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Eigenvalue Problem For A Class Of Nonlinear Operators In A Variable Exponent Sobolev Space

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In this paper, we consider an eigenvalue problem for a class of nonlinear operators containing $p(\cdot)$-Laplacian and mean curvature operator with mixed boundary conditions. More precisely, we are concerned with the problem with the Dirichlet condition on a part of the boundary and the Steklov boundary condition on an another part of the boundary. We show that the eigenvalue problem has an infinitely many eigenpairs by using the Ljusternik-Schnirelmann principle of the calculus of variation. Moreover, in a variable exponent Sobolev space, we derive some sufficient conditions that the infimum of all eigenvalues is equal to zero and remains to positive, respectively.

Keywords: eigenvalue problem, $p(\cdot)$-Laplacian, mean curvature operator, mixed boundary value problem, variable exponent Sobolev space.

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1. Introduction

In this paper, we consider the following eigenvalue problem with mixed boundary conditions:

$$\begin{cases} -\mathrm{div}\left[\boldsymbol{a}(x,\nabla u(x))\right) = \lambda f(x,u(x)) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \Gamma_1, \\ n(x)\cdot \boldsymbol{a}(x,\nabla u(x)) = 0 & \text{on } \Gamma_2. \end{cases} \tag{1.1}$$

Here Ω is a bounded domain of $\mathbb{R}^N$ ($N \geq 2$) with a Lipschitz-continuous ($C^{0,1}$ for short) boundary Γ satisfying that

$$\Gamma_1 \text{ and } \Gamma_2 \text{ are disjoint open subsets of } \Gamma \text{ such that } \overline{\Gamma_1} \cup \overline{\Gamma_2} = \Gamma \text{ and } \Gamma_1 \neq \emptyset, \tag{1.2}$$

and the vector field $\boldsymbol{n}$ denotes the unit, outer, normal vector to Γ. The function $\boldsymbol{a}(x, \boldsymbol{\xi})$ is a Carathéodory function on $\Omega\times\mathbb{R}^N$ satisfying some structure conditions associated with an anisotropic exponent function $p(x)$. Here we say that $\boldsymbol{a}(x, \boldsymbol{\xi})$ is a Carathéodory function on $\Omega \times \mathbb{R}^N$, if for a.e. $x \in \Omega$, the map $\mathbb{R}^N \ni \boldsymbol{\xi} \mapsto \boldsymbol{a}(x, \boldsymbol{\xi})$ is continuous and for every $\boldsymbol{\xi} \in \mathbb{R}^N$, the map $\Omega \ni x \mapsto \boldsymbol{a}(x, \boldsymbol{\xi})$ is measurable on Ω. The operator $u \mapsto \mathrm{div}\,[\boldsymbol{a}(x, \nabla u(x))]$ is more general than the $p(\cdot)$-Laplacian $\Delta_{p(x)}u(x) = \mathrm{div}\,[|\nabla u(x)|^{p(x)-2}\nabla u(x)]$ and the mean curvature operator $\mathrm{div}\,[(1 +|\nabla u(x)|^2)^{(p(x)-2)/2}\nabla u(x)]$. This generality brings about difficulties and requires some conditions.

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We impose the mixed boundary conditions, that is, the Dirichlet condition on Γ_1 and the Steklov condition on Γ_2. The given function $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carath´eodory function satisfying some structure conditions and λ is a real number.

The study of differential equations with $p(\cdot)$-growth conditions is a very interesting topic recently. Studying such problem stimulated its application in mathematical physics, in particular, in elastic mechanics (Zhikov [34]), in electrorheological fluids (Diening [11], Halsey [20], Mihăilescu and Rădulescu [26], Růžička [30]).

However, since we find a few papers associate with the problem with the mixed boundary condition in variable exponent Sobolev space as in (1.1) (for example, Aramaki [3, 4]). We are convinced of the reason for existence of this paper.

The purpose of this paper is to solve eigenvalue problem (1.1) for a class of operators containing $p(\cdot)$-Laplacian and the mean curvature operator. According to some assumptions on f, we use the Ljusternik-Schnirelmann principle in the constrained variational method. See Ljusternik and Schnirelmann [23] and Szulkin [31].

When $p(x)\equiv p=$ const., there are many articles for the p-Laplacian. For example, see Lê [22], Anane [2], Friedlander [19]. For the p-Laplacian Dirichlet eigenvalue problem:

$$\begin{cases}-\Delta_p u(x)=\lambda\,|\,u(x)\,|^{p-2}\,u(x) & \text{in }\Omega,\\ u(x)=0 & \text{on }\Gamma,\end{cases}$$

the following properties are well-known.

(1) There exists a nondecreasing sequence of positive eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ tending to ∞ as $n\to\infty$.

(2) The first eigenvalue λ_1 is simple and only eigenfunctions associated with λ_1 do not change sign.

(3) The set of eigenvalues is closed.

(4) The first eigenvalue λ_1 is isolated.

On the contrary, recently many authors study the eigenvalue problem for the $p(\cdot)$-Laplacian. In particular, Fan [14] has studied the eigenvalue problem for the $p(\cdot)$-Laplacian with zero Neumann boundary condition in a bounded domain, and Fan et. al. [17] has studied the eigenvalue problem for the $p(\cdot)$-Laplacian Dirichlet problem:

$$\begin{cases}-\Delta_{p(x)} u(x)=\lambda\,|\,u(x)\,|^{p(x)-2}\,u(x) & \text{in }\Omega,\\ u(x)=0 & \text{on }\Gamma,\end{cases}\tag{1.3}$$

where $p(x)$ is a continuous function on $\overline{\Omega}$ such that $\inf_\Omega p>1$. The authors of [17] derived that the infimum λ_* of all eigenvalue of (1.3) is equal to zero or remains positive, according to the properties of exponent $p(\cdot)$, respectively. Moreover, for the case $\lambda_*=0$, they proved an important theorem [17, Theorem 3.3] which will be used in this paper. The case $\lambda_*>0$ was investigated not only in [17], but also in Allegretto [1], Mihăilescu et. al. [25], Mihăilescu et al. [28].

Mihăilescu and Rădulescu [27] have studied nonhomogeneous quasilinear eigenvalue problem with variable exponent:

$$\begin{cases}-\Delta_{p(x)} u(x)=\lambda\,|\,u(x)\,|^{q(x)-2}\,u(x) & \text{in }\Omega,\\ u(x)=0 & \text{on }\Gamma,\end{cases}\tag{1.4}$$

where $p(x)$ and $q(x)$ are continuous functions on $\bar{\Omega}$ such that $1 < \inf_\Omega q < \inf_\Omega p < \sup_\Omega q$, $\sup_\Omega p < N$, and $q(x) < Np(x)/(N - p(x))$ for all $x \in \bar{\Omega}$. The authors established that under some condition, there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$ is an eigenvalue of (1.4), so $\lambda_* = 0$. See also Deng [10] and the previous paper Aramaki [8].

In this paper, we will deal with the mixed boundary value eigenvalue problem (1.1) for a class of operators involving the $p(\cdot)$-Laplacian and the mean curvature operator which is a new topic. Problem (1.3) is a special case of problem (1.1) as $\boldsymbol{a}(x, \boldsymbol{\xi}) = |\boldsymbol{\xi}|^{p(x)-2}\boldsymbol{\xi}$ and $\Gamma_2 = \emptyset$. We will show that for the problem (1.1), there exist infinitely many positive eigenvalues $\{\lambda_{(n,\alpha)}\}_{n=1}^{\infty}$ tending to ∞ as $n \to \infty$ for any fixed $\alpha > 0$. Moreover, we will derive that under some condition, the infimum λ_* is equal to zero, so there does not exist a principal eigenvalue and the set of eigenvalues is not closed. We also show that in the case where $p(\cdot) = p =$ const., λ_* is positive, and in the case where $p(\cdot)$ is a variable exponent satisfying some condition, and the given function f is of special type, λ_* is also positive. We also give a sufficient condition for $\lambda_* = 0$ in a case where $p(\cdot) \neq$ const..

The paper is organized as follows. In Section 2, we recall some results on variable exponent Lebesgue-Sobolev spaces. In Section 3, we give the setting of the problem (1.1) rigorously and a main theorem (Theorems 3.21) on the eigenvalue problem (1.1). In Section 4, we present some sufficient conditions for $\lambda_* > 0$ and $\lambda_* = 0$, respectively.

2. Preliminaries

Throughout this paper, let Ω be a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) with a $C^{0,1}$-boundary Γ and Ω is locally on the same side of Γ. Moreover, we assume that Γ satisfies (1.2).

In the present paper, we only consider vector spaces of real valued functions over $\mathbb{R}$. For any space B, we denote B^N by the boldface character $\boldsymbol{B}$. Hereafter, we use this character to denote vectors and vector-valued functions, and we denote the standard inner product of vectors $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_N)$ and $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_N)$ in $\mathbb{R}^N$ by $\boldsymbol{\xi} \cdot \boldsymbol{\eta} = \sum_{i=1}^N \xi_i \eta_i$ and $|\boldsymbol{\xi}| = (\boldsymbol{\xi} \cdot \boldsymbol{\xi})^{1/2}$. Furthermore, we denote the dual space of B by B^* and the duality bracket by $\langle \cdot, \cdot \rangle_{B^*,B}$.

We recall some well-known results on variable exponent Lebesgue and Sobolev spaces. See Fan and Zhang [16], Kováčik and Rácosník [21], Diening et al. [12] and references therein for more detail. Furthermore, we consider some new properties on variable exponent Lebesgue space. Define $C(\bar{\Omega}) = \{p; p$ is a continuous function on $\bar{\Omega}\}$, and for any $p \in C(\bar{\Omega})$, put

$$p^+ = p^+(\Omega) = \sup_{x\in\Omega} p(x) \text{ and } p^- = p^-(\Omega) = \inf_{x\in\Omega} p(x).$$

For any $p \in C(\bar{\Omega})$ with $p^- \geq 1$ and for any measurable function u on Ω, a modular $\rho_{p(\cdot)} = \rho_{p(\cdot),\Omega}$ is defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega | u(x) |^{p(x)}\, dx.$$

The variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(\Omega) = \{u; u : \Omega \to \mathbb{R} \text{ is a measurable function satisfying } \rho_{p(\cdot)}(u) < \infty\}$$

equipped with the (Luxemburg) norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\tau > 0; \rho_{p(\cdot)}\left(\frac{u}{\tau}\right) \le 1\right\}.$$

Then $L^{p(\cdot)}(\Omega)$ is a Banach space. We also define the Sobolev space

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega); |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

where ∇ is a gradient operator, that is, $\nabla u = (\partial_1 u, \ldots, \partial_N u)$, $\partial_i = \partial/\partial x_i$, endowed with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)},$$

and $\|\nabla u\|_{L^{p(\cdot)}(\Omega)} = \||\nabla u|\|_{L^{p(\cdot)}(\Omega)}$.

The following three propositions are well known (see [17], Fan and Zhao [18], Zhao et al. [33]).

Proposition 2.1. *Let $p \in C(\Omega)$ with $p^- \ge 1$, and let $u, u_n \in L^{p(\cdot)}(\Omega)$ $(n = 1, 2, \ldots)$. Then we have the following properties.*

(i) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1(=1, > 1) \Leftrightarrow \rho_{p(\cdot)}(u) < 1(=1, > 1)$.

(ii) $\|u\|_{L^{p(\cdot)}(\Omega)} > 1 \Rightarrow \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)} \le \rho_{p(\cdot)}(u) \le \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}$.

