Minimax Theory and its Applications
Volume 7 (2022), No. 1, 159–172

Equations with s-Fractional (p, q)-Laplacian and Convolution

Dumitru Motreanu
Départment de Mathématiques, Université de Perpignan, France
motreanu@univ-perp.fr

Received: December 21, 2021
Accepted: March 4, 2022

This paper deals with a Dirichlet problem on a bounded domain $\Omega \subset \mathbb{R}^N$ for an equation which is doubly nonlocal: it is driven by the (negative) s-fractional (p, q)-Laplacian for $s \in (0, 1)$ and $1 < q < p < \infty$ and has as reaction term a nonlinearity with an incorporated convolution. Such a problem is considered for the first time. Another major feature concerns the correct formulation for the notion of s-fractional (p, q)-Laplacian. The stated problem is studied through two different approaches: limit process via finite dimensional approximations and sub-supersolution in the nonlocal setting.

Keywords: Nonlocal Dirichlet problem, weak solution, s-fractional (p, q)-Laplacian, convolution, finite dimensional approximation, sub-supersolution.

2010 Mathematics Subject Classification: 35S15, 47G20, 35J92.

1. Introduction

Nonlinear differential equations driven by fractional operators of p-Laplacian type arise in various models describing nonlocal phenomena related to surface diffusion, phase transition, statistical mechanics, material sciences, image processing, population dynamics, game strategies. Such fractional operators permit to describe diffusion within Lévi type random motion with jumps, which is more accurate compared to Brownian processes. A comprehensive discussion of corresponding kernels with integral diffusion that lead to nonlinear nonlocal equations can be found in [4].

The present paper deals with equations that are governed by the fractional (p, q)-Laplacian, which is a nonlocal operator that can be introduced by means of the fractional p-Laplacian and the fractional q-Laplacian. In the well-established local case, the (p, q)-Laplacian $\Delta_p + \Delta_q$ has significant qualitative differences with respect to the the p-Laplacian Δ_p, mainly coming from the fact that $\Delta_p + \Delta_q$ is a nonhomogeneous operator contrary to Δ_p (see, e.g., [15]). A fundamentally different behavior of the fractional (p, q)-Laplacian versus the fractional p-Laplacian occurs too. In this direction, some aspects are pointed out in Section 2.

The aim of this paper is to study a class of equations driven by the s-fractional (p, q)-Laplacian and exhibiting a nonlinearity expressed through a convolution term, which makes the problem nonvariational. Such an equation is doubly nonlocal due to the fractional operator and the convolution. It is for the first time when this type of problems is investigated.

ISSN 2199-1413 (printed), ISSN 2199-1421 (electronic) / \$ 2.50

Fix real numbers s, p, q with $s \in (0,1)$ and $1 < q < p < \infty$. On a bounded domain $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary and $N > ps$ we state the nonlocal nonlinear problem

$$(-\Delta)_p^s u + (-\Delta)_q^s u = f(x, u, \rho * u) \quad \text{in } \Omega \tag{1}$$

$$u = 0 \quad \text{in } \Omega^c, \tag{2}$$

where $\Omega^c := \mathbb{R}^N \setminus \Omega$. In the left-hand side of the equation in (1) there are the (negative) s-fractional p-Laplacian $(-\Delta)_p^s$ acting on the space $W_0^{s,p}(\Omega)$ and the (negative) s-fractional q-Laplacian $(-\Delta)_q^s$ defined on the space $W_0^{s,q}(\Omega)$. The sum $(-\Delta)_p^s+(-\Delta)_q^s$ is the s-fractional analog of the (negative) (p,q)-Laplacian $-\Delta_p-\Delta_q$, so it is natural to be called the (negative) s-fractional (p,q)-Laplacian. The operator $(-\Delta)_p^s + (-\Delta)_q^s$ cannot be defined on the space $W_0^{s,p}(\Omega)$ despite the fact that $q < p$, thus contrasting to the classical case of $-\Delta_p - \Delta_q$ defined on $W_0^{1,p}(\Omega)$. As shown in [12], it is never true that $W^{s,p}(\Omega)$ is contained in $W^{s,q}(\Omega)$ whatever domain Ω is taken. Furthermore, there exist compactly supported functions in $W^{s,p}(\Omega)$ that are not in $W^{s,q}(\Omega)$.

The essential fact that $W^{s,p}(\Omega) \not\subset W^{s,q}(\Omega)$ prevents to have the (negative) s-fractional (p,q)-Laplacian $(-\Delta)_p^s + (-\Delta)_q^s$ defined on $W_0^{s,p}(\Omega)$ as incorrectly stated in certain publications [1, 21]. For instance, Lemma 2.2 in [1] is false. We also mention that the operator $(-\Delta)_p^{s_1} + (-\Delta)_q^{s_2}$ with $0 < s_2 < s_1 < 1$ termed in [8] to be the fractional (p,q)-Laplacian, which is defined on $W^{s_1,p}(\Omega)$ (due to the continuous embedding $W^{s_1,p}(\Omega) \subset W^{s_2,p}(\Omega)$, see [7, Proposition 2.1]), by its expression is not an operator like $(-\Delta)_p^s + (-\Delta)_q^s$ entering (1). In Section 2 we indicate a direct way to study the (negative) s-fractional (p,q)-Laplacian $(-\Delta)_p^s + (-\Delta)_q^s$ taking into account both spaces $W_0^{s,p}(\Omega)$ and $W_0^{s,q}(\Omega)$.

The nonlinearity in the right-hand side of the equation in (1) is nonlocal too, and in addition is nonvariational. It is the composition of a Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ (i.e., $f(x,t,\xi)$ is measurable in $x \in \Omega$ for all $(t,\xi) \in \mathbb{R} \times \mathbb{R}$ and is continuous in $(t,\xi) \in \mathbb{R} \times \mathbb{R}$ for a.e. $x \in \Omega$) and the convolution $\rho * u$ of some $\rho \in L^1(\mathbb{R}^N)$ and the solution $u : \mathbb{R}^N \to \mathbb{R}$,

$$\rho * u(x) = \int_{\mathbb{R}^N} \rho(x-y)u(y)\, dy \quad \text{for a.e. } x \in \mathbb{R}^N.$$

The local elliptic equations involving convolution were first considered in [16].

By a *(weak) solution* to problem (1)–(2) we mean any $u \in W_0^{s,p}(\Omega) \cap W_0^{s,q}(\Omega)$ such that $f(x,u,\rho * u) \in L^r(\Omega)$ for some $r > Np/(Np - N + sp)$ and

$$\langle (-\Delta)_p^s(u), v\rangle_{W_0^{s,p}(\Omega)} + \langle (-\Delta)_q^s(u), v\rangle_{W_0^{s,q}(\Omega)} = \int_\Omega f(x,u,\rho * u)v dx \tag{3}$$

for all $v \in W_0^{s,p}(\Omega) \cap W_0^{s,q}(\Omega)$. In the present paper we set forth two different approaches for studying the existence of solutions to problem (1)–(2) that are here initiated.

