

Characterizing the Convexity of Joint-Range for a Pair of Inhomogeneous Quadratic Functions and Strong Duality

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We establish various extensions of the convexity Dines theorem for a (joint-range) pair of inhomogeneous quadratic functions. If convexity fails we describe those rays for which the sum of the joint-range and the ray is convex. Afterwards, we derive a characterization of the convexity of the joint-range itself. The convexity Dines theorem for a pair of homogeneous quadratic functions and its extension for inhomogeneous functions due to Polyak are re-obtained as consequences. These results are suitable for dealing nonconvex inhomogeneous quadratic optimization problems under one quadratic equality or inequality constraint. As applications of our main results, different sufficient conditions for the validity of S-lemma (a nonstrict version of Finsler's theorem) for inhomogeneous quadratic functions, are presented, as well as a new characterization of strong duality (which is a minimax-type result) under Slater-type condition is established.

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1. Introduction

Quadratic functions have proved to be very important in mathematics because of their consequences in various subjects like calculus of variations, mathematical programming, matrix theory (related to matrix pencil), geometry and special relativity [24, 21, 42, 22, 20, 26, 4, 15], among others, and applications in Applied sciences: telecommunications, robust control, [33], trust region problems [19, 40].

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The lack of convexity always offers a nice challenge in mathematics, but sometimes, as occurs in the quadratic world, hidden convexity is present. It seems to be that one of the first results for quadratic forms is due to Finsler [16], known as (strict) Finsler's theorem, which refers to positive definiteness of a matrix pencil. The same result was proved, independently, by the Chicago's School under the guidance of Bliss. We quote Albert [1], Reid, [39], Dines [14], Calabi [11], Hestenes [23].

Perhaps the first beautiful results for a pair of quadratic forms are due to Dines [14] and Brickman [10], proving the convexity, respectively, of

$$\{(\langle Ax, x \rangle, \langle Bx, x \rangle) \in \mathbb{R}^2 : x \in \mathbb{R}^n\}, \quad (1)$$

$$\{(\langle Ax, x \rangle, \langle Bx, x \rangle) \in \mathbb{R}^2 : \langle x, x \rangle = 1, x \in \mathbb{R}^n\} \quad (n \geq 3), \quad (2)$$

provided A and B are real symmetric matrices. Actually Dines, motivated by the above result due to Finsler, searched the convexity in (1). This convexity property inspired many researchers for searching hidden convexity in the quadratic framework. Generalizations to more than two matrices were developed in [4, 37, 21, 25, 13, 36], and references therein, without being completed. It is well known that, in general, $(f, g)(\mathbb{R}^n)$ is nonconvex if f and g are inhomogeneous quadratic functions.

In the next paragraph we highlight briefly the main contributions in the paper.

Precisely, our interest is to consider a pair of inhomogeneous quadratic functions f and g , and to characterize the convexity of $(f, g)(\mathbb{R}^n)$ (Theorem 4.16), and that of $(f, g)(\mathbb{R}^n) + \mathbb{R}_+d$ (Theorem 4.14), given $d \in \mathbb{R}^2$, $d \neq 0$. As a byproduct we obtain the convexity of $(f, g)(\mathbb{R}^n) + \text{int } P$ (Theorem 4.19), or equivalently of $(f, g)(\mathbb{R}^n) + P$, for every convex cone $P \subseteq \mathbb{R}^2$ with non empty interior (for instance $P = \mathbb{R}_+^2$). Hence, Theorem 2.2 in [37] due to Polyak (a non-homogeneous case) and the Dines convexity theorem are recovered. Finally, looking at the proofs about the nonconvexity of $(f, g)(\mathbb{R}^n) + \mathbb{R}_+d$, we realize that our method is constructive and therefore may be used to describe completely such a property, see for instance Remark 4.17.

We point out that the convexity of $C \doteq (f, g)(\mathbb{R}^n) + \mathbb{R}_+^2$ was stated in Corollary 10 of [44], but its proof is not correct since the set C is not closed in general: Examples 3.5 and 5.15 show this fact. On the other hand, we mention the recent paper [29] where it is proved, under suitable assumptions, the convexity of $(f, g_0, h_1, \dots, h_m)(\mathbb{R}^n) + \mathbb{R}_+^{m+2}$ with f being any quadratic function, g_0 (quadratic) strictly convex and all the other functions h_i affine linear. Another joint-range convexity result involving Z -matrices may be found in [28].

Additionally, by exploiting the algebraic characterization of the convexity of $(f, g)(\mathbb{R}^n) + \mathbb{R}_+(1, 0)$, we derive some (new) sufficient conditions for the validity of an S-lemma with an equality constraint (a nonstrict version of Finsler's theorem for inhomogeneous quadratic functions). These conditions are expressed in a different way than that established in [47], where the fulfillment of such an

S-lemma is algebraically characterized. This characterization is applied to provide necessary and sufficient conditions for the the convexity of $(f, g)(\mathbb{R}^n)$, in the special case when g is affine by using a completely different approach. The S-lemma just mentioned is suitable for dealing with the problem

$$\inf\{f(x) : g(x) = 0, x \in \mathbb{R}^n\}, \tag{3}$$

and it is also useful for dealing with bounded generalized trust region subproblems, that is, with constraints $l \leq g(x) \leq u$, as shown in [47].

We also provide a new characterization of strong duality for (3) under a Slater-type condition, see Corollary 5.6.

A complete description (besides the convexity) of the set

$$\text{cone}((f, g)(\mathbb{R}^n) - \mu(1, 0) + \mathbb{R}_+^2),$$

where

$$\mu \doteq \inf\{f(x) : g(x) \leq 0, x \in \mathbb{R}^n\}, \tag{4}$$

for any pair of inhomogeneous quadratic functions f and g , is given in [17] by assuming μ to be finite. When f and g are any real-valued functions, strong duality for (4) implies the convexity of $\overline{\text{cone}}((f, g)(\mathbb{R}^n) - \mu(1, 0) + \mathbb{R}_+^2)$ as shown in [18]. In case an equality instead of inequality in (4) is considered, we refer to the recent work [12].

It is worthwhile mentioning that the existence of solution for (4) was fully analyzed in [5] under simultaneous diagonalization (SD).

Apart from the characterizations of strong duality, several sufficient conditions of the zero duality gap for convex programs have been established in the literature, see [2, 3, 51, 7, 8, 9, 43, 35].

The paper is structured as follows. Section 2 provides the necessary notations, definitions and some preliminaries to be used throughout the paper: in particular, the Dines theorem is recalled. Some characterizations of bi-dimensional Simultaneous Diagonalization (SD) and Non Degenerate (ND) properties for a pair of matrices are established in Section 3. Section 4 contains our main results: firstly, given $d \in \mathbb{R}^2, d \neq 0$, it is characterized the nonconvexity of $(f, g)(\mathbb{R}^n) + \mathbb{R}_+d$, and secondly, necessary and sufficient conditions for the convexity of $(f, g)(\mathbb{R}^n)$ are also established. Thus, the Dines convexity theorem for homogeneous functions is recovered, as well as Theorem 2.2 in [37] for inhomogeneous functions. An application to characterize the validity of strong duality, under a Slater condition, of nonconvex quadratic optimization under a single equality constraint is presented in Section 5 (Corollary 5.6). Finally, Section 6 presents, for reader’s convenience, a brief historical note about the appearance, in a chronological order, of the several properties arising in the study of quadratic forms. Some relationships between those properties are also outlined.

2. Basic notations and some preliminaries

In this section we introduce the basic definitions, notations and some preliminary results.

Given any nonempty set $K \subseteq \mathbb{R}^n$, its closure is denoted by \overline{K} ; its convex hull by $\text{co}(K)$ which is the smallest convex set containing K ; its topological interior by $\text{int } K$, whereas its relative interior by $\text{ri } K$, it is the interior with respect to its affine set; the (topological) boundary of K is denoted by $\text{bd } K$. We denote the complement of K by $\mathcal{C}(K)$. We set $\text{cone}(K) \doteq \bigcup_{t \geq 0} tK$, being the smallest cone

containing K , and $\overline{\text{cone}}(K) \doteq \overline{\bigcup_{t \geq 0} tK}$. In case $K = \{u\}$, we denote $\text{cone } K = \mathbb{R}_+ u$

and $\mathbb{R}u \doteq \{tu : t \in \mathbb{R}\}$, where $\mathbb{R}_+ \doteq [0, +\infty[$; thus $\mathbb{R}_+ u$ stands for the ray starting at the origin along the direction u . Furthermore, K^* stands for the (non-negative) polar cone of K which is defined by

$$K^* \doteq \{\xi \in \mathbb{R}^n : \langle \xi, a \rangle \geq 0 \ \forall a \in K\},$$

where $\langle \cdot, \cdot \rangle$ means the scalar or inner product in \mathbb{R}^n , whose elements are considered column vectors. Thus, $\langle a, b \rangle = a^\top b$ for all $a, b \in \mathbb{R}^n$. By K^\perp we mean the orthogonal subspace to K , given by $K^\perp = \{u \in \mathbb{R}^n : \langle u, v \rangle = 0 \ \forall v \in K\}$; in case $K = \{u\}$, we simply put u^\perp . We say P is a cone if $tP \subseteq P$ for all $t \geq 0$, and it is pointed if $P \cap (-P) = \{0\}$ whenever P is convex.

Throughout this paper the matrices are always with real entries. Given any matrix A or order $m \times n$, A^\top stands for the transpose of A ; whereas if A is a symmetric square matrix of order n , we say it is positive semidefinite, denoted by $A \succcurlyeq 0$, if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathbb{R}^n$; it is positive definite, denoted by $A \succ 0$ if $\langle Ax, x \rangle > 0$ for all $x \in \mathbb{R}^n$, $x \neq 0$. The set of symmetric square matrices of order n is denoted by \mathcal{S}^n .

Given any quadratic function

$$f(x) = \langle Ax, x \rangle + \langle a, x \rangle + k_1,$$

for some $A \in \mathcal{S}^n$, $a \in \mathbb{R}^n$ and $k_1 \in \mathbb{R}$, we set

$$f_H(x) \doteq \langle Ax, x \rangle, \quad f_L(x) \doteq \langle a, x \rangle.$$

If we are given another quadratic function

$$g(x) = \langle Bx, x \rangle + \langle b, x \rangle + k_2,$$

for some $B \in \mathcal{S}^n$, $b \in \mathbb{R}^n$ and $k_2 \in \mathbb{R}$. Set

$$z_{u,v} \doteq \begin{pmatrix} \langle Au, v \rangle \\ \langle Bu, v \rangle \end{pmatrix}, \quad F_H(u) \doteq \begin{pmatrix} f_H(u) \\ g_H(u) \end{pmatrix}. \quad (5)$$

An important property in matrix analysis and in the study of nonconvex quadratic programming, is that of Simultaneous Diagonalization. We say that any two

matrices A, B in \mathcal{S}^n have the Simultaneous Diagonalization (SD) property, simple simultaneous diagonalizable, if there exists a nonsingular matrix C such that both $C^\top AC$ and $C^\top BC$ are diagonal [26, Section 7.6], that is, if there are linearly independent (LI) vectors $u_i \in \mathbb{R}^n, i = 1, \dots, n$, such that $z_{u_i, u_j} = 0, i \neq j$. Such an assumption, for instance, allowed the authors in [6] to re-write the original problem in a more tractable one. The symbol LD stands for linear dependence.

It is said that A and B are Non Degenerate (ND) if

$$\langle Au, u \rangle = 0 = \langle Bu, u \rangle \implies u = 0. \tag{6}$$

One of the most important results concerning quadratic functions refers to Dine’s theorem [14], it perhaps motivated by Finsler’s theorem [16].

Theorem 2.1. [14, Theorem 1, Theorem 2] [23, Theorem 2] *The set $F_H(\mathbb{R}^n)$ is a convex cone. In addition, if (6) holds then either $F_H(\mathbb{R}^n) = \mathbb{R}^2$ or $F_H(\mathbb{R}^n)$ is closed and pointed.*

The convexity may fail for $F(\mathbb{R}^n)$ if $F(x) = (f(x), g(x))$ with f, g being not necessarily homogeneous quadratic functions, as the next example shows.

Example 2.2. Consider $f(x_1, x_2) = x_1 + x_2 - x_1^2 - x_2^2 - 2x_1x_2, g(x_1, x_2) = x_1^2 + x_2^2 + 2x_1x_2 - 1$, and define the set $M \doteq \{(f(x_1, x_2), g(x_1, x_2)) \in \mathbb{R}^2 : (x_1, x_2) \in \mathbb{R}^2\}$. Clearly $(0, 0) = (f(0, 1), g(0, 1)) \in M$ and $(-2, 0) = (f(-1, 0), g(-1, 0)) \in M$, but $(-1, 0) = \frac{1}{2}(0, 0) + \frac{1}{2}(-2, 0) \notin F(\mathbb{R}^2)$. One can actually see that

$$F_H(\mathbb{R}^2) = \mathbb{R}_+(-1, 1), \text{ and } F(\mathbb{R}^2) = \{(t - t^2, t^2 - 1) : t \in \mathbb{R}\}.$$

Another instance is Example 4.3, where

$$F(\mathbb{R}^2) = \{(0, 0) \cup [\mathbb{R}^2 \setminus (\mathbb{R} \times \{0\})], F_H(\mathbb{R}^2) = \mathbb{R} \times \{0\}.$$

Notice that $F(\mathbb{R}^2) + \mathbb{R}_+d$ still remains nonconvex whenever $-d \in \text{bd } F_H(\mathbb{R}^2) = F_H(\mathbb{R}^2)$.

We now state a simple result which will be used in the next sections. For any $u = (u_1, u_2) \in \mathbb{R}^2$, set $u_\perp \doteq (-u_2, u_1)$, so that $\|u\| = \|u_\perp\|$ and $\langle u_\perp, u \rangle = 0$.

Proposition 2.3. *Let $u, v \in \mathbb{R}^2$. The following hold*

- (a) $\langle v_\perp, u \rangle = -\langle u_\perp, v \rangle$;
- (b) $\langle v_\perp, u \rangle \neq 0 \iff \{u, v\}$ is LI.
- (c) Assume that $\{u, v\}$ is LI. Then
 - (c1) $h = t_1u + t_2v, t_2 \geq 0$ (resp. $t_2 > 0$) $\iff \langle u_\perp, v \rangle \langle u_\perp, h \rangle \geq 0$ (resp. > 0);
 - (c2) $h = t_1u + t_2v, t_1 \geq 0, t_2 \geq 0 \iff \langle u_\perp, v \rangle \langle u_\perp, h \rangle \geq 0$ and $\langle v_\perp, u \rangle \langle v_\perp, h \rangle \geq 0$.

Finally, the next lemma which is important by itself will play an important role in the subsequent sections.

Lemma 2.4. *Let $X \subseteq \mathbb{R}^n$ be a nonempty subset of \mathbb{R}^n and $h \neq 0$, h_1 , be any elements in \mathbb{R}^2 such that*

$$F(X) + \mathbb{R}h + \mathbb{R}_+h_1 \subseteq F(\mathbb{R}^n). \tag{7}$$

Then $F(\mathbb{R}^n)$ is convex under any of the following circumstances:

- (a) $\{h_1, h\}$ is LI and $\overline{X} = \mathbb{R}^n$;
- (b) $\{h_1, h\}$ is LD and $X = \mathbb{R}^n$.

Proof. Let $0 < t < 1$ and $x, y \in \mathbb{R}^n$ with $F(x) \neq F(y)$. The desired result is obtained by showing that $f_t \doteq tF(x) + (1 - t)F(y) \in F(\mathbb{R}^n)$.

(a): By assumption $\langle h_\perp, h_1 \rangle \neq 0$, and therefore, from (7) and Proposition 2.3 one gets, for all $x_0 \in X$,

$$H(x_0) \doteq \{u : \langle h_\perp, h_1 \rangle \langle h_\perp, u - F(x_0) \rangle > 0\} \subseteq F(\mathbb{R}^n). \tag{8}$$

The desired result is obtained by showing that $f_t \doteq tF(x) + (1 - t)F(y) \in H(x_0)$ for some $x_0 \in X$. We distinguish two cases.

