, \|.\|_X)$ (resp. $\{x_j^*\}_{j\in J}$ converges to $x^* \in X^*$ in $(X^*, w(X^*, X))$).

The following theorems are the key tools for deriving our main result, so they will be needed in the next section.

Theorem 2.8. (El Maghri and Laghdir [7]) *Let X, Z be two Hausdorff topological vector spaces and $f : X \to Z \cup \{+\infty_Z\}$ be a Q-convex vector mapping. Let $\overline{x} \in X$ and $\sigma \in \{w, p\}$ with Q is pointed as $\sigma = p$. Then*

$$\partial^\sigma f(\overline{x}) = \bigcup_{z^* \in Q^\sigma} \{A \in \mathrm{L}(X, Z) : z^* \circ A \in \partial(z^* \circ f)(\overline{x})\}.$$

Theorem 2.9. (Laghdir et al. [9]) *Let $(X, \|.\|_X)$ be a Banach space and the maps $f_1, ..., f_m : X \to \overline{\mathbb{R}}$, $m \geq 2$, be proper, convex and lower semicontinuous functions. Assume that $\overline{x} \in \bigcap_{i=1}^m \mathrm{dom} f_i$, then $x^* \in \partial\Big(\sum_{i=1}^m f_i\Big)(\overline{x})$ if and only if there exist nets $\{x_{i,j}\}_{j\in J} \subseteq \mathrm{dom} f_i$ and $\{x_{i,j}^*\}_{j\in J} \subseteq X^*$, $i = 1, ..., m$, satisfying*

$$x_{i,j}^* \in \partial f_i(x_{i,j}),\ x_{i,j} \xrightarrow[j\in J]{\|\cdot\|_X} \overline{x},\ \sum_{i=1}^m x_{i,j}^* \xrightarrow[j\in J]{w(X^*,X)} x^*$$

and $$f_i(x_{i,j}) - f_i(\overline{x}) - \langle x_{i,j}^*, x_{i,j} - \overline{x}\rangle \xrightarrow[j\in J]{} 0.$$

3. A sequential formula for the weak/proper Pareto subdifferential of multi-composed convex vector mappings

In what follows $(X, \|.\|_X)$ and $(Y_i, \|.\|_{Y_i})$ are Banach spaces, Z is a real Hausdorff topological vector space and $(X^*, w(X^*, X))$, $(Y_i^*, w(Y_i^*, Y_i))$ and Z^* their respective topological duals paired in duality by $\langle ., .\rangle$ with $i \in \{0, ..., m\}$ and $m \geq 2$. We assume that Y_i and Z are partially ordered by nonempty convex cones $K_i \subseteq Y_i$ and $Q \subseteq Z$ with $\mathrm{int}\, Q \neq \emptyset$ and $i \in \{0, ..., m\}$. Moreover, we use on $X \times \prod_{k=0}^m Y_k$ the following norm

$$\|(x, y_0, y_1, ..., y_m)\|_{X\times\prod_{k=0}^m Y_k} := \sqrt{\|x\|_X^2 + \sum_{k=0}^m \|y_k\|_{Y_k}^2}.$$

Our aim in this section is to give a sequential formula for the weak/proper Pareto subdifferential of the multi-composed vector mapping $f+\varphi\circ\psi+g\circ h_1\circ h_2\circ ...\circ h_m$, where

- $f : X \to Z\cup\{+\infty_Z\}$ is proper, Q-convex and lower semicontinuous,
- $\varphi : Y_0 \to Z\cup\{+\infty_Z\}$ is proper, Q-convex, (K_0, Q)-nondecreasing on domφ and lower semicontinuous with $\varphi(+\infty_{Y_0}) = +\infty_Z$,
- $\psi : X \to Y_0\cup\{+\infty_{Y_0}\}$ is proper, K_0-convex and K_0-epi closed with $\psi(\text{dom}\psi) \subseteq \text{dom}\varphi$,
- $g : Y_1 \to Z\cup\{+\infty_Z\}$ is proper, Q-convex, (K_1, Q)-nondecreasing on dom$g$ and lower semicontinuous with $g(+\infty_{Y_1}) = +\infty_Z$,
- $h_1 : Y_2 \to Y_1\cup\{+\infty_{Y_1}\}$ is proper, K_1-convex, (K_2, K_1)-nondecreasing on domh_1 and K_1-epi closed with $h_1(\text{dom}h_1) \subseteq \text{dom}g$ and $h_1(+\infty_{Y_2}) = +\infty_{Y_1}$,
- $h_i : Y_{i+1} \to Y_i\cup\{+\infty_{Y_i}\}$ is proper, K_i-convex, (K_{i+1}, K_i)-nondecreasing on domh_i and K_i-epi closed with $h_i(\text{dom}h_i) \subseteq \text{dom}h_{i-1}$ and $h_i(+\infty_{Y_{i+1}}) = +\infty_{Y_i}$, $i = 2, ..., m-1$,
- $h_m : X \to Y_m\cup\{+\infty_{Y_m}\}$ is proper, K_m-convex and K_m-epi closed satisfying $h_m(\text{dom}h_m) \subseteq \text{dom}h_{m-1}$,
- $\text{dom}f\cap\psi^{-1}(\text{dom}\varphi)\cap\text{dom}\psi\cap(h_m^{-1}\circ h_{m-1}^{-1}\circ ...\circ h_1^{-1})(\text{dom}g)\cap\text{dom}h_m \neq \emptyset$.

Remark 3.1. Let us point out that we do not need the following assumptions $h_1(\text{dom}h_1) \subseteq \text{dom}g$ and $\psi(\text{dom}\psi) \subseteq \text{dom}\varphi$ when $g : Y_1 \to Z\cup\{+\infty_Z\}$ is (K_1, Q)-nondecreasing on Y_1 and $\varphi : Y_0 \to Z\cup\{+\infty_Z\}$ is (K_0, Q)-nondecreasing on Y_0. □

Remark 3.2. By Proposition 2.4, $f+\varphi\circ\psi+g\circ h_1\circ h_2\circ ...\circ h_m$ is a Q-convex vector mapping. □

Let us consider the following vector mappings

$$F: \begin{array}{ll} X\times\prod_{k=0}^m Y_k & \to Z\cup\{+\infty_Z\} \\ (x, y_0, y_1, ..., y_m) & \to f(x) \end{array}, \quad \Phi: \begin{array}{ll} X\times\prod_{k=0}^m Y_k & \to Z\cup\{+\infty_Z\} \\ (x, y_0, y_1, ..., y_m) & \to \varphi(y_0) \end{array}$$

$$\Psi: \begin{array}{ll} X\times\prod_{k=0}^m Y_k & \to Z\cup\{+\infty_Z\} \\ (x, y_0, y_1, ..., y_m) & \to \delta^v_{\text{epi}\psi}(x, y_0) \end{array}, \quad G: \begin{array}{ll} X\times\prod_{k=0}^m Y_k & \to Z\cup\{+\infty_Z\} \\ (x, y_0, y_1, ..., y_m) & \to g(y_1) \end{array}$$

$$H_i: \begin{array}{ll} X\times\prod_{k=0}^m Y_k & \to Z\cup\{+\infty_Z\} \quad (i = 1, ..., m-1) \\ (x, y_0, y_1, ..., y_m) & \to \delta^v_{\text{epi}h_i}(y_{i+1}, y_i) \end{array}$$

and

$$H_m: \begin{array}{ll} X\times\prod_{k=0}^m Y_k & \to Z\cup\{+\infty_Z\} \\ (x, y_0, y_1, ..., y_m) & \to \delta^v_{\text{epi}h_m}(x, y_m). \end{array}$$

The following lemmas will be needed in this section.

Lemma 3.3. *Assume that*

$$\overline{x} \in \text{dom}f\cap\psi^{-1}(\text{dom}\varphi)\cap\text{dom}\psi\cap(h_m^{-1}\circ h_{m-1}^{-1}\circ ...\circ h_1^{-1})(\text{dom}g)\cap\text{dom}h_m,$$
$$\overline{y}_m = h_m(\overline{x}), \quad \overline{y}_{m-1} = h_{m-1}(\overline{y}_m), ..., \overline{y}_1 = h_1(\overline{y}_2), \ \overline{y}_0 = \psi(\overline{x})$$

and $\sigma \in \{w, p\}$ *with* Q *is pointed as* $\sigma = p$.
Let $A \in \mathrm{L}(X, Z)$ *and* $T \in \mathrm{L}(X\times\prod_{k=0}^m Y_k, Z)$ *defined by* $T(x, y_0, y_1, ..., y_m) := A(x)$, *for all* $(x, y_0, y_1, \ldots, y_m) \in X\times\prod_{k=0}^m Y_k$.

Then we have $\qquad A \in \partial^{\sigma}(f + \varphi\circ\psi + g \circ h_1 \circ h_2 \circ ... \circ h_m)(\overline{x})$

$$\iff \quad T \in \partial^{\sigma}\Big(F + \Phi + \Psi + G + \sum_{i=1}^{m} H_i\Big)(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m).$$

Proof. Let us prove the lemma for $\sigma = w$.

($\Rightarrow$) We proceed by contradiction. Let $A \in \partial^{w}(f + \varphi\circ\psi + g \circ h_1 \circ h_2 \circ ... \circ h_m)(\overline{x})$ and suppose that $T \notin \partial^{w}\Big(F + \Phi + \Psi + G + \sum_{i=1}^{m} H_i\Big)(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m)$. Then, there exists $(x, y_0, y_1, ..., y_m) \in X \times \prod_{k=0}^{m} Y_k$ such that

$$\begin{aligned}(F + \Phi + \Psi + G + &\sum_{i=1}^{m} H_i)(x, y_0, y_1, ..., y_m)\\ &- (F + \Phi + \Psi + G + \sum_{i=1}^{m} H_i)(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m) <_Q A(x - \overline{x}).\end{aligned}$$

This yields

$$\begin{cases} x \in \mathrm{dom} f,\ y_0 \in \mathrm{dom}\varphi,\ (x, y_0) \in \mathrm{epi}\psi,\ y_1 \in \mathrm{dom} g,\\ (x, y_m) \in \mathrm{epi} h_m,\ (y_{i+1}, y_i) \in \mathrm{epi} h_i, i = 1, ..., m-1, \end{cases} \tag{2}$$

and
$$f(x) + \varphi(y_0) + g(y_1) - f(\overline{x}) - \varphi(\overline{y}_0) - g(\overline{y}_1) <_Q A(x - \overline{x}). \tag{3}$$

From (2) and by using the monotonicity of φ, g and $h_1, ..., h_{m-1}$, it follows that

$$(\varphi\circ\psi)(x) \leqq_Q \varphi(y_0) \text{ and } (g \circ h_1 \circ h_2 \circ ... \circ h_m)(x) \leqq_Q g(y_1). \tag{4}$$

Hence, by taking into account that Q is a convex cone and $Q + \mathrm{int}Q \subseteq \mathrm{int}Q$, we deduce from (3) and (4) that

$$(f+\varphi\circ\psi+g\circ h_1\circ h_2\circ...\circ h_m)(x)-(f+\varphi\circ\psi+g\circ h_1\circ h_2\circ...\circ h_m)(\overline{x}) <_Q A(x-\overline{x})$$

which contradicts $A \in \partial^{w}(f + \varphi\circ\psi + g \circ h_1 \circ h_2 \circ ... \circ h_m)(\overline{x})$.

($\Leftarrow$) This is also obvious by contradiction too.

the case $\sigma = p$ is obtained similarly as $\sigma = w$ in which it is recommended to take into account that $\mathrm{int}\,\hat{Q} \subseteq \hat{Q}\backslash \mathrm{l}(\hat{Q})$ and $\hat{Q}\backslash \mathrm{l}(\hat{Q}) + \hat{Q}\backslash \mathrm{l}(\hat{Q}) \subseteq \hat{Q}\backslash \mathrm{l}(\hat{Q})$, for all convex cone $\hat{Q} \subsetneq Z$ that satisfies $Q \setminus \{0\} \subseteq \mathrm{int}\,\hat{Q}$. □

Lemma 3.4. (1) *Let* $i \in \{1, ..., m-1\}$ *and* $(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m) \in \mathrm{dom} H_i$.

Then, for all $z^* \in Q^*$ *we have*

$$(x^*, y_0^*, y_1^*, ..., y_m^*) \in \partial(z^* \circ H_i)(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m)$$

$$\iff \begin{cases} x^* = 0,\ y_k^* = 0,\ k \in \{0, ..., m\}\backslash\{i, i+1\},\\ -y_i^* \in K_i^* \text{ and } \langle -y_i^*, \overline{y}_i - h_i(\overline{y}_{i+1})\rangle = 0,\\ y_{i+1}^* \in \partial(-y_i^* \circ h_i)(\overline{y}_{i+1}). \end{cases}$$

(2) *Let* $(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m) \in \mathrm{dom} H_m$. *Then, for all* $z^* \in Q^*$ *we have*

$$(x^*, y_0^*, y_1^*, ..., y_m^*) \in \partial(z^* \circ H_m)(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m)$$

$$\iff \begin{cases} y_k^* = 0, \ k = 0, ..., m-1, \\ -y_m^* \in K_m^* \ \textit{and} \ \langle -y_m^*, \overline{y}_m - h_m(\overline{x})\rangle = 0, \\ x^* \in \partial(-y_m^* \circ h_m)(\overline{x}). \end{cases}$$

(3) *Let* $(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m) \in \mathrm{dom}\Psi$. *Then, for all* $z^* \in Q^*$ *we have*

$$(x^*, y_0^*, y_1^*, ..., y_m^*) \in \partial(z^* \circ \Psi)(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m)$$

$$\iff \begin{cases} y_k^* = 0, \ k = 1, ..., m, \\ -y_0^* \in K_0^* \ \textit{and} \ \langle -y_0^*, \overline{y}_0 - \psi(\overline{x})\rangle = 0, \\ x^* \in \partial(-y_0^* \circ \psi)(\overline{x}). \end{cases}$$

(4) *Let* $(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m) \in \mathrm{dom}F$. *Then, for all* $z^* \in Q^*$ *we have*

$$\partial(z^* \circ F)(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m) = \partial(z^* \circ f)(\overline{x}) \times \{0\} \times ... \times \{0\}.$$

(5) *Let* $(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m) \in \mathrm{dom}G$. *Then, for all* $z^* \in Q^*$ *we have*

$$\partial(z^* \circ G)(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m) = \{0\} \times \{0\} \times \partial(z^* \circ g)(\overline{y}_1) \times \{0\} \times ... \times \{0\}.$$

(6) *Let* $(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m) \in \mathrm{dom}\Phi$. *Then, for all* $z^* \in Q^*$ *we have*

$$\partial(z^* \circ \Phi)(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m) = \{0\} \times \partial(z^* \circ \varphi)(\overline{y}_0) \times \{0\} \times ... \times \{0\}.$$

Proof. We prove (1), since the proof of (2)–(6) is similar to (1).