The first approach describes the construction of a solution to (1)–(2) through an approximation process using approximating solutions on finite dimensional spaces. The finite dimensional approximating problems are resolved by applying a consequence of Brouwer's fixed point theorem. The approximation is carried out by assuming the growth condition

$$|f(x,s,\xi)| \leq \theta(x) + a_1|s|^{\alpha} + a_2|\xi|^{\beta} \tag{4}$$

for a.e. $x \in \Omega$, all $(s,\xi) \in \mathbb{R}\times\mathbb{R}$, with a function $\theta \in L^{\gamma}(\Omega)$ with $\gamma > Np/(Np-N+sp)$ and constants $a_1, a_2 \geq 0$ and $\alpha, \beta \in [0, p-1)$. Notice that, under hypothesis (4), the integral in the right-hand side of (3) exists. The main result of this part is Theorem 3.4 in Section 3.

Our second approach consists in finding a solution $u \in W_0^{s,p}(\Omega) \cap W_0^{s,q}(\Omega)$ to (1)–(2) within the ordered interval $\underline{u} \leq \overline{u}$ determined by a subsolution $\underline{u}$ and a supersolution $\overline{u}$. The location $\underline{u}(x) \leq u(x) \leq \overline{u}(x)$ for $x \in \Omega$ almost everywhere is achieved on the basis of comparison arguments assuming that $\rho(x) \geq 0$ for a.e. $x \in \Omega$ (note that then $\rho * \underline{u} \leq \rho * \overline{u}$ a.e.). We assume a growth condition for $f(x,s,\xi)$ that fits precisely the prescribed sub-supersolution $\underline{u} \leq \overline{u}$, namely

$$|f(x,t,\xi)| \leq \sigma(x) \quad \text{for a.e } x \in \Omega, \text{ all } t \in [\underline{u}(x), \overline{u}(x)],\ \xi \in [\rho * \underline{u}(x), \rho * \overline{u}(x)], \tag{5}$$

with $\sigma \in L^r(\Omega)$ for some $r > Np/(Np - N + sp)$. Notice that the growth condition (4) implies the growth in (5). The existence and enclosure result involving a sub-supersolution is the object of Theorem 4.2 in Section 4.

The rest of the paper is organized as follows. Section 2 contains preliminary material on the s-fractional (p,q)-Laplacian. Section 3 provides the existence and approximation of a solution to problem (1)–(2). Section 4 establishes the existence of a solution to problem (1)–(2) located by a sub-supersolution.

2. Preliminaries on the s-fractional (p, q)-Laplacian

In the sequel norm convergence is denoted by $\to$ and weak convergence by $\rightharpoonup$. The duality pairing between a Banach space X and its dual X^* is denoted by $\langle \cdot, \cdot \rangle_X$.

Given $s \in (0,1)$ and $p \in (1, +\infty)$, the *Gagliardo seminorm* of a measurable function $u : \mathbb{R}^N \to \mathbb{R}$ is defined by

$$[u]_{s,p} := \left(\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} dxdy \right)^{1/p}.$$

The s-fractional Sobolev space

$$W^{s,p}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N) : \ [u]_{s,p} < \infty \right\}$$

is endowed with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} := (\|u\|^p_{L^p(\mathbb{R}^N)} + [u]^p_{s,p})^{1/p}.$$

For a bounded domain $\Omega \subset \mathbb{R}^N$, the closed linear subspace of $W^{s,p}(\mathbb{R}^N)$ defined by

$$W_0^{s,p}(\Omega) := \{ u \in W^{s,p}(\mathbb{R}^N) : \ u = 0 \ \text{ a.e. in } \Omega^c \}$$

can be endowed with the equivalent norm $\| \cdot \|_{W_0^{s,p}(\Omega)} = [\cdot]_{s,p}$ becoming a uniformly convex and separable Banach space.

For the characteristic function χ_Ω of Ω we have $\chi_\Omega \notin W_0^{s,p}(\Omega)$ due to the divergent integral

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\chi_\Omega(x)-\chi_\Omega(y)|^p}{|x-y|^{N+ps}}dxdy = 2\int_\Omega\int_{\Omega^c}\frac{1}{|x-y|^{N+ps}}dxdy.$$

The dual of $W_0^{s,p}(\Omega)$ is denoted $W^{-s,p'}(\Omega)$, with $p' = p/(p-1)$. The embedding $W_0^{s,p}(\Omega) \subset L^r(\Omega)$ is continuous if $1 \le r \le p_s^*$, and compact if $1 \le r < p_s^*$, where $p_s^* = Np/(N-sp)$ stands for the critical fractional Sobolev exponent. Note that the Hölder conjugate of p_s^* is $Np/(Np-N+sp)$, which explains the choice of exponents in (4) and (5). More details on the space $W_0^{s,p}(\Omega)$ can be found in [7, 14].

The variational definition of the (negative) s-fractional p-Laplacian

$$(-\Delta)_p^s : W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$$

is given by

$$\langle(-\Delta)_p^s(u), v\rangle_{W_0^{s,p}(\Omega)} \tag{6}$$

$$:= \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}dxdy$$

for all $u, v \in W_0^{s,p}(\Omega)$. We refer to [2, 9, 10, 11, 13, 14] for equations in variational form driven by the s-fractional p-Laplacian. The nonlocal character of $(-\Delta)_p^s$ is apparent in (6) observing that are involved not only the points in Ω but in the entire space $\mathbb{R}^N$. This makes natural the Dirichlet boundary condition in (2) posed on Ω^c. The same principle is adopted in [18] for the Neumann boundary condition.

A powerful reason for the definition in (6) is the fact that $(-\Delta)_p^s$ represents the gradient of the continuously differentiable functional $J : W_0^{s,p}(\Omega) \to \mathbb{R}$ given by

$$J(u) = \frac{1}{p}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy, \quad \forall u \in W_0^{s,p}(\Omega).$$

In particular, the continuity of $(-\Delta)_p^s : W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ holds. As is often the case, we have dropped in (6) the normalizing constant $C_{N,p,s}$ that appears in some works [7, 13, 20]. This constant is only needed to recover from $(-\Delta)_p^s$ the (negative) p-Laplacian $-\Delta_p$ as $s \uparrow 1$.

It is readily seen from (6) that $(-\Delta)_p^s$ is strictly monotone on $W_0^{s,p}(\Omega)$, that is,

$$\langle(-\Delta)_p^s u - (-\Delta)_p^s v, u-v\rangle > 0, \quad \forall u, v \in W_0^{s,p}(\Omega), \qquad u \ne v. \tag{7}$$

Moreover, we have

$$\langle(-\Delta)_p^s(u), u\rangle_{W_0^{s,p}(\Omega)} = \|u\|_{W_0^{s,p}(\Omega)}^p \quad \text{and} \quad \|(-\Delta)_p^s(u)\|_{W^{-s,p'}(\Omega)} = \|u\|_{W_0^{s,p}(\Omega)}^{p-1} \tag{8}$$

for all $u \in W_0^{s,p}(\Omega)$.