(iii) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1 \Rightarrow \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} \le \rho_{p(\cdot)}(u) \le \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}$.

(iv) $\lim_{n\to\infty} \|u_n - u\|_{L^{p(\cdot)}(\Omega)} = 0 \Leftrightarrow \lim_{n\to\infty} \rho_{p(\cdot)}(u_n - u) = 0$.

(v) $\|u_n\|_{L^{p(\cdot)}(\Omega)} \to \infty$ *as* $n \to \infty \Leftrightarrow \rho_{p(\cdot)}(u_n) \to \infty$ *as* $n \to \infty$.

The following proposition is a generalized Hölder inequality.

Proposition 2.2. *Let $p \in C_+(\bar{\Omega})$, where $C_+(\bar{\Omega}) := \{p \in C(\bar{\Omega}); p^- > 1\}$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have*

$$\int_\Omega u(x)v(x)|\,dx \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)} \le 2\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)}.$$

Here and from now on, for any $p \in C_+(\bar{\Omega})$, $p'(\cdot)$ denotes the conjugate exponent of $p(\cdot)$, that is, $p'(x) = p(x)/(p(x) - 1)$ for $x \in \bar{\Omega}$.

For $p \in C_+(\bar{\Omega})$, define

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \ge N. \end{cases}$$

Proposition 2.3. *Let* Ω *be a bounded domain of* $\mathbb{R}^N$ *with* $C^{0,1}$*-boundary and let* $p \in C_+(\bar{\Omega})$. *Then we have the following properties.*

(i) *The spaces* $L^{p(\cdot)}(\Omega)$ *and* $W^{1,p(\cdot)}(\Omega)$ *are separable, reflexive and uniformly convex Banach spaces.*

(ii) *If* $q(\cdot) \in C(\bar{\Omega})$ *with* $q^- \geq 1$ *satisfies* $q(x) \leq p(x)$ *for all* $x \in \Omega$, *then* $W^{1,p(\cdot)}(\Omega) \hookrightarrow W^{1,q(\cdot)}(\Omega)$ *where* $\hookrightarrow$ *means that the embedding is continuous.*

(iii) *If* $q(x) \in C(\bar{\Omega})$ *with* $q^- \geq 1$ *satisfies that* $q(x) < p^*(x)$ *for all* $x \in \Omega$, *then the embedding* $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ *is compact.*

Next we consider the trace (cf. Fan [15]). Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$-boundary Γ and $p \in C(\bar{\Omega})$ with $p^- \geq 1$. Since $W^{1,p(\cdot)}(\Omega) \subset W^{1,1}(\Omega)$, the trace $\gamma(u) = u|_\Gamma$ to Γ of any function u in $W^{1,p(\cdot)}(\Omega)$ is well defined as a function in $L^1(\Gamma)$. We define

$$\mathrm{Tr}(W^{1,p(\cdot)}(\Omega)) = (\mathrm{Tr}W^{1,p(\cdot)})(\Gamma) = \{f\,; f \text{ is the trace to } \Gamma \text{ of a function } F \in W^{1,p(\cdot)}(\Omega)\}$$

equipped with the norm

$$\| f \|_{(TrW^{1,p(\cdot)})(\Gamma)} = \inf\{\| F \|_{W^{1,p(\cdot)}(\Omega)}; F \in W^{1,p(\cdot)}(\Omega) \text{ satisfying } F|_\Gamma = f\}$$

for $f \in (\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)$, Where the infimum can be achieved. Then we can see that $(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)$ is a Banach space. In the later, we also write $F|_\Gamma = g$ *by* $F = g$ *on* Γ. Moreover, for $i = 1, 2$, we denote

$$(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma_i) = \{f|_{\Gamma_i}; f \in (\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)\}$$

equipped with the norm

$$\| g \|_{(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma_i)} = \inf\{\| f \|_{(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)}; f \in (\mathrm{Tr}W^{1,p(\cdot)})(\Gamma) \text{ satisfying } f|_{\Gamma_i} = g\},$$

where the infimum can also be achieved, so for any $g \in (\mathrm{Tr}W^{1,p(\cdot)})(\Gamma_i)$, there exists $F \in W^{1,p(\cdot)}(\Omega)$ such that $F|_{\Gamma_i} = g$ and $\| F \|_{W^{1,p(\cdot)}(\Omega)} = \| g \|_{(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma_i)}$.

Now we consider the weighted variable exponent Lebesgue space. Let $p \in C(\bar{\Omega})$ with $p^- \geq 1$ and let $a(x)$ be a measurable function on Ω with $a(x) > 0$ a.e. $x \in \Omega$. We define a modular

$$\rho_{(p(\cdot),a(\cdot))}(u) = \int_\Omega a(x)\,|u(x)|^{p(x)}\,dx \text{ for any measurable function } u \text{ in } \Omega.$$

Then the weighted Lebesgue space is defined by

$$L^{p(\cdot)}_{a(\cdot)}(\Omega) = \{u; u \text{ is a measurable function on } \Omega \text{ satisfying } \rho_{(p(\cdot),a(\cdot))}(u) < \infty\}$$

equipped with the norm

$$\| u \|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} = \inf\left\{\tau > 0; \int_\Omega a(x)\left|\frac{u(x)}{\tau}\right|^{p(x)} dx \leq 1\right\}.$$

Then $L^{p(\cdot)}_{a(\cdot)}(\Omega)$ is a Banach space.

We have the following proposition (cf. Fan [13, Proposition 2.5]).

Proposition 2.4. *Let $p \in C(\bar{\Omega})$ with $p^- \geq 1$. For $u, u_n \in L^{p(\cdot)}_{a(\cdot)}(\Omega)$, we have the following.*

(i) *For* $u \neq 0$, $\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} = \tau \Leftrightarrow \rho_{(p(\cdot),a(\cdot))}\left(\frac{u}{\tau}\right) = 1.$

(ii) $\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} < 1 (=1, >1) \Rightarrow \rho_{(p(\cdot),a(\cdot))}(u) \leq 1 (=1, >1).$

(iii) $\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} > 1 \Rightarrow \|u\|^{p^-}_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} \leq \rho_{(p(\cdot),a(\cdot))}(u) \leq \|u\|^{p+}_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}.$

(iv) $\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} < 1 \Rightarrow \|u\|^{p^+}_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} \leq \rho_{(p(\cdot),a(\cdot))}(u) \leq \|u\|^{p^-}_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}.$

(v) $\lim_{n\to\infty} \|u_n - u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} = 0 \Leftrightarrow \lim_{n\to\infty} \rho_{(p(\cdot),a(\cdot))}(u_n - u) = 0.$

(vi) $\|u_n\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} \to \infty$ *as* $n \to \infty \Leftrightarrow \rho_{(p(\cdot),a(\cdot))}(u_n) \to \infty$ *as* $n \to \infty$.

The author of [13] also derived the following proposition (cf. [13, Theorem 2.1]).

Proposition 2.5. *Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$-boundary and $p \in C_+(\bar{\Omega})$. Moreover, let $a \in L^{\alpha(\cdot)}(\Omega)$ satisfy $a(x) > 0$ a.e. $x \in \Omega$ and $\alpha \in C_+(\bar{\Omega})$. If $q \in C(\bar{\Omega})$ satisfies*

$$1 \leq q(x) < \frac{\alpha(x)-1}{\alpha(x)} p^*(x) \text{ for all } x \in \bar{\Omega},$$

then the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}_{a(\cdot)}(\Omega)$ is compact.

Now we consider the Nemytskii operator.

Proposition 2.6. *Let $q \in C(\bar{\Omega})$ with $q^- \geq 1$ and a be a measurable function with $a(x) > 0$ for a.e. $x \in \Omega$. Assume that*

(F.1) *A function $F(x, t)$ is a Carathéodory function on $\Omega \times \mathbb{R}$.*

(F.2) *The growth condition holds: there exist $c \in L^{q_1(\cdot)}(\Omega)$ with $c(x) \geq 0$ a.e. $x \in \Omega$, $q_1 \in C(\bar{\Omega})$ with $q_1^- \geq 1$, and a constant $c_1 > 0$ such that*

$$|F(x, t)| \leq c(x) + c_1 a(x)^{1/q_1(x)} |t|^{q(x)/q_1(x)} \textit{ for a.e. } x \in \Omega \textit{ and all } t \in \mathrm{R}.$$

Then the Nemytskii operator $N_F : L^{q(\cdot)}_{a(\cdot)}(\Omega) \ni u \mapsto F(x, u(x)) \in L^{q_1(\cdot)}(\Omega)$ is continuous and there exists a constant $C > 0$ such that

$$\rho_{q_1(\cdot)}(N_F(u)) \leq C(\rho_{q_1(\cdot)}(c) + \rho_{(q(\cdot),a(\cdot))}(u)) \text{ for all } u \in L^{q(\cdot)}_{a(\cdot)}(\Omega).$$

In particular, if $q_1(x) \equiv 1$, then $N_F : L^{q(\cdot)}_{a(\cdot)}(\Omega) \hookrightarrow L^1(\Omega)$ is continuous.

For the proof, see Aramaki [7, Proposition 7].

Remark 2.7. This proposition is an extension of [4, Proposition 2.12].

Define a space by

$$X = \{v \in W^{1,p(\cdot)}(\Omega); v = 0 \text{ on } \Gamma_1\}. \tag{2.1}$$

Then it is clear to see that X is a closed subspace of $W^{1,p(\cdot)}(\Omega)$, so X is a reflexive and separable Banach space. We can see the following Poincaré-type inequality (cf. Ciarlet and Dinca [9]).

Proposition 2.8. *Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$-boundary and let $p \in C_+(\bar{\Omega})$. Then there exists a constant $C = C(\Omega, N, p) > 0$ such that*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \le C\|\nabla u\|_{L^{p(\cdot)}(\Omega)} \text{ for all } u \in X.$$

In particular, $\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$ is equivalent to $\|u\|_{W^{1,p(\cdot)}(\Omega)}$ for $u \in X$.

For the direct proof, see [3, Lemma 2.5].

Thus we can define the norm on X so that

$$\|v\|_X = \|\nabla v\|_{L^{p(\cdot)}(\Omega)} \text{ for } v \in X, \tag{2.2}$$

which is equivalent to $\|v\|_{W^{1,p(\cdot)}(\Omega)}$ from Proposition 2.8.

3. Assumptions and the main theorem

In this section, we state the rigorous set-up of the problem (1.1), the assumptions and main theorem according to the Ljusternik-Schnirelmann theory.

Let $p \in C_+(\bar{\Omega})$ be fixed. Assume that the following (A.0)-(A.5) hold.

(A.0) $A : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a function satisfying that for a.e. $x \in \Omega$, the function $A(x, \cdot) : \mathbb{R}^N \ni \boldsymbol{\xi} \mapsto A(x, \boldsymbol{\xi})$ is of C^1-class, and for all $\boldsymbol{\xi} \in \mathbb{R}^N$, the function $A(\cdot, \boldsymbol{\xi}) : \Omega \ni x \mapsto A(x, \boldsymbol{\xi})$ is measurable. Moreover, suppose that $A(x, \mathbf{0}) = 0$ and put $\boldsymbol{a}(x, \boldsymbol{\xi}) = \nabla_\xi A(x, \boldsymbol{\xi})$. Then

$\boldsymbol{a}(x, \boldsymbol{\xi})$ is a Carathéodory function.