($\Rightarrow$) Let $i \in \{1, ..., m-1\}$, $(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m) \in \mathrm{dom} H_i$ and $z^* \in Q^*$. Since

$$(z^* \circ H_i)(x, y_0, y_1, ..., y_m) = (z^* \circ \delta^v_{\mathrm{epi} h_i})(y_{i+1}, y_i) = \delta_{\mathrm{epi} h_i}(y_{i+1}, y_i),$$

for all $(x, y_0, \, y_1, ..., y_m) \in X \times \prod_{k=0}^m Y_k$, and $(\overline{y}_{i+1}, \overline{y}_i) \in \mathrm{epi} h_i$, it follows that

$$(x^*, y_0^*, y_1^*, ..., y_m^*) \in \partial(z^* \circ H_i)(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m) \iff$$

$$\delta_{\mathrm{epi} h_i}(y_{i+1}, y_i) \geq \langle x^*, x - \overline{x}\rangle + \sum_{k=0}^m \langle y_k^*, y_k - \overline{y}_k\rangle, \ \forall (x, y_0, y_1, ..., y_m) \in X \times \prod_{k=0}^m Y_k. \quad (5)$$

Set $y_k := \overline{y}_k$ in (5), $k = 0, ..., m$, then we have for all $x \in X$

$$0 \geq \langle x^*, (x + \overline{x}) - \overline{x}\rangle = \langle x^*, x\rangle \ \text{ and } \ 0 \geq \langle x^*, (-x + \overline{x}) - \overline{x}\rangle = -\langle x^*, x\rangle.$$

Thus, we deduce that $\langle x^*, x\rangle = 0$ for all $x \in X$, i.e. $x^* = 0$. Similarly, we can prove that $y_k^* = 0$ for all $k \in \{0, ..., m\} \backslash \{i, i+1\}$. As a consequence, (5) becomes

$$\delta_{\mathrm{epi} h_i}(y_{i+1}, y_i) \geq \langle y_i^*, y_i - \overline{y}_i\rangle + \langle y_{i+1}^*, y_{i+1} - \overline{y}_{i+1}\rangle, \ \forall (y_i, y_{i+1}) \in Y_i \times Y_{i+1}. \quad (6)$$

By taking $y_{i+1} := \overline{y}_{i+1}$ in (6), we obtain that

$$0 \geq \langle y_i^*, (y_i + \overline{y}_i) - \overline{y}_i\rangle = \langle y_i^*, y_i\rangle, \forall y_i \in K_i.$$

This inequality implies $-y_i^* \in K_i^*$.

Now we prove that $\langle -y_i^*, \overline{y}_i - h_i(\overline{y}_{i+1})\rangle = 0$. By setting $y_{i+1} := \overline{y}_{i+1}$ and $y_i := h_i(\overline{y}_{i+1})$ in (6), one can see easily that

$$\langle -y_i^*, \overline{y}_i - h_i(\overline{y}_{i+1})\rangle \leq 0. \tag{7}$$

On the other hand, we have

$$\langle -y_i^*, \overline{y}_i - h_i(\overline{y}_{i+1})\rangle \geq 0, \tag{8}$$

since $-y_i^* \in K_i^*$ and $\overline{y}_i - h_i(\overline{y}_{i+1}) \in K_i$. Hence, from (7) and (8), we deduce that

$$\langle -y_i^*, \overline{y}_i - h_i(\overline{y}_{i+1})\rangle = 0.$$

To complete the proof of the right implication, it remains to show that we have $y_{i+1}^* \in \partial(-y_i^* \circ h_i)(\overline{y}_{i+1})$, that is

$$(-y_i^* \circ h_i)(y) \geq (-y_i^* \circ h_i)(\overline{y}_{i+1}) + \langle y_{i+1}^*, y - \overline{y}_{i+1}\rangle, \ \forall y \in Y_{i+1}. \tag{9}$$

If $y \notin \mathrm{dom} h_i$ then (9) is obvious. Suppose that $y \in \mathrm{dom} h_i$, then by setting $y_{i+1} := y$ and $y_i := h_i(y)$ in (6), we obtain that

$$(-y_i^* \circ h_i)(y) \geq -\langle y_i^*, \overline{y}_i\rangle + \langle y_{i+1}^*, y - \overline{y}_{i+1}\rangle.$$

Since $\langle -y_i^*, \overline{y}_i - h_i(\overline{y}_{i+1})\rangle = 0$, it follows that $(-y_i^* \circ h_i)(\overline{y}_{i+1}) = -\langle y_i^*, \overline{y}_i\rangle$ and hence

$$(-y_i^* \circ h_i)(y) \geq (-y_i^* \circ h_i)(\overline{y}_{i+1}) + \langle y_{i+1}^*, y - \overline{y}_{i+1}\rangle.$$

Thus, we conclude that $y_{i+1}^* \in \partial(-y_i^* \circ h_i)(\overline{y}_{i+1})$.

($\Leftarrow$) This is obvious. □

Now, we can state and prove our main result.

Theorem 3.5. *Let*

$$\overline{x} \in \mathrm{dom} f \cap \psi^{-1}(\mathrm{dom}\varphi) \cap \mathrm{dom}\psi \cap (h_m^{-1} \circ h_{m-1}^{-1} \circ ... \circ h_1^{-1})(\mathrm{dom} g) \cap \mathrm{dom} h_m,$$

$\overline{y}_m = h_m(\overline{x})$, $\overline{y}_{m-1} = h_{m-1}(\overline{y}_m), ..., \overline{y}_1 = h_1(\overline{y}_2)$, $\overline{y}_0 = \psi(\overline{x})$ *and* $\sigma \in \{w, p\}$. *In addition, we assume that* Q *is pointed as* $\sigma = p$. *Then we have*

$$A \in \partial^\sigma(f + \varphi \circ \psi + g \circ h_1 \circ h_2 \circ ... \circ h_m)(\overline{x})$$

if and only if there exist $z^* \in Q^\sigma$ *and nets* $\{x_j\}_{j\in J} \subseteq \mathrm{dom} f$, $\{y_{0,j}\}_{j\in J} \subseteq \mathrm{dom}\varphi$, $\{(w_j, w_{0,j})\}_{j\in J} \subseteq \mathrm{epi}\psi$, $\{u_{1,j}\}_{j\in J} \subseteq \mathrm{dom} g$, $\{(v_{i+1,j}^i, v_{i,j}^i)\}_{j\in J} \subseteq \mathrm{epi} h_i$, $i = 1, ..., m-1$, $\{(v_j^m, v_{m,j}^m)\}_{j\in J} \subseteq \mathrm{epi} h_m$, $\{x_j^*\}_{j\in J} \subseteq X^*$, $\{y_{0,j}^*\}_{j\in J} \subseteq Y_0^*$, $\{(w_j^*, w_{0,j}^*)\}_{j\in J} \subseteq X^* \times Y_0^*$, $\{u_{1,j}^*\}_{j\in J} \subseteq Y_1^*$, $\{(v_{i,j}^{i*}, v_{i+1,j}^{i*})\}_{j\in J} \subseteq Y_i^* \times Y_{i+1}^*$, $i = 1, ..., m-1$, *and finally the net* $\{(v_j^{m*}, v_{m,j}^{m*})\}_{j\in J} \subseteq X^* \times Y_m^*$ *satisfying*

$$\begin{cases} x_j^* \in \partial(z^* \circ f)(x_j), \ x_j \xrightarrow[j\in J]{\|\cdot\|_X} \overline{x}, \\ (z^* \circ f)(x_j) - (z^* \circ f)(\overline{x}) - \langle x_j^*, x_j - \overline{x}\rangle \underset{j\in J}{\longrightarrow} 0, \end{cases}$$

$$\begin{cases} y_{0,j}^* \in \partial(z^* \circ \varphi)(y_{0,j}),\ y_{0,j} \xrightarrow[j\in J]{\|\cdot\|_{Y_0}} \overline{y}_0, \\ (z^* \circ \varphi)(y_{0,j}) - (z^* \circ \varphi)(\overline{y}_0) - \langle y_{0,j}^*, y_{0,j} - \overline{y}_0\rangle \xrightarrow[j\in J]{} 0, \end{cases}$$

$$\begin{cases} -w_{0,j}^* \in K_0^*,\ \langle -w_{0,j}^*, w_{0,j} - \psi(w_j)\rangle = 0, \\ w_j^* \in \partial(-w_{0,j}^* \circ \psi)(w_j),\ w_j \xrightarrow[j\in J]{\|\cdot\|_X} \overline{x},\ w_{0,j} \xrightarrow[j\in J]{\|\cdot\|_{Y_0}} \overline{y}_0, \\ \langle -w_j^*, w_j - \overline{x}\rangle + \langle -w_{0,j}^*, w_{0,j} - \overline{y}_0\rangle \xrightarrow[j\in J]{} 0, \end{cases}$$

$$\begin{cases} u_{1,j}^* \in \partial(z^* \circ g)(u_{1,j}),\ u_{1,j} \xrightarrow[j\in J]{\|\cdot\|_{Y_1}} \overline{y}_1, \\ (z^* \circ g)(u_{1,j}) - (z^* \circ g)(\overline{y}_1) - \langle u_{1,j}^*, u_{1,j} - \overline{y}_1\rangle \xrightarrow[j\in J]{} 0, \end{cases}$$

for $i = 1, ..., m-1,$

$$\begin{cases} -v_{i,j}^{i*} \in K_i^*,\ \langle -v_{i,j}^{i*}, v_{i,j}^i - h_i(v_{i+1,j}^i)\rangle = 0, \\ v_{i+1,j}^{i*} \in \partial(-v_{i,j}^{i*} \circ h_i)(v_{i+1,j}^i),\ v_{i,j}^i \xrightarrow[j\in J]{\|\cdot\|_{Y_i}} \overline{y}_i,\ v_{i+1,j}^i \xrightarrow[j\in J]{\|\cdot\|_{Y_{i+1}}} \overline{y}_{i+1}, \\ \langle -v_{i,j}^{i*}, v_{i,j}^i - \overline{y}_i\rangle + \langle -v_{i+1,j}^{i*}, v_{i+1,j}^i - \overline{y}_{i+1}\rangle \xrightarrow[j\in J]{} 0, \end{cases}$$

$$\begin{cases} -v_{m,j}^{m*} \in K_m^*,\ \langle -v_{m,j}^{m*}, v_{m,j}^m - h_m(v_j^m)\rangle = 0, \\ v_j^{m*} \in \partial(-v_{m,j}^{m*} \circ h_m)(v_j^m),\ v_j^m \xrightarrow[j\in J]{\|\cdot\|_X} \overline{x},\ v_{m,j}^m \xrightarrow[j\in J]{\|\cdot\|_{Y_m}} \overline{y}_m, \\ \langle -v_j^{m*}, v_j^m - \overline{x}\rangle + \langle -v_{m,j}^{m*}, v_{m,j}^m - \overline{y}_m\rangle \xrightarrow[j\in J]{} 0, \end{cases}$$

and

$$\begin{cases} x_j^* + w_j^* + v_j^{m*} \xrightarrow[j\in J]{w(X^*,X)} z^* \circ A,\ y_{0,j}^* + w_{0,j}^* \xrightarrow[j\in J]{w(Y_0^*,Y_0)} 0, \\ u_{1,j}^* + v_{1,j}^{1*} \xrightarrow[j\in J]{w(Y_1^*,Y_1)} 0,\ v_{i,j}^{(i-1)*} + v_{i,j}^{i*} \xrightarrow[j\in J]{w(Y_i^*,Y_i)} 0,\ i = 2, ..., m. \end{cases}$$

Proof. According to Lemma 3.3, $A \in \partial^\sigma(f + \varphi\circ\psi + g\circ h_1\circ h_2\circ ...\circ h_m)(\overline{x})$ if and only if

$$T \in \partial^\sigma\Big(F + \Phi + \Psi + G + \sum_{i=1}^m H_i\Big)(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m),$$

where $T(x, y_0, y_1, ..., y_m) := A(x)$, for all $(x, y_0, y_1, ..., y_m) \in X \times \prod_{k=0}^m Y_k$. Therefore, by Theorem 2.8, it follows that there exists $z^* \in Q^\sigma$ such that

$$z^* \circ T \in \partial\Big(z^* \circ F + z^* \circ \Phi + z^* \circ \Psi + z^* \circ G + \sum_{i=1}^m z^* \circ H_i\Big)(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m)$$

and as can be seen easily this assertion is equivalent to

$$(z^* \circ A, 0, 0, ..., 0) \in \partial\Big(z^* \circ F + z^* \circ \Phi + z^* \circ \Psi + z^* \circ G + \sum_{i=1}^m z^* \circ H_i\Big)(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m).$$