Fix real numbers $s \in (0,1)$ and $1 < q < p < \infty$. The continuous embedding $W_0^p(\Omega) \subset W_0^q(\Omega)$ allows that the (negative) (p,q)-Laplacian $-\Delta_p - \Delta_q$ be well defined on $W_0^p(\Omega)$ (see [15, 17] for the study of $-\Delta_p - \Delta_q$ including its spectrum). As already said, such an embedding is no longer valid in the nonlocal setting. It is shown in [12] that there always holds $W_0^{s,p}(\Omega) \not\subset W_0^{s,q}(\Omega)$ despite the fact that $1 < q < p < \infty$, so the operator $(-\Delta)_p^s + (-\Delta)_q^s$ does not exist on $W_0^{s,p}(\Omega)$ (neither on $W_0^{s,q}(\Omega)$).

In order to overcome this issue, we set

$$W_0^{s,p,q}(\Omega) := W_0^{s,p}(\Omega) \cap W_0^{s,q}(\Omega)$$

obtaining quite a large space that contains $W_0^p(\Omega)$.

Proposition 2.1. *The space* $W_0^{s,p,q}(\Omega)$ *endowed with the norm*

$$\|u\|_{W_0^{s,p,q}(\Omega)} := [u]_{s,p} + [u]_{s,q}$$

is a reflexive and separable Banach space.

Proof. Let $\{u_n\}_{n\geq 1}$ be a Cauchy sequence in $W_0^{s,p,q}(\Omega)$. Hence there is a $u \in L^r(\mathbb{R}^N)$ with $u_n \to u$ in $L^r(\mathbb{R}^N)$ and by Fatou's lemma,

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^r}{|x-y|^{N+ps}}dxdy \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^r}{|x-y|^{N+ps}}dxdy$$

for $r = p, q$. This establishes that $u \in W_0^{s,p,q}(\Omega)$. The completeness of $W_0^{s,p,q}(\Omega)$ follows from that of $W_0^{s,p}(\Omega)$ and $W_0^{s,q}(\Omega)$. The map

$$u \in W_0^{s,p,q}(\Omega) \mapsto (u,u) \in W_0^{s,p}(\Omega) \times W_0^{s,q}(\Omega)$$

is an isometrical isomorphism on its range. Since $W_0^{s,p}(\Omega) \times W_0^{s,q}(\Omega)$ is reflexive and separable, the same is true for $W_0^{s,p,q}(\Omega)$. □

We define the s-fractional (p,q)-Laplacian $(-\Delta)_p^s + (-\Delta)_q^s : W_0^{s,p,q}(\Omega) \to (W_0^{s,p,q}(\Omega))^*$ for all $u, v \in W_0^{s,p,q}(\Omega)$ by

$$\langle((-\Delta)_p^s + (-\Delta)_q^s)(u), v\rangle_{W_0^{s,p,q}(\Omega)} = \langle(-\Delta)_p^s u, v\rangle_{W_0^{s,p}(\Omega)} + \langle(-\Delta)_q^s u, v\rangle_{W_0^{s,q}(\Omega)} \tag{9}$$

The (S_+)-property holds true for the operator introduced in (9).

Proposition 2.2. *Any sequence* $\{u_n\}_{n\geq 1}$ *satisfying* $u_n \rightharpoonup u$ *in* $W_0^{s,p,q}(\Omega)$ *and*

$$\limsup_{n\to\infty} \langle((-\Delta)_p^s + (-\Delta)_q^s))(u_n), u_n - u\rangle_{W_0^{s,p,q}(\Omega)} \leq 0, \tag{10}$$

for some in $u \in W_0^{s,p,q}(\Omega)$, *is strongly convergent to* u *in* $W_0^{s,p,q}(\Omega)$.

Proof. By (7) and (10) we derive

$$\lim_{n\to\infty} \langle(-\Delta)_p^s(u_n) + (-\Delta)_q^s(u_n) - (-\Delta)_p^s(u) - (-\Delta)_q^s(u), u_n - u\rangle_{W_0^{s,p,q}(\Omega)} = 0, \tag{11}$$

whereas (8) and (9) yield

$$\begin{aligned}
&\langle(-\Delta)_p^s(u_n) + (-\Delta)_q^s(u_n) - (-\Delta)_p^s(u) - (-\Delta)_q^s(u), u_n - u\rangle_{W_0^{s,p,q}(\Omega)} \\
&\geq (\|u_n\|_{W_0^{s,p}(\Omega)} - \|u\|_{W_0^{s,p}(\Omega)})(\|u_n\|_{W_0^{s,p}(\Omega)}^{p-1} - \|u\|_{W_0^{s,p}(\Omega)}^{p-1}) \\
&\quad + (\|u_n\|_{W_0^{s,q}(\Omega)} - \|u\|_{W_0^{s,q}(\Omega)})(\|u_n\|_{W_0^{s,q}(\Omega)}^{q-1} - \|u\|_{W_0^{s,q}(\Omega)}^{q-1}) \geq 0.
\end{aligned}$$

Then (11) entails

$$\lim_{n\to+\infty} \|u_n\|_{W_0^{s,p}(\Omega)} = \|u\|_{W_0^{s,p}(\Omega)} \quad \text{and} \quad \lim_{n\to+\infty} \|u_n\|_{W_0^{s,q}(\Omega)} = \|u\|_{W_0^{s,q}(\Omega)}.$$

Since the spaces $W_0^{s,p}(\Omega)$ and $W_0^{s,q}(\Omega)$ are uniformly convex, we conclude that we have $u_n \to u$ in $W_0^{s,p,q}(\Omega)$. □

3. Existence and approximation

Throughout this section we assume that condition (4) is fulfilled. The definition of a (weak) solution $u \in W_0^{s,p,q}(\Omega)$ to problem (1) given in (3) says that

$$\langle((-\Delta)_p^s + (-\Delta)_q^s)(u), v\rangle_{W_0^{s,p,q}(\Omega)} = \int_\Omega f(x, u, \rho * u)v dx, \ \forall v \in W_0^{s,p,q}(\Omega).$$

Our finite-dimensional approximation makes use of a Galerkin basis of the space $W_0^{s,p,q}(\Omega)$, which exists because $W_0^{s,p,q}(\Omega)$ is separable as known from Proposition 2.1.