(a1): $\langle h_\perp, h_1 \rangle \langle h_\perp, F(y) - F(x) \rangle > 0$ (the case “ $<$ ” is entirely similar). Since

$$\langle h_\perp, h_1 \rangle \langle h_\perp, f_t - F(x) \rangle > 0,$$

by densedness and continuity, we get $\bar{x} \in X$ close to x such that $f_t \in H(\bar{x})$, and so $f_t \in F(\mathbb{R}^n)$ by (8).

(a2): $\langle h_\perp, h_1 \rangle \langle h_\perp, F(y) - F(x) \rangle = 0$. Let us consider the functions $q_1 : \mathbb{R} \rightarrow \mathbb{R}^2$ and $q : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$q_1(\lambda) \doteq F(\lambda x + (1 - \lambda)y), \quad q(\lambda) \doteq \langle h_\perp, h_1 \rangle \langle h_\perp, q_1(\lambda) - F(x) \rangle.$$

Clearly q is quadratic satisfying $q(0) = q(1) = 0$. Let us consider first that $q \equiv 0$. Due to continuity $q_1([0, 1])$ is a connected set contained in the line $F(x) + \mathbb{R}h$ passing through $F(x)$ and $F(y)$. Thus, $f_t \in q_1([0, 1]) \subseteq F(\mathbb{R}^n)$.

We now consider $q \not\equiv 0$. Then there exists $\lambda_1 \in \mathbb{R}$ satisfying $q(\lambda_1) < 0$, i. e.,

$$\langle h_\perp, h_1 \rangle \langle h_\perp, F(\lambda_1 x + (1 - \lambda_1)y) - f_t \rangle = \langle h_\perp, h_1 \rangle \langle h_\perp, F(\lambda_1 x + (1 - \lambda_1)y) - F(x) \rangle < 0.$$

Hence by taking $\bar{x} \in X$ near $\lambda_1 x + (1 - \lambda_1)y$, we obtain $\langle h_\perp, h_1 \rangle \langle h_\perp, f_t - F(\bar{x}) \rangle > 0$, and so $f_t \in F(\mathbb{R}^n)$ by (8).

(b): As $\{h_1, h\}$ is LD, then (7) means that for all $x_0 \in X$,

$$H_0(x_0) \doteq \{u \in \mathbb{R}^2 : \langle h_\perp, u - F(x_0) \rangle = 0\} \subseteq F(\mathbb{R}^n).$$

Let $q(\lambda) = \langle h_\perp, F(\lambda x + (1 - \lambda)y) - f_t \rangle$. Then q is continuous and $q(0) = t \langle h_\perp, F(y) - F(x) \rangle$, $q(1) = (1 - t) \langle h_\perp, F(x) - F(y) \rangle$. We observe that either $q(0) = 0 = q(1)$ or $q(0)q(1) < 0$. In the first case $q(\lambda) = 0$ for all $\lambda \in \mathbb{R}$, and so $f_t \in F(\mathbb{R}^n)$. In case of opposite sign, we get $\lambda_0 \in]0, 1[$ such that $q(\lambda_0) = 0$, which implies that $f_t \in H_0(\lambda_0 x + (1 - \lambda_0)y) \subseteq F(\mathbb{R}^n)$. \square

3. Characterizing SD and ND in two dimensional spaces

This section is devoted to characterizing the simultaneous diagonalization and non degenerate properties for a pair of matrices in terms of its homogeneous quadratic forms. As one may found in the literature, the study in \mathbb{R}^2 deserves a special treatment from \mathbb{R}^n , $n \geq 3$, and to the best knowledge of these authors the following characterizations are new. As said before, here $A, B \in \mathcal{S}^2$.

We start by a simple proposition appearing elsewhere whose proof is presented here just for reader's convenience.

Proposition 3.1. *Let us consider the assertions:*

- (a) $F_H(\mathbb{R}^2) = \mathbb{R}^2$;
- (b) ND holds for A and B ;
- (c) $F_H(\mathbb{R}^2)$ is closed.

Then (a) \implies (b) \implies (c).

Proof. (a) \implies (b): Let $u \in \mathbb{R}^2$ satisfying $F_H(u) = 0$. If on the contrary $u \neq 0$, then by taking $v \in \mathbb{R}^2$ such that $\{u, v\}$ is linearly independent, we obtain for $\alpha, \beta \in \mathbb{R}$,

$$F_H(\alpha u + \beta v) = \alpha^2 F_H(u) + \beta^2 F_H(v) + 2\alpha\beta z_{u,v}.$$

Thus $F_H(\mathbb{R}^2) \subseteq \mathbb{R}_+ F_H(v) + \mathbb{R} z_{u,v}$, which is impossible if $F_H(\mathbb{R}^2) = \mathbb{R}^2$.

(b) \implies (c): Let $F_H(x_k)$ be a sequence such that $F_H(x_k) \rightarrow z$. In case $\|x_k\|$ is bounded, there is nothing to do. If $\|x_k\|$ is unbounded, up to a subsequence, we may suppose that $\|x_k\| \rightarrow +\infty$ and $\frac{x_k}{\|x_k\|} \rightarrow u$. Thus $\|u\| = 1$ and

$$\frac{1}{\|x_k\|^2} F_H(x_k) = F_H\left(\frac{x_k}{\|x_k\|}\right) \rightarrow F_H(u) = 0,$$

which yields, by assumption, $u = 0$, a contradiction. □

Example 3.2 below shows that (a) \implies (b) may fail in higher dimension. However, for $n \geq 3$, one obtains that (a) implies the existence of $u \in \mathbb{R}^n$, $u \neq 0$, such that $F_H(u) = 0$, as Corollary in [23, p. 401] shows. We also point out that the proof for proving (b) implies (c) remains valid for any dimension, see also Theorem 6 in [23].

Example 3.2. Take

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

Then, $F_H(\mathbb{R}^3) = \mathbb{R}^2$, but ND does not hold for A and B .

The next result provides a new characterization for SD in two dimension.

Theorem 3.3. *The following statements are equivalent:*

- (a) SD holds for A and B ;
- (b) $\exists u, v \in \mathbb{R}^2$ such that $F_H(\mathbb{R}^2) = \mathbb{R}_+u + \mathbb{R}_+v$;
- (c) $F_H(\mathbb{R}^2)$ is closed and $F_H(\mathbb{R}^2) \neq \mathbb{R}^2$.

Proof. (a) \implies (b): By assumption, there exist LI vectors $x, y \in \mathbb{R}^2$, such that $z_{x,y} = 0$. Thus, $F_H(\mathbb{R}^2) = \{F_H(\alpha x + \beta y) : \alpha, \beta \in \mathbb{R}\}$. From the equality $F_H(\alpha x + \beta y) = \alpha^2 F_H(x) + \beta^2 F_H(y)$ the desired conclusion is obtained.

(b) \implies (c): is straightforward.

(c) \implies (a): We already know that $F_H(\mathbb{R}^2)$ is a convex cone. We first check that $F_H(\mathbb{R}^2)$ cannot be a halfspace. Indeed, suppose that $F_H(\mathbb{R}^2) = \{y \in \mathbb{R}^2 : \langle p, y \rangle \geq 0\}$ for some $p \in \mathbb{R}^2, p \neq 0$. Then there exist $u, v \in \mathbb{R}^2$ such that $F_H(u) = p_\perp$ and $F_H(v) = -p_\perp$, which imply that $\{u, v\}$ is LI. Since $F_H(\alpha u + \beta v) = \alpha^2 F_H(u) + \beta^2 F_H(v) + 2\alpha\beta z_{u,v}$ for all $\alpha, \beta \in \mathbb{R}$, we get $2\alpha\beta \langle p, z_{u,v} \rangle \geq 0$ for all $\alpha, \beta \in \mathbb{R}$. Hence $\langle p, z_{u,v} \rangle = 0$, and therefore $F_H(\mathbb{R}^2) = \{y \in \mathbb{R}^2 : \langle p, y \rangle = 0\}$. Thus, the set $F_H(\mathbb{R}^2)$ may be (i) the origin $\{0\}$; (ii) a ray; (iii) a pointed cone, (iv) a straight line.

(i): We simply take any two LI vectors u and v . Indeed, since $F_H(u) = F_H(v) = F_H(u+v) = 0$, we obtain $z_{u,v} = 0$.

(ii): Assume that $F_H(\mathbb{R}^2) = \mathbb{R}_+p$, and take $u \in \mathbb{R}^2$ such that $F_H(u) = p$, and choose $v \in \mathbb{R}^2$ so that $\{u, v\}$ is LI. In case $z_{u,v} \neq 0$, we proceed as follows. Since $F_H(u+v) = F_H(u) + F_H(v) + 2z_{u,v}$, we obtain $0 = \langle p_\perp, z_{u,v} \rangle$, which implies that $z_{u,v} = \lambda p$ for some $\lambda \in \mathbb{R}$. It follows that $z_{u,v-\lambda u} = 0$ with $\{u, v-\lambda u\}$ being LI, and therefore SD holds.

(iii): We have, for some LI vectors p, q (see Proposition 2.3)

$$F_H(\mathbb{R}^2) = \mathbb{R}_+p + \mathbb{R}_+q = \{z \in \mathbb{R}^2 : \langle p_\perp, q \rangle \langle p_\perp, z \rangle \geq 0, \langle q_\perp, p \rangle \langle q_\perp, z \rangle \geq 0\}, \quad (9)$$

with the property $\langle p_\perp, q \rangle = -\langle q_\perp, p \rangle \neq 0$. Take u, v in \mathbb{R}^2 satisfying $F_H(u) = p, F_H(v) = q$. It follows that u and v are LI. From (9), we get in particular, $\langle p_\perp, q \rangle \langle p_\perp, F_H(tu+v) \rangle \geq 0$, for all $t \in \mathbb{R}$. This implies that $\langle p_\perp, z_{u,v} \rangle = 0$. Similarly one obtains $\langle q_\perp, z_{u,v} \rangle = 0$. Thus $z_{u,v} = 0$, which is the desired result.

(iv): This case is similar to (ii). Take $u, v \in \mathbb{R}^2$ such that $F_H(u) = p_\perp, F_H(v) = -p_\perp$, which imply that $\{u, v\}$ is LI. Hence $\{u, v-\lambda u\}$ is LI for some $\lambda \in \mathbb{R}$ and $z_{u,v-\lambda u} = 0$. □

The next example illustrates that u and v need not to be LI in the previous theorem; Example 3.2 shows that (a) does not imply (b) in higher dimension, since we get $\mathbb{R}^2 = F_H(\mathbb{R}^3) = \mathbb{R}_+(1, 0) + \mathbb{R}_+(0, 1) + \mathbb{R}_+(-1, -1)$, and clearly SD holds for A and B ; whereas Example 3.5 exhibits an instance where without the closedness of $F_H(\mathbb{R}^2)$ the implication (c) \implies (a) may fail.

Example 3.4. Take

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = 0, \quad C = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

After some computations, we get that $C^T AC$ is diagonal. It is easy to see that

$$F_H(\mathbb{R}^2) = \mathbb{R}_+(1, 0) + \mathbb{R}_+(-1, 0) = \mathbb{R} \times \{0\}.$$

Example 3.5. Consider

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then, even if

$$F_H(x_1, x_2) = (x_1 + x_2)^2(1, 0) + (x_1^2 - x_2^2)(0, 1),$$

one obtains $F_H(\mathbb{R}^2) = (\mathbb{R}_{++} \times \mathbb{R}) \cup \{(0, 0)\}$, which is not closed and clearly SD does not hold for A and B .

We are now in a position to establish a new characterization for ND in \mathbb{R}^2 .

Theorem 3.6. *The following assertions are equivalent:*

- (a) ND holds for A and B ;
- (b) $\ker A \cap \ker B = \{0\}$ and $F_H(\mathbb{R}^2)$ is a closed set different from a line.

Proof. (a) \implies (b): The first part of (b) is straightforward, and the closedness of $F_H(\mathbb{R}^2)$ is a consequence of Proposition 3.1. It remains only to prove that $F_H(\mathbb{R}^2)$ is different from a line. In case $F_H(\mathbb{R}^2) = \mathbb{R}^2$, we are done; thus suppose that $F_H(\mathbb{R}^2) \neq \mathbb{R}^2$. By Theorem 3.3, we have SD, that is, there exist $u, v \in \mathbb{R}^2$, LI, such that $z_{u,v} = 0$. This means $F_H(\alpha u + \beta v) = \alpha^2 F_H(u) + \beta^2 F_H(v)$ for all $\alpha, \beta \in \mathbb{R}$. Obviously $F_H(u) \neq 0 \neq F_H(v)$, and if $F_H(u) = -\lambda^2 F_H(v)$ for some $\lambda \neq 0$, then $F_H(u + \lambda v) = 0$. This implies that $u + \lambda v = 0$ which is impossible, therefore $F_H(u) \neq -\lambda^2 F_H(v)$ for all $\lambda \neq 0$. Thus $F_H(\mathbb{R}^2) = \{\alpha^2 F_H(u) + \beta^2 F_H(v) : \alpha, \beta \in \mathbb{R}\}$ is not a line.

(b) \implies (a): Since $F_H(\mathbb{R}^2)$ is closed, by Theorem 3.3, either $F_H(\mathbb{R}^2) = \mathbb{R}^2$ or SD holds. In the first case, Proposition 3.1 implies that (a) is satisfied. Assume that SD holds, as before, there exist $u, v \in \mathbb{R}^2$, LI, such that $z_{u,v} = 0$. Let $w \in \mathbb{R}^2$ satisfying $F_H(w) = 0$, we claim that $w = 0$. By writing $w = \lambda_1 u + \lambda_2 v$ for some $\lambda_i \in \mathbb{R}$, $i = 1, 2$, we get $F_H(w) = \lambda_1^2 F_H(u) + \lambda_2^2 F_H(v)$. We distinguish various situations. If $F_H(u) = 0$ (resp. $F_H(v) = 0$), then $\langle Au, u \rangle = 0$ and $\langle Bu, u \rangle = 0$ (resp. $\langle Av, v \rangle$ and $\langle Bv, v \rangle = 0$), which along with $\langle Au, v \rangle = 0$ and $\langle Bu, v \rangle = 0$, allow us to infer $Au = 0 = Bu$ (resp. $Av = 0 = Bv$). It follows that $u = 0$ (resp. $v = 0$), which is impossible.

We now consider $F_H(u) \neq 0 \neq F_H(v)$. Suppose, on the contrary, that $\lambda_i \neq 0$ for $i = 1, 2$. Then, from $F_H(w) = \lambda_1^2 F_H(u) + \lambda_2^2 F_H(v) = 0$, we obtain $F_H(u) = -\lambda F_H(v)$ for some $\lambda > 0$. This yields that $F_H(\mathbb{R}^2) = \{\alpha^2 F_H(u) + \beta^2 F_H(v) : \alpha, \beta \in \mathbb{R}\}$ is a line, a contradiction. Hence $\lambda_i = 0$ for $i = 1, 2$, and so $w = 0$, completing the proof. \square

The same proof of the previous theorem allows us to obtain the next result which establishes a relationship between ND and SD.

Corollary 3.7. *The following assertions are equivalent:*

- (a) $F_H(\mathbb{R}^2) \neq \mathbb{R}^2$ and ND holds;
- (b) $\ker A \cap \ker B = \{0\}$, $F_H(\mathbb{R}^2)$ is different from a line and SD holds;
- (c) $\exists (\lambda_1, \lambda_2) \in \mathbb{R}^2$ such that $\lambda_1 A + \lambda_2 B \succ 0$.

Proof. (a) \implies (b) follows from Theorems 3.3 and 3.6; whereas the reverse implication is derived from the proof of the previous theorem. The equivalence between (a) and (c) is Corollary 1 in [14, page 498], valid for all $n \geq 2$. \square

4. Dines-type theorem for inhomogeneous quadratic functions and relatives

This section is devoted to proving a generalization of Dines theorem for inhomogeneous quadratic functions, see Theorem 4.16. Set

$$f(x) \doteq \langle Ax, x \rangle + \langle a, x \rangle, \quad g(x) \doteq \langle Bx, x \rangle + \langle b, x \rangle, \tag{10}$$

and, as before $F(x) = (f(x), g(x))$, so that $F(0) = (0, 0)$.