By Proposition 2.6, the functions $z^* \circ F$, $z^* \circ \Phi$, $z^* \circ \Psi$, $z^* \circ G$ and $z^* \circ H_1, ..., z^* \circ H_m$ are all proper, convex and lower semicontinuous since F, Φ, Ψ, G and $H_1, ..., H_m$ are proper, Q-convex and lower semicontinuous and $z^* \in Q^\sigma$. Therefore, by applying Theorem 2.9, we assert that there exist nets

$$\{(x_j, x_{0,j}, x_{1,j}, ..., x_{m,j})\}_{j\in J} \subseteq \mathrm{dom}(z^* \circ F) = \mathrm{dom}f \times \prod_{k=0}^{m} Y_k,$$

$$\{(x_j^*, x_{0,j}^*, x_{1,j}^*, ..., x_{m,j}^*)\}_{j\in J} \subseteq X^* \times \prod_{k=0}^{m} Y_k^*,$$

$$\{(y_j, y_{0,j}, y_{1,j}, ..., y_{m,j})\}_{j\in J} \subseteq \mathrm{dom}(z^* \circ \Phi) = X \times \mathrm{dom}\varphi \times \prod_{k=1}^{m} Y_k,$$

$$\{(y_j^*, y_{0,j}^*, y_{1,j}^*, ..., y_{m,j}^*)\}_{j\in J} \subseteq X^* \times \prod_{k=0}^{m} Y_k^*,$$

$$\{(w_j, w_{0,j}, w_{1,j}, ..., w_{m,j})\}_{j\in J} \subseteq \mathrm{dom}(z^* \circ \Psi)$$

$\big($i.e. $\{w_j\}_{j\in J} \subseteq X$ and $\{w_{k,j}\}_{j\in J} \subseteq Y_k$, $k = 0, ..., m$, with $\{(w_j, w_{0,j})\}_{j\in J} \subseteq \mathrm{epi}\psi\big)$,

$$\{(w_j^*, w_{0,j}^*, w_{1,j}^*, ..., w_{m,j}^*)\}_{j\in J} \subseteq X^* \times \prod_{k=0}^{m} Y_k^*,$$

$$\{(u_j, u_{0,j}, u_{1,j}, ..., u_{m,j})\}_{j\in J} \subseteq \mathrm{dom}(z^* \circ G) = X \times Y_0 \times \mathrm{dom}g \times \prod_{k=2}^{m} Y_k,$$

$$\{(u_j^*, u_{0,j}^*, u_{1,j}^*, ..., u_{m,j}^*)\}_{j\in J} \subseteq X^* \times \prod_{k=0}^{m} Y_k^*,$$

$$\{(v_j^i, v_{0,j}^i, v_{1,j}^i, ..., v_{m,j}^i)\}_{j\in J} \subseteq \mathrm{dom}(z^* \circ H_i),\ i = 1, ..., m-1$$

$\big($i.e. for $i = 1, ..., m-1$, $\{v_j^i\}_{j\in J} \subseteq X$ and $\{v_{k,j}^i\}_{j\in J} \subseteq Y_k$, $k = 0, ..., m$, with $\{(v_{i+1,j}^i, v_{i,j}^i)\}_{j\in J} \subseteq \mathrm{epi}h_i\big)$,

$$\{(v_j^{i*}, v_{0,j}^{i*}, v_{1,j}^{i*}, ..., v_{m,j}^{i*})\}_{j\in J} \subseteq X^* \times \prod_{k=0}^{m} Y_k^*,\ i = 1, ..., m-1,$$

$$\{(v_j^m, v_{0,j}^m, v_{1,j}^m, ..., v_{m,j}^m)\}_{j\in J} \subseteq \mathrm{dom}(z^* \circ H_m)$$

$\big($i.e. $\{v_j^m\}_{j\in J} \subseteq X$ and $\{v_{k,j}^m\}_{j\in J} \subseteq Y_k$, $k = 0, ..., m$, with $\{(v_j^m, v_{m,j}^m)\}_{j\in J} \subseteq \mathrm{epi}h_m\big)$,

$$\{(v_j^{m*}, v_{0,j}^{m*}, v_{1,j}^{m*}, ..., v_{m,j}^{m*})\}_{j\in J} \subseteq X^* \times \prod_{k=0}^{m} Y_k^*$$

satisfying

$$\begin{cases} (x_j^*, x_{0,j}^*, x_{1,j}^*, ..., x_{m,j}^*) \in \partial(z^* \circ F)(x_j, x_{0,j}, x_{1,j}, ..., x_{m,j}), \\ (x_j, x_{0,j}, x_{1,j}, ..., x_{m,j}) \xrightarrow[j\in J]{\|\cdot\|_{X\times\prod_{k=0}^{m} Y_k}} (\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m), \\ (z^* \circ F)(x_j, x_{0,j}, x_{1,j}, ..., x_{m,j}) - (z^* \circ F)(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m) \\ -\langle x_j^*, x_j - \overline{x}\rangle - \sum_{k=0}^{m}\langle x_{k,j}^*, x_{k,j} - \overline{y}_k\rangle \xrightarrow[j\in J]{} 0, \end{cases} \tag{10}$$

$$\begin{cases} (y_j^*, y_{0,j}^*, y_{1,j}^*, ..., y_{m,j}^*) \in \partial(z^* \circ \Phi)(y_j, y_{0,j}, y_{1,j}, ..., y_{m,j}), \\ (y_j, y_{0,j}, y_{1,j}, ..., y_{m,j}) \xrightarrow[j\in J]{\|\cdot\|_{X\times\prod_{k=0}^m Y_k}} (\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m), \\ (z^* \circ \Phi)(y_j, y_{0,j}, y_{1,j}, ..., y_{m,j}) - (z^* \circ \Phi)(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m) \\ -\langle y_j^*, y_j - \overline{x}\rangle - \sum_{k=0}^{m} \langle y_{k,j}^*, y_{k,j} - \overline{y}_k\rangle \xrightarrow[j\in J]{} 0, \end{cases} \tag{11}$$

$$\begin{cases} (w_j^*, w_{0,j}^*, w_{1,j}^*, ..., w_{m,j}^*) \in \partial(z^* \circ \Psi)(w_j, w_{0,j}, w_{1,j}, ..., w_{m,j}), \\ (w_j, w_{0,j}, w_{1,j}, ..., w_{m,j}) \xrightarrow[j\in J]{\|\cdot\|_{X\times\prod_{k=0}^m Y_k}} (\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m), \\ (z^* \circ \Psi)(w_j, w_{0,j}, w_{1,j}, ..., w_{m,j}) - (z^* \circ \Psi)(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m) \\ -\langle w_j^*, w_j - \overline{x}\rangle - \sum_{k=0}^{m} \langle w_{k,j}^*, w_{k,j} - \overline{y}_k\rangle \xrightarrow[j\in J]{} 0, \end{cases} \tag{12}$$

$$\begin{cases} (u_j^*, u_{0,j}^*, u_{1,j}^*, ..., u_{m,j}^*) \in \partial(z^* \circ G)(u_j, u_{0,j}, u_{1,j}, ..., u_{m,j}), \\ (u_j, u_{0,j}, u_{1,j}, ..., u_{m,j}) \xrightarrow[j\in J]{\|\cdot\|_{X\times\prod_{k=0}^m Y_k}} (\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m), \\ (z^* \circ G)(u_j, u_{0,j}, u_{1,j}, ..., u_{m,j}) - (z^* \circ G)(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m) \\ -\langle u_j^*, u_j - \overline{x}\rangle - \sum_{k=0}^{m} \langle u_{k,j}^*, u_{k,j} - \overline{y}_k\rangle \xrightarrow[j\in J]{} 0, \end{cases} \tag{13}$$

for $i = 1, ..., m-1$,

$$\begin{cases} (v_j^{i*}, v_{0,j}^{i*}, v_{1,j}^{i*}, ..., v_{m,j}^{i*}) \in \partial(z^* \circ H_i)(v_j^i, v_{0,j}^i, v_{1,j}^i, ..., v_{m,j}^i), \\ (v_j^i, v_{0,j}^i, v_{1,j}^i, ..., v_{m,j}^i) \xrightarrow[j\in J]{\|\cdot\|_{X\times\prod_{k=0}^m Y_k}} (\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m), \\ (z^* \circ H_i)(v_j^i, v_{0,j}^i, v_{1,j}^i, ..., v_{m,j}^i) - (z^* \circ H_i)(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m) \\ -\langle v_j^{i*}, v_j^i - \overline{x}\rangle - \sum_{k=0}^{m} \langle v_{k,j}^{i*}, v_{k,j}^i - \overline{y}_k\rangle \xrightarrow[j\in J]{} 0, \end{cases} \tag{14}$$

$$\begin{cases} (v_j^{m*}, v_{0,j}^{m*}, v_{1,j}^{m*}, ..., v_{m,j}^{m*}) \in \partial(z^* \circ H_m)(v_j^m, v_{0,j}^m, v_{1,j}^m, ..., v_{m,j}^m), \\ (v_j^m, v_{0,j}^m, v_{1,j}^m, ..., v_{m,j}^m) \xrightarrow[j\in J]{\|\cdot\|_{X\times\prod_{k=0}^m Y_k}} (\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m), \\ (z^* \circ H_m)(v_j^m, v_{0,j}^m, v_{1,j}^m, ..., v_{m,j}^m) - (z^* \circ H_m)(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m) \\ -\langle v_j^{m*}, v_j^m - \overline{x}\rangle - \sum_{k=0}^{m} \langle v_{k,j}^{m*}, v_{k,j}^m - \overline{y}_k\rangle \xrightarrow[j\in J]{} 0, \end{cases} \tag{15}$$

and

$$\begin{cases} x_j^* + y_j^* + w_j^* + u_j^* + \sum_{k=1}^{m} v_j^{k*} \xrightarrow[j\in J]{w(X^*,X)} z^* \circ A, \\ x_{i,j}^* + y_{i,j}^* + w_{i,j}^* + u_{i,j}^* + \sum_{k=1}^{m} v_{i,j}^{k*} \xrightarrow[j\in J]{w(Y_i^*,Y_i)} 0, \ i = 0, ..., m. \end{cases} \tag{16}$$

By Lemma 3.4, (10)–(15) are equivalent respectively to

$$\begin{cases} x_j^* \in \partial(z^* \circ f)(x_j),\ x_j \xrightarrow[j\in J]{\|\cdot\|_X} \overline{x},\ x_{k,j} \xrightarrow[j\in J]{\|\cdot\|_{Y_k}} \overline{y}_k,\ k=0,...,m, \\ (z^* \circ f)(x_j) - (z^* \circ f)(\overline{x}) - \langle x_j^*, x_j - \overline{x}\rangle \xrightarrow[j\in J]{} 0, \end{cases}$$

$$\begin{cases} y_{0,j}^* \in \partial(z^* \circ \varphi)(y_{0,j}),\ y_j \xrightarrow[j\in J]{\|\cdot\|_X} \overline{x},\ y_{k,j} \xrightarrow[j\in J]{\|\cdot\|_{Y_k}} \overline{y}_k,\ k=0,...,m, \\ (z^* \circ \varphi)(y_{0,j}) - (z^* \circ \varphi)(\overline{y}_0) - \langle y_{0,j}^*, y_{0,j} - \overline{y}_0\rangle \xrightarrow[j\in J]{} 0, \end{cases}$$

$$\begin{cases} -w_{0,j}^* \in K_0^*,\ \langle -w_{0,j}^*, w_{0,j} - \psi(w_j)\rangle = 0, \\ w_j^* \in \partial(-w_{0,j}^* \circ \psi)(w_j),\ w_j \xrightarrow[j\in J]{\|\cdot\|_X} \overline{x},\ w_{k,j} \xrightarrow[j\in J]{\|\cdot\|_{Y_k}} \overline{y}_k,\ k=0,...,m, \\ \langle -w_j^*, w_j - \overline{x}\rangle + \langle -w_{0,j}^*, w_{0,j} - \overline{y}_0\rangle \xrightarrow[j\in J]{} 0, \end{cases}$$

$$\begin{cases} u_{1,j}^* \in \partial(z^* \circ g)(u_{1,j}),\ u_j \xrightarrow[j\in J]{\|\cdot\|_X} \overline{x},\ u_{k,j} \xrightarrow[j\in J]{\|\cdot\|_{Y_k}} \overline{y}_k,\ k=0,...,m, \\ (z^* \circ g)(u_{1,j}) - (z^* \circ g)(\overline{y}_1) - \langle u_{1,j}^*, u_{1,j} - \overline{y}_1\rangle \xrightarrow[j\in J]{} 0, \end{cases}$$

for $i=1,...,m-1$,

$$\begin{cases} -v_{i,j}^{i*} \in K_i^*,\ \langle -v_{i,j}^{i*}, v_{i,j}^i - h_i(v_{i+1,j}^i)\rangle = 0, \\ v_{i+1,j}^{i*} \in \partial(-v_{i,j}^{i*} \circ h_i)(v_{i+1,j}^i),\ v_j^i \xrightarrow[j\in J]{\|\cdot\|_X} \overline{x},\ v_{k,j}^i \xrightarrow[j\in J]{\|\cdot\|_{Y_k}} \overline{y}_k,\ k=0,...,m, \\ \langle -v_{i,j}^{i*}, v_{i,j}^i - \overline{y}_i\rangle + \langle -v_{i+1,j}^{i*}, v_{i+1,j}^i - \overline{y}_{i+1}\rangle \xrightarrow[j\in J]{} 0, \end{cases}$$

and

$$\begin{cases} -v_{m,j}^{m*} \in K_m^*,\ \langle -v_{m,j}^{m*}, v_{m,j}^m - h_m(v_j^m)\rangle = 0, \\ v_j^{m*} \in \partial(-v_{m,j}^{m*} \circ h_m)(v_j^m),\ v_j^m \xrightarrow[j\in J]{\|\cdot\|_X} \overline{x},\ v_{k,j}^m \xrightarrow[j\in J]{\|\cdot\|_{Y_k}} \overline{y}_k,\ k=0,...,m, \\ \langle -v_j^{m*}, v_j^m - \overline{x}\rangle + \langle -v_{m,j}^{m*}, v_{m,j}^m - \overline{y}_m\rangle \xrightarrow[j\in J]{} 0, \end{cases}$$

with

$$\begin{cases} x_{k,j}^* = 0,\ k=0,...,m,\quad y_j^* = 0,\ y_{k,j}^* = 0,\ k=1,...,m, \\ w_{k,j}^* = 0,\ k=1,...,m,\quad u_j^* = 0,\ u_{0,j}^* = 0,\ u_{k,j}^* = 0,\ k=2,...,m, \\ v_j^{i*} = 0,\ v_{k,j}^{i*} = 0,\ k \in \{0,...,m\}\backslash\{i,i+1\},\ i=1,...,m-1, \\ v_{k,j}^{m*} = 0,\ k=0,...,m-1. \end{cases} \tag{17}$$