In the following (A.1)-(A.3), $c, k_0, k_1 > 0$ denote some constants, a function $h_0 \in L^{p'(\cdot)}(\Omega)$ is non-negative and $h_1 \in L^1_{\text{loc}}(\Omega)$ with $h_1(x) \ge 1$ for a.e. $x \in \Omega$.

(A.1) $|\boldsymbol{a}(x, \boldsymbol{\xi})| \le c(h_0(x) + h_1(x)|\boldsymbol{\xi}|^{p(x)-1})$ for all $\boldsymbol{\xi} \in \mathbb{R}^N$ and a.e. $x \in \Omega$.

(A.2) A is $p(\cdot)$-uniformly convex, that is,

$$A\left(x, \frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{2}\right) + k_1 h_1(x)|\boldsymbol{\xi}-\boldsymbol{\eta}|^{p(x)} \le \frac{1}{2}A(x,\boldsymbol{\xi}) + \frac{1}{2}A(x,\boldsymbol{\eta}) \text{ for all } \boldsymbol{\xi},\boldsymbol{\eta} \in \mathbb{R}^N \text{and a.e. } x \in \Omega.$$

(A.3) $k_0 h_1(x)|\boldsymbol{\xi}|^{p(x)} \le \boldsymbol{a}(x, \boldsymbol{\xi}) \cdot \boldsymbol{\xi} \le p(x)A(x, \boldsymbol{\xi})$ for all $\boldsymbol{\xi} \in \mathbb{R}^N$ and a.e. $x \in \Omega$.

(A.4) $(\boldsymbol{a}(x, \boldsymbol{\xi}) - \boldsymbol{a}(x, \boldsymbol{\eta})) \cdot (\boldsymbol{\xi} - \boldsymbol{\eta}) > 0$ for all $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^N$ with $\boldsymbol{\xi} \ne \boldsymbol{\eta}$ and a.e. $x \in \Omega$.

(A.5) $A(x, -\boldsymbol{\xi}) = A(x, \boldsymbol{\xi})$ for all $\boldsymbol{\xi} \in \mathbb{R}^N$ and a.e. $x \in \Omega$.

Remark 3.1. (i) The condition (A.1) is more general than that of Mashiyev et al. [24] who considered the case $h_1(x) \equiv 1$. In our case, to overcome this we have to consider the space Y defined by (3.1) later as a basic space rather than the space X defined by (2.1).

(ii) (A.3) implies that A is $p(\cdot)$-sub-homogeneous, that is,

$$A(x, s\boldsymbol{\xi}) \le A(x, \boldsymbol{\xi})s^{p(x)} \text{ for any } \boldsymbol{\xi} \in \mathbb{R}^N, \text{ a.e. } x \in \Omega \text{ and } s \ge 1. \tag{3.1}$$

For the proof, see Aramaki [5, (4.14)].

Example 3.2. (i) $A(x,\boldsymbol{\xi}) = \dfrac{h(x)}{p(x)}|\boldsymbol{\xi}|^{p(x)}$ with $p^- \ge 2$, $h \in L^1_{\text{loc}}(\Omega)$ satisfying $h(x) \ge 1$ a.e. $x \in \Omega$.

(ii) $A(x,\boldsymbol{\xi}) = \dfrac{h(x)}{p(x)}\left(\left(1+|\boldsymbol{\xi}|^2\right)^{p(x)/2} - 1\right)$ with $p^- \ge 2$, $h \in L^{p'(\cdot)}(\Omega)$ satisfying $h(x) \ge 1$ a.e. $x \in \Omega$.

Then $A(x, \boldsymbol{\xi})$ and $\boldsymbol{a}(x, \boldsymbol{\xi}) = \nabla_\xi A(x, \boldsymbol{\xi})$ of (i) and (ii) satisfy (A.0)-(A.5).

Remark 3.3. In Example 3.2, when $h(x) \equiv 1$, (i) corresponds to the $p(\cdot)$-Laplacian and (ii) corresponds to the prescribed mean curvature operator for nonparametric surface.

For the function $h_1 \in L^1_{loc}(\Omega)$ with $h_1(x) \geq 1$ for a.e. $x \in \Omega$, we define a modular on X by

$$\tilde{\rho}_{(p(\cdot),h_1(\cdot))}(v) = \int_\Omega h_1(x)|\nabla v(x)^{p(x)} dx \text{ for } v \in X, \tag{3.2}$$

where the space X is defined by (2.1). Define our basic space

$$Y = \{v \in X; \tilde{\rho}_{(p(\cdot),h_1(\cdot))}(v) < \infty\}$$

equipped with the norm

$$\|v\|_Y = \inf\left\{\tau > 0; \tilde{\rho}_{(p(\cdot),h_1(\cdot))}\left(\frac{v}{\tau}\right) \leq 1\right\}.$$

Proposition 3.4. *The space* $(Y, \|\cdot\|_Y)$ *is a separable and reflexive Banach space.*

For the proof, see Aramaki [6, Proposition 3.4].

We note that $C_0^\infty(\Omega) \subset Y$. Since $h_1(x) \geq 1$ a.e. $x \in \Omega$, it follows that

$$\tilde{\rho}_{(p(\cdot),h_1(\cdot))}(v) = \rho_{p(\cdot)}\left(h_1^{1/p(\cdot)}\nabla v\right) \geq \rho_{p(\cdot)}(\nabla v) \text{ for } v \in Y,$$

and so

$$\|v\|_Y = \|h_1^{1/p(\cdot)}\nabla v\|_{L^{p(\cdot)}(\Omega)} \geq \|\nabla v\|_{L^{p(\cdot)}(\Omega)} = \|v\|_X \text{ for } v \in Y. \tag{3.3}$$

From (3.3) and Proposition 2.1, we have the following proposition.

Proposition 3.5. *Let* $p \in C_+(\bar{\Omega})$ *and let* $u, u_n \in Y$ $(n = 1, 2, \ldots)$. *Then the following properties hold.*

(i) $Y \hookrightarrow X$ *and* $\|u\|_X \leq \|u\|_Y$.
(ii) $\|u\|_Y > 1(= 1, < 1) \Leftrightarrow \tilde{\rho}_{(p(\cdot),h_1(\cdot))}(u) > 1(= 1, < 1)$.
(iii) $\|u\|_Y > 1 \Rightarrow \|u\|_Y^{p-} \leq \tilde{\rho}_{(p(\cdot),h_1(\cdot))}(u) \leq \|u\|_Y^{p+}$.
(iv) $\|u\|_Y < 1 \Rightarrow \|u\|_Y^{p+} \leq \tilde{\rho}_{(p(\cdot),h_1(\cdot))}(u) \leq \|u\|_Y^{p-}$.
(v) $\lim_{n\to\infty}\|u_n - u\|_Y = 0 \Leftrightarrow \lim_{n\to\infty}\tilde{\rho}_{(p(\cdot),h_1(\cdot))}(u_n - u) = 0$.
(vi) $\|u_n\|_Y \to \infty$ *as* $n \to \infty \Leftrightarrow \tilde{\rho}_{(p(\cdot),h_1(\cdot))}(u_n) \to \infty$ *as* $n \to \infty$.

We assume the following (f.0)-(f.2) on the function f in (1.1).

(f.0) $f(x, t)$ is a Carathéodory function on $\Omega \times \mathbb{R}$ and there exist $1 \leq a \in L^{\alpha(\cdot)}(\Omega)$ with $\alpha \in C_+(\bar{\Omega})$ and $q \in C_+(\bar{\Omega})$ satisfying

$$q(x) < \frac{\alpha(x)-1}{\alpha(x)}p^*(x) \text{ for all } x \in \bar{\Omega}$$

such that

$$|f(x, t)| \leq d(1 + a(x)|t|^{q(x)-1}) \text{ for a.e. } x \in \Omega \text{ and } t \in \mathbb{R}$$

for some constant $d > 0$.

(f.1) For $x \in \Omega$, $f(x, t)$ is an odd function with respect to $t \in \mathrm{R}$.
(f.2) $0 < f(x, t)t = q(x)F(x, t)$ for a.e. $x \in \Omega$ and $t > 0$, where

$$F(x,t) = \int_0^t f(x,s)ds \text{ for a.e. } x \in \Omega \text{ and } t \in \mathbb{R}.$$

We note that from (f.1) and (f.2), we obtain the homogeneity of F with respect to t, that is,

$$F(x, t) = F(x, 1)|t|^{q(x)}. \tag{3.4}$$

Indeed, from (f.2), we have $\frac{f(x,s)}{F(x,s)} = \frac{q(x)}{s}$ for a.e. $x \in \Omega$ and $s > 0$. Taking that $F(x, t)$ is an even function with respect to t into consideration, for $t \geq 0$, if we integrate this equality from 1 to t, we easily get (3.4).

For example, a function $f(x, t) = a(x)|t|^{q(x)-2}t$, where a is a function as in (f.0), satisfies (f.0)-(f.2).

Here we introduce the notions of a weak solution and an eigenfunction for the problem (1.1).

Definition 3.6. (i) We say that a pair $(u, \lambda) \in Y \times \mathbb{R}$ is a weak solution of (1.1), if

$$\int_\Omega \boldsymbol{a}(x,\nabla u(x)) \cdot \nabla v(x)dx = \lambda \int_\Omega f(x,u(x))v(x)dx \text{ for all } v \in Y. \tag{3.5}$$

(ii) Such a pair $(u, \lambda) \in Y \times \mathbb{R}$ with $u \neq 0$ is called an eigenpair, λ is called an eigenvalue and u is called an associated eigenfunction.

Define functionals on Y by

$$\Phi(u) = \int_\Omega A\big(x,\nabla u(x)\big)dx, \quad J(u) = \int_\Omega F(x,u(x))dx \text{ for } u \in Y. \tag{3.6}$$

It follows from (A.5) and (f.1) that Φ and J are even functionals, that is, $\Phi(-u) = \Phi(u)$ and $J(-u) = J(u)$ for all $u \in Y$.

Lemma 3.7. (i) *We have*

$$\frac{k_0}{p^+}\tilde{\rho}_{p((\cdot),h_1(\cdot))}(u) \leq \Phi(u) \leq c(2\|h_0\|_{L^{p'(\cdot)}(\Omega)}\|\nabla u\|_{L^{p(\cdot)}(\Omega)} + \tilde{\rho}_{(p(\cdot),h_1(\cdot))}(u))$$

for $u \in Y$, where c and k_0 are the constants in (A.1) *and* (A.3).

(ii) *We have*

$$\Phi\left(\frac{u+v}{2}\right) + k_1\tilde{\rho}_{(p(\cdot),h_1(\cdot))}(u-v) \leq \frac{1}{2}\Phi(u) + \frac{1}{2}\Phi(v)$$

for all $u, v \in Y$, where k_1 is the constant in (A.2), *in particular, Φ is convex, that is, $\Phi((1-\tau)u + \tau v) \leq (1-\tau)\Phi(u) + \tau\Phi(v)$ for all $u, v \in Y$ and $\tau \in [0, 1]$.*

Proof. (i) easily follows from (A.3) and the Hölder inequality (Proposition 2.2). (ii) easily follows from (A.2) and the continuity of $A(x, \boldsymbol{\xi})$ with respect to $\boldsymbol{\xi}$.

Proposition 3.8. (i) Φ *is coercive, that is,* $\Phi(u) \to \infty$ *as* $\|u\|_Y \to \infty$.