We first deal with the one-dimensional case and afterward the general situation.

4.1. The case of one-dimension

We begin with the following useful simple result.

Proposition 4.1. *Let $u \in \mathbb{R}^n$, $u \neq 0$. Then*

- (a) $F(\mathbb{R}u) = \{\alpha^2 F_H(u) + \alpha F_L(u) : \alpha \in \mathbb{R}\}$;
- (b) $\text{co } F(\mathbb{R}u) = F(\mathbb{R}u) + \mathbb{R}_+ F_H(u)$.

Proof. (a) is straightforward and (b) is a consequence of the following equalities:

$$\begin{aligned} tF(\alpha u) + (1-t)F(\beta u) &= [t\alpha^2 + (1-t)\beta^2]F_H(u) + [t\alpha + (1-t)\beta]F_L(u) \\ &= [(t\alpha + (1-t)\beta)^2 + (t-t^2)(\alpha-\beta)^2]F_H(u) + [t\alpha + (1-t)\beta]F_L(u) \\ &= F((t\alpha + (1-t)\beta)u) + (t-t^2)(\alpha-\beta)^2 F_H(u). \end{aligned} \tag{11}$$

Hence, the proof is complete. \square

The one-dimensional version of (inhomogeneous) Dines-type theorem is expressed in the following

Lemma 4.2. *Let $u \in \mathbb{R}^n$, $u \neq 0$ and $0 \neq d \in \mathbb{R}^2$. The following hold:*

- (a) *Assume that $\{F_H(u), F_L(u)\}$ is LD then $F(\mathbb{R}u)$ is convex.*
- (b) *Assume that $\{F_H(u), F_L(u)\}$ is LI. Then*
 - (b1) *if $d = F_H(u)$ one has $F(\mathbb{R}u) + \mathbb{R}_+ d = \text{co } F(\mathbb{R}u) + \mathbb{R}_+ d = \text{co } F(\mathbb{R}u)$;*

(b2) if $d = -F_H(u)$ then

$$F(\mathbb{R}u) + \mathbb{R}_+d = F(\mathbb{R}u) \cup \mathcal{C}(\text{co } F(\mathbb{R}u)) = \overline{\mathcal{C}(\text{co } F(\mathbb{R}u))} \neq \text{co } F(\mathbb{R}u) + \mathbb{R}_+d;$$

(b3) if $\{d, F_H(u)\}$ is LI, one has $F(\mathbb{R}u) + \mathbb{R}_+d = \text{co } F(\mathbb{R}u) + \mathbb{R}_+d \neq \text{co } F(\mathbb{R}u)$.

Similar results hold for the set $F(x + \mathbb{R}u) + \mathbb{R}_+d$ for any fixed $x \in \mathbb{R}^n$ since

$$F(x + tu) = t^2 F_H(u) + t \begin{pmatrix} \langle 2Ax + a, u \rangle \\ \langle 2Bx + b, u \rangle \end{pmatrix} + F(x).$$

Proof. We write $F(tu) = t^2 F_H(u) + t F_L(u)$.

(a): In this case the set $F(\mathbb{R}u)$ is either a point or ray or a line, so convex.

(b1): From Proposition 4.1, we obtain

$$\text{co}(F(\mathbb{R}u) + \mathbb{R}_+d) = \text{co } F(\mathbb{R}u) + \mathbb{R}_+d = F(\mathbb{R}u) + \mathbb{R}_+F_H(u) + \mathbb{R}_+d, \tag{12}$$

from which the convexity of $F(\mathbb{R}u) + \mathbb{R}_+d$ follows if $d = F_H(u)$.

(b2): We obtain the following equalities, thanks to the LI of $\{F_H(u), F_L(u)\}$:

$$\begin{aligned} \text{co } F(\mathbb{R}u) &= \left\{ \sum_{i=1}^3 \lambda_i F(\alpha_i u) : \sum_{i=1}^3 \lambda_i = 1, \lambda_i \geq 0, \alpha_i \in \mathbb{R} \right\} \\ &= \left\{ \sum_{i=1}^3 \lambda_i \alpha_i^2 F_H(u) + \sum_{i=1}^3 \lambda_i \alpha_i F_L(u) : \lambda_i \geq 0, \sum_{i=1}^3 \lambda_i = 1, \alpha_i \in \mathbb{R} \right\} \\ &= \{ \alpha F_H(u) + \beta F_L(u) : \alpha \geq \beta^2, \alpha, \beta \in \mathbb{R} \}. \end{aligned} \tag{13}$$

Thus

$$\mathcal{C}(\text{co } F(\mathbb{R}u)) = \{ \alpha F_H(u) + \beta F_L(u) : \alpha < \beta^2, \alpha, \beta \in \mathbb{R} \}, \text{ and so}$$

$$\overline{\mathcal{C}(\text{co } F(\mathbb{R}u))} = \{ \alpha F_H(u) + \beta F_L(u) : \alpha \leq \beta^2, \alpha, \beta \in \mathbb{R} \} = F(\mathbb{R}u) + \mathbb{R}_+d,$$

since $F(\mathbb{R}u) + \mathbb{R}_+d = \{ (\beta^2 - t)F_H(u) + \beta F_L(u) : \beta \in \mathbb{R}, t \geq 0 \}$ by Propos. 4.1.

(b3): We write $F_L(u) = \lambda_1 F_H(u) + \lambda_2 d$ with $\lambda_2 \neq 0$. By virtue of (12), we need to check that $F(\mathbb{R}u) + \mathbb{R}_+F_H(u) + \mathbb{R}_+d \subseteq F(\mathbb{R}u) + \mathbb{R}_+d$. This requires to solve a quadratic equation, which is always possible. Indeed, take $\alpha \in \mathbb{R}, \lambda_+ \geq 0, \gamma_+ \geq 0$, we must find $\beta \in \mathbb{R}$ and $r_+ > 0$ such that

$$\alpha \lambda_2 + \lambda_+ = \beta \lambda_2 + r_+, \quad \alpha^2 + \alpha \lambda_1 + \gamma_+ = \beta^2 + \beta \lambda_1. \tag{14}$$

We can solve this system by substituting β from the first equation of (14) into the second one, proving the convexity of $F(\mathbb{R}u) + \mathbb{R}_+d$.

Let us check the last assertion. By assumption, we can write $d = \sigma_1 F_H(u) + \sigma_2 F_L(u)$ with $\sigma_2 \neq 0$. From (13), $x \in \text{co } F(\mathbb{R}u)$ if and only if $x = \alpha^2 F_H(u) + \beta F_L(u)$ with $\alpha^2 \geq \beta^2$. By taking $\gamma > 0$ sufficiently large such that $y \doteq F(tu) + \gamma d = [t^2 + \sigma_1 \gamma] u_H + [t + \sigma_2 \gamma] u_L$ with $t^2 + \sigma_1 \gamma < (t + \sigma_2 \gamma)^2$, we get $y \in F(\mathbb{R}u) + \mathbb{R}_+d$ and $y \notin \text{co } F(\mathbb{R}u)$. \square

The next example shows that in fact $F(\mathbb{R}u)$ may be nonconvex for some u , but it becomes convex once a particular direction is added.

Example 4.3. Take

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

with all other data vanishing. Let $u = (u_1, u_2)$, $u_1 u_2 \neq 0$. Then, $F_H(u) = (2u_1 u_2, 0)$, $F_L(u) = (0, u_1)$ and $F(\mathbb{R}u) = \{(x, y) \in \mathbb{R}^2 : x = \frac{2u_2}{u_1} y^2\}$ is nonconvex, but certainly, $F(\mathbb{R}u) + \mathbb{R}_+ d$ is convex if, and only, if $d \neq (-2u_1 u_2, 0)$. Here,

$$F(\mathbb{R}^2) = \{(0, 0\} \cup [\mathbb{R}^2 \setminus (\mathbb{R} \times \{0\})], \quad F_H(\mathbb{R}^2) = \mathbb{R} \times \{0\}.$$

We note that, due to convexity,

$$\overline{F_H(\mathbb{R}^n)} = \mathbb{R}^2 \iff F_H(\mathbb{R}^n) = \mathbb{R}^2 \iff \text{int } F_H(\mathbb{R}^n) = \mathbb{R}^2. \tag{15}$$

As a consequence of the previous lemma we get the characterization of convexity.

Theorem 4.4. *Let $u \in \mathbb{R}^n$, $u \neq 0$, and f, g as above. Then,*

- (a) $F(\mathbb{R}u)$ is convex $\iff \{F_H(u), F_L(u)\}$ is LD;
- (b) in case $\{F_H(u), F_L(u)\}$ is LI and $d \neq 0$, one has

$$F(\mathbb{R}u) + \mathbb{R}_+ d \text{ is convex} \iff -d \notin \mathbb{R}_+ F_H(u).$$

4.2. The case of higher dimension

We first recall the following result due to Polyak:

Theorem 4.5. [37, Theorem 2.2] *If $n \geq 2$ and there exist $\alpha, \beta \in \mathbb{R}$ such that $\alpha A + \beta B \succ 0$, then $F(\mathbb{R}^n)$ is convex (also closed).*

The next theorem is an extension of the previous result. Indeed, Corollary 1 in [14, page 498] establishes

$$\exists \alpha, \beta \in \mathbb{R}, \alpha A + \beta B \succ 0 \iff \text{ND holds and } F_H(\mathbb{R}^n) \neq \mathbb{R}^2. \tag{16}$$

Observe also that in case $F_H(\mathbb{R}^n) = \mathbb{R}^2$, one obtains $F(\mathbb{R}^n) = \mathbb{R}^2$ by Lemma 4.11. We present a self-contained proof of the next theorem, but another proof is as follows. In case $F_H(\mathbb{R}^n) = \mathbb{R}^2$, by Lemma 4.11, $F(\mathbb{R}^n) = \mathbb{R}^2$ as well; if $F_H(\mathbb{R}^n) \neq \mathbb{R}^2$, then from (16) it follows that $\alpha A + \beta B \succ 0$ for some $\alpha, \beta \in \mathbb{R}$, and thus Theorem 4.5 implies the convexity of $F(\mathbb{R}^n)$.

Theorem 4.6. *Assume that ND holds for A and B . Then*

- (a) either $F_H(\mathbb{R}^2) = \mathbb{R}^2$ or $F(\mathbb{R}^2)$ is convex;
- (b) if $n \geq 3$ then $F(\mathbb{R}^n)$ is convex.

Proof. (a): Assume that $F_H(\mathbb{R}^2) \neq \mathbb{R}^2$. From Proposition 3.1 and Theorem 3.3, we get SD for A and B , which means that there exist $\{u, v\}$ LI satisfying $z_{u,v} = 0$. Thus $F(\mathbb{R}^2) = F(\mathbb{R}u) + F(\mathbb{R}v)$. In addition, $F_H(u) \neq 0 \neq F_H(v)$ and by the choice of u and v , $F_H(u) \neq -\rho F_H(v)$ for all $\rho > 0$. We claim that

$$\text{co } F(\mathbb{R}u) + F(\mathbb{R}v) \subseteq F(\mathbb{R}u) + F(\mathbb{R}v). \tag{17}$$

By virtue of Lemma 4.2, we need only to consider $\{F_H(u), F_L(u)\}$ to be LI. We can write for some μ_i and σ_i , $i = 1, 2$, $F_H(v) = \mu_1 F_H(u) + \mu_2 F_L(u)$ and $F_L(v) = \sigma_1 F_H(u) + \sigma_2 F_L(u)$. Take any $x \in \text{co } F(\mathbb{R}u) + F(\mathbb{R}v)$; then, by Lemma 4.1, $x = F(\alpha u) + \gamma^2 F_H(u) + F(\beta v)$ for some $\alpha, \beta \in \mathbb{R}$ and $\gamma \in \mathbb{R}$.

We search for $\lambda_i \in \mathbb{R}$, $i = 1, 2$ satisfying $x = F(\lambda_1 u) + F(\lambda_2 v)$. From the last two equalities, we get

$$\begin{aligned} \lambda_1^2 + \lambda_2^2 \mu_1 + \lambda_2 \sigma_1 - \alpha^2 - \gamma^2 - \beta^2 \mu_1 - \beta \sigma_1 &= 0 \\ \lambda_1 + \lambda_2^2 \mu_2 + \lambda_2 \sigma_2 - \alpha - \beta^2 \mu_2 - \beta \sigma_2 &= 0 \end{aligned}$$

From the second equation, we obtain $\lambda_1 = \alpha + \beta^2 \mu_2 + \beta \sigma_2 - \lambda_2^2 \mu_2 - \lambda_2 \sigma_2$, which is substituted on the left-hand side of the first equation to get a polynomial in λ_2 , say $p(\lambda_2)$. Our goal is to find a zero of p . Observe that $\lambda_2 = \beta$ implies $\lambda_1 = \alpha$ and so $p(\beta) = -\gamma^2 \leq 0$. If $\mu_2 \neq 0$, the higher degree term of p is $\mu_2^2 \lambda^4$ which goes to $+\infty$ as $\lambda_2 \rightarrow +\infty$; if $\mu_2 = 0$, the higher degree term of p is $(\sigma_2^2 + \mu_1) \lambda_2^2$, with μ_1 being positive by the choice of u and v . Thus, in both cases, $p(\lambda_2) > 0$ for λ_2 sufficiently large. Hence, there exists $p(\lambda_2) = 0$, and so (17) is proved. We now check that $\text{co } F(\mathbb{R}^2) = F(\mathbb{R}^2)$. Indeed, it is obtained from the following chain of equalities:

$$\begin{aligned} \text{co } F(\mathbb{R}^2) &= \text{co } F(\mathbb{R}u) + \text{co } F(\mathbb{R}v) = \text{co } F(\mathbb{R}u) + F(\mathbb{R}v) + \mathbb{R}_{++} F_H(v) \\ &= F(\mathbb{R}u) + F(\mathbb{R}v) + \mathbb{R}_{++} F_H(v) = F(\mathbb{R}u) + \text{co } F(\mathbb{R}v) \\ &\subseteq F(\mathbb{R}u) + F(\mathbb{R}v) = F(\mathbb{R}^2). \end{aligned}$$

(b): We will see now how we can reduce to the case $n = 2$, so that (a) is applicable. Let $x, y \in \mathbb{R}^n$ and $t \in]0, 1[$, we have

$$tF(x) + (1 - t)F(y) \in tF(\mathbb{R}x + \mathbb{R}y) + (1 - t)(F(\mathbb{R}x + \mathbb{R}y)). \tag{18}$$

Thus, it suffices to prove the convexity of $F(\mathbb{R}x + \mathbb{R}y)$ whenever $\{x, y\}$ is LI. Take any $\lambda_i \in \mathbb{R}$, $i = 1, 2$, then

$$\begin{aligned} f(\lambda_1 x + \lambda_2 y) &= \lambda_1^2 \langle Ax, x \rangle + 2\lambda_1 \lambda_2 \langle Ax, y \rangle + \lambda_2^2 \langle Ay, y \rangle + \lambda_1 \langle a, x \rangle + \lambda_2 \langle a, y \rangle \\ &= (\lambda_1 \ \lambda_2) \begin{pmatrix} \langle Ax, x \rangle & \langle Ax, y \rangle \\ \langle Ax, y \rangle & \langle Ay, y \rangle \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} + (\langle a, x \rangle \ \langle a, y \rangle) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}. \end{aligned}$$

A similar expression is obtained for g . By denoting

$$\tilde{A}(x, y) \doteq \begin{pmatrix} \langle Ax, x \rangle & \langle Ax, y \rangle \\ \langle Ax, y \rangle & \langle Ay, y \rangle \end{pmatrix}, \quad \tilde{B}(x, y) \doteq \begin{pmatrix} \langle Bx, x \rangle & \langle Bx, y \rangle \\ \langle Bx, y \rangle & \langle By, y \rangle \end{pmatrix},$$

$$\tilde{a}(x, y) \doteq \begin{pmatrix} \langle a, x \rangle \\ \langle a, y \rangle \end{pmatrix}, \quad \tilde{b}(x, y) \doteq \begin{pmatrix} \langle b, x \rangle \\ \langle b, y \rangle \end{pmatrix},$$

we can write

$$\begin{aligned} F(\mathbb{R}x + \mathbb{R}y) &= \left\{ \tilde{F}(\lambda) \doteq \begin{pmatrix} \langle \tilde{A}(x, y)\lambda, \lambda \rangle \\ \langle \tilde{B}(x, y)\lambda, \lambda \rangle \end{pmatrix} + \begin{pmatrix} \langle \tilde{a}(x, y), \lambda \rangle \\ \langle \tilde{b}(x, y), \lambda \rangle \end{pmatrix} : \lambda \in \mathbb{R}^2 \right\} \\ &= \tilde{F}(\mathbb{R}^2). \end{aligned} \tag{19}$$

We want to apply the result in (a) to the set on the right-hand side of (19). It is not difficult to verify that if ND holds for A and B , then ND also holds for $\tilde{A}(x, y)$ and $\tilde{B}(x, y)$ provided $\{x, y\}$ is LI. Furthermore, since $F_H(\mathbb{R}^n) \neq \mathbb{R}^2$, we get $\tilde{F}_H(\mathbb{R}^2) \neq \mathbb{R}^2$. By applying (a), we conclude that $\tilde{F}(\mathbb{R}^2) = F(\mathbb{R}x + \mathbb{R}y)$ is convex, and therefore the convexity of $F(\mathbb{R}^n)$. \square

In order to establish our second main result without ND, some preliminaries are needed.