Taking (17) into account, then we deduce that

$$(16) \iff \begin{cases} x_j^* + w_j^* + v_j^{m*} \xrightarrow[j\in J]{w(X^*,X)} z^* \circ A,\ y_{0,j}^* + w_{0,j}^* \xrightarrow[j\in J]{w(Y_0^*,Y_0)} 0, \\ u_{1,j}^* + v_{1,j}^{1*} \xrightarrow[j\in J]{w(Y_1^*,Y_1)} 0,\ v_{i,j}^{(i-1)*} + v_{i,j}^{i*} \xrightarrow[j\in J]{w(Y_i^*,Y_i)} 0,\ i=2,...,m. \end{cases}$$

Therefore the proof is complete since $\{x_{k,j}\}_{j\in J}$, $k=0,...,m$, $\{y_j\}_{j\in J}$, $\{y_{k,j}\}_{j\in J}$, $k=1,...,m$, $\{w_{k,j}\}_{j\in J}$, $k=1,...,m$, $\{u_j\}_{j\in J}$, $\{u_{0,j}\}_{j\in J}$, $\{u_{k,j}\}_{j\in J}$, $k=2,...,m$, $\{v^i_j\}_{j\in J}$, $\{v^i_{k,j}\}_{j\in J}$, $k\in\{0,...,m\}\backslash\{i,i+1\}$, $i=1,...,m-1$, and $\{v^m_{k,j}\}_{j\in J}$, $k=0,...,m-1$, are superfluous nets. □

As a particular case of Theorem 3.5, we get the following corollary when $g\equiv 0$, $h_i\equiv 0$ and $K_0=Y_0$, ..., $K_m=Y_m$ (see the reflexive case in [10]).

Corollary 3.6. *Let $\overline{x}\in \operatorname{dom} f\cap\psi^{-1}(\operatorname{dom}\varphi)\cap\operatorname{dom}\psi$, $\overline{y}_0=\psi(\overline{x})$ and $\sigma\in\{w,p\}$. In addition we assume that Q is pointed as $\sigma=p$. Then, $A\in\partial^\sigma(f+\varphi\circ\psi)(\overline{x})$ if and only if there exist $z^*\in Q^\sigma$ and nets $\{x_j\}_{j\in J}\subseteq\operatorname{dom} f$, $\{y_{0,j}\}_{j\in J}\subseteq\operatorname{dom}\varphi$, $\{(w_j,w_{0,j})\}_{j\in J}\subseteq\operatorname{epi}\psi$, $\{x^*_j\}_{j\in J}\subseteq X^*$, $\{y^*_{0,j}\}_{j\in J}\subseteq Y^*_0$ and $\{(w^*_j,w^*_{0,j})\}_{j\in J}\subseteq X^*\times Y^*_0$ satisfying*

$$\begin{cases} x^*_j\in\partial(z^*\circ f)(x_j),\ x_j\xrightarrow[j\in J]{\|\cdot\|_X}\overline{x},\\ (z^*\circ f)(x_j)-(z^*\circ f)(\overline{x})-\langle x^*_j,x_j-\overline{x}\rangle\xrightarrow[j\in J]{}0,\end{cases}$$

$$\begin{cases} y^*_{0,j}\in\partial(z^*\circ\varphi)(y_{0,j}),\ y_{0,j}\xrightarrow[j\in J]{\|\cdot\|_{Y_0}}\overline{y}_0,\\ (z^*\circ\varphi)(y_{0,j})-(z^*\circ\varphi)(\overline{y}_0)-\langle y^*_{0,j},y_{0,j}-\overline{y}_0\rangle\xrightarrow[j\in J]{}0,\end{cases}$$

$$\begin{cases} -w^*_{0,j}\in K^*_0,\ \langle -w^*_{0,j},w_{0,j}-\psi(w_j)\rangle=0,\\ w^*_j\in\partial(-w^*_{0,j}\circ\psi)(w_j),\ w_j\xrightarrow[j\in J]{\|\cdot\|_X}\overline{x},\ w_{0,j}\xrightarrow[j\in J]{\|\cdot\|_{Y_0}}\overline{y}_0,\\ \langle -w^*_j,w_j-\overline{x}\rangle+\langle -w^*_{0,j},w_{0,j}-\overline{y}_0\rangle\xrightarrow[j\in J]{}0,\end{cases}$$

and
$$x^*_j+w^*_j\xrightarrow[j\in J]{w(X^*,X)}z^*\circ A,\quad y^*_{0,j}+w^*_{0,j}\xrightarrow[j\in J]{w(Y^*_0,Y_0)}0.$$

4. Applications

In this section, we present two multiobjective optimization problems to illustrate our main result. The first is a multiobjective minimax problem with infimal distances and the second is a multiobjective bilevel programming problem with extremal value function.

4.1. Sequential σ-efficient optimality conditions for multiobjective minimax problems with infimal distances

In order to illustrate the main result of this work, we propose deriving, in the absence of any qualification condition, sequential optimality conditions for σ-efficient solutions of a multiobjective minimax problem with infimal distances. To do this, we will first write the considered problem as a multi-composed vector optimization problem.

We consider the following multiobjective minimax problem with infimal distances treated (from a duality point of view) in [16]

$$(\mathcal{MMP})\qquad \text{v-}\min_{x\in X}\left(\max_{1\le i\le q} w^1_i d_{A_i}(x),...,\max_{1\le i\le q} w^\alpha_i d_{A_i}(x)\right)$$

where $A_1, ..., A_q$ are nonempty convex subsets of a reflexive Banach space X with $\cap_{i=1}^q \mathrm{cl}A_i = \emptyset$ (here $\mathrm{cl}A_i$ stands for the closure of A_i, $i = 1, ..., q$), $w_i^k > 0$, $i = 1, ..., q$, $k = 1, ..., \alpha$, are positive weights and $d_{A_1}, ..., d_{A_q} : X \to \mathbb{R}$ are distance functions defined by

$$d_{A_i}(x) := \inf_{a_i \in A_i} \gamma_i(x - a_i),\ \forall x \in X,\ i = 1, ..., q,$$

with $\gamma_1, ..., \gamma_q$ are continuous norms on X.

To apply our main result, we define $Z := \mathbb{R}^\alpha$, $Q := \mathbb{R}^\alpha_+$, $Y_2 := \mathbb{R}^q$, $K_2 := \mathbb{R}^q_+$, $Y_1 := (\mathbb{R}^q)^\alpha := \underbrace{\mathbb{R}^q \times ... \times \mathbb{R}^q}_{\alpha-times}$ and $K_1 := (\mathbb{R}^q_+)^\alpha := \underbrace{\mathbb{R}^q_+ \times ... \times \mathbb{R}^q_+}_{\alpha-times}$ with

$$\|(x_1^1, ..., x_q^1, ..., x_1^\alpha, ..., x_q^\alpha)\|_{(\mathbb{R}^q)^\alpha} := \sqrt{\sum_{k=1}^{\alpha} \|(x_1^k, ..., x_q^k)\|^2_{\mathbb{R}^q}}.$$

Besides, we introduce the following vector functions

- $g : (\mathbb{R}^q)^\alpha \to \mathbb{R}^\alpha$ defined by
  $g(x_1^1, ..., x_q^1, ..., x_1^\alpha, ..., x_q^\alpha) := \Big(l_1^+(x_1^1, ..., x_q^1), ..., l_\alpha^+(x_1^\alpha, ..., x_q^\alpha)\Big)$,
  where $l_k^+(x_1^k, ..., x_q^k) := l_k\Big((x_1^k)^+, ..., (x_q^k)^+\Big) := \max_{1 \le i \le q} \Big|(x_i^k)^+\Big|$, $k = 1, ..., \alpha$,
  $(x_i^k)^+ := \max\{0, x_i^k\}$, $i = 1, ..., q$, $k = 1, ..., \alpha$, $l_k^+(+\infty_{\mathbb{R}^q}) = +\infty$, $k = 1, ..., \alpha$,
- $h_1 : \mathbb{R}^q \to (\mathbb{R}^q)^\alpha \cup \{+\infty_{(\mathbb{R}^q)^\alpha}\}$ defined by
  $$h_1(x_1, ..., x_q) := \begin{cases} (w_1^1 x_1, ..., w_q^1 x_q, ..., w_1^\alpha x_1, ..., w_q^\alpha x_q), & \text{if } (x_1, ..., x_q) \in \mathbb{R}^q_+, \\ +\infty_{(\mathbb{R}^q)^\alpha}, & \text{otherwise}, \end{cases}$$
  and
- $h_2 : X \to \mathbb{R}^q$ defined by $h_2(x) := (d_{A_1}(x), ..., d_{A_q}(x))$, $x \in X$.

From here it follows that the problem $(\mathcal{MMP})$ can equivalently be written as a multi-composed vector optimization problem

$$(\mathcal{MMP}) \qquad \text{v-}\min_{x \in X} (g \circ h_1 \circ h_2)(x).$$

Clearly the map $g : (\mathbb{R}^q)^\alpha \to \mathbb{R}^\alpha$ is proper, $\mathbb{R}^\alpha_+$-convex, $((\mathbb{R}^q_+)^\alpha, \mathbb{R}^\alpha_+)$-nondecreasing on $\mathrm{dom}g = (\mathbb{R}^q)^\alpha$ and lower semicontinuous, since l_k^+, $k = 1, ..., \alpha$, are convex, continuous and $\mathbb{R}^q_+$-nondecreasing on $\mathbb{R}^q$ with $\mathrm{dom}l_k^+ = \mathbb{R}^q$, $k = 1, ..., \alpha$ (see [17]).

Moreover, it is easy to observe that $h_1 : \mathbb{R}^q \to (\mathbb{R}^q)^\alpha \cup \{+\infty_{(\mathbb{R}^q)^\alpha}\}$ is proper, $(\mathbb{R}^q_+)^\alpha$-convex, $(\mathbb{R}^q_+, (\mathbb{R}^q_+)^\alpha)$-nondecreasing on $\mathrm{dom}h_1 = \mathbb{R}^q_+$ and $(\mathbb{R}^q_+)^\alpha$-epi closed with $h_1(\mathrm{dom}h_1) \subseteq (\mathbb{R}^q_+)^\alpha \subseteq \mathrm{dom}g$. As $d_{A_1}, ..., d_{A_q}$ are convex and continuous with $\mathrm{dom}d_{A_i} = X$, $i = 1, ..., q$, it follows that $h_2 : X \to \mathbb{R}^q$ is proper, $\mathbb{R}^q_+$-convex and $\mathbb{R}^q_+$-epi closed with $\mathrm{dom}h_2 = X$ and $h_2(\mathrm{dom}(h_2)) \subseteq \mathbb{R}^q_+ = \mathrm{dom}h_1$.

Remark 4.1. Let us note that

- $$(\mathbb{R}^\alpha_+)^\sigma = \begin{cases} \mathbb{R}^\alpha_+ \setminus \{0\}, & \text{if } \sigma = w, \\ \mathrm{int}\,\mathbb{R}^\alpha_+, & \text{if } \sigma = p, \end{cases}$$

- $\mathrm{epi}h_1 = \mathcal{E}_1 := \big\{(x_1,...,x_q,y_1^1,...,y_q^1,...,y_1^\alpha,...,y_q^\alpha) \in \mathbb{R}_+^q \times (\mathbb{R}_+^q)^\alpha : w_i^k x_i \le y_i^k,\ i=1,...,q,\ k=1,...,\alpha\big\}$,
- $\mathrm{epi}h_2 = \mathcal{E}_2 := \big\{(x,y_1,...,y_q) \in X \times \mathbb{R}_+^q :\ d_{A_i}(x) \le y_i,\ i=1,...,q\big\}$.

The following lemma will be needed.