Let $\{X_n\}_{n\geq 1}$ be a Galerkin basis of $W_0^{s,p,q}(\Omega)$, that is, X_n is a finite-dimensional subspace of $W_0^{s,p,q}(\Omega)$ with $X_n \subset X_{n+1}$ for all n and

$$\overline{\bigcup_{n=1}^{\infty} X_n} = W_0^{s,p,q}(\Omega). \tag{12}$$

We formulate an approximating problem on X_n: find $u_n \in X_n$ such that

$$\langle((-\Delta)_p^s + (-\Delta)_q^s)(u_n), v\rangle_{W_0^{s,p,q}(\Omega)} - \int_\Omega f(x, u_n, \rho * u_n(x))v(x)dx = 0, \ \forall v \in X_n. \tag{13}$$

Lemma 3.1. *For each $n \geq 1$ there exists a solution $u_n \in X_n$ to problem (13).*

Proof. We introduce the map $A_n : X_n \to X_n^*$ by

$$\langle A_n(u), v\rangle_{X_n} := \langle((-\Delta)_p^s + (-\Delta)_q^s)u, v\rangle_{W_0^{s,p,q}(\Omega)} - \int_\Omega f(x, u(x), \rho * u(x))v(x)dx$$

for all $u, v \in X_n$. Due to hypothesis (4), the map A_n is well defined. By (9), (8), (4), and Hölder's inequality, we note that

$$\begin{aligned}\langle A_n(v), v\rangle_{X_n} &= \|v\|^p_{W_0^{s,p}(\Omega)} + \|v\|^q_{W_0^{s,q}(\Omega)} - \int_\Omega f(x, v, \rho * v)v dx \\ &\geq \|v\|^p_{W_0^{s,p}(\Omega)} + \|v\|^q_{W_0^{s,q}(\Omega)} - \|\theta\|_{L^\gamma(\Omega)}\|v\|_{L^{\gamma/(\gamma-1)}(\Omega)} \\ &\quad - a_1\|v\|^{\alpha+1}_{L^{\alpha+1}(\Omega)} - a_2\|\rho * v\|^\beta_{L^{\beta+1}(\mathbb{R}^N)}\|v\|_{L^{\beta+1}(\Omega)}, \ \forall v \in X_n.\end{aligned}$$

Recall the Young's inequality for convolution (see, e.g., [3, p. 104])

$$\|\rho * u\|_{L^{\beta+1}(\mathbb{R}^N)} \leq \|\rho\|_{L^1(\mathbb{R}^N)}\|u\|_{L^{\beta+1}(\mathbb{R}^N)}, \ \forall u \in L^{\beta+1}(\mathbb{R}^N). \tag{14}$$

By (14), in conjunction with the continuous embeddings of $W_0^{s,p}(\Omega)$ into the spaces $L^{\gamma/(\gamma-1)}(\Omega)$, $L^{\alpha+1}(\Omega)$ and $L^{\beta+1}(\Omega)$, we find the estimate

$$\langle A_n(v), v\rangle_{X_n} \geq \|v\|^p_{W_0^{s,p}(\Omega)} + \|v\|^q_{W_0^{s,q}(\Omega)} - c_1(\|v\|_{W_0^{s,p}(\Omega)} + \|v\|^{\alpha+1}_{W_0^{s,p}(\Omega)} + \|v\|^{\beta+1}_{W_0^{s,p}(\Omega)}) \tag{15}$$

for all $v \in X_n$, with a constant $c_1 > 0$.

Since on the finite-dimensional space X_n all the norms are equivalent, we infer from (15) that a constant $c_0 = c_0(n) > 0$ exists such that, for all $v \in X_n$,

$$\langle A_n(v), v\rangle_{X_n} \geq c_0\|v\|^p_{W_0^{s,p,q}(\Omega)} - c_1(\|v\|_{W_0^{s,p,q}(\Omega)} + \|v\|^{\alpha+1}_{W_0^{s,p,q}(\Omega)} + \|v\|^{\beta+1}_{W_0^{s,p,q}(\Omega)}).$$

Thanks to $p > \max\{\alpha+1, \beta+1\}$, it turns out that

$$\langle A_n(v), v\rangle_{X_n} \geq 0 \text{ for all } v \in X_n \text{ with } \|v\|_{W_0^{s,p,q}(\Omega)} = R$$

provided $R > 0$ is sufficiently large. Then a well-known consequence of Brouwer's fixed point theorem (see, e.g., [19, Proposition 2.1]) guarantees the existence of $u_n \in X_n$ for which $A_n(u_n) = 0$, thereby (13) is achieved. □

Next we focus on the sequence of approximate solutions u_n.

Lemma 3.2. *The sequence* $\{u_n\}_{n\geq 1}$ *with* $u_n \in X_n$ *constructed in Lemma* 3.1 *is bounded in* $W_0^{s,p,q}(\Omega)$.

Proof. Choose $v = u_n$ as test function in (13), which gives

$$\|u_n\|^p_{W_0^{s,p}(\Omega)} + \|u_n\|^q_{W_0^{s,q}(\Omega)} = \int_\Omega f(x, u_n, \rho * u_n)u_n dx. \tag{16}$$

We derive from (16), hypothesis (4), Hölder's inequality, (14) with u_n in place of u, and the continuous embeddings of $W_0^{s,p}(\Omega)$ into $L^{\gamma/(\gamma-1)}(\Omega)$, $L^{\alpha+1}(\Omega)$ and $L^{\beta+1}(\Omega)$ that

$$\|u_n\|^p_{W_0^{s,p}(\Omega)} \leq c_2(\|u_n\|_{W_0^{s,p}(\Omega)} + \|u_n\|^{\alpha+1}_{W_0^{s,p}(\Omega)} + \|u_n\|^{\beta+1}_{W_0^{s,p}(\Omega)}),$$

with a constant $c_2 > 0$. Since $p > \max\{\alpha+1, \beta+1\}$, the boundedness of $\{u_n\}_{n\geq 1}$ in $W_0^{s,p}(\Omega)$ follows. Going back to (16), we note that the sequence $\{u_n\}_{n\geq 1}$ is bounded in $W_0^{s,q}(\Omega)$, too. We conclude that $\{u_n\}_{n\geq 1}$ is bounded in $W_0^{s,p,q}(\Omega)$. □

Lemma 3.3. *The sequence* $\{u_n\}_{n\geq 1}$ *constructed in Lemma 3.1 fulfills*

$$((-\Delta)^s_p + (-\Delta)^s_q)u_n - f(\cdot, u_n, \rho * u_n) \rightharpoonup 0 \quad \textit{in } (W_0^{s,p,q}(\Omega))^* \textit{ as } n \to \infty. \tag{17}$$

Proof. Lemma 3.2 ensures that the sequence $\{u_n\}_{n\geq 1}$ is bounded in $W_0^{s,p,q}(\Omega)$. By (8) we find that the sequence $\{((-\Delta)^s_p + (-\Delta)^s_q)u_n\}_{n\geq 1}$ is bounded in $(W_0^{s,p,q}(\Omega))^*$. We also observe that (4), (14) and Hölder's inequality entail

$$\|f(\cdot, u_n, \rho * u_n)\|_{(W_0^{s,p,q}(\Omega))^*} \leq c_3(\|\theta\|_{L^\gamma(\Omega)} + a_1\|u_n\|^{\alpha+1}_{L^{\alpha+1}(\Omega)} + a_2\|u_n\|^{\beta}_{L^{\beta+1}(\Omega)}),$$

with a constant $c_3 > 0$, thus we can infer that $\{f(\cdot, u_n, \rho * u_n)\}_{n\geq 1}$ is bounded in $(W_0^{s,p,q}(\Omega))^*$. In view of the reflexivity of the space $(W_0^{s,p,q}(\Omega))^*$ demonstrated in Proposition 2.1, along a subsequence there holds

$$((-\Delta)^s_p + (-\Delta)^s_q)u_n - f(\cdot, u_n, \rho * u_n) \rightharpoonup \xi \text{ in } (W_0^{s,p,q}(\Omega))^* \tag{18}$$

for some $\xi \in (W_0^{s,p,q}(\Omega))^*$.