Proposition 4.7. *Let $n \geq 2$ and $0 \neq v \in \mathbb{R}^n$ such that $F_H(v) = 0$. The following assertions hold:*

- (a) $F_H(\mathbb{R}^2) \neq \mathbb{R}^2$ and $\{Av, Bv\}$ is LD;
- (b) if $n \geq 3$ then either $F_H(\mathbb{R}^n) = \mathbb{R}^2$ or $\{Av, Bv\}$ is LD.
- (c) The set $Z \doteq \{z_{u,v} : u \in \mathbb{R}^n\}$ is a vector subspace, and if $F_H(\mathbb{R}^n) \neq \mathbb{R}^2$ then for all $u \in \mathbb{R}^n$ satisfying $z_{u,v} \neq 0$,

$$\text{bd } F_H(\mathbb{R}^n) = \mathbb{R}z_{u,v}. \tag{20}$$

In particular, if $Av \neq 0$ and $Bv = \lambda Av$ (resp. $Bv \neq 0$ and $Av = \lambda Bv$) for some $\lambda \in \mathbb{R}$, then

$$\text{bd } F_H(\mathbb{R}^n) = \mathbb{R}(1, \lambda) \quad (\text{resp. } \text{bd } F_H(\mathbb{R}^n) = \mathbb{R}(\lambda, 1)). \tag{21}$$

Proof. (a): The first part follows from Proposition 3.1. By assumption $\{Av, Bv\} \subseteq v^\perp$, thus $\{Av, Bv\}$ is LD.

(b): Again $\{Av, Bv\} \subseteq v^\perp$. Let $v, x, y \in \mathbb{R}^n$ be LI vectors. We consider first the case where $\{z_{v,x}, z_{v,y}\}$ is LD. In this case there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ not both null such that $\lambda_1 z_{v,x} + \lambda_2 z_{v,y} = 0$. The latter means $z_{v, \lambda_1 x + \lambda_2 y} = 0$, which implies that

$$\{Av, Bv\} \subseteq [\text{span}\{v, \lambda_1 x + \lambda_2 y\}]^\perp.$$

The latter subspace has dimension $n - 2$. If $n - 2$ equals 1, we are done; if $n - 2 \geq 2$, we proceed in the same manner until reaching dimension 1, in which case we conclude that $\{Av, Bv\}$ is LD.

Now consider the case where $\{z_{v,x}, z_{v,y}\}$ is LI. Take any $w \in \mathbb{R}^2$ and write $w = \alpha z_{v,x} + \beta z_{v,y}$ for some $\alpha, \beta \in \mathbb{R}$. We easily obtain for all $\varepsilon > 0$:

$$F_H\left(\frac{1}{\varepsilon}x + \alpha\frac{\varepsilon}{2}v\right) = \frac{1}{\varepsilon^2}F_H(x) + \alpha z_{v,x}, \quad F_H\left(\frac{1}{\varepsilon}y + \beta\frac{\varepsilon}{2}v\right) = \frac{1}{\varepsilon^2}F_H(y) + \beta z_{v,y}.$$

By Dines theorem $F_H(\mathbb{R}^n)$ is a convex cone, therefore

$$\frac{1}{\varepsilon^2}(F_H(x) + F_H(y)) + w \in F_H(\mathbb{R}^n), \quad \forall \varepsilon > 0.$$

Letting $\varepsilon \rightarrow +\infty$, we get $w \in \overline{F_H(\mathbb{R}^n)}$, proving that $\overline{F_H(\mathbb{R}^n)} = \mathbb{R}^2$, and the result follows from (15).

(c): Obviously Z is a vector subspace. Let $u \in \mathbb{R}^n, z_{u,v} \neq 0$. $F_H(u \pm tv) = F_H(u) \pm 2tz_{u,v}$ for all $t \in \mathbb{R}$, which implies that $\pm z_{u,v} \in \overline{F_H(\mathbb{R}^n)}$. Since the latter set is a convex cone different from \mathbb{R}^2 , $\overline{F_H(\mathbb{R}^n)}$ is either a halfspace or the straightline $\mathbb{R}z_{u,v}$. In either case we obtain (20).

For the last part simply observe that $z_{Av,v} = \|Av\|^2(1, \lambda) \neq (0, 0)$ if $Av \neq 0$. \square

When ND does not hold, the next result asserts the convexity of $F(\mathbb{R}^2)$ under nonemptiness of the interior of the homogeneous part.

Lemma 4.8. *Assume that ND does not hold. If $\text{int } F_H(\mathbb{R}^2) \neq \emptyset$, then $F(\mathbb{R}^2)$ is convex.*

Proof. Let $v \neq 0$ satisfying $F_H(v) = 0$. Take $u \in \mathbb{R}^2$ such that $\{u, v\}$ is LI. It follows that $F_H(\mathbb{R}^2) = F_H(\mathbb{R}u + \mathbb{R}v) = \mathbb{R}_+F_H(u) + \mathbb{R}z_{u,v}$. Since $\text{int } F_H(\mathbb{R}^2) \neq \emptyset$, one gets $\{F_H(u), z_{u,v}\}$ is LI. We will check that $F(\mathbb{R}^2) = F(\mathbb{R}u + \mathbb{R}v)$ is convex. From Theorem 2 in [38], it suffices to prove that $F(\mathbb{R}u + \mathbb{R}v) = F_H(\mathbb{R}u + \mathbb{R}v) + F(\mathbb{R}u + \mathbb{R}v)$.

Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $s = (s_u, s_v), h = (h_u, h_v) \in \mathbb{R}^2$ such that $F_L(u) = s_u z_{u,v} + h_u F_H(u)$ and $F_L(v) = s_v z_{u,v} + h_v F_H(u)$. We must find $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ satisfying $F(\alpha u + \beta v) + F_H(\gamma u + \delta v) = F(\lambda_1 u + \lambda_2 v)$. This equality along with the LI of $\{F_H(u), z_{u,v}\}$ lead to the following two equations:

$$\begin{aligned} 2\lambda_1\lambda_2 - 2(\alpha\beta + \gamma\delta) + s_u(\lambda_1 - \alpha) + s_v(\lambda_2 - \beta) &= 0 \\ \lambda_2^2 - (\beta^2 + \delta^2) + h_u(\lambda_1 - \alpha) + h_v(\lambda_2 - \beta) &= 0 \end{aligned}$$

If $h_u \neq 0$, from the second equation we get a expression for λ_1 and substitutes it into the first one. The obtained equation is polynomial of third degree in the variable λ_2 , so it admits at least one real zero. Thus a solution (λ_1, λ_2) of the above equations is found.

Now consider the case $h_u = 0$. The second equation is quadratic in λ_2 with discriminant $\Delta = (h_v + 2\beta)^2 + 4\delta^2 \geq 0$. Thus the second equation always admits a solution $\lambda_2 \in \mathbb{R}$. Since the first equation is linear in λ_1 , it will be solvable in λ_1 provided its coefficient $2\lambda_2 + s_u$ is non zero. This is satisfied if $\Delta > 0$. If $\Delta = 0$ and the worst case $2\lambda_2 + s_u = 0$ fulfills, we easily see that the first equation is satisfied vacuously. \square

In what follows, in view of

$$F(\mathbb{R}^n) = F((\ker A \cap \ker B)^\perp) + F_L(\ker A \cap \ker B),$$

we show that there is no loss of generality in assuming $\ker A \cap \ker B = \{0\}$. In fact, set $K \doteq \ker A \cap \ker B$ with $\dim K^\perp = m$. Take a basis $\{u_i : 1 \leq i \leq m\}$ of K^\perp . Thus $F(K^\perp) = \tilde{F}(\mathbb{R}^m)$ is a pair of quadratic functions having the following data: $\tilde{A} = (\langle u_i, Au_j \rangle)_{ij}$, $\tilde{B} = (\langle u_i, Bu_j \rangle)_{ij}$, $\tilde{a} = (\langle a, u_1 \rangle, \dots, \langle a, u_m \rangle)$ and $\tilde{b} = (\langle b, u_1 \rangle, \dots, \langle b, u_m \rangle)$. Let us prove that $\tilde{K} \doteq \ker \tilde{A} \cap \ker \tilde{B} = \{0\}$. Take $z \in \tilde{K}$. Then

$$\langle u_i, A(\sum_{j=1}^m z_j u_j) \rangle = 0 \quad \text{and} \quad \langle u_i, B(\sum_{j=1}^m z_j u_j) \rangle = 0 \quad \forall i = 1, \dots, m.$$

This means $\sum_{j=1}^m z_j u_j \in K^\perp \cap K^{\perp\perp} = \{0\}$, and so $\tilde{K} = \{0\}$. This condition will be assumed in (b) of the following lemma, which is the second main Dines-type result without ND property.

Lemma 4.9. *The set $F(\mathbb{R}^n)$ is convex under any of the following conditions:*

- (a) $F_L(\ker A \cap \ker B) \neq \{0\}$, or equivalently, $\{a, b\} \not\subseteq (\ker A \cap \ker B)^\perp$;
- (b) $\emptyset \neq \text{int } F_H(\mathbb{R}^n) \neq \mathbb{R}^2$.

Proof. (a) Let $u \in \ker A \cap \ker B$ and set $0 \neq h \doteq F_L(u)$. Then, for all $x \in \mathbb{R}^n$, $F(x + tu) = F(x) + th \in F(\mathbb{R}^n)$. Lemma 2.4 yields the desired result.

(b): We apply the procedure as above to consider $F(K^\perp) = \tilde{F}(\mathbb{R}^m)$ and $F(\mathbb{R}^n) = \tilde{F}(\mathbb{R}^m) + F_L(K)$ with $K \doteq \ker A \cap \ker B$. Obviously $\dim K^\perp = m \geq 2$ since $\emptyset \neq \text{int } F_H(\mathbb{R}^n) = \text{int } \tilde{F}_H(\mathbb{R}^m)$. Then, if ND holds for \tilde{A} and \tilde{B} , by Lemma 4.6, $\tilde{F}(\mathbb{R}^m)$ is convex, and so $F(\mathbb{R}^n)$ as well. In case ND does not hold, we proceed on F , by assuming now that $\ker A \cap \ker B = \{0\}$.

Let $v \neq 0$ satisfying $F_H(v) = 0$. It is not difficult to check that $\{z_{u,v} : u \in \mathbb{R}^n\}$ is contained in a line passing through the origin; actually it is the entire line since $z_{-u,v} = -z_{u,v}$ and $\ker A \cap \ker B = \{0\}$. Thus, $\{z_{u,v} : u \in \mathbb{R}^n\} = \mathbb{R}\pi$ with $\pi \neq 0$. Let us define

$$X_v \doteq \{u \in \mathbb{R}^n : z_{u,v} \neq 0\}, \quad Y_v \doteq \{u \in \mathbb{R}^n : F_H(u) \notin \mathbb{R}\pi\},$$

and consider $C_v \doteq X_v \cap Y_v$. Besides X_v is nonempty since $\ker A \cap \ker B = \{0\}$, it is also dense ($u_0 \in X_v$ implies $u + \frac{1}{k}u_0 \in X_v$ for any $u \in \mathbb{R}^n$ and $k \in \mathbb{N}$); Y_v is nonempty in view of $\emptyset \neq \text{int } F_H(\mathbb{R}^n)$, and open by continuity. It is also dense (take $u_0 \in Y_v$ and note that for every $u \notin Y_v$, one has

$$\langle \pi_\perp, F_H(u + \frac{1}{k}u_0) \rangle = \frac{2}{k} \langle \pi_\perp, z_{u,u_0} \rangle + \frac{1}{k^2} \langle \pi_\perp, F_H(u_0) \rangle \neq 0$$

for all $k \in \mathbb{N}$ sufficiently large, implying $u + \frac{1}{k}u_0 \in Y_v$ for all $k \in \mathbb{N}$) sufficiently large. Consequently, C_v is nonempty and dense since it is the intersection of two

dense sets being one of them open. Notice that for all $u \in C_v$, $\{u, v\}$ is LI and therefore $F(\mathbb{R}u + \mathbb{R}v)$ satisfies all the assumptions of Lemma 4.8, so it is convex. Moreover

$$F_H(\mathbb{R}u + \mathbb{R}v) = \{0\} \cup \{\mathbb{R}\pi + \mathbb{R}_{++}F_H(u)\} = \{0\} \cup \{h \in \mathbb{R}^2 : \langle \pi_\perp, F_H(u) \rangle \langle \pi_\perp, h \rangle > 0\},$$

where the second equality follows from Proposition 2.3. On the other hand, all the elements of the form $\langle \pi_\perp, F_H(u) \rangle$ have the same sign since $F_H(\mathbb{R}^n) \neq \mathbb{R}^2$. Hence, by using Theorem 2 in [38], we obtain

$$\begin{aligned} F(\mathbb{R}^n) &\supseteq F(\mathbb{R}u + \mathbb{R}v) = F(\mathbb{R}u + \mathbb{R}v) + F_H(\mathbb{R}u + \mathbb{R}v) \\ &\supseteq F(\mathbb{R}u + \mathbb{R}v) + \mathbb{R}\pi + \mathbb{R}_{++}F_H(u) \\ &= F(\mathbb{R}u + \mathbb{R}v) + \{h \in \mathbb{R}^2 : r \langle \pi_\perp, h \rangle > 0\} \end{aligned}$$

with $F_H(u) \notin \mathbb{R}\pi$ for all $u \in C_v$ and some constant $r \neq 0$. Thus, by Lemma 2.4 (with $X = \mathbb{R}C_v + \mathbb{R}v$), $F(\mathbb{R}^n)$ is convex. \square

Remark 4.10. Observe that Example 3.5 provides an instance where

$$\emptyset \neq \text{int } F_H(\mathbb{R}^2) \neq \mathbb{R}^2.$$

The next lemma is also new in the literature.