Lemma 4.2. (1) *Let* $(x_1^1,...,x_q^1,...,x_1^\alpha,...,x_q^\alpha) \in (\mathbb{R}^q)^\alpha$ *and* $(z_1^*,...,z_\alpha^*) \in (\mathbb{R}_+^\alpha)^\sigma$. *Then we have*

$$\partial l_k^+(x_1^k,...,x_q^k) = \left\{ (x_1^{k*},...,x_q^{k*}) \in \mathbb{R}_+^q : \begin{array}{c} \displaystyle\sum_{i=1}^q x_i^{k*} \le 1 \ \textit{and} \\ \displaystyle\max_{1\le i\le q}(x_i^k)^+ = \sum_{i=1}^q x_i^{k*} x_i^k \end{array} \right\},\ k=1,...,\alpha,$$

$$\textit{and} \quad \partial\Big((z_1^*,...,z_\alpha^*)\circ g\Big)(x_1^1,...,x_q^1,...,x_1^\alpha,...,x_q^\alpha) = \prod_{k=1}^\alpha z_k^* \partial l_k^+(x_1^k,...,x_q^k).$$

(2) *Let* $(x_1,...,x_q) \in \mathbb{R}_+^q$ *and* $(y_1^1,...,y_q^1,...,y_1^\alpha,...,y_q^\alpha) \in (\mathbb{R}_+^q)^\alpha$. *Then we have*

$$(x_1^*,...,x_q^*) \in \partial\Big((y_1^1,...,y_q^1,...,y_1^\alpha,...,y_q^\alpha)\circ h_1\Big)(x_1,...,x_q)$$

$$\Longleftrightarrow \quad x_i^* \in \mathcal{S}_1\Big(\sum_{k=1}^\alpha w_i^k y_i^k, x_i\Big) := \left\{ x\in\mathbb{R} : \begin{array}{c} \displaystyle x - \sum_{k=1}^\alpha w_i^k y_i^k \le 0 \ \textit{and} \\ \displaystyle x_i\Big(x - \sum_{k=1}^\alpha w_i^k y_i^k\Big) = 0 \end{array} \right\},\ i=1,...,q.$$

(3) *Let* $x \in X$ *and* $(y_1,...,y_q) \in \mathbb{R}_+^q$. *Then, we have*

$$\partial\Big((y_1,...,y_q)\circ h_2\Big)(x) = \mathcal{S}_2(y_1,...,y_q,x) := y_1\partial d_{A_1}(x) + ... + y_q \partial d_{A_q}(x).$$

Proof. (1) Let $(x_1^1,...,x_q^1,...,x_1^\alpha,...,x_q^\alpha) \in (\mathbb{R}^q)^\alpha$ and $(z_1^*,...,z_\alpha^*) \in (\mathbb{R}_+^\alpha)^\sigma$. To compute the subdifferential of l_k^+ at $(x_1^k,...,x_q^k) \in \mathbb{R}^q$, $k=1,...,\alpha$, it suffices to see by Proposition 4.2 in [17] that

$$\Big(l_k^+\Big)^*(x_1^{k*},...,x_q^{k*}) = \begin{cases} 0, \text{ if } (x_1^{k*},...,x_q^{k*}) \in \mathbb{R}_+^q \text{ and } \displaystyle\sum_{i=1}^q |x_i^{k*}| \le 1, \\ +\infty, \text{ otherwise,} \end{cases}$$

for all $(x_1^{k*},...,x_q^{k*}) \in \mathbb{R}^q$, $k=1,...,\alpha$. On the other hand, since l_k^+ is finite, convex and continuous at $(x_1^k,...,x_q^k) \in \mathbb{R}^q$, $k=1,...,\alpha$, and also the fact that

$$\Big((z_1^*,...,z_\alpha^*)\circ g\Big)(t_1^1,...,t_q^1,...,t_1^\alpha,...,t_q^\alpha) = \sum_{k=1}^\alpha z_k^* l_k^+(t_1^k,...,t_q^k),$$

for all $(t_1^1,...,t_q^1,...,t_1^\alpha,...,t_q^\alpha) \in (\mathbb{R}^q)^\alpha$.

It results by Corollary 2.4.5 in [19] that

$$\partial\Big((z_1^*,...,z_\alpha^*)\circ g\Big)(x_1^1,...,x_q^1,...,x_1^\alpha,...,x_q^\alpha) = \prod_{k=1}^{\alpha}\partial(z_k^* l_k^+)(x_1^k,...,x_q^k)$$
$$= \prod_{k=1}^{\alpha} z_k^*\partial l_k^+(x_1^k,...,x_q^k).$$

(2) Let $(x_1,...,x_q)\in\mathbb{R}_+^q$ and $(y_1^1,...,y_q^1,...,y_1^\alpha,...,y_q^\alpha)\in\mathbb{R}_+^q$. Then, we have

$$(x_1^*,...,x_q^*)\in\partial\Big((y_1^1,...,y_q^1,...,y_1^\alpha,...,y_q^\alpha)\circ h_1\Big)(x_1,...,x_q)$$
$$\Longleftrightarrow \quad \sum_{i=1}^{q} t_i\Big(\sum_{k=1}^{\alpha} w_i^k y_i^k\Big) \geq \sum_{i=1}^{q} x_i\Big(\sum_{k=1}^{\alpha} w_i^k y_i^k\Big)+\sum_{i=1}^{q} x_i^*(t_i-x_i),\ \forall(t_1,...,t_q)\in\mathbb{R}_+^q$$
$$\Longleftrightarrow \quad t_i\Big(\sum_{k=1}^{\alpha} w_i^k y_i^k\Big) \geq x_i\Big(\sum_{k=1}^{\alpha} w_i^k y_i^k\Big) + x_i^*(t_i-x_i),\ \forall t_i\geq 0,\ i=1,...,q$$
$$\Longleftrightarrow \quad x_i^* - \sum_{k=1}^{\alpha} w_i^k y_i^k \in N_{\mathbb{R}_+}(x_i),\ i=1,...,q$$
$$\Longleftrightarrow \quad x_i^* - \sum_{k=1}^{\alpha} w_i^k y_i^k \leq 0 \ \text{ and } \ x_i\Big(x_i^* - \sum_{k=1}^{\alpha} w_i^k y_i^k\Big) = 0,\ i=1,...,q.$$

(3) The proof is immediate because the finite functions $d_{A_1},...,d_{A_q}$ are convex and continuous on X. □

We are now ready to state sequential σ-efficient optimality conditions for the problem $(\mathcal{MMP})$.

Theorem 4.3. *$\overline{x}\in X$ is a σ-efficient solution of the problem $(\mathcal{MMP})$ if and only if there exist $(z_1^*,...,z_\alpha^*)\in(\mathbb{R}_+^\alpha)^\sigma$ and sequences $\{(x_{1,n}^k,...,x_{q,n}^k)\}_{n\in\mathbb{N}}\subseteq\mathbb{R}^q$, $k=1,...,\alpha$, $\{(t_{1,n},...,t_{q,n},r_{1,n}^1,...,r_{q,n}^1,...,r_{1,n}^\alpha,...,r_{q,n}^\alpha)\}_{n\in\mathbb{N}}\subseteq\mathcal{E}_1$, $\{(x_n,s_{1,n},...,s_{q,n})\}_{n\in\mathbb{N}}\subseteq\mathcal{E}_2$, $\{(x_{1,n}^{k*},...,x_{q,n}^{k*})\}_{n\in\mathbb{N}}\subseteq\mathbb{R}_+^q$, $k=1,...,\alpha$, $\{(t_{1,n}^*,...,t_{q,n}^*)\}_{n\in\mathbb{N}}\subseteq\mathbb{R}^q$, $\{(r_{1,n}^{k*},...,r_{q,n}^{k*})\}_{n\in\mathbb{N}}$ $\subseteq\mathbb{R}_+^q$, $k=1,...,\alpha$, $\{x_n^*\}_{n\in\mathbb{N}}\subseteq X^*$, $\{(s_{1,n}^*,...,s_{q,n}^*)\}_{n\in\mathbb{N}}\subseteq\mathbb{R}_+^q$ satisfying*

$$\begin{cases}\displaystyle\sum_{i=1}^{q} x_{i,n}^{k*}\leq 1, \max_{1\leq i\leq q}(x_{i,n}^k)^+ = \sum_{i=1}^{q} x_{i,n}^{k*}x_{i,n}^k,\ k=1,...,\alpha,\\ (x_{1,n}^k,...,x_{q,n}^k)\xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}}(w_1^k d_{A_1}(\overline{x}),...,w_q^k d_{A_q}(\overline{x})),\ k=1,...,\alpha,\\ z_k^*\Big(\displaystyle\sum_{i=1}^{q} x_{i,n}^{k*}w_i^k d_{A_i}(\overline{x})\Big)\xrightarrow[n\to+\infty]{} z_k^*\max_{1\leq i\leq q} w_i^k d_{A_i}(\overline{x}),\ k=1,...,\alpha,\end{cases}$$

$$\begin{cases} t_{i,n}^* \in \mathcal{S}_1\Big(\sum_{k=1}^{\alpha} w_i^k r_{i,n}^k, t_{i,n}\Big),\ i=1,...,q, \\ r_{i,n}^{k*}(r_{i,n}^k - w_i^k t_{i,n}) = 0,\ i=1,...,q,\ k=1,...,\alpha, \\ (t_{1,n},...,t_{q,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} (d_{A_1}(\overline{x}),...,d_{A_q}(\overline{x})), \\ (r_{1,n}^k,...,r_{q,n}^k) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} (w_1^k d_{A_1}(\overline{x}),...,w_q^k d_{A_q}(\overline{x})),\ k=1,...,\alpha, \\ \sum_{k=1}^{\alpha}\sum_{i=1}^{q} r_{i,n}^{k*}(r_{i,n}^k - w_i^k d_{A_i}(\overline{x})) - \sum_{i=1}^{q} t_{i,n}^*(t_{i,n} - d_{A_i}(\overline{x})) \xrightarrow[n\to+\infty]{} 0, \end{cases}$$

$$\begin{cases} x_n^* \in \mathcal{S}_2(s_{1,n}^*,...,s_{q,n}^*, x_n), \\ s_{i,n}^*(s_{i,n} - d_{A_i}(x_n)) = 0,\ i=1,...,q, \\ x_n \xrightarrow[n\to+\infty]{\|\cdot\|_X} \overline{x},\ (s_{1,n},...,s_{q,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} (d_{A_1}(\overline{x}),...,d_{A_q}(\overline{x})), \\ \sum_{i=1}^{q} s_{i,n}^*(s_{i,n} - d_{A_i}(\overline{x})) - \langle x_n^*, x_n - \overline{x}\rangle \xrightarrow[n\to+\infty]{} 0, \end{cases}$$

and

$$\begin{cases} x_n^* \xrightarrow[n\to+\infty]{\|\cdot\|_{X^*}} 0, \\ (z_k^* x_{1,n}^{k*} - r_{1,n}^{k*},...,z_k^* x_{q,n}^{k*} - r_{q,n}^{k*}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} (0,...,0),\ k=1,...,\alpha, \\ (t_{1,n}^* - s_{1,n}^*,...,t_{q,n}^* - s_{q,n}^*) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} (0,...,0). \end{cases}$$

Proof. Clearly $\overline{x} \in X$ is a σ-efficient solution of the problem $(\mathcal{MMP})$ if and only if $0 \in \partial^\sigma(g \circ h_1 \circ h_2)(\overline{x})$. Therefore, by applying Theorem 3.5 and also Lemma 4.2, we assert that there exist $(z_1^*,...,z_\alpha^*) \in (\mathbb{R}_+^\alpha)^\sigma$ and sequences

$$\begin{aligned} &\{(x_{1,n}^1,...,x_{q,n}^1,...,x_{1,n}^\alpha,...,x_{q,n}^\alpha)\}_{n\in\mathbb{N}} \subseteq \mathrm{dom} g = (\mathbb{R}^q)^\alpha, \\ &\{(t_{1,n},...,t_{q,n}, r_{1,n}^1,...,r_{q,n}^1,...,r_{1,n}^\alpha,...,r_{q,n}^\alpha)\}_{n\in\mathbb{N}} \subseteq \mathrm{epi} h_1 = \mathcal{E}_1, \\ &\{(x_n, s_{1,n},...,s_{q,n})\}_{n\in\mathbb{N}} \subseteq \mathrm{epi} h_2 = \mathcal{E}_2, \\ &\{(x_{1,n}^{1*},...,x_{q,n}^{1*},...,x_{1,n}^{\alpha*},...,x_{q,n}^{\alpha*})\}_{n\in\mathbb{N}} \subseteq (\mathbb{R}_+^q)^\alpha, \\ &\{(t_{1,n}^*,...,t_{q,n}^*)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^q, \quad \{(r_{1,n}^{1*},...,r_{q,n}^{1*},...,r_{1,n}^{\alpha*},...,r_{q,n}^{\alpha*})\}_{n\in\mathbb{N}} \subseteq (\mathbb{R}_+^q)^\alpha, \\ &\{x_n^*\}_{n\in\mathbb{N}} \subseteq X^*, \{(s_{1,n}^*,...,s_{q,n}^*)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}_+^q \end{aligned}$$

satisfying

$$\begin{cases} (x_{1,n}^{k*},...,x_{q,n}^{k*}) \in \partial l_k^+(x_{1,n}^k,...,x_{q,n}^k),\ k=1,...,\alpha, \\ (x_{1,n}^k,...,x_{q,n}^k) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} (w_1^k d_{A_1}(\overline{x}),...,w_q^k d_{A_q}(\overline{x})),\ k=1,...,\alpha, \\ \sum_{k=1}^{\alpha} z_k^* l_k^+(x_{1,n}^k,...,x_{q,n}^k) - \sum_{k=1}^{\alpha} z_k^* l_k^+(w_1^k d_{A_1}(\overline{x}),...,w_q^k d_{A_q}(\overline{x})) \\ \qquad\qquad - \sum_{k=1}^{\alpha} z_k^*\Big(\sum_{i=1}^{q} x_{i,n}^{k*}(x_{i,n}^k - w_i^k d_{A_i}(\overline{x}))\Big) \xrightarrow[n\to+\infty]{} 0, \end{cases}$$

$$\begin{cases} t^*_{i,n} \in \mathcal{S}_1\Big(\sum_{k=1}^{\alpha} w_i^k r^k_{i,n}, t_{i,n}\Big),\ i=1,...,q, \\ \sum_{k=1}^{\alpha}\sum_{i=1}^{q} r^{k*}_{i,n}(r^k_{i,n} - w_i^k t_{i,n}) = 0, \\ (t_{1,n},...,t_{q,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} (d_{A_1}(\overline{x}),...,d_{A_q}(\overline{x})), \\ (r^k_{1,n},...,r^k_{q,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} (w_1^k d_{A_1}(\overline{x}),...,w_q^k d_{A_q}(\overline{x})),\ k=1,...,\alpha, \\ \sum_{k=1}^{\alpha}\sum_{i=1}^{q} r^{k*}_{i,n}(r^k_{i,n} - w_i^k d_{A_i}(\overline{x})) - \sum_{i=1}^{q} t^*_{i,n}(t_{i,n} - d_{A_i}(\overline{x})) \xrightarrow[n\to+\infty]{} 0, \end{cases}$$

$$\begin{cases} x^*_n \in \mathcal{S}_2(s^*_{1,n},...,s^*_{q,n},x_n),\ \sum_{i=1}^{q} s^*_{i,n}(s_{i,n} - d_{A_i}(x_n)) = 0, \\ x_n \xrightarrow[n\to+\infty]{\|\cdot\|_X} \overline{x},\ (s_{1,n},...,s_{q,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} (d_{A_1}(\overline{x}),...,d_{A_q}(\overline{x})), \\ \sum_{i=1}^{q} s^*_{i,n}(s_{i,n} - d_{A_i}(\overline{x})) - \langle x^*_n, x_n - \overline{x}\rangle \xrightarrow[n\to+\infty]{} 0, \end{cases}$$

and

$$\begin{cases} x^*_n \xrightarrow[n\to+\infty]{\|\cdot\|_{X^*}} 0,\ (z^*_k x^{k*}_{1,n} - r^{k*}_{1,n},...,z^*_k x^{k*}_{q,n} - r^{k*}_{q,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} (0,...,0),\ k=1,...,\alpha, \\ (t^*_{1,n} - s^*_{1,n},...,t^*_{q,n} - s^*_{q,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} (0,...,0). \end{cases}$$