We are going to show that $\xi = 0$. Let $v \in \bigcup_{n\geq 1} X_n$ and choose an integer $m \geq 1$ such that $v \in X_m$. The definition of Galerkin basis postulates that $X_m \subset X_n$ for all $n \geq m$. Letting $n \to \infty$ in (13) we obtain from (18) that $\langle \xi, v\rangle_{W_0^{s,p,q}(\Omega)} = 0$. The density of $\bigcup_{n\geq 1} X_n$ in $W_0^{s,p,q}(\Omega)$ according to (12) renders $\xi = 0$. Invoking (18), the proof of (17) is achieved upon the entire sequence because the reasoning runs for every subsequence of $\{u_n\}_{n\geq 1}$. $\square$

The main result of this section reads as follows.

Theorem 3.4. *Under assumption* (4) *there exists a (weak) solution* $u \in W_0^{s,p,q}(\Omega)$ *to problem* (1)–(2) *in the sense of* (3) *which is the limit in* $W_0^{s,p,q}(\Omega)$ *of a strongly convergent sequence* $\{u_n\}_{n\geq 1}$ *with* u_n *solution of* (13).

Proof. Consider the sequence $\{u_n\}_{n\geq 1}$ constructed in Lemma 3.1. We shall prove that there is a subsequence converging strongly in $W_0^{s,p,q}(\Omega)$ to a weak solution of problem (1)–(2). Proposition 2.1 shows that the space $W_0^{s,p,q}(\Omega)$ is reflexive, whereas Lemma 3.2 asserts that the sequence $\{u_n\}_{n\geq 1}$ is bounded. In consequence, along a relabeled subsequence we have $u_n \rightharpoonup u$ in $W_0^{1,p,q}(\Omega)$ as $n \to \infty$, with some $u \in W_0^{1,p,q}(\Omega)$.

In view of the compact embeddings of $W_0^{s,p}(\Omega)$ into $L^{\alpha+1}(\Omega)$ and $L^{\beta+1}(\mathbb{R}^N)$, we infer that $u_n \to u$ in $L^{\alpha+1}(\Omega)$ and $L^{\beta+1}(\mathbb{R}^N)$. Moreover, thanks to (14), we have that $\rho * u_n \to \rho * u$ in $L^{\beta+1}(\Omega)$. As a result, hypothesis (4) enables to apply Lebesgue's dominated convergence theorem leading to

$$f(\cdot, u_n, \rho * u_n) \rightharpoonup f(\cdot, u, \rho * u) \text{ in } (W_0^{s,p,q}(\Omega))^* \text{ as } n \to \infty \tag{19}$$

and

$$\lim_{n\to\infty} \int_\Omega f(x, u_n, \rho * u_n) u_n dx = \int_\Omega f(x, u, \rho * u) u dx. \tag{20}$$

Combining (19) and Lemma 3.3 confirms that

$$((-\Delta)_p^s + (-\Delta)_q^s) u_n \rightharpoonup f(\cdot, u, \rho * u) \;\text{ in } (W_0^{s,p,q}(\Omega))^* \text{ as } n \to \infty. \tag{21}$$

Let us note from (8) and (13) that

$$\begin{aligned}\langle((-\Delta)_p^s + (-\Delta)_q^s))(u_n), u_n - u\rangle_{W_0^{s,p,q}(\Omega)} &= \int_\Omega f(x, u_n, \rho * u_n) u_n dx \\ - \Big(\langle((-\Delta)_p^s + (-\Delta)_q^s)(u_n), u\rangle_{W_0^{s,p,q}(\Omega)} &- \int_\Omega f(x, u_n, \rho * u_n) u dx\Big) \\ - \int_\Omega f(x, u_n, \rho * u_n) u dx.&\end{aligned}$$

In the limit as $n \to \infty$, (19), (20) and (21) imply

$$\lim_{n\to\infty} \langle((-\Delta)_p^s + (-\Delta)_q^s))(u_n), u_n - u\rangle_{W_0^{s,p,q}(\Omega)} = 0.$$

Hence Proposition 2.2 can be applied ensuring the strong convergence $u_n \to u$ in $W_0^{s,p,q}(\Omega)$ as $n \to \infty$. By the continuity of the operator $(-\Delta)_p^s + (-\Delta)_q^s$ on $W_0^{s,p,q}(\Omega)$ it turns out that

$$((-\Delta)_p^s + (-\Delta)_q^s) u_n \to ((-\Delta)_p^s + (-\Delta)_q^s) u \;\text{ in } (W_0^{s,p,q}(\Omega))^* \text{ as } n \to \infty. \tag{22}$$

Finally, (21) and (22) allow us to get the equality

$$((-\Delta)_p^s + (-\Delta)_q^s)u = f(\cdot, u, \rho * u) \text{ in } (W_0^{s,p,q}(\Omega))^*$$

establishing that $u \in W_0^{s,p,q}(\Omega)$ is a (weak) solution to problem (1)–(2). The proof is complete. $\square$

Remark 3.5. A careful reading of the preceding arguments reveals that the existence of a solution to (1)–(2) cannot be derived by applying the main theorem for pseudomonotone operators (see, e.g., [5, Theorem 2.99])). Condition (4) does not ensure the needed coercivity with respect to the $W_0^{s,q}(\Omega)$-component of the norm on $W_0^{s,p,q}(\Omega)$ (note $W^{s,p}(\Omega) \not\subset W^{s,q}(\Omega)$).

4. A sub-supersolution approach

For the general method of sub-supersolution in the study of local elliptic problems we refer to [5] (see also [6] for a recent use regarding equations with convection). This section implements a sub-supersolution approach in the case of the nonlocal problem (1)–(2) supposing that $\rho \in L^1(\mathbb{R}^N)$ verifies $\rho \geq 0$ almost everywhere. We start by defining the appropriate notions of subsolution and supersolution.

A *subsolution* for problem (1)–(2) is a function $\underline{u} \in W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N)$ such that $\underline{u} \leq 0$ on Ω^c a. e., $f(\cdot, \underline{u}, \rho * \underline{u}) \in L^{r_1}(\Omega)$ with some $r_1 > Np/(Np - N + sp)$ and

$$\begin{aligned}
&\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\underline{u}(x) - \underline{u}(y)|^{p-2}(\underline{u}(x) - \underline{u}(y))(v(x) - v(y))}{|x-y|^{N+sp}} dxdy \\
&\qquad + \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\underline{u}(x) - \underline{u}(y)|^{q-2}(\underline{u}(x) - \underline{u}(y))(v(x) - v(y))}{|x-y|^{N+sq}} dxdy \\
&\leq \int_\Omega f(x, \underline{u}, \rho * \underline{u}) v dx, \ \forall v \in W_0^{s,p,q}(\Omega), \ v \geq 0 \text{ a.e. in } \Omega. \qquad (23)
\end{aligned}$$

A *supersolution* for problem (1)–(2) is a function $\overline{u} \in W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N)$ such that $\overline{u} \geq 0$ on Ω^c a. e., $f(\cdot, \overline{u}, \rho * \overline{u}) \in L^{r_2}(\Omega)$ with some $r_2 > Np/(Np - N + sp)$ and

$$\begin{aligned}
&\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\overline{u}(x) - \overline{u}(y)|^{p-2}(\overline{u}(x) - \overline{u}(y))(v(x) - v(y))}{|x-y|^{N+sp}} dxdy \\
&\qquad + \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\overline{u}(x) - \overline{u}(y)|^{q-2}(\overline{u}(x) - \overline{u}(y))(v(x) - v(y))}{|x-y|^{N+sq}} dxdy \\
&\geq \int_\Omega f(x, \overline{u}, \rho * \overline{u}) v dx, \ \forall v \in W_0^{s,p,q}(\Omega), \ v \geq 0 \text{ a.e. in } \Omega. \qquad (24)
\end{aligned}$$

Notice that the integrals in (23) and (24) exist.