Lemma 4.11. *Assume that $F_H(\mathbb{R}^n) = \mathbb{R}^2$. Then $n \geq 2$ and $F(\mathbb{R}^n) = \mathbb{R}^2$.*

Proof. The fact that $n \geq 2$ is obvious. Consider first $n = 2$ and let $L_1 \in \mathbb{R}^2$ be any non-zero vector. Take u and v satisfying $F_H(u) = -F_H(v) = L_1$. Thus $\{u, v\}$ is LI. Since $F_H(\mathbb{R}^n) = \mathbb{R}^2$, $\{z_{u,v}, L_1\}$ is LI. Set $L_2 \doteq z_{u,v}$. Then, there exist $\sigma_i, \rho_i, i = 1, 2$, such that $F_L(u) = 2\sigma_1L_1 + \rho_1L_2$ and $F_L(v) = -2\sigma_2L_1 + \rho_2L_2$. Given any $x \in \mathbb{R}^2$, we will find $\lambda_i \in \mathbb{R}, i = 1, 2$, satisfying

$$x = F((\lambda_1 - \sigma_1)u + (\lambda_2 - \sigma_2)v). \tag{22}$$

On the other hand, we obtain

$$F((\lambda_1 - \sigma_1)u + (\lambda_2 - \sigma_2)v) = [\lambda_1^2 - \lambda_2^2]L_1 + [2\lambda_1\lambda_2 + (\rho_1 - \sigma_2)\lambda_1 + (\rho_2 - \sigma_1)\lambda_2]L_2 - C,$$

with $C = (\sigma_1^2 - \sigma_2^2)L_1 + (\sigma_1\rho_1 + \sigma_2\rho_2)L_2$. By writing $x = x_1L_1 + x_2L_2 - C$ and setting $\pi_1 = \rho_1 - \sigma_2, \pi_2 = \rho_2 - \sigma_1$, (22) yields

$$x_1 = \lambda_1^2 - \lambda_2^2, \quad x_2 = 2\lambda_1\lambda_2 + \pi_1\lambda_1 + \pi_2\lambda_2. \tag{23}$$

We distinguish two cases.

Suppose first that the set $\{(2, \pi_1), (\pi_2, -x_2)\}$ is LD. Then, there exists t_0 such that $t_0(2, \pi_1) = (\pi_2, -x_2)$. Thus, the second equation in (23) reduces to $0 = (\lambda_1 + t_0)(2\lambda_2 + \pi_1)$. If $0 = \lambda_1 + t_0$ then $x_1 = t_0^2 - \lambda_2^2$ for any $\lambda_2 \in \mathbb{R}$; if $0 = 2\lambda_2 + \pi_1$ then $x_1 = \lambda_1^2 - (\frac{\pi_1}{2})^2$ for any $\lambda_1 \in \mathbb{R}$. From this we infer that the first equation in (23) is always satisfied as well.

Suppose now that the set $\{(2, \pi_1), (\pi_2, -x_2)\}$ is LI, which is equivalent to $2x_2 + \pi_1\pi_2 \neq 0$ by Proposition 2.3. From the second equation in (23), we obtain, by assuming additionally $2\lambda_2 + \pi_1 \neq 0$ (since otherwise we are done)

$$\lambda_1 = \frac{x_2 - \pi_2\lambda_2}{2\lambda_2 + \pi_1} = -\frac{\pi_2}{2} + \frac{2x_2 + \pi_1\pi_2}{2(2\lambda_2 + \pi_1)}.$$

Thus,

$$x_1 = \lambda_1^2 - \lambda_2^2 = \left(-\frac{\pi_2}{2} + \frac{2x_2 + \pi_1\pi_2}{2(2\lambda_2 + \pi_1)}\right)^2 - \lambda_2^2 \doteq p(\lambda_2).$$

Since $p(-\infty, -\frac{\pi_1}{2}] = \mathbb{R} = p(-\frac{\pi_1}{2}, +\infty)$, we conclude that system (23) admits a solution, proving that $F(\mathbb{R}^2) = \mathbb{R}^2$.

We consider now that $n \geq 3$. Take any u and v satisfying $F_H(u) = (1, 0)$ and $F_H(v) = (0, 1)$. Then $\mathbb{R}_+^2 \subseteq F_H(\mathbb{R}u + \mathbb{R}v)$, which implies $\text{int } F_H(\mathbb{R}u + \mathbb{R}v) \neq \emptyset$. In case $F_H(\mathbb{R}u + \mathbb{R}v) = \mathbb{R}^2$, we apply the above result to conclude that $\mathbb{R}^2 = F(\mathbb{R}u + \mathbb{R}v)$ and therefore $F(\mathbb{R}^n) = \mathbb{R}^2$. If on the contrary, $F_H(\mathbb{R}u + \mathbb{R}v) \neq \mathbb{R}^2$, from Lemma 4.9, we get the convexity of $F(\mathbb{R}u + \mathbb{R}v)$. By Theorem 2 in [38], $\mathbb{R}_+^2 \subseteq F(\mathbb{R}u + \mathbb{R}v) + F_H(\mathbb{R}u + \mathbb{R}v) = F(\mathbb{R}u + \mathbb{R}v) \subseteq F(\mathbb{R}^n)$. Similarly, we also get the sets $-\mathbb{R}_+^2$, $\mathbb{R}_+ \times \mathbb{R}_-$ and $\mathbb{R}_- \times \mathbb{R}_+$ are contained in $F(\mathbb{R}^n)$, and therefore $F(\mathbb{R}^n) = \mathbb{R}^2$. \square

By using the previous two lemmas and Theorem 4.6, the following theorem is obtained.

Theorem 4.12. *Let $n \geq 2$. If either $\text{int } F_H(\mathbb{R}^n) \neq \emptyset$ or ND holds for A and B then $F(\mathbb{R}^n)$ is convex.*

We now describe a procedure to find a suitable change of variable to be used presently.

Lemma 4.13. *Let $d = (d_1, d_2) \neq 0$, and consider F as in (10) with $A = d_1I$ and $B = d_2I$ with I being the identity matrix of order n and a, b any vectors in \mathbb{R}^n . Then, there exist $t_0 \geq 0$, $k \in \mathbb{R}^2$, $\bar{x} \in \mathbb{R}^2$ and a square matrix C satisfying $C^\top C = I$ such that, if $x = Cy - \bar{x}$ one obtains*

- (a) $F(x) = \tilde{F}(y) - k$ where \tilde{F} is defined in terms of $\tilde{A} = A$, $\tilde{B} = B$, $\tilde{a} = -d_2t_0e_1$, $\tilde{b} = d_1t_0e_1$ with $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$;
- (b) if $n \geq 2$ then $F(\mathbb{R}^n) = \tilde{F}(\mathbb{R}e_1) + \mathbb{R}_+d - k = \text{co } \tilde{F}(\mathbb{R}e_1) - k$, and there exists $\bar{y} \in \mathbb{R}^n$ with $\tilde{F}_H(\bar{y}) = d$ and $\tilde{F}_L(\bar{y}) = 0$;
- (c) the following statements are equivalent (or both fail if $n \geq 2$):
 - (c1) $\{d, F_L(x)\}$ is LI for all $x \in F_H^{-1}(d)$;
 - (c2) $\{d, \tilde{F}_L(y)\}$ is LI for all $y \in \tilde{F}_H^{-1}(d)$.

Proof. (a): As d and d_\perp are LI, there exist $\bar{x}, \bar{y} \in \mathbb{R}^n$ satisfying

$$\begin{pmatrix} a_i \\ b_i \end{pmatrix} = 2\bar{x}_i d + \bar{y}_i d_\perp$$

for all i . For any $x \in \mathbb{R}^n$, we write

$$\begin{aligned} F(x) &= \langle x, x \rangle d + 2\langle \bar{x}, x \rangle d + \langle \bar{y}, x \rangle d_{\perp} \\ &= \langle x + \bar{x}, x + \bar{x} \rangle d + \langle \bar{y}, x + \bar{x} \rangle d_{\perp} - \langle \bar{x}, \bar{x} \rangle d - \langle \bar{x}, \bar{y} \rangle d_{\perp}. \end{aligned}$$

If $\bar{y} = 0$, we choose $t_0 = 0$, $C = I$ and the conclusion follows; otherwise take $x + \bar{x} = Cy$ with $C = \begin{pmatrix} \frac{\bar{y}}{\|\bar{y}\|} & W \end{pmatrix}$ where W is any matrix having as columns a orthonormal basis of \bar{y}^{\perp} . Clearly $C^{\top}C = I$ and, by choosing $t_0 = \|\bar{y}\|$, we get

$$F(x) = \tilde{F}(y) - k, \quad \tilde{F}(y) = \langle y, y \rangle d + t_0 y_1 d_{\perp}, \quad k \doteq \|\bar{x}\|^2 d + \langle \bar{x}, \bar{y} \rangle d_{\perp}. \quad (24)$$

(b): From the last equality, we obtain

$$F(x) = y_1^2 d + \|\bar{y}\| y_1 d_{\perp} + d \sum_{i \geq 2} y_i^2 - k,$$

which implies that $F(\mathbb{R}^n) = \tilde{F}(\mathbb{R}e_1) + \mathbb{R}_+ d - k$; the second equality in (b) follows from Proposition 4.1 since $\tilde{F}_H(e_1) = d$. In addition, we obtain $\tilde{F}_H(e_2) = d$ and $\tilde{F}_L(e_2) = 0$.

(c1) \Rightarrow (c2): From above we deduce

$$F(x) = F(Cy - \bar{x}) = F(Cy) + F(-\bar{x}) - 2z_{Cy, \bar{x}} = \tilde{F}(y) - k,$$

with $k = -F(-\bar{x})$, $\tilde{F}_H(y) = F_H(y) = F_H(Cy)$ and $\tilde{F}_L(y) = F_L(Cy) - 2z_{Cy, \bar{x}}$. Let $y \in \tilde{F}_H^{-1}(d)$. Then $F_H(Cy) = d$, and by (c1) $\{F_L(Cy), d\}$ is LI. Thus $\{\tilde{F}_L(y), d\}$ is LI as well, since $\tilde{F}_L(y) = F_L(Cy) - 2z_{Cy, \bar{x}}$ and $z_{Cy, \bar{x}} = \begin{pmatrix} \langle \bar{x}, ACy \rangle \\ \langle \bar{x}, BCy \rangle \end{pmatrix} = \langle \bar{x}, Cy \rangle d$.

(c2) \Rightarrow (c1): it is similar.

In case $n \geq 2$, both expressions (c1) and (c2) fail in view of (b). □

Next theorem characterizes those directions d under which $F(\mathbb{R}^n) + \mathbb{R}_+ d$ is non-convex.

Theorem 4.14. *Let f, g be any quadratic functions as above and $d = (d_1, d_2) \in \mathbb{R}^2$, $d \neq 0$. The following assertions are equivalent:*

- (a) $F(\mathbb{R}^n) + \mathbb{R}_+ d$ is nonconvex;
- (b) The following hold:
 - (b1) $\{a, b\} \subseteq (\ker A \cap \ker B)^{\perp}$;
 - (b2) $d_2 A = d_1 B$;
 - (b3) $-d \in F_H(\mathbb{R}^n)$;
 - (b4) $[\langle Au, u \rangle = -d_1, \langle Bu, u \rangle = -d_2] \implies d_1 \langle b, u \rangle \neq d_2 \langle a, u \rangle$.

Proof. (a) \Rightarrow (b): From Lemma 4.9, we get $F_L(\ker A \cap \ker B) = \{0\}$ and so (b1) holds, and additionally $\text{int } F_H(\mathbb{R}^n) = \emptyset$. We now introduce the function \tilde{F} which has the same form as F , but on \mathbb{R}^{n+1} , with $\tilde{F}(0) = 0$ and data

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & d_1 \end{pmatrix}, \tilde{B} = \begin{pmatrix} B & 0 \\ 0 & d_2 \end{pmatrix}, \tilde{a} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \tilde{b} = \begin{pmatrix} b \\ 0 \end{pmatrix}.$$

Then, we get $\tilde{F}(\mathbb{R}^{n+1}) = F(\mathbb{R}^n) + \mathbb{R}_+d$ and $\tilde{F}_H(\mathbb{R}^{n+1}) = F_H(\mathbb{R}^n) + \mathbb{R}_+d$. Since (b2) holds if and only if $F_H(\mathbb{R}^n) \subseteq \mathbb{R}d$, one gets $\text{int } \tilde{F}(\mathbb{R}^{n+1}) \neq \emptyset$ if $F_H(\mathbb{R}^n) \not\subseteq \mathbb{R}d$. Hence, if (b2) does not hold $\tilde{F}(\mathbb{R}^{n+1})$ is convex by Lemma 4.9, that is, $F(\mathbb{R}^n) + \mathbb{R}_+d$ is convex, proving (a) implies (b2).

(b3): We now check that $F_H^{-1}(-d) \neq \emptyset$. If on the contrary $-d \notin F_H(\mathbb{R}^n)$, we get $F_H(\mathbb{R}^n) \subseteq \mathbb{R}_+d$ by (b2). Thus, either $F_H(\mathbb{R}^n) = \{0\}$ or $F_H(\mathbb{R}^n) = \mathbb{R}_+d$. In the first case $A = 0$ and $B = 0$, implying the convexity of $F(\mathbb{R}^n)$, which is not possible if (a) is assumed. The second case is also impossible due to (c) of Proposition 4.7, proving the desired result.

(b4): Take $u \in \mathbb{R}^n$ such that $F_H(u) = -d$ and $F_L(u) = \lambda_0 d$ for some $\lambda_0 \in \mathbb{R}$. From (b2) for all $x \in \mathbb{R}^n$, $z_{x,u} \in \mathbb{R}d$. This along with the fact that $F(x + tu) = F(x) + 2z_{x,u} - t^2d + t\lambda_0 d$, yield $F(x) + \mathbb{R}d \in F(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$. Thus, the convexity of $F(\mathbb{R}^n)$ follows from Lemma 2.4, which contradicts (a).

(b) \Rightarrow (a): By a spectral theorem, we can find a non singular matrix D satisfying

$$D^T A D = d_1 \begin{pmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{pmatrix},$$

where I_l denotes the identity matrix of order l (in view of (b1) we may ignore the null eigenvalues if any), and $m_2 \geq 1$ by (b3). From

$$(b2), \text{ we also get } D^T B D = d_2 \begin{pmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{pmatrix}.$$

We apply the preceding lemma to both blocks corresponding to the matrices A and B . Thus, we obtain $m_2 = 1$ since otherwise (b3) would be impossible by virtue of (c) in Lemma 4.13. Hence we may assume from now on $m_1 = m$ and $m_2 = 1$. From Lemma 4.13, there exist $0 \leq t_1, t_2, k \in \mathbb{R}^2, \bar{x} \in \mathbb{R}^n$ and a

square matrix $C = \begin{pmatrix} C_{m_1} & 0 \\ 0 & C_{m_2} \end{pmatrix}$ such that $C^T C = I$, and if $x = Cy - \bar{x}$, one

obtains $F(x) = \tilde{F}(y) - k$, where \tilde{F} is as F with data $\tilde{A} = D^T A D, \tilde{B} = D^T B D, \tilde{a} = -d_2 t_1 e_1 - d_2 t_2 e_{m+1}$, and $\tilde{b} = d_1 t_1 e_1 + d_1 t_2 e_{m+1}$.

By (b3), $\{d, \tilde{F}_L(y)\}$ is LI for all $y \in \tilde{F}_H^{-1}(-d)$. This last expression means $\tilde{F}_H(y) = (\sum_{i=1}^m y_i^2 - y_{m+1}^2)d = -d$, which reduces to $y_{m+1}^2 = 1 + \sum_{i=1}^m y_i^2$.

On the other hand, $\tilde{F}_L(y) = (t_1 y_1 + t_2 y_{m+1})d_\perp$. Then $\{d, \tilde{F}_L(y)\}$ is LI if and only if $t_1 y_1 + t_2 y_{m+1} \neq 0$.

Let us show that $F(\mathbb{R}^n) + \mathbb{R}_+d$ is nonconvex. First, observe that $\tilde{F}(\pm \gamma e_{m+1}) = -\gamma^2 d \pm \gamma t_1 d_\perp$, so that $-\gamma^2 d \pm \gamma t_1 d_\perp - k \in F(\mathbb{R}^n)$ for all $\gamma > 0$. We now check

that for all $\gamma > 0$,

$$-\gamma^2 d - k = \frac{-\gamma^2 d + \gamma t_1 d_\perp - k - \gamma^2 d - \gamma t_1 d_\perp - k}{2} \notin F(\mathbb{R}^n),$$

which turn out that $-\gamma^2 d - k \notin F(\mathbb{R}^n) + \mathbb{R}_+ d$ for all γ sufficiently large. Assume that there exists $y \in \mathbb{R}^{m+1}$ such that $-\gamma^2 d = \tilde{F}(y)$. But

$$\tilde{F}(y) = \left(\sum_{i=1}^m y_i^2 - y_{m+1}^2 \right) d + (t_1 y_1 + t_2 y_{m+1}) d_\perp,$$

so

$$y_{m+1}^2 = \gamma^2 + \sum_{i=1}^m y_i^2 \quad \text{and} \quad t_1 y_1 + t_2 y_{m+1} = 0. \tag{25}$$

This yields a contradiction, since the second equality implies that

$$\tilde{F}\left(\frac{1}{\gamma}y\right) = \tilde{F}_H\left(\frac{1}{\gamma}y\right) = -d$$

and therefore $\{d, \tilde{F}_L(\frac{1}{\gamma}y)\}$ must be LI, that is, see above, $t_1 y_1 + t_2 y_{m+1} \neq 0$. \square

From the preceding result the following theorem follows.