By Lemma 4.2, it is clear that

$$\begin{aligned} & (x^{k*}_{1,n},...,x^{k*}_{q,n}) \in \partial l_k^+(x^k_{1,n},...,x^k_{q,n}),\ k=1,...,\alpha \\ \iff \quad & \sum_{i=1}^{q} x^{k*}_{i,n} \le 1,\ \max_{1\le i\le q}(x^k_{i,n})^+ = \sum_{i=1}^{q} x^{k*}_{i,n} x^k_{i,n},\ k=1,...,\alpha. \end{aligned}$$

As $(x^{k*}_{1,n},...,x^{k*}_{q,n}) \in \partial l_k^+(x^k_{1,n},...,x^k_{q,n})$, $k=1,...,\alpha$ and $(z^*_1,...,z^*_\alpha) \in (\mathbb{R}^\alpha_+)^\sigma$ we have

$$\begin{aligned} & z^*_k l_k^+(w_1^k d_{A_1}(\overline{x}),...,w_q^k d_{A_q}(\overline{x})) \\ & \qquad \ge z^*_k l_k^+(x^k_{1,n},...,x^k_{q,n}) + z^*_k\Big(\sum_{i=1}^{q} x^{k*}_{i,n}(w_i^k d_{A_i}(\overline{x}) - x^k_{i,n})\Big),\ k=1,...,\alpha,\ n\in\mathbb{N}. \end{aligned}$$

Therefore, from these inequalities, we obtain

$$\begin{aligned} & \sum_{k=1}^{\alpha} z^*_k l_k^+(x^k_{1,n},...,x^k_{q,n}) - \sum_{k=1}^{\alpha} z^*_k l_k^+(w_1^k d_{A_1}(\overline{x}),...,w_q^k d_{A_q}(\overline{x})) \\ & \qquad - \sum_{k=1}^{\alpha} z^*_k\Big(\sum_{i=1}^{q} x^{k*}_{i,n}(x^k_{i,n} - w_i^k d_{A_i}(\overline{x}))\Big) \xrightarrow[n\to+\infty]{} 0 \\ \iff \quad & z^*_k l_k^+(x^k_{1,n},...,x^k_{q,n}) - z^*_k l_k^+(w_1^k d_{A_1}(\overline{x}),...,w_q^k d_{A_q}(\overline{x})) \\ & \qquad - z^*_k\Big(\sum_{i=1}^{q} x^{k*}_{i,n}(x^k_{i,n} - w_i^k d_{A_i}(\overline{x}))\Big) \xrightarrow[n\to+\infty]{} 0,\ k=1,...,\alpha. \end{aligned} \tag{18}$$

Since $l_k^+(x_{1,n}^k, ..., x_{q,n}^k) = \max_{1\leq i\leq q}(x_{i,n}^k)^+ = \sum_{i=1}^{q} x_{i,n}^{k*}x_{i,n}^k,\ k = 1, ..., \alpha,\ n\in\mathbb{N}$, we deduce

$$(18) \iff z_k^*\Big(\sum_{i=1}^{q} x_{i,n}^{k*}w_i^k d_{A_i}(\overline{x})\Big) \xrightarrow[n\to+\infty]{} z_k^* \max_{1\leq i\leq q} w_i^k d_{A_i}(\overline{x}),\ k = 1, ..., \alpha.$$

Finally, since $r_{i,n}^{k*} \geq 0,\ r_{i,n}^k - w_i^k t_{i,n} \geq 0,\ s_{i,n}^* \geq 0,\ s_{i,n} - d_{A_i}(x_n) \geq 0,\ i = 1, ..., q,$ $k = 1, ..., \alpha$, we see easily that

$$\sum_{k=1}^{\alpha}\sum_{i=1}^{q} r_{i,n}^{k*}(r_{i,n}^k - w_i^k t_{i,n}) = 0$$

$$\iff \qquad r_{i,n}^{k*}(r_{i,n}^k - w_i^k t_{i,n}) = 0,\ i = 1, ..., q,\ k = 1, ..., \alpha,$$

and $\displaystyle\sum_{i=1}^{q} s_{i,n}^*(s_{i,n} - d_{A_i}(x_n)) = 0 \iff s_{i,n}^*(s_{i,n} - d_{A_i}(x_n)) = 0,\ i = 1, ..., q.$ □

4.2. Sequential σ-efficient optimality conditions for multiobjective bilevel programming problems with extremal value function

In this subsection, we present an example of multiobjective programming problems where the standard Lagrange multipliers conditions can not be derived due to the lack of constraint qualification and the sequential conditions hold. For this, let us consider the following multiobjective bilevel programming problem with an extremal value function

$$(\mathcal{MBP}) \qquad \text{v-}\min_{x\in\mathcal{A}} \Big(f_1(x, v(x)), ..., f_q(x, v(x))\Big)$$

where $\mathcal{A} := \{x \in A : g_i(x, v(x)) \leq 0,\ i = 1, ..., s\} \neq \emptyset$, $v(x)$ is the optimal value function of the following problem parametrized by x

$$(\mathcal{P}_x) \qquad \min_{y\in B} \mathcal{F}(x, y),$$

A is a nonempty closed convex subset of $\mathbb{R}^m$, B is a nonempty compact convex subset of $\mathbb{R}^d$, $\mathcal{F}\colon \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ is a convex function, $f_i\colon \mathbb{R}^{m+1} \to \mathbb{R}$ are convex and $\mathbb{R}_+^{m+1}$-nondecreasing functions, $i = 1, ..., q$, $g_i\colon \mathbb{R}^{m+1} \to \mathbb{R}$ are convex and $\mathbb{R}_+^{m+1}$-nondecreasing functions, $i = 1, ..., s$.

We note that the functions $f_1, ..., f_q$ and $g_1, ..., g_s$ are all continuous. Also, we point out that the function $v\colon \mathbb{R}^m \to \mathbb{R}$ is finite, convex and continuous (see [1, Remark 3.1]). Moreover, one can see that for each $x \in \mathbb{R}^m$, there exists $y \in B$ such that $v(x) = \mathcal{F}(x, y)$.

In the sequel, we derive sequential optimality conditions characterizing σ-efficient solutions of the problem $(\mathcal{MBP})$ via Theorem 3.1. For this aim, we set $Z := \mathbb{R}^q$, $Q := \mathbb{R}_+^q$, $X = \mathbb{R}^m$, $Y_2 := \mathbb{R}^{m+1}$, $K_2 := \mathbb{R}_+^{m+1}$, $Y_1 := \mathbb{R}^s$, $K_1 := \mathbb{R}_+^s$, $Y_0 := \mathbb{R}^{m+1}$, $K_0 := \mathbb{R}_+^{m+1}$ and consider the following functions

- $F : \mathbb{R}^{m+1} \to \mathbb{R}^q$ defined by
$$F(x, y) := (f_1(x, y), ..., f_q(x, y)),\ (x, y) \in \mathbb{R}^m \times \mathbb{R},$$
- $G : \mathbb{R}^{m+1} \to \mathbb{R}^s$ defined by
$$G(x, y) := (g_1(x, y), ..., g_s(x, y)),\ (x, y) \in \mathbb{R}^m \times \mathbb{R},$$

- $V : \mathbb{R}^m \to \mathbb{R}^{m+1}$ defined by $V(x) = (x, v(x)),\ x \in \mathbb{R}^m$.

With these auxiliary functions we can write $(\mathcal{MBP})$ as a multi-composed convex vector optimization problem

$$(\mathcal{MBP}) \qquad \text{v-}\min_{x\in\mathbb{R}^m}\Big\{\delta^v_A(x) + (F\circ V)(x) + (\delta^v_{-\mathbb{R}^s_+}\circ G\circ V)(x)\Big\}. \tag{19}$$

Remark 4.4. It is easy to see that

- $F\colon \mathbb{R}^{m+1} \to \mathbb{R}^q$ is proper, $\mathbb{R}^q_+$-convex, $\mathbb{R}^q_+$-epi closed and $(\mathbb{R}^{m+1}_+, \mathbb{R}^q_+)$-nondecreasing on $\mathbb{R}^{m+1}$,
- $G\colon \mathbb{R}^{m+1} \to \mathbb{R}^s$ is proper, $\mathbb{R}^s_+$-convex, $\mathbb{R}^s_+$-epi closed and $(\mathbb{R}^{m+1}_+, \mathbb{R}^s_+)$-nondecreasing on $\mathbb{R}^{m+1}$ with
$$\text{epi}G = \mathbb{E} := \Big\{(x, r_1, ..., r_s) \in \mathbb{R}^{m+1}\times\mathbb{R}^s : g_i(x) \le r_i, i = 1, ..., s\Big\},$$
- $V\colon \mathbb{R}^m \to \mathbb{R}^{m+1}$ is proper, $\mathbb{R}^{m+1}_+$-convex, $\mathbb{R}^{m+1}_+$-epi closed and $V(\text{dom}V) \subseteq \mathbb{R}^{m+1}$ with
$$\text{epi}V = \mathbb{E}_v := \Big\{(x, y, r) \in \mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R} : x \leqq_{\mathbb{R}^m_+} y \text{ and } v(x) \le r\Big\},$$
- $\delta^v_A\colon \mathbb{R}^m \to \mathbb{R}^q$ and $\delta^v_{-\mathbb{R}^s_+} : \mathbb{R}^s \to \mathbb{R}^q$ are proper, $\mathbb{R}^q_+$-convex and lower semicontinuous with $\delta^v_{-\mathbb{R}^s_+}$ is $(\mathbb{R}^s_+, \mathbb{R}^q_+)$-nondecreasing on $\mathbb{R}^s$ (see [7]).

Lemma 4.5. *Let $(\overline{x}, \overline{y}) \in \mathbb{R}^m\times B$ such that $v(\overline{x}) = \mathcal{F}(\overline{x}, \overline{y})$. Then for any $y^* \in \mathbb{R}^m_+$ and $t \ge 0$ we have*
$$\partial\Big((y^*, t)\circ V\Big)(\overline{x}) = \{y^*\} + \mathcal{S}_t(\overline{x}, \overline{y})$$
where $\mathcal{S}_t(\overline{x}, \overline{y}) := \begin{cases}\Big\{z^* \in \mathbb{R}^m : (z^*, 0) \in t\partial\mathcal{F}(\overline{x}, \overline{y}) + \{0\}\times N_B(\overline{y})\Big\}, & \textit{if } t > 0,\\ \{0\}, & \textit{if } t = 0.\end{cases}$

Proof. Let $id_{\mathbb{R}^m} : \mathbb{R}^m \to \mathbb{R}^m$ be the identity function (i.e. $id_{\mathbb{R}^m}(x) = x$, for all $x \in \mathbb{R}^m$) and $(\overline{x}, \overline{y}) \in \mathbb{R}^m\times B$ such that $v(\overline{x}) = \mathcal{F}(\overline{x}, \overline{y})$. Since $id_{\mathbb{R}^m}$ and v are finite, convex and continuous, then for any $y^* \in \mathbb{R}^m_+$ and $t \ge 0$ we have
$$\begin{aligned}\partial\Big((y^*, t)\circ V\Big)(\overline{x}) &= \partial(y^*\circ id_{\mathbb{R}^m} + tv)(\overline{x})\\ &= \partial(y^*\circ id_{\mathbb{R}^m})(\overline{x}) + \partial(tv)(\overline{x}) = \{y^*\} + \partial(tv)(\overline{x}).\end{aligned}$$
For $t = 0$ we have $\partial(0.v)(\overline{x}) = \{0\}$.
If $t > 0$ then it follows that (see [2])
$$\begin{aligned}z^* \in \partial(tv)(\overline{x}) &\iff (z^*, 0) \in \partial(t\mathcal{F} + \delta_{(\mathbb{R}^m\times B)})(\overline{x}, \overline{y})\\ &\iff (z^*, 0) \in \partial(t\mathcal{F})(\overline{x}, \overline{y}) + \partial\delta_{(\mathbb{R}^m\times B)}(\overline{x}, \overline{y})\\ &\iff (z^*, 0) \in t\partial\mathcal{F}(\overline{x}, \overline{y}) + N_{(\mathbb{R}^m\times B)}(\overline{x}, \overline{y})\\ &\iff (z^*, 0) \in t\partial\mathcal{F}(\overline{x}, \overline{y}) + N_{\mathbb{R}^m}(\overline{x})\times N_B(\overline{y})\\ &\iff (z^*, 0) \in t\partial\mathcal{F}(\overline{x}, \overline{y}) + \{0\}\times N_B(\overline{y}).\end{aligned}$$
Let us mention here that $\partial(t\mathcal{F} + \delta_{(\mathbb{R}^m\times B)})(\overline{x}, \overline{y}) = \partial(t\mathcal{F})(\overline{x}, \overline{y}) + \partial\delta_{(\mathbb{R}^m\times B)}(\overline{x}, \overline{y})$ since $(\text{int}\,\text{dom}\mathcal{F}) \cap(\mathbb{R}^m\times B) \neq \emptyset$. Hence the proof is complete. □

Now, we are able to state the main result of this subsection.