Remark 4.1. A measurable function u on $\mathbb{R}^N$ is a solution to problem (1)–(2) if and only if it is simultaneously a subsolution and a supersolution.

The main result of this section is the following.

Theorem 4.2. *Assume that condition* (5) *holds. Then there exists a (weak) solution* $u \in W_0^{s,p,q}(\Omega)$ *to problem* (1)–(2) *satisfying* $\underline{u}(x) \le u(x) \le \overline{u}(x)$ *for a.e.* $x \in \Omega$, *where* $\underline{u} \le \overline{u}$ *is the sub-supersolution postulated in assumption* (5).

Proof. For each $u \in W^{s,p,q}(\mathbb{R}^N)$ and a.e. $x \in \Omega$ we define the truncation

$$(Tu)(x) = \begin{cases} \underline{u}(x) & \text{if} \quad u(x) < \underline{u}(x) \\ u(x) & \text{if} \quad \underline{u}(x) \le u(x) \le \overline{u}(x) \\ \overline{u}(x) & \text{if} \quad u(x) > \overline{u}. \end{cases} \tag{25}$$

Since $\underline{u} \le Tu \le \overline{u}$ a.e. in Ω whenever $u \in W^{s,p,q}(\mathbb{R}^N)$, assumption (5) allows us to consider the Nemytskii type operator $N : W_0^{s,p,q}(\Omega) \to (W_0^{s,p,q}(\Omega))^*$ defined by

$$N(u) = f(\cdot, Tu, \rho * (Tu)), \qquad \forall u \in W_0^{s,p,q}(\Omega). \tag{26}$$

Specifically, through assumption (5) and Hölder's inequality, we have

$$\left| \int_\Omega f(x, Tu, \rho * (Tu)) v(x) dx \right| \le \int_\Omega \sigma(x) |v(x)| dx \le \|\sigma\|_{L^r(\Omega)} \|v\|_{L^{r'}(\Omega)}$$

for all $u, v \in W_0^{s,p,q}(\Omega)$. Actually, the map $N : W_0^{s,p,q}(\Omega) \to (W_0^{s,p,q}(\Omega))^*$ in (26) is completely continuous as the composition of the continuous map

$$u \in W_0^{s,p,q}(\Omega) \mapsto f(\cdot, Tu, \rho * (Tu)) \in L^r(\Omega)$$

(see (5)) and the inclusion $L^r(\Omega) \subset (W_0^{s,p,q}(\Omega))^*$ which is compact being the adjoint of the compact embedding $W_0^{s,p,q}(\Omega) \subset L^{r/(r-1)}(\Omega)$ (note that $r/(r-1) < p_s^*$ due to assumption (5)).

Let the map $A : W_0^{s,p,q}(\Omega) \to (W_0^{s,p,q}(\Omega))^*$ be defined by

$$A = (-\Delta)_p^s + (-\Delta)_q^s - N, \tag{27}$$

where N is given in (26).

The operators $(-\Delta)_p^s : W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ and $(-\Delta)_q^s : W_0^{s,q}(\Omega) \to W^{-s,q'}(\Omega)$ are continuous, monotone and bounded (in the sense that they map bounded sets to bounded sets), so the operator $(-\Delta)_p^s + (-\Delta)_q^s : W_0^{s,p,q}(\Omega) \to (W_0^{s,p,q}(\Omega))^*$ in (9) inherits the same properties. In particular, $A : W_0^{s,p,q}(\Omega) \to (W_0^{s,p,q}(\Omega))^*$ is a bounded operator.

Next we show that the operator $A : W_0^{s,p,q}(\Omega) \to (W_0^{s,p,q}(\Omega))^*$ in (27) is pseudomonotone. To this end, let $u_n \rightharpoonup u$ in $W_0^{s,p,q}(\Omega)$ and

$$\limsup_{n\to\infty} \langle A(u_n), u_n - u \rangle_{W_0^{s,p,q}(\Omega)} \le 0. \tag{28}$$

By virtue of assumption (5), the sequence $\{N(u_n)\}$ is bounded in $L^r(\Omega)$. On the other hand, owing to $r > Np/(Np-N+sp)$ we have $r/(r-1) < p_s^*$, which guarantees the compact embedding $W_0^{s,p}(\Omega) \subset L^{r/(r-1)}(\Omega)$, whence $u_n \to u$ in $L^{r/(r-1)}(\Omega)$ and

$$\lim_{n\to\infty} \langle N(u_n), u_n - u \rangle_{W_0^{s,p,q}(\Omega)} = \langle N(u_n), u_n - u \rangle_{L^r(\Omega)} = 0.$$

Consequently, (28) takes the form (10). Then the (S_+)-property proven in Proposition 2.2 implies the strong convergence $u_n \to u$ in $W_0^{s,p,q}(\Omega)$. The continuity of the operator $A : W_0^{s,p,q}(\Omega) \to (W_0^{s,p,q}(\Omega))^*$ in (27) entails $A(u_n) \to A(u)$ in $(W_0^{s,p,q}(\Omega))^*$ and $\langle A(u_n), u_n\rangle_{W_0^{s,p,q}(\Omega)} \to \langle A(u), u\rangle_{W_0^{s,p,q}(\Omega)}$. According to the definition of pseudomonote operator (see, e.g., [5, Definition 2.97]), we have that $A : W_0^{s,p,q}(\Omega) \to (W_0^{s,p,q}(\Omega))^*$ is pseudomonotone.

We claim that the operator $A : W_0^{s,p,q}(\Omega) \to (W_0^{s,p,q}(\Omega))^*$ in (27) is coercive meaning

$$\lim_{\|u\|_{W_0^{s,p,q}(\Omega)}\to+\infty} \frac{\langle A(u), u\rangle_{W_0^{s,p,q}(\Omega)}}{\|u\|_{W_0^{s,p,q}(\Omega)}} = +\infty. \tag{29}$$

Indeed, from (27), (26), assumption (5) and Hölder's inequality, we infer the estimate

$$\begin{aligned}\langle\langle A(u), u\rangle_{W_0^{s,p,q}(\Omega)} &= \|u\|^p_{W_0^{s,p}(\Omega)} + \|u\|^q_{W_0^{s,q}(\Omega)} - \int_\Omega f(x, Tu, \rho * (Tu))u dx\\ &\geq \|u\|^p_{W_0^{s,p}(\Omega)} + \|u\|^q_{W_0^{s,q}(\Omega)} - C\|\sigma\|_{L^r(\Omega)}\|u\|_{W_0^{s,p}(\Omega)}\end{aligned}$$

for all $u \in W_0^{s,p,q}(\Omega)$, with a constant $C > 0$. Since $p > 1$ and $q > 1$, the coercivity property (29) ensues.