Theorem 4.15. *Let $n \geq 1$ and f, g be any quadratic functions as above. If $F(\mathbb{R}^n) + \mathbb{R}_+ d$ is convex for all $d \in \mathbb{R}^2, d \neq 0$, then $F(\mathbb{R}^n)$ is convex.*

Proof. If $F(\mathbb{R}^n)$ is nonconvex then $F_H(\mathbb{R}^n) \neq \{0\}$, and by Lemma 4.9, $F_L(\ker A \cap \ker B) = \{0\}$ and $\text{int } F_H(\mathbb{R}^n) = \emptyset$. From the latter condition, $F_H(\mathbb{R}^n) \subseteq \mathbb{R}d$ for some $d \in \mathbb{R}^2$, which is equivalent, as seen in the proof of the previous theorem, to $d_2 A = d_1 B$. Actually either $F_H(\mathbb{R}^n) = \mathbb{R}d$ or $F_H(\mathbb{R}^n) = \mathbb{R}_+ d$ or $F_H(\mathbb{R}^n) = -\mathbb{R}_+ d$. In case $F_H^{-1}(-d) \neq \emptyset$, we proceed as follows. By Theorem 4.14, $\{d, F_L(u)\}$ is LD for some (all) $u \in F_H^{-1}(-d)$. Then, for such u , $F_L(u) = \gamma d$ for some $\gamma \in \mathbb{R}$. On the other hand, for all $x \in \mathbb{R}^n$, all $t \in \mathbb{R}$, assuming $d_2 \neq 0$, one has

$$F(x + d_2 t u) = F(x) - d_2^2 t^2 d + \gamma d_2 t d + 2t d_2 z_{x,u} = F(x) - d_2^2 t^2 d + \gamma d_2 t d + 2t \langle Bx, u \rangle d$$

(In case $d_1 \neq 0$, one has $F(x + d_1 t u) = F(x) - d_1^2 t^2 d + \gamma d_1 t d + 2t \langle Ax, u \rangle d$). From this, we infer $F(\mathbb{R}^n) - \mathbb{R}_+ d \subseteq F(\mathbb{R}^n)$. If $F_H^{-1}(-d) = \emptyset$ but $F_H^{-1}(d) \neq \emptyset$, we work with $\tilde{d} = -d$ to conclude with the same equality as above, implying $F(\mathbb{R}^n) - \mathbb{R}_+ \tilde{d} \subseteq F(\mathbb{R}^n)$. Thus $F(\mathbb{R}^n) + \mathbb{R}_+ d \subseteq F(\mathbb{R}^n)$. The previous reasoning proves, in any of the three situations for $F_H(\mathbb{R}^n)$, that $F(\mathbb{R}^n) + F_H(\mathbb{R}^n) \subseteq F(\mathbb{R}^n)$. Hence, $F(\mathbb{R}^n)$ is convex as a consequence of Theorem 2 in [38], so a contradiction is reached, establishing that in fact $F(\mathbb{R}^n)$ is convex. \square

By combining the last two theorems, we obtain the next result which characterizes the convexity of joint-range for a pair of quadratic functions, and contains the Dines theorem, [14] ($a = b = 0$).

Theorem 4.16. *Let $n \geq 1$ and f, g be any quadratic functions as above. Then, $F(\mathbb{R}^n)$ is convex if, and only if for all $d \in \mathbb{R}^2$, $d \neq 0$, any of the following conditions hold:*

- (C1) $F_L(\ker A \cap \ker B) \neq \{0\}$;
- (C2) $d_1 B \neq d_2 A$;
- (C3) $-d \notin F_H(\mathbb{R}^n)$;
- (C4) $\exists u \in F_H^{-1}(-d): d_1 \langle b, u \rangle = d_2 \langle a, u \rangle$.

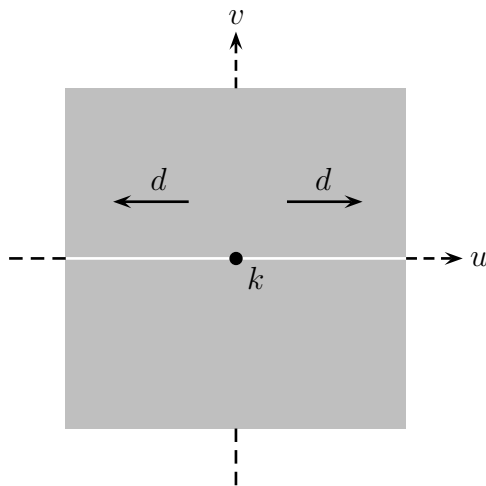


Figure 4.1: $n - l \geq 2$

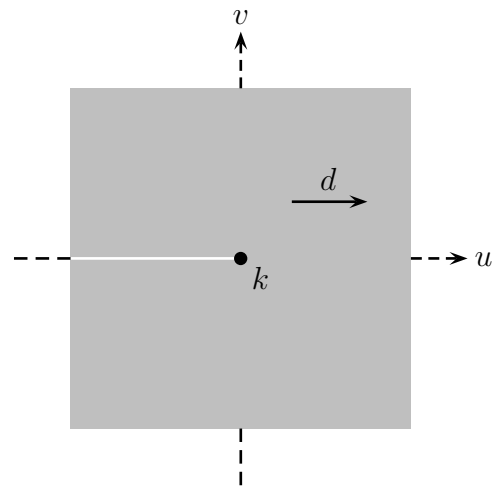


Figure 4.2: $n - l \geq 3$

Remark 4.17. (The nonconvexity of the joint-range set: a complete description)
 If $F(\mathbb{R}^n)$ is not convex then Theorems 4.15 and 4.14, and its proof, imply the existence of $d = (d_1, d_2) \neq (0, 0)$, a change of variable $x = Cy - \bar{x}$ and $k \in \mathbb{R}^2$ such that for all $x \in \mathbb{R}^n$, one has $F(x) = \tilde{F}(y) - k$ with $\tilde{F}_H(y) = \left(\sum_{i=1}^m y_i^2 - y_{m+1}^2 \right) d$ and $\tilde{F}_L(y) = (t_1 y_1 + t_2 y_{m+1}) d_\perp$, where m may possibly be zero; moreover, there it holds

$$y_{m+1}^2 = 1 + \sum_{i=1}^m y_i^2 \implies t_1 y_1 + t_2 y_{m+1} \neq 0. \tag{26}$$

In particular from (26) it follows (using $y_{m+1} = 1$ and $y_i = 0, i \neq m + 1$) that $t_2 \neq 0$. Furthermore, if $t_1^2 > t_2^2$, setting $t_3 \doteq \sqrt{t_1^2 - t_2^2} > 0$, the vector y whose components are $y_1 = \frac{t_2}{t_3}, y_{m+1} = -\frac{t_1}{t_3}$ and $y_i = 0, i \neq 1, m + 1$, yields a contradiction with (26); proving that $t_1^2 \leq t_2^2 \neq 0$. Thus, two possibilities arise:

- $t_1^2 = t_2^2$, in which case, two sets come out as shown in Figures 4.1 and 4.2, up to translations and/or rotations. Consider $l \doteq \dim(\ker A \cap \ker B)$.
- $t_1^2 < t_2^2$, in which case we may assume $t_1 = 0$ up to the change of variables $y'_1 = \frac{t_2}{t_3} y_1 + \frac{t_1}{t_3} y_{m+1}, y'_{m+1} = \frac{t_1}{t_3} y_1 + \frac{t_2}{t_3} y_{m+1}, t_3 = \sqrt{t_2^2 - t_1^2}$; thus the set may have

two possible forms as well, see Figures 4.3 and 4.4, up to translations and/or rotations.

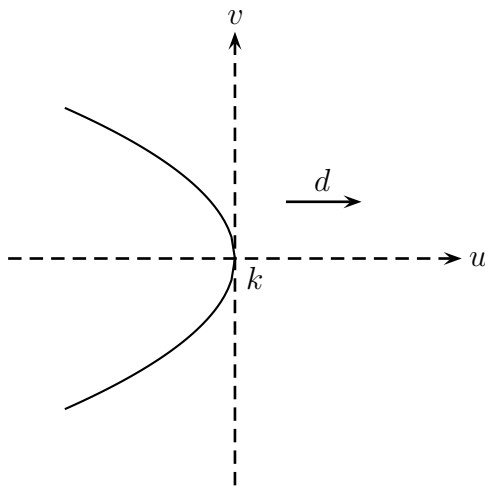


Figure 4.3: $n - l = 1$

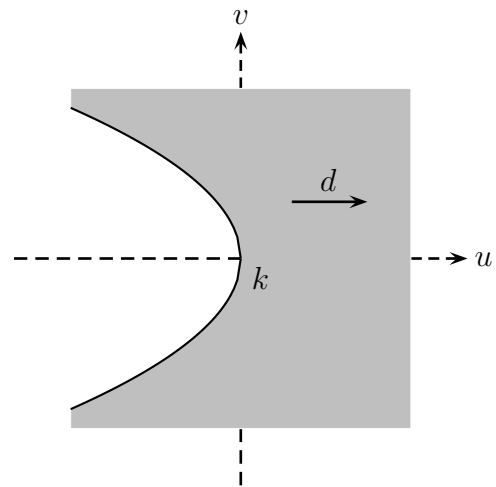


Figure 4.4: $n - l \geq 2$

Remark 4.18. Assume that $F(\mathbb{R}^n) + \mathbb{R}_+d$ is nonconvex for some $d \neq 0$. Then, by Theorems 4.12 and 4.14 one concludes that $\text{int } F_H(\mathbb{R}^n) = \emptyset$, ND does not hold for A and B (so Proposition 4.7 is applicable), and $-d \in F_H(\mathbb{R}^n)$. Consequently, there are at most two directions $d, -d \in \text{bd } F_H(\mathbb{R}^n)$, for which $F(\mathbb{R}^n) + \mathbb{R}_+d$ is nonconvex.

From the previous description, we immediately obtain (a) of the next theorem. From (b) one concludes, in particular, that $C \doteq F(\mathbb{R}^n) + \mathbb{R}_+^2$ is convex. Such a result was asserted in Corollary 10 of [44], but unfortunately the proof is not correct, since C is not closed in general, as Examples 3.5 and 5.15 show.

Theorem 4.19. *Let f, g be any quadratic functions as above. Then,*

- (a) $F(\mathbb{R}^n) + \mathbb{R}_{++}d$ is convex for all non-null directions d except possibly for $-d \in \text{bd } F_H(\mathbb{R}^n)$;
- (b) $F(\mathbb{R}^n) + Q$ is convex for all convex cone with nonempty interior $Q \subseteq \mathbb{R}^2$. Consequently $F(\mathbb{R}^n) + \text{int } Q$ is also convex.

Proof. (a): It is a consequence of the preceding remark and the following equalities:

$$F(\mathbb{R}^n) + \mathbb{R}_{++}d = F(\mathbb{R}^n) + (\mathbb{R}_+d + \mathbb{R}_{++}d) = (F(\mathbb{R}^n) + \mathbb{R}_+d) + \mathbb{R}_{++}d.$$

(b): Since $\text{int } Q \neq \emptyset$, we can choose $d \in Q$ such that $F(\mathbb{R}^n) + \mathbb{R}_+d$ is convex. The result follows by noting that

$$F(\mathbb{R}^n) + Q = F(\mathbb{R}^n) + (\mathbb{R}_+d + Q) = (F(\mathbb{R}^n) + \mathbb{R}_+d) + Q.$$

Thus, $\text{int}(F(\mathbb{R}^n) + Q) = F(\mathbb{R}^n) + \text{int } Q$ is also convex. □

Theorem 4.20. *Let $d \in \mathbb{R}^2$, $d \neq 0$. Then either $d = (d_1, d_2) \notin -\text{bd } F_H(\mathbb{R}^n)$ or $d_2A - d_1B$ is semidefinite (positive or negative).*

Proof. Assume that $d_2A - d_1B$ is not semidefinite, that is, there exist $x_1, x_2 \in \mathbb{R}^n$ such that $\langle d_\perp, F_H(x_1) \rangle < 0 < \langle d_\perp, F_H(x_2) \rangle$. Then, it is not difficult to check that either $-d \in \text{int } F_H(\mathbb{R}^n)$ or $-d \notin F_H(\mathbb{R}^n)$, which mean $-d \notin \text{bd } F_H(\mathbb{R}^n)$. \square

5. Nonconvex quadratic programming with one single inequality or equality constraint

In this section we are concerned with the following quadratic minimization problem:

$$\mu \doteq \inf \{ f(x) : g(x) \in -P, x \in \mathbb{R}^n \}, \tag{27}$$

where P is either \mathbb{R}_+ or $\{0\}$, and $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ are any quadratic functions given by

$$f(x) \doteq \frac{1}{2} \langle Ax, x \rangle + \langle a, x \rangle + k_1; \quad g(x) \doteq \frac{1}{2} \langle Bx, x \rangle + \langle b, x \rangle + k_2, \tag{28}$$

with $A, B \in \mathcal{S}^n$, $a, b \in \mathbb{R}^n$ and $k_1, k_2 \in \mathbb{R}$.

The (Lagrangian) dual problem associated to (27) is defined by

$$\nu \doteq \sup_{\lambda \in P^*} \inf_{x \in \mathbb{R}^n} \{ f(x) + \lambda g(x) \}. \tag{29}$$

Clearly we obtain

$$\inf_{x \in C} \{ f(x) + \lambda g(x) \} \leq \mu, \quad \forall \lambda \in P^*. \tag{30}$$

It is said that (27) has the strong duality property, or simply that strong duality holds for (27), if $\mu = \nu$ and problem (29) admits any solution.

Thus, in case $\mu = -\infty$, there is no duality gap since $\nu = -\infty$ as well, and from (30), we conclude that any element in P^* is a solution for the problem (29). Hence, strong duality always holds for (27) provided $\mu = -\infty$.

Setting $F \doteq (f, g)$, $\mu \in \mathbb{R}$ means

$$[F(\mathbb{R}^n) - \mu(1, 0)] \cap -(\mathbb{R}_{++} \times P) = \emptyset, \tag{31}$$

or equivalently,

$$[F(\mathbb{R}^n) + (\mathbb{R}_+ \times P) - \mu(1, 0)] \cap -(\mathbb{R}_{++} \times \{0\}) = \emptyset. \tag{32}$$

Hence, in case of one inequality constraint, i. e., $P = \mathbb{R}_+$, (32) becomes

$$[F(\mathbb{R}^n) + \mathbb{R}_+^2 - \mu(1, 0)] \cap -(\mathbb{R}_{++} \times \{0\}) = \emptyset;$$

whereas in case $P = \{0\}$, that is, under one single equality constraint, (32) reduces to

$$[F(\mathbb{R}^n) + \mathbb{R}_+(1, 0) - \mu(1, 0)] \cap -(\mathbb{R}_{++} \times \{0\}) = \emptyset.$$

Thus, we are interested only in the convexity of $F(\mathbb{R}^n) + \mathbb{R}_+(1, 0)$ since $F(\mathbb{R}^n) + \mathbb{R}^2$ is always convex by Theorem 4.19.

By particularizing $d = (1, 0)$ in Theorem 4.14, it yields the following corollary.