(ii) Φ *is sequentially weakly lower-semicontinuous on Y.*

(iii) $\Phi \in C^1(Y, \mathbb{R})$ *and we have*

$$\langle \Phi'(u) - v \rangle_{Y^*,Y} = \int_\Omega \boldsymbol{a}(x, \nabla u(x)) \cdot \nabla v(x) dx \text{ for } u, v \in Y. \tag{3.7}$$

Proof. (i) follows from Lemma 3.7 (i) and Proposition 3.5 (vi). (ii) follows from [5, Proposition 4.4 (iii)]. (iii) follows from [5, Proposition 4.1].

Proposition 3.9. *If* $u_n \to u$ *weakly in Y and* $\Phi(u_n) \to \Phi(u)$ *as* $n \to \infty$, *then we have* $\Phi\left(\frac{u_n - u}{2}\right) \to 0$ *as* $n \to \infty$, so $u_n \to u$ *strongly in Y.*

Proof. Assume that the conclusion is false. Then there exists $\varepsilon_0 > 0$ and a subsequence $\{u_{n'}\}$ of $\{u_n\}$ such that $\Phi\left(\frac{u_{n'} - u}{2}\right) \geq \varepsilon_0$ for all n'. From Lemma 3.7 (i), there exists a constant $C(\varepsilon_0) = C(\varepsilon_0, p) > 0$ such that

$$\varepsilon_0 \leq \Phi\left(\frac{u_{n'} - u}{2}\right) \leq c\left(\| h_0 \|_{L^{p'(\cdot)}(\Omega)} \left\| \frac{u_{n'} - u}{2} \right\|_Y + \left\| \frac{u_{n'} - u}{2} \right\|_Y^{p^+} \vee \left\| \frac{u_{n'} - u}{2} \right\|_Y^{p^-} \right)$$
$$\leq \frac{\varepsilon_0}{2} + C(\varepsilon_0)(\| u_{n'} - u \|_Y^{p^+} \vee \| u_{n'} - u \|_Y^{p^-}).$$

Here and from now on, $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$ for real numbers a and b. Hence $C(\varepsilon_0) \| u_{n'} - u \|_Y^{p^+} \vee \| u_{n'} - u \|_Y^{p^-} \geq \varepsilon_0/2$, so by Lemma 3.7 (v), there exists a subsequence of $\{u_{n'}\}$ (still denoted by $\{u_{n'}\}$) and $c_1 > 0$ such that $k_1 \tilde{\rho}_{p(\cdot),h_1(\cdot)}(u_{n'} - u) \geq c_1$.

Hence it follows from Lemma 3.7 (ii) that

$$\Phi\left(\frac{u_{n'} + u}{2}\right) + c_1 \leq \frac{1}{2}\Phi(u_{n'}) + \frac{1}{2}\Phi(u).$$

Since $(u_{n'} + u)/2 \to u$ weakly in Y and Φ is sequencially weakly lower semi-continuous, we have

$$\Phi(u) + c_1 \leq \liminf_{n' \to \infty} \Phi\left(\frac{u_{n'} + u}{2}\right) + c_1 \leq \frac{1}{2} \liminf_{n' \to \infty} \Phi(u_{n'}) + \frac{1}{2}\Phi(u) = \Phi(u).$$

This is a contradiction. Thus $\Phi((u_n - u)/2) \to 0$ as $n \to \infty$. By Lemma 3.7 (i), $\tilde{\rho}_{(p(\cdot),h_1(\cdot))}\left(\frac{u_n - u}{2}\right) \to 0$ as $n \to \infty$. Thus from Proposition 3.5 (v), $u_n \to u$ strongly in Y.

Proposition 3.10. (i) $\Phi \in \mathcal{W}_Y$, *that is, if* $u_n \to u$ *weakly in Y as* $n \to \infty$ *and* $\liminf_{n\to\infty} \Phi(u_n) \leq \Phi(u)$, *then there exists a subsequence* $\{u_{n'}\}$ *of* $\{u_n\}$ *such that* $u_{n'} \to u$ *strongly in Y as* $n' \to \infty$.

(ii) Φ *is bounded on every bounded subset of* Y.

Proof. (i) Let $u_n \to u$ weakly in Y as $n \to \infty$ and $\liminf_{n\to\infty}\Phi(u_n) \leq \Phi(u)$. Since Φ is sequentially weakly lower semi-continuous from Proposition 3.8 (ii), we have

$\Phi(u) \le \liminf_{n\to\infty} \Phi(u_n)$, so $\liminf_{n\to\infty} \Phi(u_n) = \Phi(u)$. Hence there exists a subsequence $\{u_{n'}\}$ of $\{u_n\}$ such that $\lim_{n'\to\infty} \Phi(u_{n'}) = \Phi(u)$. By Proposition 3.9, $u_{n'} \to u$ strongly in Y.

(ii) easily follows from Lemma 3.7 (i).

Proposition 3.11. (i) Φ' *is strictly monotone in* Y, *that is,*

$$\langle \Phi'(u) - \Phi'(v), u - v\rangle_{Y^*,Y} > 0 \text{ for all } u, v \in Y \text{ with } u \ne v.$$

Moreover, Φ' *is bounded on every bounded subset of* Y *and coercive in the sense that*

$$\lim_{\|u\|_Y\to\infty} \frac{\left\langle \Phi'(u), u\right\rangle_{Y^*,Y}}{\|u\|_Y} = \infty.$$

(ii) Φ' *is of* (S_+)*-type, that is, if* $u_n \to u$ *weakly in* Y *and*

$$\limsup_{n\to\infty} \left\langle \Phi'(u_n), u_n - u\right\rangle_{Y^*,Y} \le 0,$$

then $u_n \to u$ *strongly in* Y.

(iii) *The mapping* $\Phi' : Y \to Y^*$ *is a homeomorphism.*

For the proof, see [7, Proposition 21].

For the functional J defined by (3.6), we have the following proposition.

Proposition 3.12. *Under the hypotheses* (f.0)*, we have the following.*

(i) $J \in C^1(Y, \mathbb{R})$ *and*

$$\left\langle J'(u), v\right\rangle_{Y^*,Y} = \int_\Omega f(x,u(x))v(x)dx \text{ for } u,v \in Y. \tag{3.8}$$

(ii) *J is sequentially weakly continuous in* Y.

(iii) $J' : Y \to Y^*$ *is weakly-strongly continuous, that is, if* $u_n \to u$ *weakly in* Y *as* $n \to \infty$, *then* $J'(u_n) \to J'(u)$ *strongly in* Y^* *as* $n \to \infty$.

Proof. Since (i) and (ii) follows from Aramaki [5, Proposition 4.2, Proposition 4.4], we only derive (iii). Let $u_n \to u$ weakly in Y. Then

$$\left\langle J'(u_n) - J'(u), v\right\rangle_{Y^*,Y} = \int_\Omega (f(x,u_n(x) - f(x,u(x)))v(x)dx \text{ for } v \in Y.$$

From Proposition 2.5 and (f.0), the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}_{a(\cdot)}(\Gamma_2)$ is compact. Since $Y \hookrightarrow X \hookrightarrow W^{1,p(\cdot)}(\Omega)$, there exists a constant $C > 0$ such that

$$\|v\|_{L^{q(\cdot)}_{a(\cdot)}(\Gamma_2)} \le C\|v\|_Y \text{ for all } v \in Y.$$

By the Hölder inequality (Proposition 2.2), for any $v \in Y$,

$$\begin{aligned} |\left\langle J'(u_n) - J'(u), v\right\rangle_{Y^*,Y}| &\le \int_\Omega a(x)^{-1/q(x)} |f(x,u_n(x)) - f(x,u(x))| a(x)^{1/q(x)} |v(x)| d\sigma_x \\ &\le 2\| a(\cdot)^{-1/q(\cdot)} |f(\cdot,u_n(\cdot)) - f(\cdot,u(\cdot))| \|_{L^{q'(\cdot)}(\Omega)} \| a^{1/q(\cdot)} |v(\cdot)| \|_{L^{q(\cdot)}(\Omega)}. \end{aligned}$$

Since

$$\| a^{1/q(\cdot)} v(\cdot) \|_{L^{q(\cdot)}(\Omega)} = \| v \|_{L^{q(\cdot)}_{a(\cdot)}(\Omega)} \le C \| v \|_Y ,$$

we have

$$\| J'(u_n) - J'(u) \|_{Y^*} \le 2C \| a^{-1/q(\cdot)} | f(\cdot, u_n(\cdot)) - f(\cdot, u(\cdot)) | \|_{L^{q'(\cdot)}(\Omega)} .$$

We want to show that $\|J'(u_n) - J'(u)\|_{Y^*} \to 0$ as $n \to \infty$. By Proposition 2.1 (iv) and the above inequality, it suffices to show that

$$\rho_{q'(\cdot)}(a^{-1/q(\cdot)} f(\cdot, u_n(\cdot)) - a(\cdot)^{-1/q(\cdot)} f(\cdot, u(\cdot))) \to 0 \text{ as } n \to \infty. \tag{3.9}$$

We can see that

$$\begin{aligned} &\rho_{q'(\cdot)}(a^{-1/q(\cdot)} f(\cdot, u_n(\cdot)) - a^{-1/q(\cdot)} f(\cdot, u(\cdot))) \\ &= \int_\Omega a(x)^{-q'(x)/q(x)} | f(x, u_n(x)) - f(x, u(x)) |^{q'(x)} \, dx. \end{aligned}$$

Since $u_n \to u$ weakly in Y and the embedding $Y \hookrightarrow L^{q(\cdot)}_{a(\cdot)}(\Omega)$ is compact, we can see that $u_n \to u$ strongly in $L^{q(\cdot)}_{a(\cdot)}(\Omega)$. From [4, Theorem A.1], there exist a subsequence $\{u_{n'}\}$ of $\{u_n\}$ and $g \in L^{q(\cdot)}(\Omega)$ such that $a(x)^{1/q(x)} u_{n'}(x) \to a(x)^{1/q(x)} u(x)$ a.e. $x \in \Omega$ and $|a(x)^{1/q(x)} u_{n'}(x)| \le g(x)$ for a.e. $x \in \Omega$. Since $a(x) > 0$ and f is a Carathéodory function, $f(x, u_{n'}(x)) \to f(x, u(x))$ a.e. $x \in \Omega$. From (f.0) and $a(x) \ge 1$, we have

$$\begin{aligned} &a(x)^{-q'(x)/q(x)} |f(x, u_{n'}(x) - f(x, u(x))|^{q'(x)} \\ &\le C_1 a(x)^{-q'(x)/q(x)} (1 + a(x)|u_{n'}(x)|^{q(x)-1} + a(x)|u(x)|^{q(x)-1})^{q'(x)} \\ &\le C_1 (a(x)^{-q'(x)/q(x)} + a(x)^{q'(x) - q'(x)/q(x)} (|u_{n'}(x)|^{q(x)} + |u(x)|^{q(x)})) \\ &\le C_1 (1 + a(x)(|u_{n'}(x)|^{q(x)} + |u(x)|^{q(x)})) \\ &\le 2C_1 (1 + g(x)^{q(x)}). \end{aligned}$$

The last term is an integrable function in Ω independent of n'. Thus by the Lebesgue dominated convergence theorem, we have

$$\rho_{q'(\cdot)}(a(\cdot)^{-1/q(\cdot)} f(\cdot, u_{n'}(\cdot)) - a(\cdot)^{-1/q(\cdot)} f(\cdot, u(\cdot))) \to 0 \text{ as } n' \to \infty.$$

From the convergent principle (Zeidler [32, Proposition 10.13]), we see that (3.9) holds, so $\|J'(u_n) - J'(u)\|_{Y^*} \to 0$ as $n \to \infty$.