Theorem 4.6. *Assume* $\overline{x} := (\overline{x}_1, ..., \overline{x}_m) \in \mathcal{A}$, $\overline{y} := (\overline{x}, v(\overline{x})) \in \mathbb{R}^{m+1}$ *and finally* $\overline{z} := (g_1(\overline{y}), ..., g_s(\overline{y})) \in \mathbb{R}^s$. *Then* $\overline{x}$ *is a* σ*-efficient solution of* $(\mathcal{MBP})$ *if and only if there exist* $(z_1^*, ..., z_q^*) \in (\mathbb{R}^q_+)^\sigma$ *and sequences*

$$\{(x_{1,n}, ..., x_{m,n}, y_{1,n}, ..., y_{m+1,n})\}_{n\in\mathbb{N}} \subseteq \mathbb{E}_v, \ \ \{b_n\}_{n\in\mathbb{N}} \subseteq B,$$
$$\mathcal{F}(x_{1,n}, ..., x_{m,n}, b_n) = v(x_{1,n}, ..., x_{m,n}), \ \ \{(x^*_{1,n}, ..., x^*_{m,n})\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^m,$$
$$\{(y^*_{1,n}, ..., y^*_{m+1,n})\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^{m+1}_+, \ \ \{(\alpha_{1,n}, ..., \alpha_{s,n})\}_{n\in\mathbb{N}} \subseteq -\mathbb{R}^s_+,$$
$$\{(\alpha^*_{1,n}, ..., \alpha^*_{s,n})\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^s_+, \ \ \{(r_{1,n}, ..., r_{m+1,n}, t_{1,n}, ..., t_{s,n})\}_{n\in\mathbb{N}} \subseteq \mathbb{E},$$
$$\{(r^*_{1,n}, ..., r^*_{m+1,n})\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^{m+1}, \ \ \{(t^*_{1,n}, ..., t^*_{s,n})\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^s_+,$$
$$\{(w_{1,n}, ..., w_{m,n})\}_{n\in\mathbb{N}} \subseteq A, \ \{(w^*_{1,n}, ..., w^*_{m,n})\ \}_{n\in\mathbb{N}} \subseteq \mathbb{R}^m,$$
$$\{(u_{1,n}, ..., u_{m+1,n})\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^{m+1}, \ \ \{(u^*_{1,n}, ..., u^*_{m+1,n})\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^{m+1}$$

satisfying

$$\begin{cases} (w^*_{1,n}, ..., w^*_{m,n}) \in N_A(w_{1,n}, ..., w_{m,n}), \\ (w_{1,n}, ..., w_{m,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^m}} \overline{x}, \quad \displaystyle\sum_{i=1}^m w^*_{i,n}(w_{i,n} - \overline{x}_i) \xrightarrow[n\to+\infty]{} 0, \end{cases}$$

$$\begin{cases} (u^*_{1,n}, ..., u^*_{m+1,n}) \in \partial\Big(\displaystyle\sum_{i=1}^q z_i^* f_i\Big)(u_{1,n}, ..., u_{m+1,n}), \\ (u_{1,n}, ..., u_{m+1,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} \overline{y}, \\ \displaystyle\sum_{i=1}^q (z_i^* f_i)(u_{1,n}, ..., u_{m+1,n}) - \sum_{i=1}^q (z_i^* f_i)(\overline{x}, v(\overline{x})) \\ \qquad\qquad - \displaystyle\sum_{i=1}^m u^*_{i,n}(u_{i,n} - \overline{x}_i) - u^*_{m+1,n}(u_{m+1,n} - v(\overline{x})) \xrightarrow[n\to+\infty]{} 0, \end{cases}$$

$$\begin{cases} (x^*_{1,n} - y^*_{1,n}, ..., x^*_{m,n} - y^*_{m,n}) \in \mathcal{S}_{y^*_{m+1,n}}(x_{1,n}, ..., x_{m,n}, b_n), \\ \displaystyle\sum_{i=1}^m y^*_{i,n}(y_{i,n} - x_{i,n}) + y^*_{m+1,n}(y_{m+1,n} - v(x_{1,n}, ..., x_{m,n})) = 0, \\ (x_{1,n}, ..., x_{m,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^m}} \overline{x}, \ (y_{1,n}, ..., y_{m+1,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} \overline{y}, \\ -\displaystyle\sum_{i=1}^m x^*_{i,n}(x_{i,n} - \overline{x}_i) + \sum_{i=1}^m y^*_{i,n}(y_{i,n} - \overline{x}_i) + y^*_{m+1,n}(y_{m+1,n} - v(\overline{x})) \xrightarrow[n\to+\infty]{} 0, \end{cases}$$

$$\begin{cases} \displaystyle\sum_{i=1}^s \alpha^*_{i,n}\alpha_{i,n} = 0, \quad (\alpha_{1,n}, ..., \alpha_{s,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^s}} \overline{z}, \\ \displaystyle\sum_{i=1}^s \alpha^*_{i,n}(\alpha_{i,n} - g_i(\overline{y})) \xrightarrow[n\to+\infty]{} 0, \end{cases}$$

$$\begin{cases}(r^*_{1,n},...,r^*_{m+1,n}) \in \partial\Big(\sum_{i=1}^{s} t^*_{i,n} g_i\Big)(r_{1,n},...,r_{m+1,n}),\\ \sum_{i=1}^{s} t^*_{i,n}(t_{i,n} - g_i(r_{1,n},...,r_{m+1,n})) = 0,\\ (r_{1,n},...,r_{m+1,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} \overline{y}, \quad (t_{1,n},...,t_{s,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{s}}} \overline{z},\\ \sum_{i=1}^{s} t^*_{i,n}(t_{i,n} - g_i(\overline{y})) - \sum_{i=1}^{m} r^*_{i,n}(r_{i,n} - \overline{x}_i) - r^*_{m+1,n}(r_{m+1,n} - v(\overline{x})) \xrightarrow[n\to+\infty]{} 0,\end{cases}$$

and

$$\begin{cases}(w^*_{1,n} + 2x^*_{1,n},...,w^*_{m,n} + 2x^*_{m,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m}}} (0,...,0),\\ (u^*_{1,n} - y^*_{1,n},...,u^*_{m+1,n} - y^*_{m+1,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} (0,...,0),\\ (\alpha^*_{1,n} - t^*_{1,n},...,\alpha^*_{s,n} - t^*_{s,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{s}}} (0,...,0),\\ (r^*_{1,n} - y^*_{1,n},...,r^*_{m+1,n} - y^*_{m+1,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} (0,...,0).\end{cases}$$

Proof. Assume that $\overline{x} := (\overline{x}_1,...,\overline{x}_m) \in \mathcal{A}$ and define $\overline{y} := (\overline{x}, v(\overline{x})) \in \mathbb{R}^{m+1}$ and $\overline{z} := (g_1(\overline{y}),...,g_s(\overline{y})) \in \mathbb{R}^s$. By (19), it follows that $\overline{x}$ is a σ-efficient solution of the problem $(\mathcal{MBP})$ if and only if $0 \in \partial^\sigma(\delta^v_A + F\circ V + \delta^v_{-\mathbb{R}^s_+}\circ G\circ V)(\overline{x})$. Therefore, by taking into account Remark 4.4 and Remark 3.1 and by applying Theorem 3.5 to $f := \delta^v_A$, $\varphi := F$, $g := \delta^v_{-\mathbb{R}^s_+}$, $h_1 := G$, $\psi = h_2 := V$, one can assert that there exist $(z^*_1,...,z^*_q) \in (\mathbb{R}^q_+)^\sigma$ and sequences $\{(x_{1,n},...,x_{m,n},y_{1,n},...,y_{m+1,n})\}_{n\in\mathbb{N}} \subseteq \mathrm{epi}V = \mathbb{E}_v$, $\{b_n\}_{n\in\mathbb{N}} \subseteq B$, $\mathcal{F}(x_{1,n},...,x_{m,n},b_n) = v(x_{1,n},...,x_{m,n})$, $\{(x^*_{1,n},...,x^*_{m,n})\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^m$, $\{(y^*_{1,n},...,y^*_{m+1,n})\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^{m+1}_+$, $\{(\alpha_{1,n},...,\alpha_{s,n})\}_{n\in\mathbb{N}} \subseteq \mathrm{dom}\delta^v_{-\mathbb{R}^s_+} = -\mathbb{R}^s_+$, $\{(\alpha^*_{1,n},...,\alpha^*_{s,n})\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^s$, $\{(r_{1,n},...,r_{m+1,n},t_{1,n},...,t_{s,n})\}_{n\in\mathbb{N}} \subseteq \mathrm{epi}G = \mathbb{E}$, $\{(r^*_{1,n},...,r^*_{m+1,n})\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^{m+1}$, $\{(t^*_{1,n},...,t^*_{s,n})\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^s_+$, $\{(w_{1,n},...,w_{m,n})\}_{n\in\mathbb{N}} \subseteq \mathrm{dom}\delta^v_A = A$, $\{(w^*_{1,n},...,w^*_{m,n})\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^m$, $\{(u_{1,n},...,u_{m+1,n})\}_{n\in\mathbb{N}} \subseteq \mathrm{dom}F = \mathbb{R}^{m+1}$, $\{(u^*_{1,n},...,u^*_{m+1,n})\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^{m+1}$ satisfying

$$\begin{cases}(w^*_{1,n},...,w^*_{m,n}) \in \partial\Big((z^*_1,...,z^*_q)\circ\delta^v_A\Big)(w_{1,n},...,w_{m,n}),\\ (w_{1,n},...,w_{m,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m}}} \overline{x}, \quad \sum_{i=1}^{m} w^*_{i,n}(w_{i,n} - \overline{x}_i) \xrightarrow[n\to+\infty]{} 0,\end{cases} \tag{20}$$

$$\begin{cases}(u^*_{1,n},...,u^*_{m+1,n}) \in \partial\Big(\sum_{i=1}^{q} z^*_i f_i\Big)(u_{1,n},...,u_{m+1,n}),\\ (u_{1,n},...,u_{m+1,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} \overline{y},\\ \sum_{i=1}^{q}(z^*_i f_i)(u_{1,n},...,u_{m+1,n}) - \sum_{i=1}^{q}(z^*_i f_i)(\overline{x}, v(\overline{x}))\\ \qquad - \sum_{i=1}^{m} u^*_{i,n}(u_{i,n} - \overline{x}_i) - u^*_{m+1,n}(u_{m+1,n} - v(\overline{x})) \xrightarrow[n\to+\infty]{} 0,\end{cases}$$

$$\begin{cases} (x^*_{1,n}, ..., x^*_{m,n}) \in \partial\Big((y^*_{1,n}, ..., y^*_{m+1,n}) \circ V\Big)(x_{1,n}, ..., x_{m,n}), \\ \displaystyle\sum_{i=1}^{m} y^*_{i,n}(y_{i,n} - x_{i,n}) + y^*_{m+1,n}(y_{m+1,n} - v(x_{1,n}, ..., x_{m,n})) = 0, \\ (x_{1,n}, ..., x_{m,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^m}} \overline{x}, \quad (y_{1,n}, ..., y_{m+1,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} \overline{y}, \\ \displaystyle -\sum_{i=1}^{m} x^*_{i,n}(x_{i,n}-\overline{x}_i) + \sum_{i=1}^{m} y^*_{i,n}(y_{i,n}-\overline{x}_i) + y^*_{m+1,n}(y_{m+1,n}-v(\overline{x})) \xrightarrow[n\to+\infty]{} 0, \end{cases} \tag{21}$$

$$\begin{cases} (\alpha^*_{1,n}, ..., \alpha^*_{s,n}) \in \partial\Big((z^*_1, ..., z^*_q) \circ \delta^v_{-\mathbb{R}^s_+}\Big)(\alpha_{1,n}, ..., \alpha_{s,n}), \\ (\alpha_{1,n}, ..., \alpha_{s,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^s}} \overline{z}, \quad \displaystyle\sum_{i=1}^{s} \alpha^*_{i,n}(\alpha_{i,n} - g_i(\overline{y})) \xrightarrow[n\to+\infty]{} 0, \end{cases} \tag{22}$$

$$\begin{cases} (r^*_{1,n}, ..., r^*_{m+1,n}) \in \partial\Big(\displaystyle\sum_{i=1}^{s} t^*_{i,n} g_i\Big)(r_{1,n}, ..., r_{m+1,n}), \\ \displaystyle\sum_{i=1}^{s} t^*_{i,n}(t_{i,n} - g_i(r_{1,n}, ..., r_{m+1,n})) = 0, \\ (r_{1,n}, ..., r_{m+1,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} \overline{y}, \ (t_{1,n}, ..., t_{s,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^s}} \overline{z}, \\ \displaystyle\sum_{i=1}^{s} t^*_{i,n}(t_{i,n} - g_i(\overline{y})) - \sum_{i=1}^{m} r^*_{i,n}(r_{i,n} - \overline{x}_i) - r^*_{m+1,n}(r_{m+1,n} - v(\overline{x})) \xrightarrow[n\to+\infty]{} 0, \end{cases}$$

and

$$\begin{cases} (w^*_{1,n} + 2x^*_{1,n}, ..., w^*_{m,n} + 2x^*_{m,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^m}} (0, ..., 0), \\ (u^*_{1,n} - y^*_{1,n}, ..., u^*_{m+1,n} - y^*_{m+1,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} (0, ..., 0), \\ (\alpha^*_{1,n} - t^*_{1,n}, ..., \alpha^*_{s,n} - t^*_{s,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^s}} (0, ..., 0), \\ (r^*_{1,n} - y^*_{1,n}, ..., r^*_{m+1,n} - y^*_{m+1,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^{m+1}}} (0, ..., 0). \end{cases}$$

Since $(z^*_1, ..., z^*_q) \circ \delta^v_A = \delta_A$ and $\partial(\delta_A)(w_{1,n}, ..., w_{m,n}) = N_A(w_{1,n}, ..., w_{m,n})$, we have

$$(20) \iff (w^*_{1,n}, ..., w^*_{m,n}) \in N_A(w_{1,n}, ..., w_{m,n}).$$

By Lemma 4.5, it follows that

$$(21) \iff (x^*_{1,n} - y^*_{1,n}, ..., x^*_{m,n} - y^*_{m,n}) \in \mathcal{S}_{y^*_{m+1,n}}(x_{1,n}, ..., x_{m,n}, b_n).$$

Finally, since $\mathbb{R}^s_+$ is a convex cone and

$$\partial\Big((z^*_1, ..., z^*_q) \circ \delta^v_{-\mathbb{R}^s_+}\Big)(\alpha_{1,n}, ..., \alpha_{s,n}) = N_{-\mathbb{R}^s_+}(\alpha_{1,n}, ..., \alpha_{s,n})$$

we have

$$\begin{aligned} (22) &\iff (\alpha^*_{1,n}, ..., \alpha^*_{s,n}) \in N_{-\mathbb{R}^s_+}(\alpha_{1,n}, ..., \alpha_{s,n}) \\ &\iff (\alpha^*_{1,n}, ..., \alpha^*_{s,n}) \in \mathbb{R}^s_+ \ \text{ and } \sum_{i=1}^{s} \alpha^*_{i,n}\alpha_{i,n} = 0. \end{aligned}$$

Hence the proof is complete. □

Next, we establish the standard Lagrange multipliers conditions characterizing σ-efficient solutions of the problem $(\mathcal{MBP})$ under the following Slater-type constraint qualification

$$(CQ) \qquad \exists\, \hat{x} \in \operatorname{ri} A \text{ such that } g_i(\hat{x}, v(\hat{x})) < 0,\ i = 1, ..., s,$$

where $\operatorname{ri} A$ is the relative interior of A.