Corollary 5.1. *Let f, g be quadratic functions as in (28). Then,*

(a) $F(\mathbb{R}^n) + \mathbb{R}_+(1, 0)$ is nonconvex if and only if

$$B = 0, \{a, b\} \subseteq (\ker A)^\perp = A(\mathbb{R}^n), A \not\geq 0, \text{ and} \\ \langle Au, u \rangle < 0 \implies \langle b, u \rangle \neq 0; \tag{33}$$

(b) if $F(\mathbb{R}^n) + \mathbb{R}_+(1, 0)$ is nonconvex, one obtains

(b1) $\nexists (\lambda, \rho) \in \mathbb{R}^2, f(x) + \lambda g(x) \geq \rho, \forall x \in \mathbb{R}^n$ and therefore

$$\inf_{x \in \mathbb{R}^n} [f(x) + \lambda g(x)] = -\infty, \forall \lambda \in \mathbb{R};$$

(b2) $\exists x_i \in \mathbb{R}^n, i = 1, 2, g(x_1) < 0 < g(x_2)$.

Proof. (a) follows from Theorem 4.14 with $d = (1, 0)$.

Assume now that (b1) does not hold, then $A \geq 0$, contradicting (a). Then, the second part immediately follows.

(b2) It follows from (33). □

Remark 5.2. As a counterpart to the preceding corollary we deduce that $F(\mathbb{R}^n) + \mathbb{R}_+(1, 0)$ is convex if, and only if any of the following conditions is satisfied:

(C1) $B = 0$ and $\exists u \in \ker A$ such that either $\langle a, u \rangle \neq 0$ or $\langle b, u \rangle \neq 0$;

(C2) $B \neq 0$;

(C3) $B = 0$ and $A \geq 0$;

(C4) $B = 0$ and $\exists u \in \mathbb{R}^n: \langle Au, u \rangle < 0$ and $\langle b, u \rangle = 0$.

The latter condition implies $\mu = -\infty$, provided (27) is feasible for $P = \{0\}$.

5.1. Characterizing the strong duality under Slater condition

We now characterize the validity of strong duality of problem (27) under a Slater-type condition. This will be a consequence of an important result well-known as S-lemma or (non-strict) Finsler theorem. The validity of S-lemma with equality ($P = \{0\}$) is characterized in Theorems 1 and 3 in [47] by a completely different approach. Our purpose here is to provide some sufficient conditions for that validity as a consequence of our results from Section 4. These conditions will be expressed in a different way than that in [47].

The case $P = \mathbb{R}_+$ already appears in [48, 49] known as the S -procedure, see also [41, Corollary 5], [32, Proposition 3.1], [36, Theorem 2.2], [28, Corollary 3.7], and a slight variant in [17, Theorem 3.4]. Some extensions of the S -procedure in a different direction may be found in [13].

We now establish some sufficient conditions for the validity of S-lemma for inhomogeneous quadratic functions. Set $K_P \doteq \{x \in \mathbb{R}^n : g(x) \in -P\}$.

Theorem 5.3. (*S-lemma*) *Let P be either \mathbb{R}_+ or $\{0\}$, $K_P \neq \emptyset$ and $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ be any quadratic functions as in (28), satisfying $0 \in \text{ri}(g(\mathbb{R}^n) + P)$. In case $P = \{0\}$, assume additionally that $g \not\equiv 0$ and that any of the conditions (Ci), $i = 1, 2, 3$, holds. Then, (a) and (b) are equivalent:*

- (a) $x \in \mathbb{R}^n, g(x) \in -P \implies f(x) \geq 0.$
- (b) *There is $\lambda \in P^*$ such that $f(x) + \lambda g(x) \geq 0, \forall x \in \mathbb{R}^n.$*

Proof. Obviously (b) \implies (a) always holds. Assume therefore that (a) is satisfied. This means that $0 \leq \mu \doteq \inf_{g(x) \in -P} f(x).$ It follows that (32) holds. By our previous discussion $F(\mathbb{R}^n) + (\mathbb{R}_+ \times P)$ is convex, and so by a separation theorem, there exist $(\gamma, \lambda) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ and $\alpha \in \mathbb{R}$ such that

$$\gamma(f(x) - \mu + p) + \lambda(g(x) + q) \geq \alpha \geq \gamma u, \forall x \in \mathbb{R}^n, \forall p \in \mathbb{R}_+, \forall q \in P, \forall u < 0.$$

This yields $\alpha \geq 0, \gamma \geq 0$ and $\lambda \in P^*$, which imply $\gamma(f(x) - \mu) + \lambda g(x) \geq 0, \forall x \in \mathbb{R}^n,$ that is, $\gamma f(x) + \lambda g(x) \geq \gamma \mu \geq 0, \forall x \in \mathbb{R}^n.$ The Slater-type condition gives $\gamma > 0,$ completing the proof of the theorem. □

Remark 5.4. (Comparison with the S-lemma with equality given in [41, 32, 36]) Here, our discussion refers to $P = \{0\}.$ The S-lemma in [41, Corollary 6], see also [32, Proposition 3.2], or [36, Proposition 3.1], asserts that (a) and (b) are equivalent under the assumptions (i) and (ii) :

- (i) g is strictly convex (or strictly concave) and
- (ii) there exist $x_i \in \mathbb{R}^n, i = 1, 2$ such that $g(x_1) < 0 < g(x_2).$

We first observe that such a result cannot be applied to homogeneous quadratic functions (which only requires (ii), see [34, Theorem 2.3] or [21]), as one can notice it directly. On the contrary, our S-lemma recovers that result, since (i) implies (C3): indeed $\langle Bu, u \rangle = 0$ implies $u = 0,$ and so $F_H^{-1}(-1, 0) = \emptyset.$ Secondly, it is easy to check that (ii) is equivalent to:

- (ii') $0 \in \text{ri } g(\mathbb{R}^n)$ and $g \not\equiv 0.$

On the other hand, our Theorem 5.3 applies to Example 5.15 but Proposition 3.1 in [36] does not, since g in this case is neither strictly convex nor strictly concave.

A characterization of the validity of the S-lemma, with $P = \mathbb{R}_+,$ for each quadratic function $f,$ for fixed $g,$ is given in [27, Theorem 3.1]; contrary to our purpose which looks for conditions on the pair f and g jointly. When $P = \{0\},$ the next result is new.

Theorem 5.5. *Let P be either \mathbb{R}_+ or $\{0\}; f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ be any quadratic functions, as above, satisfying $0 \in \text{ri}(g(\mathbb{R}^n) + P)$ with $\mu \in \mathbb{R}.$ In case $P = \{0\},$ assume additionally that $g \not\equiv 0$ and that any of the conditions (Ci), $i = 1, 2, 3,$ holds. Then, strong duality holds for the problem (27).*

Proof. From $\mu \in \mathbb{R},$ we infer that there is no $x \in \mathbb{R}^n$ such that $f(x) - \mu < 0, g(x) \in -P.$ Then, we apply Theorem 5.3 to conclude with the proof. □

We single out the case $P = \{0\}$ to obtain a new characterization of the validity of strong duality for inhomogeneous quadratic functions under Slater-type condition. Its proof follows from the previous theorem and Corollary 5.1.

Corollary 5.6. *Let $P = \{0\}$ in problem (27); $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ be as above satisfying $g(x_1) < 0 < g(x_2)$ for some $x_1, x_2 \in \mathbb{R}^n$. Assume that $\mu \in \mathbb{R}$. Then, the following are equivalent:*

- (a) *strong duality holds for (27);*
- (b) *any of the conditions below holds:*
 - (C1) *$B = 0$ and $\exists u \in \ker A$ such that either $\langle a, u \rangle \neq 0$ or $\langle b, u \rangle \neq 0$;*
 - (C2) *$B \neq 0$;*
 - (C3) *$B = 0$ and $A \succcurlyeq 0$.*

In case we have strong duality with $\mu = -\infty$ it is possible that $F(\mathbb{R}^n) + \mathbb{R}_+(1, 0)$ may be nonconvex. The following example shows this fact.

Example 5.7. Take $f(x_1, x_2) = x_1x_2$, $g(x_1, x_2) = x_1 + 1$. Then $\mu = -\infty$. Moreover, since $(-1, 2) = F(1, -1)$, $(-1, 0) = F(-1, 1)$ but $(-1, 1) \notin F(\mathbb{R}^n) + \mathbb{R}_+(1, 0)$, the latter set is nonconvex.

In connection to the previous result, we must point out that when $F(\mathbb{R}^n) + \mathbb{R}_+(1, 0)$ is not convex, then $g(x_1) < 0 < g(x_2)$ for some $x_1, x_2 \in \mathbb{R}^n$ and $\nu = -\infty$ by Corollary 5.1.

The next example shows that a Slater-type condition is necessary.

Example 5.8. Let P be either \mathbb{R}_+ or $\{0\}$ and consider $f(x_1, x_2) = x_1 + x_2$, $g(x_1, x_2) = (x_1 + x_2)^2$. One can deduce that there is no duality gap. It is easy to get

$$F_H(\mathbb{R}^2) = \mathbb{R}_+(0, 1), \quad F(\mathbb{R}^2) = \{(u, v) \in \mathbb{R}^2 : v = u^2\}, \quad g(\mathbb{R}^2) = \mathbb{R}_+.$$

Thus $F_H^{-1}(-1, 0) = \emptyset$ (implying that $F(\mathbb{R}^2) + \mathbb{R}_+(1, 0)$ is convex), but $0 \notin \text{ri}(g(\mathbb{R}^2) + P)$. Moreover, the strong duality does not hold, since for any $\lambda \in \mathbb{R}$, the inequality

$$x_1 + x_2 + \lambda(x_1 + x_2)^2 \geq 0, \quad \forall (x_1, x_2) \in \mathbb{R}^2,$$

is impossible.

Strong duality results (with $P = \mathbb{R}_+$) were also derived in [27, Theorem 3.2] and [30, Theorem 3.2], with a different perspective: in both papers it is characterized the validity of such a result for each quadratic function f .

5.2. Characterizing minimizers

By applying Corollary 5.6, we obtain a necessary and sufficient optimality condition, which is an extension of Theorem 3.2 in [34], where the assumption $B \neq 0$ (which is our condition (C2)) is imposed when $P = \{0\}$. The case $P = \mathbb{R}_+$ was already considered in [31, Proposition 3.3], [34, Theorem 3.4], [28, Theorem 3.8], [17, Theorem 3.15].

Corollary 5.9. *Let P be either \mathbb{R}_+ or $\{0\}$, $\bar{x} \in K_P$ and $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ be any quadratic functions, as above, satisfying $0 \in \text{ri}(g(\mathbb{R}^n) + P)$. In case $P = \{0\}$, assume additionally that $g \not\equiv 0$ and that any of the conditions (Ci), $i = 1, 2, 3$, holds. Then, the following assertions are equivalent:*

- (a) $\bar{x} \in \underset{g(x) \in -P}{\text{argmin}} f$;
- (b) $\exists \lambda^* \in P^*$ such that $\nabla f(\bar{x}) + \lambda^* \nabla g(\bar{x}) = 0$, $\lambda^* g(\bar{x}) = 0$ and $A + \lambda^* B \succcurlyeq 0$.

Proof. It follows a standard reasoning by applying the previous corollary. □

The last corollary deserves to make some remarks.

Remark 5.10. We consider $P = \{0\}$.

(i) Next example, taken from [34], shows that our set of assumptions (Ci), $i = 1, 2, 3$ is, in some sense, optimal. Consider

$$\min\{x_1^2 - x_2^2 : x_2 = 0\}.$$

Then, $F_H(\mathbb{R}^2) = \mathbb{R}(1, 0)$. Observe that $B = 0$, $\ker A = \{(0, 0)\}$, $F_L(\ker A) = \{(0, 0)\}$, $F_H^{-1}(-1, 0) \neq \emptyset$ and $\{(1, 0), F_L(u)\}$ is LI for all $u \in F_H^{-1}(-1, 0)$. Hence (C1), (C2), (C3) and (C4) do not hold, in other words, $F(\mathbb{R}^n) + \mathbb{R}_+(1, 0)$ is nonconvex. We easily see that (b) of Corollary 5.9 does not hold at the only optimal solution $\bar{x} = (0, 0)$. In fact, the KKT conditions is satisfied at \bar{x} for $\lambda^* = 0$, but in this case $A \not\prec 0$.

(ii) Our Corollary 5.9 applies to situations that are not covered by Theorem 3.2 in [34]. In fact, let us consider $\min\{x_1^2 : x_2 = 0\}$. Then $F_H(\mathbb{R}^2) = \mathbb{R}_+(1, 0)$, which gives $F_H^{-1}(-1, 0) = \emptyset$. Thus our previous corollary is applicable, but not that in [34] since $B = 0$.

For completeness we establish a characterization of solutions when $P = \{0\}$ and the Slater-type condition: $0 \in \text{ri } g(\mathbb{R}^n)$ and $g \not\equiv 0$ (which is equivalent to (ii) in Remark 5.4) fails. We only consider $g(x) \geq 0$ for all $x \in \mathbb{R}^n$, the case $g(x) \leq 0$ for all $x \in \mathbb{R}^n$ is similar. This implies that

$$K_P = K_0 = \{x \in \mathbb{R}^n : g(x) = 0\} = \underset{\mathbb{R}^n}{\text{argmin}} g, \tag{34}$$

provided $K_P \neq \emptyset$. It is known that

$$\bar{x} \in \underset{\mathbb{R}^n}{\text{argmin}} g \iff [B \succcurlyeq 0 \text{ and } B\bar{x} + b = 0]. \tag{35}$$

This leads to the following corollary.

Corollary 5.11. *(Slater condition fails) Let f, g be any quadratic functions and $\bar{x} \in K_P$ with $P = \{0\}$. Assume that $g(x) \geq 0$ for all $x \in \mathbb{R}^n$. Then $F(\mathbb{R}^n) + \mathbb{R}_+(1, 0)$ is convex, and the following statements are equivalent:*

- (a) $\bar{x} \in \underset{K_P}{\text{argmin}} f$;

(b) $B \succcurlyeq 0$, A is positive semidefinite on $\ker B$, and $\exists v \in \mathbb{R}^n$ such that

$$A\bar{x} + a + Bv = 0, \quad B\bar{x} + b = 0.$$

5.3. The ND property and the minimization problem

Next result describes some necessary conditions for having the optimal value of problem (27) to be finite.

Proposition 5.12. *Assume that μ is finite. The following assertions hold.*

(a) if $P = \mathbb{R}_+$ then

$$v \neq 0, \quad \langle Bv, v \rangle \leq 0 \implies \langle Av, v \rangle \geq 0. \tag{36}$$

(b) if $P = \{0\}$ and there exists $v \in \mathbb{R}^n$ satisfying $\langle Bv, v \rangle = 0$ and $\langle Av, v \rangle < 0$, then $Bv = 0$ and either

$$\langle b, v \rangle > 0 \text{ and } f(x \pm tv) \rightarrow -\infty \text{ as } t \rightarrow +\infty, \quad \forall x \in \mathbb{R}^n,$$

or

$$\langle b, v \rangle < 0 \text{ and } f(x \pm tv) \rightarrow -\infty \text{ as } t \rightarrow +\infty, \quad \forall x \in \mathbb{R}^n.$$

Proof. (a) This is Proposition 3.6 in [17].

(b): We obtain, given any $x \in \mathbb{R}^n$, $f(x + tv) \rightarrow -\infty$ for all $|t| \rightarrow +\infty$. Then there exists $t_1 > 0$ such that $g(x + tv) = g(x) + t\langle \nabla g(x), v \rangle \neq 0$ for all $|t| > t_1$. By splitting both expressions, we obtain either

$$f(x + tv) \rightarrow -\infty \text{ as } t \rightarrow +\infty, \quad f(x - tv) \rightarrow -\infty \text{ as } t \rightarrow +\infty, \quad \text{and } \langle \nabla g(x), v \rangle > 0,$$

or

$$f(x + tv) \rightarrow -\infty \text{ as } t \rightarrow +\infty, \quad f(x - tv) \rightarrow -\infty \text{ as } t \rightarrow +\infty, \quad \text{and } \langle \nabla g(x), v \rangle < 0,$$

from which the desired results follow. □

We have to point out that (b) of the preceding proposition could be used to avoid some cases in the proof of the necessity part of Theorem 3 in [47].