Remark 3.13. From (3.7), (3.8) and Definition 3.6, we can see that $(u, \lambda) \in Y \times \mathbb{R}$ is a weak solution of the problem (1.1) if and only if

$$\Phi'(u) = \lambda J'(u). \tag{3.10}$$

In particular, we have $\langle \Phi'(u), u \rangle_{Y^*,Y} = \lambda \langle J'(u), u \rangle_{Y^*,Y}$. If $u \ne 0$, then it follows from (A.3) and (f.2) that

$$\begin{aligned} \langle \Phi'(u), u \rangle_{Y^*,Y} &= \int_\Omega a(x, \nabla u(x)) \cdot \nabla u(x) dx \ge k_0 \int_\Omega h_1(x) | \nabla u(x) |^{p(x)} \, dx \\ &\ge k_0 \| u \|_Y^{p^+} \wedge \| u \|_Y^{p^-} > 0 \end{aligned}$$

and

$$\left\langle J'(u),u\right\rangle_{Y^*,Y} = \int_\Omega f(x,u(x))u(x)dx > 0,$$

so we have

$$\lambda = \frac{\langle \Phi'(u),u\rangle_{Y^*,Y}}{\langle J'(u),u\rangle_{Y^*,Y}} > 0. \tag{3.11}$$

This means that any eigenvalue of the problem (1.1) is positive.

In order to solve the eigenvalue problem (3.10), we apply the constrained variational method. We take Φ as an objective functional and J as a constraint functional. For any fixed $r > 0$, put

$$M_r = \{u \in Y\,;\, J(u) = r\}. \tag{3.12}$$

If $u \in M_r$, then from (f.2),

$$\langle J'(u),u\rangle_{Y^*,Y} = \int_\Omega f(x,u(x))u(x)dx \geq q^- \int_\Omega F(x,u(x))dx = q^- J(u) = q^- r > 0, \tag{3.13}$$

so $J'(u) \neq 0$. Hence M_r is a C^1-submanifold of Y with codimension one. Moreover, M_r is weakly closed subset of Y. Indeed, let $u_j \in M_\alpha$ and $u_j \to u$ weakly in Y as $j \to \infty$. Since J is sequentially weakly continuous from Proposition 3.12 (ii), $r = J(u_j) \to J(u)$, so $u \in M_r$. If we denote the tangent space of M_r at $u \in M_r$ by T_uM_r, then we can see that

$$T_uM_r = \mathrm{Ker}(J'(u)) := \{v \in Y\,;\, \langle J'(u), v\rangle_{Y^*,Y} = 0\}.$$

Let $P : Y \to T_uM_r$ be the natural projection. Note that the bounded linear map $J'(u) : Y \to \mathbb{R}$ is surjective. We denote the restriction of Φ to M_r by $\tilde{\Phi} = \Phi|_{M_r}$ and the derivative $d\tilde{\Phi}(u) \in Y^*$ of $\tilde{\Phi}$ at $u \in M_r$ can be defined by $\langle d\tilde{\Phi}(u), v\rangle_{Y^*,Y} = \langle \Phi'(u), Pv\rangle_{Y^*,Y}$ for $v \in Y$. We note that $\langle d\tilde{\Phi}(u), h\rangle_{Y^*,Y} = \langle \Phi'(u), h\rangle_{Y^*,Y}$ for any $h \in T_uM_r$.

It is well known that when $u \in M_r$, there exists $\lambda \in \mathbb{R}$ such that $(u, \lambda) \in M_r \times \mathbb{R}$ solves (3.10) if and only if u is a critical point of $\tilde{\Phi}$ with respect to M_r, that is,

$$\langle \Phi'(u), h\rangle_{Y^*,Y} = 0 \text{ for all } h \in T_uM_r,$$

(for example, see [32, Proposition 43.21]).

For $u \in M_r$, put $w = (\Phi')^{-1}(J'(u))$. Then since we have (3.13), we see that $J'(u) \neq 0$. From (A.5), the functional Φ is even, so Φ' is odd and so $\Phi'(0) = 0$. Since $(\Phi')^{-1}$ is injective, we have $w \neq 0$. From strict monotonicity of Φ' (Proposition 3.11 (i)),

$$\langle J'(u), w\rangle_{Y^*,Y} = \langle J'(u), (\Phi')\,(J'(u))\rangle_{Y^*,Y} = \langle \Phi'(w), w\rangle_{Y^*,Y} > 0. \tag{3.14}$$

Hence since $w = (\Phi')^{-1}(J'(u)) \notin T_uM_r$, we can see that

$$Y = T_uM_r \oplus \{\beta(\Phi')^{-1}(J'(u));\, \beta \in \mathbb{R}\}.$$

For every $v \in Y$, there exists unique $\beta \in \mathbb{R}$ such that $v = Pv + \beta(\Phi')^{-1}(J'(u))$. Since $Pv \in T_uM_r = \mathrm{Ker}(J'(u))$, we have

$$\langle J'(u), v\rangle_{Y^*,Y} = \beta'\langle J(u), (\Phi')^{-1}(J'(u))\rangle_{Y^*,Y}.$$

Thus from (3.14), we can write

$$\beta = \frac{\langle J'(u), v\rangle_{Y^*,Y}}{\langle J'(u), (\Phi')^{-1}(J'(u))\rangle_{Y^*,Y}}.$$

Now we have

$$\begin{aligned}\langle d\tilde{\Phi}(u), v\rangle_{Y^*,Y} &= \langle \Phi'(u), P(v)\rangle_{Y^*,Y} \\ &= \langle \Phi'(u), v\rangle_{Y^*,Y} - \left\langle \Phi'(u), \frac{\langle J'(u), v\rangle_{Y^*,Y}}{\langle J'(u), (\Phi')^{-1}(J'(u))\rangle_{Y^*,Y}} (\Phi')^{-1}(J'(u)) \right\rangle_{Y^*,Y} \\ &= \left\langle \Phi'(u), \frac{\langle \Phi'(u), (\Phi')^{-1}(J'(u))\rangle_{Y^*,Y}}{\langle J'(u), (\Phi')^{-1}(J'(u))\rangle_{Y^*,Y}} J'(u), v \right\rangle_{Y^*,Y} \quad \text{for any } v \in Y.\end{aligned}$$

Thus we have

$$d\tilde{\Phi}(u) = \Phi'(u) - \lambda(\mathrm{u})J'(u),$$

where

$$\lambda(u) = \frac{\langle \Phi'(u), (\Phi')^{-1}(J'(u))\rangle_{Y^*,Y}}{\langle J'(u), (\Phi')^{-1}(J'(u))\rangle_{Y^*,Y}}.$$

Proposition 3.14. *For any* $r > 0$*, the functional* $\Phi : M_r \to \mathbb{R}$ *verifies* $(\mathrm{PS})_c$*-condition for any* $c \in \mathbb{R}$*, that is, if any sequence* $\{u_n\} \subset M_r$ *satisfies that* $\tilde{\Phi}(u_n) \to c$ *and* $\|d\tilde{\Phi}(u_n)\|_{Y^*} \to 0$ *as* $n \to \infty$*, then* $\{u_n\}$ *contains a convergent subsequence.*

Proof. Let $\{u_n\} \subset M_r$ satisfies that $\tilde{\Phi}(u_n) \to c$ and $d\tilde{\Phi}(u_n) \to 0$ in Y^* as $n \to \infty$. Then since from (A.3),

$$\tilde{\Phi}(u_n) = \Phi(u_n) \geq \frac{k_0}{p^+}\int_\Omega h_1(x) \,|\nabla u_n(x)|^{p(x)}\, dx \geq \frac{k_0}{p^+} \|u\|_Y^{p+} \wedge \|u_n\|_Y^{p^-},$$

$\{u_n\}$ is bounded in Y. Since Y is a reflexive Banach space, there exist a subsequence $\{u_{n'}\}$ of $\{u_n\}$ and $u_0 \in Y$ such that $u_{n'} \to u_0$ weakly in Y. By Proposition 3.12 (ii) and (iii), $J'(u_{n'}) \to J'(u_0)$ in Y^* and $J(u_{n'}) \to J(u_0)$ as $n \to \infty$. Thereby, $u_0 \in M_r$. Put $w_{n'} = (\Phi')^{-1}(J'(u_{n'}))$. Since $J'(u_{n'}) \to J'(u_0) \neq 0$ in Y^*, it follows from Proposition 3.11 (iii) that $w_{n'} \to w_0 \neq 0$ in Y, where $w_0 = (\Phi')^{-1}(J'(u_0))$. Thus

$$\langle J'(u_{n'}), (\Phi')^{-1}(J'(u_{n'}))\rangle_{Y^*,Y} = \langle \Phi'(w_{n'}), w_{n'}\rangle_{Y^*,Y} \to \langle \Phi'(w_0), w_0\rangle_{Y^*,Y} > 0. \tag{3.15}$$

On the other hand, we have

$$|\langle \Phi'(u_{n'}), (\Phi')^{-1}(J'(u_{n'}))\rangle_{Y^*,Y}| = |\langle \Phi'(u_{n'}), w_{n'}\rangle_{Y^*,Y}| \leq \|\Phi'(u_{n'})\|_{Y^*} \|w_{n'}\|_Y.$$

Since $\{u_{n'}\}$ is bounded in Y, it follows from Proposition 3.11 (i) that $\|\Phi'(u_{n'})\|_{Y^*}$ is bounded. Hence, there exists a constant $c_2 > 0$ such that

$$|\langle \Phi'(u_{n'}), (\Phi')^{-1}(J'(u_{n'}))\rangle_{Y^*,Y}| \leq c_2. \tag{3.16}$$

From (3.15) and (3.16), $\{\lambda(u_{n'})\}$ is bounded in $\mathbb{R}$. Passing to a subsequence, we may assume that $\lambda(u_{n'}) \to \lambda_0$ for some $\lambda_0 \in \mathbb{R}$. Since $d\tilde{\Phi}(u_{n'}) \to 0$ in Y^*, we see that $\Phi'(u_{n'}) - \lambda(u_{n'})J'(u_{n'}) \to 0$ as $n' \to \infty$. Hence, since $J'(u_{n'}) \to J'(u_0)$ in Y^*,

$$\Phi'(u_{n'}) = (\Phi'(u_{n'}) - \lambda(u_{n'})J'(u_{n'})) + \lambda(u_{n'})J'(u_{n'}) \to \lambda_0 J'(u_0) \text{ in } Y^* \text{ as } n' \to \infty.$$

Therefore, using again Proposition 3.11 (iii), we can see that $u_{n'} \to (\Phi')^{-1}(\lambda_0 J'(u_0))$ strongly in Y as $n' \to \infty$.

Here we recall the notion of "genus" which is introduced in Rabinowitz [29, Chapter 7] or [32, Section 44.3]. Let E be a real Banach space and let $\mathcal{E}$ denote the family of subsets $A \subset E \setminus \{0\}$ such that A is closed in E and symmetric with respect to 0, that is, $x \in A$ implies $-x \in A$. For $\emptyset \neq A \in \mathcal{E}$, define the genus of A to be $n \geq 1$ (denoted by $\gamma(A) = n$) if there is a map $\varphi \in C(A, \mathbb{R}^n \setminus \{\mathbf{0}\})$ with φ odd and n is the smallest integer with this property. When there does not exist a finite such n, set $\gamma(A) = \infty$. Finally set $\gamma(\emptyset) = 0$.

The main properties of genus are listed in the next proposition.