Theorem 4.7. *Let $(\overline{x}, \overline{y}) \in \mathcal{A} \times B$ such that $v(\overline{x}) = \mathcal{F}(\overline{x}, \overline{y})$ and suppose that the Slater-type constraint qualification (CQ) is fulfilled. Then $\overline{x}$ is σ-efficient solution of $(\mathcal{MBP})$ if and only if there exist $(z_1^*, ..., z_q^*) \in (\mathbb{R}^q_+)^\sigma$ and $(\xi_1^*, ..., \xi_s^*) \in \mathbb{R}^s_+$ such that we have*

$$0 \in \sum_{i=1}^{q} z_i^* \Big(\bigcup_{\substack{x^{i*} \in \mathbb{R}^m_+, t_i \geq 0 \\ (x^{i*}, t_i) \in \partial f_i(\overline{x}, v(\overline{x}))}} \{x^{i*}\} + \mathcal{S}_{t_i}(\overline{x}, \overline{y}) \Big) + \sum_{i=1}^{s} \xi_i^* \Big(\bigcup_{\substack{y^{i*} \in \mathbb{R}^m_+, r_i \geq 0 \\ (y^{i*}, r_i) \in \partial g_i(\overline{x}, v(\overline{x}))}} \{y^{i*}\} + \mathcal{S}_{r_i}(\overline{x}, \overline{y}) \Big) + N_A(\overline{x})$$

and $\xi_i^* g_i(\overline{x}, v(\overline{x})) = 0,\ i = 1, ..., s.$

Proof. By [4, Proposition 2.4.18], $\overline{x}$ is σ-efficient of $(\mathcal{MBP})$ if and only if there exists $(z_1^*, ..., z_q^*) \in (\mathbb{R}^q_+)^\sigma$ such that $\overline{x}$ is an optimal solution of the following scalar convex optimization problem

$$(\mathcal{SOP}) \qquad \min_{\substack{x \in A \\ g_i(V(x)) \leq 0, i=1,...,s}} \Big\{ \sum_{i=1}^{q} z_i^* f_i(V(x)) \Big\}.$$

As the Slater-type constraint qualification (CQ) holds for the problem $(\mathcal{SOP})$, then it follows by applying [6, Theorem 3.11] that there exists $(\xi_1^*, ..., \xi_s^*) \in \mathbb{R}^s_+$ such that

$$0 \in \sum_{i=1}^{q} z_i^* \partial (f_i \circ V)(\overline{x}) + \sum_{i=1}^{s} \xi_i^* \partial (g_i \circ V)(\overline{x}) + N_A(\overline{x})$$

and $\xi_i^* g_i(V(\overline{x})) = \xi_i^* g_i(\overline{x}, v(\overline{x})) = 0,\ i = 1, ..., s.$ On the other hand, we have

$$\begin{aligned} \partial (f_i \circ V)(\overline{x}) &= \bigcup_{\substack{x^{i*} \in \mathbb{R}^m_+, t_i \geq 0 \\ (x^{i*}, t_i) \in \partial f_i(V(\overline{x}))}} \partial \Big((x^{i*}, t^i) \circ V \Big)(\overline{x}) \\ &= \bigcup_{\substack{x^{i*} \in \mathbb{R}^m_+, t_i \geq 0 \\ (x^{i*}, t_i) \in \partial f_i(\overline{x}, v(\overline{x}))}} \{x^{i*}\} + \mathcal{S}_{t_i}(\overline{x}, \overline{y}),\ i = 1, ..., q \end{aligned}$$

and

$$\begin{aligned} \partial (g_i \circ V)(\overline{x}) &= \bigcup_{\substack{y^{i*} \in \mathbb{R}^m_+, r_i \geq 0 \\ (y^{i*}, r_i) \in \partial g_i(V(\overline{x}))}} \partial \Big((y^{i*}, r_i) \circ V \Big)(\overline{x}) \\ &= \bigcup_{\substack{y^{i*} \in \mathbb{R}^m_+, r_i \geq 0 \\ (y^{i*}, r_i) \in \partial g_i(\overline{x}, v(\overline{x}))}} \{y^{i*}\} + \mathcal{S}_{r_i}(\overline{x}, \overline{y}),\ i = 1, ..., s, \end{aligned}$$

because $f_1, ..., f_q$ and $g_1, ..., g_s$ are all finite and continuous at $V(\overline{x}) = (\overline{x}, v(\overline{x}))$ (see [5]). Hence the proof is complete. □

We close this subsection by presenting an example illustrating our sequential optimality conditions given in Theorem 4.6 where the Slater-type constraint qualification (CQ) fails.

Example 4.8. Let

- $d := 1$, $m := 1$, $q := 2$, $s := 1$, $A := \left[0, \frac{1}{2}\right]$, $B := [0,1]$,
- $\mathcal{F} : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\mathcal{F}(x,y) := x + y$,
- $f_1 : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f_1(x,y) := \begin{cases} x^2+1, & \text{if } x \geq 0 \\ 1, & \text{if } x < 0 \end{cases}$
- $f_2 : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f_2(x,y) := \begin{cases} y^2, & \text{if } y \geq 0 \\ 0, & \text{if } y < 0 \end{cases}$
- $g_1 : \mathbb{R}^2 \to \mathbb{R}$ be defined by $g_1(x,y) := e^x + e^y - 2$.

Then, the problem $(\mathcal{MBP})$ is nothing else than

$$(\mathcal{MBP}) \qquad \operatorname*{v-min}_{x \in \mathcal{A}} \left(x^2+1, x^2\right)$$

where $\mathcal{A} := \{x \in \left[0, \frac{1}{2}\right] : g_1(x, v(x)) = 2e^x - 2 \leq 0\} \neq \emptyset$ and $\min(\mathcal{P}_x) = v(x) = x$, for all $x \in \mathbb{R}$. Clearly, $\mathcal{F}$ is convex, f_1 and f_2 are convex and $\mathbb{R}^2_+$-nondecreasing and g_1 is convex and $\mathbb{R}^2_+$-nondecreasing. Moreover, it is easy to check that $\overline{x} = 0$ is a σ-efficient solution of $(\mathcal{MBP})$ and that the Slater-type constraint qualification (CQ) fails, i.e.

$$\nexists \hat{x} \in \operatorname{ri} A = \operatorname{int} A = \left]0, \frac{1}{2}\right[\text{ such that } g_1(\hat{x}, v(\hat{x})) = 2e^{\hat{x}} - 2 < 0.$$

On the other hand, we claim that the sequential optimality conditions given in Theorem 4.6 are all satisfied. Indeed, we set $\overline{y} := (0,0)$, $\overline{z} := 0$, $(z_1^*, z_2^*) := (1,1)$, $x_{1,n} := \frac{1}{n+1}$, $(y_{1,n}, y_{2,n}) := (\frac{1}{n+1}, \frac{1}{n+1})$, $b_n := 0$, $x_n^* := \frac{1}{n+1}$, $(y_{1,n}^*, y_{2,n}^*) := (\frac{1}{n+1}, 0)$, $\alpha_{1,n} := 0$, $\alpha_{1,n}^* := \frac{1}{n+1}$, $(r_{1,n}, r_{2,n}) := (\frac{1}{n+1}, \frac{1}{n+1})$, $t_{1,n} := 2e^{\frac{1}{n+1}} - 2$, $(r_{1,n}^*, r_{2,n}^*) := (0,0)$, $t_{1,n}^* := 0$, $w_{1,n} = w_{1,n}^* := 0$, $(u_{1,n}, u_{2,n}) := (\frac{1}{n+1}, \frac{1}{n+1})$, $(u_{1,n}^*, u_{2,n}^*) := (\frac{2}{n+1}, \frac{2}{n+1})$, for all $n \in \mathbb{N}$. Thus, we can see easily that

$$\begin{cases} w_{1,n}^* = 0 \in N_A(0), \quad w_{1,n} = 0 \xrightarrow[n \to +\infty]{} \overline{x} = 0, \\ w_{1,n}^*(w_{1,n} - \overline{x}) = 0 \xrightarrow[n \to +\infty]{} 0, \end{cases}$$

$$\begin{cases} (u_{1,n}^*, u_{2,n}^*) = (\frac{2}{n+1}, \frac{2}{n+1}) \in \partial\left(z_1^* f_1 + z_2^* f_2\right)(\frac{1}{n+1}, \frac{1}{n+1}), \\ (u_{1,n}, u_{2,n}) = (\frac{1}{n+1}, \frac{1}{n+1}) \xrightarrow[n \to +\infty]{\|\cdot\|_{\mathbb{R}^2}} \overline{y} = (0,0), \\ \displaystyle\sum_{i=1}^{2} (z_i^* f_i)(u_{1,n}, u_{2,n}) - \sum_{i=1}^{2} (z_i^* f_i)(\overline{x}, v(\overline{x})) - u_{1,n}^*(u_{1,n} - \overline{x}) \\ \qquad -u_{2,n}^*(u_{2,n} - v(\overline{x})) = \frac{-2}{(n+1)^2} \xrightarrow[n \to +\infty]{} 0, \end{cases}$$

$$\begin{cases} (x_{1,n}, y_{1,n}, y_{2,n}) = (\frac{1}{n+1}, \frac{1}{n+1}, \frac{1}{n+1}) \in \mathbb{E}_v, \\ x^*_{1,n} - y^*_{1,n} = 0 \in \mathcal{S}_0(\frac{1}{n+1}, 0) = \{0\}, \\ y^*_{1,n}(y_{1,n} - x_{1,n}) + y^*_{2,n}(y_{2,n} - v(x_{1,n})) = 0, \\ x_{1,n} = \frac{1}{n+1} \xrightarrow[n\to+\infty]{} \overline{x} = 0, \ (y_{1,n}, y_{2,n}) = (\frac{1}{n+1}, \frac{1}{n+1}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^2}} \overline{y} = (0,0), \\ -x^*_{1,n}(x_{1,n} - \overline{x}) + y^*_{1,n}(y_{1,n} - \overline{x}) + y^*_{2,n}(y_{2,n} - v(\overline{x})) = 0 \xrightarrow[n\to+\infty]{} 0, \end{cases}$$

$$\begin{cases} \alpha^*_{1,n}\alpha_{1,n} = 0, \\ \alpha_{1,n} = 0 \xrightarrow[n\to+\infty]{} \overline{z} = 0, \\ \alpha^*_{1,n}(\alpha_{1,n} - g_1(\overline{y})) = 0 \xrightarrow[n\to+\infty]{} 0, \end{cases}$$

$$\begin{cases} (r_{1,n}, r_{2,n}, t_{1,n}) = (\frac{1}{n+1}, \frac{1}{n+1}, 2e^{\frac{1}{n+1}} - 2) \in \mathbb{E}, \\ (r^*_{1,n}, r^*_{2,n}) = (0,0) \in \partial\Big(t^*_{1,n} g_1\Big)(\frac{1}{n+1}, \frac{1}{n+1}), \\ t^*_{1,n}(t_{1,n} - g_1(r_{1,n}, r_{2,n})) = 0, \\ (r_{1,n}, r_{2,n}) = (\frac{1}{n+1}, \frac{1}{n+1}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^2}} \overline{y} = (0,0), \ t_{1,n} = 2e^{\frac{1}{n+1}} - 2 \xrightarrow[n\to+\infty]{} \overline{z} = 0, \\ t^*_{1,n}(t_{1,n} - g_1(\overline{y})) - r^*_{1,n}(r_{1,n} - \overline{x}) - r^*_{2,n}(r_{2,n} - v(\overline{x})) = 0 \xrightarrow[n\to+\infty]{} 0, \end{cases}$$

and

$$\begin{cases} w^*_{1,n} + 2x^*_{1,n} = \frac{2}{n+1} \xrightarrow[n\to+\infty]{} 0, \\ (u^*_{1,n} - y^*_{1,n}, u^*_{2,n} - y^*_{2,n}) = (\frac{1}{n+1}, \frac{2}{n+1}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^2}} (0,0), \\ \alpha^*_{1,n} - t^*_{1,n} = \frac{1}{n+1} \xrightarrow[n\to+\infty]{} 0, \\ (r^*_{1,n} - y^*_{1,n}, r^*_{2,n} - y^*_{2,n}) = (\frac{-1}{n+1}, 0) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^2}} (0,0). \end{cases}$$

Acknowledgements. The authors would like to express their sincere thanks to an anonymous reviewer for his/her valuable comments and suggestions.

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