Next result provides a description of the behaviour of minimizing sequences under ND property, when either a single inequality or equality constraint is considered.

Theorem 5.13. *Consider problem (27) and let $\mu \in \mathbb{R}$. Assume that (6) holds for A and B .*

(a) If $P = \{0\}$ then every minimizing sequence is bounded, and so $\underset{g(x)=0}{\operatorname{argmin}} f$ is nonempty and compact.

(b) If $P = \mathbb{R}_+$ then $\operatorname{argmin}_{g(x) \leq 0} f$ is nonempty. More precisely, every unbounded minimizing sequence $x_k \in K_P$ satisfying $\|x_k\| \rightarrow +\infty$, $\frac{x_k}{\|x_k\|} \rightarrow v$, yields the existence of $\bar{x} \in \operatorname{argmin}_{\mathbb{R}^n} f$ such that, for some $t_0 > 0$,

$$\bar{x} + tv \in \operatorname{argmin}_{g(x) \leq 0} f, \quad \forall |t| > t_0. \tag{37}$$

Furthermore, $Av = 0$ and $\langle a, v \rangle = 0$.

Proof. (a): Take any minimizing sequence $x_k \in K_P$. Suppose that $\sup_k \|x_k\| = +\infty$. Up to a subsequence, we may assume that $\|x_k\| \rightarrow +\infty$ and $\frac{x_k}{\|x_k\|} \rightarrow v$. From $g(x_k) = 0$ and $f(x_k) \rightarrow \mu$ it follows that $\langle Bv, v \rangle = 0$ and $\langle Av, v \rangle = 0$. By assumption, $v = 0$, reaching a contradiction. Hence every minimizing sequence is bounded.

(b): Take any minimizing sequence $x_k \in K_P$. If $\sup_k \|x_k\| < +\infty$, we get that every limit point of $\{x_k\}$ yields a solution to (27), as usual.

Take now any minimizing sequence x_k such that $\|x_k\| \rightarrow +\infty$ and $\frac{x_k}{\|x_k\|} \rightarrow v$. From $g(x_k) \leq 0$ and $f(x_k) \rightarrow \mu$ it follows that $\langle Bv, v \rangle \leq 0$ and $\langle Av, v \rangle = 0$. By assumption, $\langle Bv, v \rangle < 0$ and $\langle Av, v \rangle = 0$.

Thus, by writing, for any $x \in \mathbb{R}^n$, $g(x + tv) = g(x) + t\langle \nabla g(x), v \rangle + \frac{1}{2}t^2\langle Bv, v \rangle$, we conclude that $g(x + tv) < 0$ for all $|t| > t_1$, for some t_1 depending of x , and therefore $f(x + tv) \geq \mu$ for all $|t| \geq t_1$. Since $\mu \leq f(x + tv) = f(x) + t\langle \nabla f(x), v \rangle$, we deduce that $\langle \nabla f(x), v \rangle = 0$, and so $\mu \leq f(x + tv) = f(x)$ for all $t \in \mathbb{R}$. The former implies $Av = 0$ and $\langle a, v \rangle = 0$, and the latter gives that $\mu = \inf_{x \in \mathbb{R}^n} f(x)$.

Hence $A \succcurlyeq 0$ and there exists $\bar{x} \in \operatorname{argmin}_{\mathbb{R}^n} f$ such that $A\bar{x} + a = 0$. Moreover, since $f(\bar{x} + tv) = f(\bar{x}) = \mu$ for all $t \in \mathbb{R}$, we infer that $g(\bar{x} + tv) < 0$ for all $|t| > t_0$, and so (37) is satisfied. \square

Remark 5.14. Part (b) of the previous theorem provides explicit solutions to (27). Indeed, it is well known that $\bar{x} \in \operatorname{argmin}_{\mathbb{R}^n} f$ if, and only if $A \succcurlyeq 0$ and

$A\bar{x} + a = 0$. By using the pseudoinverse of More-Penrose of any matrix, one obtains that $x_0 = -A^\dagger a$, with A^\dagger being such a pseudoinverse of A , is the unique solution with minimal norm. Thus, by taking t sufficiently large, $x_0 + tv$ is a solution for the problem (27).

The next instance shows that without assumption (6) the set of minima may be empty.

Example 5.15. Let P be either $\{0\}$ or \mathbb{R}_+ and take

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad a = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad k_1 = 0, \quad k_2 = 1.$$

Then $F(\mathbb{R}^2) = (0, 1) + F_H(\mathbb{R}^2)$, $F_H(\mathbb{R}^2) = \{(0, 0)\} \cup (\mathbb{R}_{++} \times \mathbb{R})$. In addition, one can check that $0 = \mu \doteq \min\{x_1^2 : 2x_1x_2 + 1 \in -P\}$, (6) is not satisfied and $\operatorname{argmin}_{g(x) \in -P} f = \emptyset$.

6. Some historical notes for a pair of quadratic forms

We discuss only a pair of quadratic forms in \mathbb{R}^n , and use the notation introduced in Section 2. It seems that the Dines convexity theorem was conceived once Dines got aware of the following result, known as (strict) Finsler’s theorem: if

$$[0 \neq v, \langle Bv, v \rangle = 0] \implies \langle Av, v \rangle > 0 \tag{38}$$

then

$$\exists \lambda \in \mathbb{R}, A + \lambda B \succ 0, \tag{39}$$

and believed that convexity must be present. The previous result was proved first, as far as we know, by Finsler in [16], and re-proved in [1, 39, 23, 21] (an extension to more than two matrices appears in [24]). That result is a kind of S-lemma which originally read as follows: assuming that $\langle B\bar{v}, \bar{v} \rangle < 0$ for some \bar{v} , then

$$\langle Bv, v \rangle \leq 0 \implies \langle Av, v \rangle \geq 0 \tag{40}$$

is equivalent to

$$\exists \lambda \geq 0, A + \lambda B \succcurlyeq 0. \tag{41}$$

This lemma was proved by Yakubovich [48, 49]. Since then, several variants of it and possible connections with well-known properties of matrices have appeared. A nice survey about the S-lemma is presented in [36]; whereas the mentioned properties treated in detail may be found, for instance, in [20, 26], see also [46].

In what follows we list some of the main properties that are useful in the study of quadratic forms.

- (a) SD;
- (b) $\exists t_1, t_2 \in \mathbb{R}, t_1A + t_2B \succ 0$;
- (c) $\exists t \in \mathbb{R}, A + tB \succ 0$;
- (d) $[0 \neq v, \langle Bv, v \rangle = 0] \implies \langle Av, v \rangle > 0$;
- (e) ND;
- (f) $\langle Bv, v \rangle = 0 \implies \langle Av, v \rangle \geq 0$;
- (g) $\exists t \in \mathbb{R}, A + tB \succcurlyeq 0$;
- (h) $F_H(\mathbb{R}^n) = \mathbb{R}^2$.

The relationships between these properties are given below:

- (b) \implies (a), see [26, Theorem 7.6.4];
- (c) \iff (d), see [16], also [1], [14, Corollary 2, page 498], [11, 21]; a different proof may be found in [34, Theorem 2.2];
- ($n \geq 3$) (e) \implies (a) it is attributed to Milnor, [20, page 256], see also [45, Theorem 2.1];

- $[F_H(\mathbb{R}^n) \neq \mathbb{R}^2 \text{ and } (e)] \iff (b)$, see [14, Corollary 1, page 498];
- $(n \geq 3) (h) \implies \text{ND is not true}$, see [23, Corollary, page 401];
- $(n \geq 3) (e) \iff (b)$, see [16], also [11];
- $(B \text{ indefinite}) (f) \iff (g)$, see [34, Theorem 2.3] and [21].

Finally, in [50] some relationships between (f) , (d) and (e) and the Yakuvovich S -lemma (for a pair of quadratic forms), are established. They are related to the non-strict Finsler's, strict Finsler's and Finsler-Calabi's theorem, respectively.

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References

- [1] A. A. Albert: *A quadratic form problem in the Calculus of Variations*, Bull. Amer. Math. Soc. 44 (1938) 250–253.
- [2] A. Auslender: *Existence of optimal solutions and duality results under weak conditions*, Math. Program., Ser. A 88 (2000) 45–59.
- [3] A. Auslender, M. Teboulle: *Asymptotic cones and functions in optimization and variational inequalities*, Springer Monographs Math., Springer-Verlag, New York (2003).
- [4] A. I. Barvinok: *Problems of distance geometry and convex properties of quadratic maps*, Discrete Comput. Geom. 13 (1995) 189–202.
- [5] A. Ben-Tal, D. Hertog: *Hidden conic quadratic representation of some nonconvex quadratic optimization problems*, Math. Program., Ser. A 143 (2014) 1–29.
- [6] A. Ben-Tal, M. Teboulle: *Hidden convexity in some nonconvex quadratically constrained quadratic programming*, Math. Program., Ser. A 72 (1996) 51–63.
- [7] R. I. Bot, G. Wanka: *An alternative formulation for a new closed cone constraint qualification*, Nonlinear Analysis 64 (2006) 1367–1381.
- [8] R. I. Bot, E. R. Csetnek, A. Moldovan: *Revisiting some duality theorems via the quasirelative interior in convex optimization*, J. Optim. Theory Appl. 139 (2008) 67–84.
- [9] R. I. Bot, E. R. Csetnek, G. Wanka: *Regularity conditions via quasi-relative interior in convex programming*, SIAM J. Optim. 19 (2008) 217–233.
- [10] L. Brickman: *On the field of values of a matrix*, Proc. Amer. Math. Soc. 12 (1961) 61–66.
- [11] E. Calabi: *Linear systems of real quadratic forms*, Proc. Amer. Math. Soc. 15 (1964) 844–846.
- [12] G. Cárcamo, F. Flores-Bazán: *Strong duality and KKT conditions in nonconvex optimization with a single equality and geometric constraints*, Math. Program., Ser. B (submitted).
- [13] K. Derinkuyu, M. C. Pinar: *On the S -procedure and some variants*, Math. Meth. Oper. Res. 64 (2006) 55–77.

- [14] L. L. Dines: *On the mapping of quadratic forms*, Bull. Am. Math. Soc. 47 (1941) 494–498.
- [15] J. H. Elton: *Indefinite quadratic forms and the invariance of the interval in special relativity*, Amer. Math. Monthly 117 (2010) 540–547.
- [16] P. Finsler: *Über das Vorkommen definiter und semi-definiter Formen in Scharen quadratischer Formen*, Commentarii Mathematici Helvetici 9 (1937) 188–192.
- [17] F. Flores-Bazán, G. Cárcamo: *A geometric characterization of strong duality in nonconvex quadratic programming with linear and nonconvex quadratic constraints*, Math. Program., Ser. A 145 (2014) 263–290.
- [18] Fabián Flores-Bazán, Fernando Flores-Bazán, C. Vera: *A complete characterization of strong duality in nonconvex optimization with a single constraint*, J. Global Optim. 53 (2012) 185–201.
- [19] D. M. Gay: *Computing optimal locally constrained steps*, SIAM J. Sci. Statist. Comput. 2 (1981) 186–197.
- [20] W. H. Greub: *Linear Algebra*, 3rd edn., Springer-Verlag, Berlin-New York-Heidelberg (1967).
- [21] C. Hamburger: *Two extensions to Finsler’s recurring theorem*, Appl. Math. Optim. 40 (1999) 183–190.
- [22] F. Hausdorff: *Der Wertvorrat einer Bilinearform*, Math. Z. 3 (1919) 314–316.
- [23] M. R. Hestenes: *Pairs of quadratic forms*, Linear Alg. Appl. 1 (1968) 397–407.
- [24] M. R. Hestenes, E. J. McShane: *A theorem on quadratic forms and its application in the calculus of variations*, Trans. Amer. Math. Soc. 47 (1940) 501–512.
- [25] J. B. Hiriart-Urruty, M. Torki: *Permanently going back and forth between the “quadratic world” and the “convexity world” in optimization*, Appl. Math. Optim. 45 (2002) 169–184.
- [26] R. E. Horn, C. R. Johnson: *Matrix Analysis*, Cambridge University Press, New York (1985).
- [27] V. Jeyakumar, N. Q. Huy, G. Y. Li: *Necessary and sufficient conditions for S-lemma and nonconvex quadratic optimization*, Optim. Eng. 10 (2009) 491–503.
- [28] V. Jeyakumar, G. M. Lee, G. Y. Li: *Alternative theorems for quadratic inequality systems and global quadratic optimization*, SIAM J. Optim. 20 (2009) 983–1001.
- [29] V. Jeyakumar, G. Y. Li: *Trust-region problems with linear inequality constraints: exact SDP relaxation, global optimality and robust optimization*, Math. Program., Ser. A, DOI 10.1007/s10107-013-0716-2.
- [30] V. Jeyakumar, G. Y. Li: *Regularized Lagrangian duality for linearly constrained quadratic optimization and the trust-region problems*, J. Global Optim. 49 (2011) 1–14.
- [31] V. Jeyakumar, A. M. Rubinov, Z. Y. Wu: *Non-convex quadratic minimization problems with quadratic constraints: global optimality conditions*, Math. Program., Ser. A 110 (2007) 521–541.
- [32] Z. Q. Luo, J. F. Sturm, S. Zhang: *Multivariate nonnegative quadratic mappings*, SIAM J. Optim. 14 (2004) 1140–1162.

- [33] E. Manskani, N. D. Sidiropoulos, Z. Q. Luo, L. Tassiulas: *Convex approximations techniques for joint multiuser downlink beamforming and admission control*, IEEE Trans. Wireless Comm. 7 (2008) 2682–2693.
- [34] J. J. Moré: *Generalizations of the trust region problem*, Optim. Methods, Softw. 2 (1993) 189–209.
- [35] A. Ozdaglar, P. Tseng: *Existence of global minima for constrained optimization*, J. Optim. Theory Appl. 128 (2006) 523–546.
- [36] I. Pólik, T. Terlaky: *A survey of the S-Lemma*, SIAM Review 49 (2007) 371–418.
- [37] B. T. Polyak: *Convexity of quadratic transformations and its use in control and optimization*, J. Optim. Theory Appl. 99 (1998) 553–583.
- [38] M. Ramana, A. J. Goldman: *Quadratic maps with convex images*, Manuscript November 07 (1995); <http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.54.3622>.
- [39] W. T. Reid: *A theorem on quadratic forms*, Bull. Amer. Math. Soc. 44 (1938) 437–440.
- [40] D. C. Sorensen: *Newton’s method with a model trust region modification*, SIAM Numer. Anal. 19 (1982) 409–426.
- [41] J. F. Sturm, S. Zhang: *On cones of nonnegative quadratic functions*, Math. Oper. Res. 28 (2003) 246–267.
- [42] O. Toeplitz: *Das algebraische Analogon zu einem Satze von Fejér*, Math. Z. 2 (1918) 187–197.
- [43] P. Tseng: *Some convex programs without a duality gap*, Math. Program., Ser. B 116 (2009) 553–578.
- [44] H. Tuy, H. D. Tuan: *Generalized S-Lemma and strong duality in nonconvex quadratic programming*, J. Global Optim. 56 (2013) 1045–1072.
- [45] F. Uhlig: *Definite and semidefinite matrices in a real symmetric matrix pencil*, Pac. J. Math. 49 (1972) 561–568.
- [46] F. Uhlig: *A recurring theorem about pairs of quadratic forms and extension: a survey*, Linear Algebra Appl. 25 (1979) 219–237.
- [47] Y. Xia, S. Wang, R. L. Sheu: *S-Lemma with equality and its applications*, Math. Program., Ser. A 156 (2016) 513–547.
- [48] V. A. Yakubovich: *S-procedure in nonlinear control theory*, Vestnik Leningrad. Univ. 1 (1971) 62–77 (in Russian).
- [49] V. A. Yakubovich: *S-procedure in nonlinear control theory*, Vestnik Leningrad. Univ. 4 (1977) 73–93 (English translation).
- [50] Z. Z. Yan, J. H. Guo: *Some equivalent results with Yakubovich S-lemma*, SIAM J. Control Optim. 48 (2010) 4474–4480.
- [51] C. Zalinescu: *Convex analysis in general vector spaces*, World Scientific, Singapore (2002).