Proposition 3.15. *Let A, $B \in \mathcal{E}$. Then the following properties hold.*

(i) *If there exists an odd map $f \in C(A, B)$, then $\gamma(A) \leq \gamma(B)$.*

(ii) *If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.*

(iii) $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.

(iv) *If A is compact, then $\gamma(A) < \infty$ and there exists $\delta > 0$ such that if we put $N_\delta(A) = \{x \in E; \|x - A\| := \inf\{\|x - y\|; y \in A\} \leq \delta\}$, then $N_\delta(A) \in \mathcal{E}$ and $\gamma(N_\delta(A)) = \gamma(A)$.*

(v) *If Ω is a bounded neighborhood of $\mathbf{0}$ in $\mathbb{R}^n$, and there exists a mapping $h \in C(A, \partial\Omega)$ with h an odd homeomorphism, then $\gamma(A) = n$.*

For the proof, see [29, Lemma 7.5 and Proposition 7.7] or [31, Proposition 2.3]. We note that it can be easily seen that when $A \in \mathcal{E}$, $A \neq \emptyset$ if and only if $\gamma(A) \geq 1$.

Let $\Sigma_r = \{H \subset M_r; H$ is compact and symmetric$\}$, $\gamma(H)$ be the genus of $H \in \Sigma_r$, and define

$$c_{(n,r)} = \inf_{H \in \Sigma_r, \gamma(H) \geq n} \sup_{u \in H} \tilde{\Phi}(u) \; (n = 1, 2, \ldots). \tag{3.17}$$

The following proposition is due to [31, Corollary 4.3].

Proposition 3.16 (Ljusternik-Schnirelmann principle). *Assume that M is a closed symmetric C^1-submanifold of a real Banach space B and $0 \notin M$. Let $f \in C^1(M, \mathrm{R})$ be an even functional and bounded from below. Define*

$$c_j = \inf_{H \in \Gamma_j} \sup_{u \in H} f(u),$$

where

$$\Gamma_j = \{H \subset M; H \text{ is compact, symmetric and } \gamma(H) \geq j\}.$$

If $\Gamma_k \neq \emptyset$ for some $k \geq 1$ and f satisfies $(\mathrm{PS})_c$*-condition for $c = c_j$, $j = m, \ldots, k$ with $1 \leq m \leq k$, then f has at least $k - m + 1$ distinct pairs of critical points.*

Since Y is a separable reflexive Banach space, it is well known that there exist $\{e_n\}_{n=1}^{\infty} \subset Y$ and $\{f_n\}_{n=1}^{\infty} \subset Y^*$ such that $\langle f_n, e_m\rangle_{Y^*,Y} = \delta_{nm}$, where δ_{nm} is the Kronecker delta and

$$Y = \overline{\mathrm{span}\{e_1, e_2, \ldots\}} \text{ and } Y^* = \overline{\mathrm{span}\{f_1, f_2, \ldots\}}.$$

Define spaces

$$Y_j = \mathrm{span}\{e_j\}, Z_n = \bigoplus_{j=1}^{n} Y_j, W_n = \overline{\bigoplus_{j=n}^{\infty} Y_j}.$$

If we apply Proposition 3.16 with $B = Y$, $M = M_r$ and $f = \tilde{\Phi}$, then we obtain the following lemma. We note that $\tilde{\Phi}$ is bounded from below on M_r and satisfies $(\mathrm{PS})_c$-condition for any $c \in \mathbb{R}$ by Proposition 3.14.

Lemma 3.17. *For any $m \in \mathrm{N}$, we have $\Gamma_m \neq \emptyset$. Thus we see that all $c_{(n,r)}$ defined by* (3.17) *are critical values of $\tilde{\Phi}$ with respect to M_r and*

$$-\infty < c_{(n,r)} \le c_{(n+1,r)} < \infty \textit{ for every } n \in \mathbb{N}.$$

Proof. For any $m \in \mathbb{N}$, we claim that $Z_m \cap M_r$ is bounded and symmetric. In fact, if $u \in Z_m \cap M_r$, then from (3.4) we can see that

$$r = J(u) = \int_\Omega F(x, u(x))dx = \int_\Omega F(x,1)\,|\,u(x)\,|^{q(x)}\,dx.$$

If we define

$$\hat{\rho}(u) = \int_\Omega F(x, u(x))dx \text{ for } u \in Z_m,$$

then $\hat{\rho}$ is a modular on Z_m and $\|u\| := \inf\left\{\tau > 0; \hat{\rho}\left(\frac{u}{\tau}\right) \le 1\right\}$ is a norm on Z_m. From Proposition 2.5 and (f.0), we have $\|u\|^{q+} \wedge \|u\|^{q-} \le \hat{\rho}(u) = r$ for all $u \in Z_m \cap M_r$. Since Z_m is of finitely dimensional, all the norms are equivalent, so there exists a constant $C > 0$ such that $\|u\|_Y \le C$ for all $u \in Z_m \cap M_r$. Clearly $Z_m \cap M_r$ is symmetric and closed. Since $Z_m \cap M_r$ is bounded and closed subset in a finitely dimensional space Z_m, we see that $Z_m \cap M_r$ is compact. Let $G = \{u = u_1e_1 + \cdots + u_me_m \in Z_m; J(u) < r\}$. Then G can be identified with an open neighborhood of $\mathbf{0}$ in R^m by a trivial odd homeomorphism. Since the identity map: $Z_m \cap M_r \to \partial G$ is an odd homeomorphism, from Proposition 3.15 (v), we have $\gamma(Z_m \cap M_r) = m$, so $\Gamma_m \neq \emptyset$. By the definition of $c_{(n,r)}$, it is clear that $-\infty < c_{(n,r)} \le c_{(n+1,r)} < \infty$.

It follows that the following lemma holds.

Lemma 3.18. *Assume that a functional $\Psi : Y \to \mathbb{R}$ is sequentially weakly continuous and satisfies $\Psi(0) = 0$. Then for any fixed $s > 0$,*

$$\lim_{n\to\infty} \sup_{u \in W_n, \|u\|_Y \le s} |\Psi(u)| = 0. \tag{3.18}$$

Proof. Put $d_n = \sup_{u \in W_n, \|u\|_Y \le s} |\Psi(u)|$. Then there exists $u_j \in W_n$ with $\|u_j\|_Y \le s$ such that $\lim_{j\to\infty} |\Psi(u_j)| = d_n$. Since Y is a reflexive Banach space, there exist a subsequence $\{u_{j'}\}$ of $\{u_j\}$

and $u^{(n)} \in Y$ such that $u_{j'} \to u^{(n)}$ weakly in Y. Hence $\|u^{(n)}\|_Y \le \liminf_{j' \to \infty} \|u_{j'}\|_Y \le s$. Since W_n is a closed subspace of Y, W_n is weakly closed, so $u^{(n)} \in W_n$. Since Ψ is sequentially weakly continuous, we have $|\Psi(u_{j'})| \to |\Psi(u^{(n)})|$ as $j' \to \infty$. Thereby $|\Psi(u^{(n)})| = d_n$. Since clearly $d_{n+1} \le d_n$, we see that $\lim_{n\to\infty} d_n = d_0 \ge 0$ exists. Since $\{u^{(n)}\}$ satisfies $\|u^{(n)}\|_Y \le s$, there exists a subsequence $\{u^{(n')}\}$ of $\{u^{(n)}\}$ and $u_0 \in Y$ such that $u^{(n')} \to u_0$ weakly in Y. So $\|u_0\|_Y \le s$. Since again Ψ is sequentially weakly continuous, $|\Psi(u^{(n')})| = d_{n'} \to |\Psi(u_0)| = d_0$. Since Y is reflexive, we can look upon $u_0 \in Y^{**} = Y$. Therefore, for any $f_j \in Y^*$, since $u^{(n')} \in W_{n'}$, we have

$$\langle u_0, f_j \rangle_{Y^{**},Y^*} = \langle f_j, u_0 \rangle_{Y^*,Y} = \lim_{n'\to\infty} \langle f_j, u^{(n')} \rangle_{Y^*,Y} = 0.$$

Thus we have $u_0 = 0$, so $d_0 = 0$, that is, (3.18) holds.

Proposition 3.19. *We have* $\lim_{n\to\infty} \inf_{u \in W_n \cap M_r} \|u\|_Y = \infty$.

Proof. Suppose that the conclusion is false. Then there exist $c_1 > 0$ and a sequence $\{u_n\} \subset W_n \cap M_r$ such that $\|u_n\|_Y \le c_1$ for all $n \in \mathbb{N}$. Then

$$\sup_{u \in W_n, \|u\|_Y \le c_1} |J(u)| \ge J(u_n) = r.$$

Therefore,

$$\lim_{n\to\infty} \sup_{u \in W_n, \|u\|_Y \le c_1} |J(u)| \ge \lim_{n\to\infty} J(u_n) = r > 0.$$

If we apply Lemma 3.18 with $\Psi = J$, this is a contradiction.

Proposition 3.20. *For any fixed* $r > 0$, *we have*

$$\lim_{n\to\infty} c_{(n,r)} = \infty. \tag{3.19}$$

Proof. By Proposition 3.19, for any $c > 1$, there exists $n_0 \in \mathbb{N}$ such that for any $n \ge n_0$ and $u \in W_n \cap M_r$, $\|u\|_Y > c$. For any $H \in \Sigma_r$, we have $\gamma(H \cap Z_{n-1}) \le n - 1$. On the other hand, we have codim $W_n \le n - 1$. Hence for $H \in \Sigma_r$ with $\gamma(H) \ge n$, $H \cap W_n$ is non-empty. Indeed, since $H = (H \cap Z_{n-1}) \cup (H \cap W_n)$, it follows from Proposition 3.15 (iii) that

$$n \le \gamma(H) \le \gamma(H \cap Z_{n-1}) + \gamma(H \cap W_n) \le n - 1 + \gamma(H \cap W_n),$$

so $\gamma(H \cap W_n) \ge 1$. Hence $H \cap W_n \ne \emptyset$. For $n \ge n_0$, we have

$$\begin{aligned}
c_{(n,r)} &= \inf_{H\in\Sigma_r, \gamma(H)\ge n} \sup_{u\in H} \tilde{\Phi}(u) \\
&= \inf_{H\in\Sigma_r, \gamma(H)\ge n} \max\left\{ \sup_{u\in H\cap(Y\setminus Z_{n-1})} \tilde{\Phi}(u), \sup_{u\in H\cap Z_{n-1}} \tilde{\Phi}(u) \right\} \\
&\ge \inf_{H\in\Sigma_r, \gamma(H)\ge n} \sup_{u\in H\cap(Y\setminus Z_{n-1})} \tilde{\Phi}(u) \\
&= \inf_{H\in\Sigma_r, \gamma(H)\ge n} \max\left\{ \sup_{u\in H\cap(Y\setminus Z_{n-1})\setminus W_n} \tilde{\Phi}(u), \sup_{u\in H\cap W_n} \tilde{\Phi}(u) \right\} \\
&\ge \inf_{H\in\Sigma_r, \gamma(H)\ge n} \sup_{u\in H\cap W_n} \tilde{\Phi}(u) \\
&\ge \inf_{H\in\Sigma_r, \gamma(H)\ge n} \sup_{u\in H\cap W_n} k_0 \|u_0\|_Y^{p^-} \ge k_0 c^{p^-}.
\end{aligned}$$

Thus we have $\lim_{n\to\infty} c_{(n,r)} = \infty$.

We are in a position to state the main theorem.