Summarizing, the operator $A : W_0^{s,p,q}(\Omega) \to (W_0^{s,p,q}(\Omega))^*$ introduced in (27) is bounded, pseudomonotone and coercive. This enables to apply the main theorem for pseudomonotone operators (see, e.g., [5, Theorem 2.99])). It provides the existence of a solution $u \in W_0^{s,p,q}(\Omega)$ of the equation $A(u) = 0$, that is,

$$\langle((-\Delta)^s_p + (-\Delta)^s_q)(u), v\rangle_{W_0^{s,p,q}(\Omega)} = \langle(f(\cdot, Tu, \rho * (Tu)), v\rangle_{W_0^{s,p,q}(\Omega)} \tag{30}$$

for all $v \in W_0^{s,p,q}(\Omega)$.

We will show that $u \in W_0^{s,p,q}(\Omega)$ found in (30) satisfies the enclosure $\underline{u} \leq u \leq \overline{u}$ a.e. in Ω. This will be the result of a comparison argument. For proving the inequality $u \leq \overline{u}$ a.e. in Ω we rely on the function $(u - \overline{u})^+ = \max\{u - \overline{u}, 0\}$. Since $\overline{u}$ is a supersolution, we have $\overline{u} \geq 0$ on Ω^c, which ensures that $(u-\overline{u})^+ \in W_0^{s,p,q}(\Omega)$. Hence $v = (u - \overline{u})^+$ can be inserted in (24) and (30). By substraction and taking into account the definition of the truncation Tu in (25), we get

$$\begin{aligned}&\langle((-\Delta)^s_p + (-\Delta)^s_q)(u) - ((-\Delta)^s_p + (-\Delta)^s_q)(\overline{u}), (u - \overline{u})^+\rangle_{W_0^{s,p,q}(\Omega)}\\ &\leq \langle f(\cdot, Tu, \rho * (Tu)) - f(\cdot, \overline{u}, \rho * \overline{u}), (u - \overline{u})^+\rangle_{W_0^{s,p,q}(\Omega)} = 0.\end{aligned}$$

It follows that $\langle(-\Delta)^s_p u - (-\Delta)^s_p \overline{u}, (u - \overline{u})^+\rangle_{W_0^{s,p}(\Omega)} \leq 0$, which by (6) reads as

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{S(x, y)((u - \overline{u})^+(x) - (u - \overline{u})^+(y))}{|x - y|^{N+sp}} dxdy \leq 0, \tag{31}$$

where

$$S(x, y) := |u(x) - u(y)|^{p-2}(u(x) - u(y)) - |\overline{u}(x) - \overline{u}(y)|^{p-2}(\overline{u}(x) - \overline{u}(y)).$$

Denote $D := \{x \in \mathbb{R}^N : u(x) > \overline{u}(x)\}$, with the Lebesgue measure $|D|$ and its complement $D^c = \mathbb{R}^N \setminus D$. We note that $\Omega^c \subset D^c$.

It is clear that

$$\int_{D^c\times D^c}\frac{S(x,y)((u-\overline{u})^+(x)-(u-\overline{u})^+(y))}{|x-y|^{N+sp}}dxdy=0$$

and
$$\int_{D\times D}\frac{S(x,y)((u-\overline{u})^+(x)-(u-\overline{u})^+(y))}{|x-y|^{N+sp}}dxdy$$
$$=\int_{D\times D}\frac{S(x,y)((u(x)-\overline{u}(x))-(u(y)-\overline{u}(y)))}{|x-y|^{N+sp}}dxdy\geq 0$$

since the function $t\mapsto |t|^{p-2}t$ is increasing on $\mathbb{R}$. Then (31) yields

$$\int_{(D\times D^c)\cup(D^c\times D)}\frac{S(x,y)((u-\overline{u})^+(x)-(u-\overline{u})^+(y))}{|x-y|^{N+sp}}dxdy\leq 0. \tag{32}$$

Suppose that $|D|>0$. We show that a contradiction is reached. If $(x,y)\in D\times D^c$, there hold $u(x)>\overline{u}(x)$ and $u(y)\leq\overline{u}(y)$, so $u(x)-u(y)-(\overline{u}(x)-\overline{u}(y))>0$. This results in $S(x,y)>0$, thus

$$S(x,y)((u-\overline{u})^+(x)-(u-\overline{u})^+(y))=S(x,y)(u(x)-\overline{u}(x))>0 \tag{33}$$

for all $(x,y)\in D\times D^c$. If $(x,y)\in D^c\times D$, we have $u(x)\leq\overline{u}(x)$ and $u(y)>\overline{u}(y)$, ensuring $u(x)-u(y)-(\overline{u}(x)-\overline{u}(y))<0$, therefore $S(x,y)<0$. It turns out that

$$S(x,y)((u-\overline{u})^+(x)-(u-\overline{u})^+(y))=-S(x,y)(u(y)-\overline{u}(y))>0 \tag{34}$$

for all $(x,y)\in D^c\times D$. Gathering (32), (33) and (34) gives rise to a contradiction. In this way we find that $|D|=0$, which proves the inequality $u\leq\overline{u}$ a.e. in $\mathbb{R}^N$.

The proof of the inequality $\underline{u}\leq u$ a.e in $\mathbb{R}^N$ can be carried out by means of the function $(\underline{u}-u)^+=\max\{\underline{u}-u,0\}$, noticing $(\underline{u}-u)^+\in W_0^{s,p,q}(\Omega)$ as seen from $\underline{u}\leq 0$ on Ω^c. Inserting $v=(\underline{u}-u)^+$ in (23) and (30), by (25) we arrive at

$$\langle((-\Delta)_p^s+(-\Delta)_q^s)(\underline{u})-((-\Delta)_p^s+(-\Delta)_q^s)(u),(\underline{u}-u)^+\rangle_{W_0^{s,p,q}(\Omega)}$$
$$\leq\langle f(\cdot,T\underline{u},\rho*(T\underline{u}))-f(\cdot,u,\rho*u),(\underline{u}-u)^+\rangle_{W_0^{s,p,q}(\Omega)}=0.$$

Then we get $\langle(-\Delta)_p^s\underline{u}-(-\Delta)_p^s u,(\underline{u}-u)^+\rangle_{W_0^{s,p}(\Omega)}\leq 0$ or equivalently

$$\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{\tilde{S}(x,y)((\underline{u}-u)^+(x)-(\underline{u}-u)^+(y))}{|x-y|^{N+sp}}dxdy\leq 0, \tag{35}$$

where

$$\tilde{S}(x,y):=|\underline{u}(x)-\underline{u}(y)|^{p-2}(\underline{u}(x)-\underline{u}(y))-|u(x)-u(y)|^{p-2}(u(x)-u(y)).$$