Theorem 3.21. *Assume that* (A.0)-(A.5) *and* (f.0)-(f.2) *hold and fix* $r > 0$. *Then for every* $n \in \mathbb{N}$, $c_{(n,r)}$ *defined by* (3.17) *is a critical value of* $\tilde{\Phi}$ *with respect to the submanifold* M_r *such that*

$$0 < c_{(n,r)} \leq c_{(n+1,r)} < \infty \text{ and } c_{(n,r)} \to \infty \text{ as } n \to \infty.$$

Moreover, the problem (1.1) *has infinitely many eigenpair sequence* $\{(u_{(n,r)}, \lambda_{(n,r)})\}$ *such that*

$$J(\pm u_{(n,r)}) = r,\ \Phi(\pm u_{(n,r)}) = c_{(n,r)} \text{ and } 0 < \lambda_{(n,r)} \to \infty \text{ as } n \to \infty.$$

Proof. Taking Proposition 3.16, (3.11) and Proposition 3.20 into consideration, it suffices to derive that $\lambda_{(n,r)} \to \infty$ as $n \to \infty$. It follows from (f.2) that

$$\langle J'(u_{(n,r)}), u_{(n,r)}\rangle_{Y^*,Y} \leq q^+ J(u_{(n,r)}) = q^+ r.$$

Hence

$$\lambda_{(n,r)} = \frac{\langle \Phi'(u_{(n,r)}), u_{(n,r)}\rangle_{Y^*,Y}}{\langle J'(u_{(n,r)}), u_{(n,r)}\rangle_{Y^*,Y}} > \frac{\langle \Phi'(u_{(n,r)}), u_{(n,r)}\rangle_{Y^*,Y}}{q^+ r}. \tag{3.20}$$

Assume that $\lambda_{(n,r)} \leq M$ for all $n \in \mathbb{N}$. Then by (3.20), $\langle \Phi'(u_{(n,r)}), u_{(n,r)}\rangle_{Y^*,Y} \leq Mq^+ r =: c_2$. From (A.3), we have

$$\tilde{\rho}_{(p(\cdot),h_1(\cdot))}(u_{(n,r)}) = \int_\Omega h_1(x)|\nabla u_{(n,r)}|^{p(x)}\,dx \leq \frac{1}{k_0}\langle \Phi'(u_{(n,r)}), u_{(n,r)}\rangle_{Y^*,Y} \leq \frac{c_2}{k_0}.$$

In particular, $\|u_{(n,r)}\|_Y \leq c_3$ for some constant $c_3 > 0$. Then from Lemma 3.7 (i),

$$c_{(n,r)} = \Phi(u_{(n,r)}) \leq c\left(2\|h_0\|_{L^{p'(\cdot)}(\Omega)}\|u_{(n,r)}\|_Y + \tilde{\rho}_{(p(\cdot),h_1(\cdot))}(u_{(n,r)})\right) \leq c_4$$

for some constant $c_4 > 0$. This contradicts Proposition 3.20.

Remark 3.22. We do not know whether the problem (1.1) only has eigenvalue sequence of the form $\{\lambda_{(n,r)}\}$.

Remark 3.23. We assume the following more restrictive condition (A.3′) instead of (A.3).
(A.3′) $k_0 h_1(x)|\boldsymbol{\xi}|^{p(x)} \leq \boldsymbol{a}(x,\boldsymbol{\xi}) \cdot \boldsymbol{\xi} = p(x)A(x,\boldsymbol{\xi})$ for a.e. $x \in \Omega$ and all $\boldsymbol{\xi} \in \mathbb{R}^N$.

We note that (i) in Example 3.2 satisfies this condition, but (ii) does not satisfy this condition. Under the hypotheses (A.1)-(A.2), (A.3′), (A.4)-(A.5) and (f.0)-(f.2), we have

$$\lambda_{(n+1,r)} \geq \frac{p^- q^-}{p^+ q^+}\lambda_{(n,r)} \text{ for } n = 1,2,\ldots \tag{3.21}$$

In particular, if $p(x) = p =$ const. and $q(x) = q =$ const., then it follows that $\lambda_{(n,r)} \leq \lambda_{(n+1,r)}$ for $n = 1, 2, \ldots$.

Proof. Let u_n be the eigenfunction of the problem (1.1) associated with $\lambda_{(n,r)}$ ($n = 1, 2, \ldots$). From (A.3′), (f.2) and Theorem 3.21, we have

$$\lambda_{(n+1,r)} = \frac{\langle \Phi'(u_{n+1}), u_{n+1}\rangle_{Y^*,Y}}{\langle J'(u_{n+1}), u_{n+1}\rangle_{Y^*,Y}} = \frac{\int_\Omega \boldsymbol{a}(x,\nabla u_{n+1}(x))\cdot\nabla u_{n+1}(x)dx}{\int_\Omega f(x,u_{n+1}(x))u_{n+1}(x)dx}$$
$$= \frac{\int_\Omega p(x)A(x,\nabla u_{n+1}(x))dx}{\int_\Omega q(x)F(x,u_{n+1}(x))dx} \geq \frac{p^-\Phi(u_{n+1})}{q^+J(u_{n+1})} = \frac{p^-}{q^+r}c_{(n+1,r)} \geq \frac{p^-}{q^+r}c_{(n.r)}.$$

On the other hand, from (f.2) we have

$$c_{(n,r)} = \Phi(u_n) = r\frac{\Phi(u_n)}{J(u_n)} = r\frac{\int_\Omega A(x,\nabla u_n(x))dx}{\int_\Omega F(x,u_n(x))dx}$$
$$\geq r\frac{\int_\Omega \frac{1}{p(x)}\boldsymbol{a}(x,\nabla u_n(x))\cdot\nabla u_n(x)dx}{\int_\Omega \frac{1}{q(x)} f(x,u_x(x))u_n(x)dx} \geq \frac{q^-r\langle \Phi'(u_n),u_n\rangle_{Y^*,Y}}{p^+\langle J'(u_n),u_n\rangle_{Y^*,Y}} = \frac{q^-r}{p^+}\lambda_{(n,r)}$$

Thus we have the conclusive estimate (3.21).

4. The infimum of all the eigenvalues

In this section, we consider the infimum of all the eigenvalues of the problem (1.1). We show that there exist two cases where the infimum is equal to zero or positive according to the hypoetheses on the variable exponent.

For a subset $A \subset \Omega$ and $\delta > 0$, put $B(A, \delta) = \{x \in \mathbb{R}^N; \operatorname{dist}(x, A) < \delta\}$. In particular, for $x_0 \in \Omega$, if $A = \{x_0\}$, then we write $B(\{x_0\}, \delta)$ by simply $B(x_0, \delta)$.

Put $\Lambda = \{\lambda; \lambda$ is an eigenvalue of the problem (1.1)$\}$ and $\lambda_* = \inf \Lambda$.

Lemma 4.1. *For $r > 0$, the following inequality holds.*

$$\beta_r := \inf_{u\in M_r} \tilde{\rho}_{(p(\cdot),h_1(\cdot))}(u) > 0,$$

where $\tilde{\rho}_{(p(\cdot),h_1}(\cdot))$ is defined by (3.2).

Proof. Arguing by contradiction, assume that $\beta_r = 0$. Then there exist $\{u_n\} \subset M_r$ such that $\tilde{\rho}_{(p(\cdot),h_1(\cdot))}(u_n) \to 0$ as $n \to \infty$. Hence by Proposition 3.5 (v), $u_n \to 0$ in Y as $n \to \infty$.

On the other hand, $J(u_n) = r > 0$, and since J is continuous in Y, $J(u_n) = r \to J(0) = 0$. This is a contradiction.

Lemma 4.2. *For $r > 0$, let u_0 be an eigenfunction of the problem* (1.1) *associated with $\lambda_{(1,r)}$ defined in Theorem 3.21. Then*

$$\Phi(u_0) = c_{(1,r)} = \inf\{\Phi(u); u \in M_r\}.$$

Proof. Put $b_r = \inf\{\Phi(u);\ u \in M_r\}$. Since $c_{(1,r)} = \inf_{H\in\Sigma_r,\gamma(H)\ge 1}\sup_{u\in H}\tilde{\Phi}(u)$, if $u \in H$ with $H \in \Sigma_r$ and $\gamma(H) \ge 1$, then we have $\Phi(u) = \tilde{\Phi}(u) \ge b_r$ from $u \in M_r$, so $c_{(1,r)} \ge b_r$.

On the other hand, by the definition of b_r, there exists a sequence $\{u_n\} \subset M_r$ such that $b_r = \lim_{n\to\infty}\Phi(u_n)$. For large n, $b_r + 1 \ge \Phi(u_n) \ge k_0\|u_n\|_Y^{p+} \wedge \|u\|_Y^{p-}$. Thus $\{u_n\}$ is bounded in Y. So there exist a subsequence $\{u_{n'}\}$ of $\{u_n\}$ and $u_* \in Y$ such that $u_{n'} \to u_*$ weakly in Y. Thereby, $\Phi(u_*) \le \liminf_{n'\to\infty}\Phi(u_{n'}) = b_r$. Since M_r is a weakly closed subset of Y, $u_* \in M_r$, so $\Phi(u_*) \ge b_r$. Thus we have $\Phi(u_*) = b_r$. By (A.5), $\Phi(\pm u_*) = b_r$. Let $H_0 = \{\pm u_*\}$, then $H_0 \in \Sigma_r$ and $\gamma(H_0) = 1$. Therefore, $c_{(1,r)} \le \sup_{u\in H_0}\Phi(u) = \Phi(\pm u_*) = b_r$.

Thus we have $c_{(1,r)} = b_r$.

From now on, we suppose that more restrictive assumptions on the function f in (1.1).

(f.0′) (f.0) holds with $q(x) = p(x)$.

(f.2′) (f.2) holds with $q(x) = p(x)$, that is, $0 < f(x, t)t = p(x)F(x, t)$ for a.e. $x \in \Omega$ and all $0 \ne t \in \mathbb{R}$.

For example, a function $f(x, t) = a(x)|t|^{p(x)-2}t$ with a function $a(x)$ satisfying the condition in (f.0) with $q(x) \equiv p(x)$ verifies (f.0′), (f.1) and (f.2′).

Theorem 4.3. *Assume that* (A.0)-(A.5), (f.0′), (f.1) *and* (f.2′) *with* $p(x) = p =$ const. *for all* $x \in \Omega$ *hold. Then we have* $\lambda_* > 0$.

Proof. Let u be the eigenfuction of (1.1) associated with λ. Then clearly $J(u) > 0$. We show that there exists $t_0 > 0$ such that $u_1 := \frac{1}{t_0}u \in M_1$. Indeed, since $f(x, t)t = pF(x, t)$ for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$, we can write $F(x, t) = F(x, 1)t^p$ from (3.4). Hence $J(u) = \int_\Omega F(x, 1)|u(x)|^p d\sigma_x$. Thus $J\left(\frac{u}{t}\right) = t^{-p}J(u)$ for $t > 0$. Here we can see that to $J\left(\frac{u}{t}\right) \to 0$ as $t \to \infty$ and $J\left(\frac{u}{t}\right) \to \infty$ as $t \to +0$. Since $J\left(\frac{u}{t}\right)$ is continues with respect to $t \in (0, \infty)$, there exists $t_0 > 0$ such that $J\left(\frac{u}{t}\right) = 1$, so $u_1 := \frac{u}{t_0} \in M_1$.