Now denote $\tilde{D}:=\{x\in\mathbb{R}^N:\underline{u}(x)>u(x)\}$, with the Lebesgue measure $|\tilde{D}|$ and its complement $\tilde{D}^c=\mathbb{R}^N\setminus\tilde{D}$. Notice that $\Omega^c\subset\tilde{D}^c$. It is readily seen that

$$\int_{\tilde{D}^c\times\tilde{D}^c}\frac{\tilde{S}(x,y)((\underline{u}-u)^+(x)-(\underline{u}-u)^+(y))}{|x-y|^{N+sp}}dxdy=0$$

and $$\int_{\tilde{D}\times\tilde{D}} \frac{\tilde{S}(x,y)((\underline{u}-u)^+(x)-(\underline{u}-u)^+(y))}{|x-y|^{N+sp}}dxdy \geq 0.$$

Then (35) implies

$$\int_{(\tilde{D}\times\tilde{D}^c)\cup(\tilde{D}^c\times\tilde{D})} \frac{\tilde{S}(x,y)((\underline{u}-u)^+(x)-(\underline{u}-u)^+(y))}{|x-y|^{N+sp}}dxdy \leq 0, \tag{36}$$

Admit that $|\tilde{D}| > 0$. Our aim is to reach a contradiction to (36). Let $(x, y) \in \tilde{D}\times\tilde{D}^c$, that is, $\underline{u}(x) > u(x)$ and $\underline{u}(y) \leq u(y)$, hence $\underline{u}(x) - \underline{u}(y) - (u(x) - u(y)) > 0$. It follows that $\tilde{S}(x, y) > 0$, which in turn gives

$$\tilde{S}(x,y)((\underline{u}-u)^+(x)-(\underline{u}-u)^+(y)) = \tilde{S}(x,y)(\underline{u}(x)-u(x)) > 0 \tag{37}$$

for all $(x, y) \in D \times D^c$. Let $(x, y) \in \tilde{D}^c \times \tilde{D}$. We have $\underline{u}(x) \leq u(x)$ and $\underline{u}(y) > u(y)$ that reflect in $\underline{u}(x) - \underline{u}(y) - (u(x) - u(y)) < 0$, and $\tilde{S}(x, y) < 0$. We find that

$$\tilde{S}(x,y)((\underline{u}-u)^+(x)-(\underline{u}-u)^+(y)) = -\tilde{S}(x,y)(\underline{u}(y)-u(y)) > 0 \tag{38}$$

for all $(x, y) \in \tilde{D}^c \times \tilde{D}$. On the basis of (36), (37) and (38), a contradiction arises. It implies $|\tilde{D}| = 0$, thereby $\underline{u} \leq u$ a.e in $\mathbb{R}^N$.

Altogether we have $\underline{u} \leq u \leq \overline{u}$ a.e. in $\mathbb{R}^N$. Then using (25) we are able to replace Tu by u in (30). As a result, the solution u of the auxiliary problem (30) becomes a (weak) solution to the original problem (1)–(2) in the sense of (3). The proof is complete. □

References

[1] V. Ambrosio, T. Isernia: *On a fractional p&q Laplacian problem with critical Sobolev-Hardy exponents*, Mediterranean J. Math. 50 (2018), art. no. 219, 17 pp.

[2] Z. Binlin, G. Molica Bisci, R. Servadei: *Superlinear nonlocal fractional problems with infinitely many solutions*, Nonlinearity 28 (2015) 2247–2264.

[3] H. Brezis: *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer, New York (2011).

[4] L. Caffarelli: *Non-local diffusions, drifts and games*, in: *Nonlinear Partial Differential Equations*, Abel Symposia vol. 7, Springer, Heidelberg (2012) 37–52.

[5] S. Carl, V. K. Le, D. Motreanu: *Nonsmooth Variational Problems and Their Inequalities. Comparison Principles and Applications*, Springer Monographs in Mathematics, Springer, New York (2007).

[6] A. Chinni, A. Sciammetta, E. Tornatore: *On the sub-supersolution approach for Dirichlet problems driven by a $(p(x), q(x))$-Laplacian operator with convection term*, Minimax Theory Appl. 6 (2021) 155–172.

[7] E. Di Nezza, G. Palatucci, E. Valdinoci: *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. 136 (2012) 521–573.

[8] D. Goel, D. Kumar, K. Sreenadh: *Regularity and multiplicity results for fractional (p,q)-Laplacian equations*, Comm. Contemp. Math. 22 (2020), art. no. 1950065.

[9] A. Iannizzotto, S. Liu, K. Perera, M. Squassina: *Existence results for fractional p-Laplacian problems via Morse theory*, Adv. Calc. Var. 9 (2016) 101–125.

[10] A. Iannizzotto, S. J. N. Mosconi, M. Squassina: *Global Hölder regularity for the fractional p-Laplacian*, Rev. Mat. Iberoam. 32 (2016) 1353–1392.

[11] A. Li, C. Wei: *On fractional p-Laplacian problems with local conditions*, Adv. Nonlinear Analysis 7 (2018) 485–496.

[12] P. Mironescu, W. Sickel: *A Sobolev non embedding*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 26 (2015) 291–298.

[13] O. H. Miyagaki, D. Motreanu, F. R. Pereira: *Multiple solutions for a fractional elliptic problem with critical growth*, J. Diff. Equations 269 (2020) 5542–5572.

[14] G. Molica Bisci, V. D. Radulescu, R. Servadei: *Variational Methods for Nonlocal Fractional Problems*, Encyclopedia of Mathematics and its Applications 162, Cambridge University Press, Cambridge (2016).

[15] D. Motreanu: *Nonlinear Differential Problems with Smooth and Nonsmooth Constraints*, Mathematical Analysis and its Applications, Academic Press, London (2018).

[16] D. Motreanu, V. V. Motreanu: *Non-variational elliptic equations involving (p, q)-Laplacian, convection and convolution*, Pure Appl. Funct. Analysis 5 (2020) 1205–1215.

[17] D. Motreanu, M. Tanaka: *On a positive solution for (p, q)-Laplace equation with indefinite weight*, Minimax Theory Appl. 1 (2016) 1–20.

[18] D. Mugnai, E. Proietti Lippi: *Neumann fractional p-Laplacian: eigenvalues and existence results*, Nonlinear Analysis 188 (2019) 455–474.

[19] R. E. Showalter: *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, Mathematical Surveys and Monographs 49, American Mathematical Society, Providence (1997).

[20] M. Warma: *Local Lipschitz continuity of the inverse of the fractional p-Laplacian, Hölder type continuity and continuous dependence of solutions to associated parabolic equations on bounded domains*, Nonlinear Analysis 135 (2016) 129–157.

[21] Z. Zhi, Z. Yang: *On a fractional $p - q$ Laplacian equation with critical nonlinearity*, J. Inequalities Appl. (2020), art. no. 183, 13 pp.