Now since $J(u_1) = 1$, it follows from Lemma 4.1 that we have

$$\lambda = \frac{\int_\Omega \boldsymbol{a}(x,\nabla u(x))\cdot\nabla u(x)dx}{\int_\Omega f(x,u(x))u(x)dx} \ge \frac{k_0\int_\Omega h_1(x)|\nabla u(x)|^p\,dx}{\int_\Omega f(x,u(x))u(x)dx} \ge \frac{k_0\int_\Omega h_1(x)|t_0\nabla u_1(x)|^p\,dx}{p\int_\Omega F(x,t_0u_1(x)dx}$$
$$= \frac{k_0t_0^p\int_\Omega h_1(x)|\nabla u_1(x)|^p\,dx}{pt_0^p\int_\Omega F(x,u_1(x))dx} = \frac{k_0}{p}\int_\Omega h_1(x)|\nabla u_1(x)|^p\,dx \ge \frac{k_0}{p}\beta_1 > 0.$$

Thus we have $\lambda_* = \inf\Lambda \ge \frac{k_0}{p}\beta_1 > 0$.

Remark 4.4. Though this theorem is well known for p-Laplacian, our theorem also contains the result for a class of operators which also contains the mean curvature operator.

Next we consider the following assumption.

(f.3) $f(x, t) = a(x)|t|^{p(x)-2}t$ with $1 \le a \in L^\infty(\Omega)$.

Then we can derive a sufficient condition for $\lambda_* > 0$ for the case $p(x) \ne$ const.

Theorem 4.5. *Assume that* (A.0)-(A.5) *and* (f.3) *hold. Moreover, suppose that* $\Gamma_2 = \varnothing$ *and there exists a vector* $\boldsymbol{l} \in \mathbb{R}^N \setminus \{\boldsymbol{0}\}$ *such that for all* $x \in \Omega$, *the function*

$$k : I_x := \{t;\ x + t\boldsymbol{l} \in \Omega\} \ni t \mapsto p(x + t\boldsymbol{l})$$

is monotone. Then we have $\lambda_* > 0$.

Proof. Since $\Gamma_2 = \varnothing$, we note that $X = W_0^{1,p(\cdot)}(\Omega)$. By the proof of [17, Theorem 3.3], we have

$$c := \inf_{v \in X \setminus \{0\}} \frac{\int_\Omega |\nabla v(x)|^{p(x)}\, dx}{\int_\Omega |v(x)|^{p(x)}\, dx} > 0.$$

Since $Y \hookrightarrow X$ and $h_1(x) \geq 1$, for any $v \in Y \setminus \{0\}$,

$$\frac{\int_\Omega h_1(x) |\nabla v(x)|^{p(x)}\, dx}{\int_\Omega |v(x)|^{p(x)}\, dx} \geq \frac{\int_\Omega |\nabla v(x)^{p(x)} dx}{\int_\Omega |v(x)|^{p(x)}\, dx} \geq c. \tag{4.1}$$

Let (u, λ) be any eigenpair of (1.1). Then from (A.3), (f.3) and (4.1),

$$\lambda = \frac{\langle \Phi'(u), u \rangle_{Y^*,Y}}{\langle J'(u), u \rangle_{Y^*,Y}} = \frac{\int_\Omega \boldsymbol{a}(x, \nabla u(x)) \cdot \nabla u(x) dx}{\int_\Omega f(x, u(x)) u(x) dx}$$
$$\geq \frac{k_0 \int_\Omega h_1(x) |\nabla u(x)^{p(x)} dx}{\|a\|_{L^\infty(\Omega)} \int_\Omega u(x)^{p(x)} dx} \geq \frac{k_0}{\|a\|_{L^\infty(\Omega)}} c > 0.$$

Thus $\lambda_* \geq \dfrac{k_0}{\|a\|_{L^\infty(\Omega)}} c > 0.$

Remark 4.6. This conclusion holds not only for $p(\cdot)$-Laplacian, but also for mean curvature operator, so this theorem is an extension of [17, Theorem 3.3].

Theorem 4.7. *Let* (A.0)-(A.5), (f.0′), (f.1) *and* (f.2′) *hold. Moreover, assume that there exist an open subset* $U \subset \Omega$ *and* $x_0 \in U$ *such that* $p(x) < p(x_0)$ *for all* $x \in \partial U$. *Then we have* $\lambda_* = 0$.

Proof. Without loss of generality, we may assume that $\bar{U} \subset \Omega$. Then there exist $\varepsilon_0, \varepsilon_1, \varepsilon_2 > 0$ such that $p(x) < p(x_0) - 4\varepsilon_0$ for all $x \in \partial U$, $p(x) < p(x_0) - 2\varepsilon_0$ for all $x \in B(\partial U, \varepsilon_1)$, $B(x_0, \varepsilon_2) \subset U \setminus B(\partial U, \varepsilon_1)$ and $|p(x) - p(x_0)| < \varepsilon_0$ for all $x \in B(x_0, \varepsilon_2)$. Choose $u_0 \in C_0^\infty(\Omega)$ such that $0 \leq u_0(x) \leq 1$ for all $x \in \Omega$,

$$u_0(x) = \begin{cases} 1 & x \in U \setminus B(\partial U, \varepsilon_1), \\ 0 & x \notin U \cup B(\partial U, \varepsilon_1), \end{cases}$$

and $|\nabla u_0(x)| \leq C$ for all $x \in \Omega$.
For $t > 0$, define

$$h(t) = J(tu_0) = \int_\Omega F(x, tu_0(x)) dx = \int_\Omega t^{p(x)} F(x, u_0(x)) dx.$$

Then h is differentiable in $(0, \infty)$ and

$$h'(t) = \int_\Omega p(x) t^{p(x)-1} F(x, u_0(x)) dx > 0 \text{ for } t > 0.$$

Thus h is strictly monotone increasing in $(0, \infty)$, $h(t) \to 0$ as $t \to +0$ and $h(t) \to \infty$ as $t \to \infty$. Hence for any $r > 0$, there exists unique $t(r) > 0$ such that $t(r)u_0 \in M_r$. We note that

$t(r) \to \infty$ as $r \to \infty$. Thus there exists $r_0 > 0$ such that $t(r) > 1$ for all $r > r_0$. For $r > r_0$, let $(u_{(1,r)}, \lambda_{(1,r)})$ be the eigenpair of the problem (1.1). Using (A.3) and Lemma 4.2, we have

$$\lambda_{(1,r)} = \frac{\langle \Phi'(u_{(1,r)}), u_{(1,r)} \rangle_{Y^*,Y}}{\langle J'(u_{(1,r)}), u_{(1,r)} \rangle_{Y^*,Y}} = \frac{\int_\Omega \boldsymbol{a}(x, \nabla u_{(1,r)}(x)) \cdot \nabla u_{(1,r)}(x) dx}{\int_\Omega f(x, u_{(1,r)}(x)) u_{(1,r)}(x) dx}$$

$$\leq \frac{\int_\Omega p(x) A(x, \nabla u_{(1,r)}(x)) dx}{\int_\Omega p(x) F(x, u_{(1,r)}(x)) dx} \leq \frac{p^+ \Phi(u_{(1,r)})}{p^- r} = \frac{p^+}{p^-} \frac{c_{(1,r)}}{r}$$

$$= \frac{p^+}{p^- r} \inf\{\Phi(u); u \in M_r\} = \frac{p^+}{p^-} \inf\left\{\frac{\Phi(u)}{J(u)}; u \in M_r\right\} \leq \frac{p^+}{p^-} \frac{\Phi(u)}{J(u)} \text{ for all } u \in M_r.$$

Therefore, using Remark 3.1 (ii), since $t(r) > 1$, we obtain

$$\lambda_* \leq \lambda_{(1,r)} \leq \frac{p^+}{p^-} \frac{\Phi(t(r)u_0)}{J(t(r))u_0)} = \frac{p^+}{p^-} \frac{\int_\Omega A(x, t(r) \nabla u_0(x)) dx}{\int_\Omega F(x, t(r) u_0(x)) dx}$$

$$\leq \frac{p^+}{p^-} \frac{\int_\Omega t(r)^{p(x)} A(x, \nabla u_0(x)) dx}{\int_\Omega t(r)^{p(x)} F(x, u_0(x)) dx} \leq \frac{p^+}{p^-} \frac{\int_{B(\partial U, \varepsilon_1)} t(r)^{p(x)} A(x, \nabla u_0(x)) dx}{\int_{B(x_0, \varepsilon_2)} t(r)^{p(x)} F(x, 1) dx}$$

$$\leq \frac{p^+}{p^-} \frac{t(r)^{p(x) - 2\varepsilon_0} \int_\Omega A(x, \nabla u_0(x)) dx}{t(r)^{p(x_0) - \varepsilon_0} \int_{B(x_0, \varepsilon_2)} F(x, 1) dx} \leq \frac{p^+}{p^-} t(r)^{-\varepsilon_0} \frac{\int_\Omega A(x, \nabla u_0(x)) dx}{\int_{B(x_0, \varepsilon_2)} F(x, 1) dx} \to 0 \text{ as } r \to \infty.$$

Thereby we have $\lim_{r \to \infty} \lambda_{(1,r)} = 0$, so $\lambda_* = 0$.

Remark 4.8. In particular, if $p(x)$ has a strictly local maximum in Ω, then from Theorem 4.7, we see that $\lambda_* = 0$. Under (A.3′) instead of (A.3), if $p(x)$ has strictly local minimum, then we can similarly derive $\lambda_* = 0$.

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References

[1] W. Allegretto, *Form estimates for the p(x)-Laplacian*, Proc. Amer. Math. Soc. 135 (2007), 2177–2185.

[2] A. Anane, *Simplicité et isolation de la premiére valeur propre de p-Laplacian avec poids*, C. R. Acad. Sci. Paris, Sér. J. Math. 305 (1987), 725–728.

[3] J. Aramaki, *Existence of three weak solutions for a class of nonlinear operators involving p(x)-Laplacian with mixed boundary conditions.* Nonlinear Funct. Anal. Appl. 26(3) (2021), 531–551.

[4] J. Aramaki, *Mixed boundary value problem for a class of quasi-linear elliptic operators containing p(·)-Laplacian in a variable exponent Sobolev space*, Adv. Math. Sci. Appl. 31(2) (2022), 207–239.

[5] J. Aramaki, *Existence of nontrivial weak solutions for nonuniformly elliptic equation with mixed boundary condition in a variable exponent Sobolev space*, Electronic J. Qualitative Theory Differ. Eq. 2023 (12) (2023), 1–22.

[6] J. Aramaki, *Existence of three weak solutions for the Kirchhoff-type problem with mixed boundary condition in a variable exponent Sobolev space*, East-West J. Math. 24(2) (2023), 89–117.

[7] J. Aramaki, *Existence of three weak solutions for a nonlinear problem with mixed boundary condition in a variable exponent Soboev space*, J. Anal. 32(2) (2024), 733–755.

[8] J. Aramaki, *Eigenvalue problem for a class of nonlinear operators containing p(·)-Laplacian in a variable exponent Sobolev space,* to appear in J. Partial Diff. Eqs.

[9] P. G. Ciarlet and G. Dinca, *A Poincaré inequality in a Sobolev space with a variable exponent*, Chin. Ann. Math. 32B(3) (2011), 333–342.

[10] S. G. Deng, *Eigenvalues of the p(x)-Laplacian Steklov problem*, J. Math. Anal. Appl. 339 (2008), 925–937.

[11] L. Diening, *Theoretical and numerical results for electrorheological fluids*, ph. D. thesis, University of Frieburg, Germany 2002.

[12] L. Diening, P. Harjulehto, P. Hästö and M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponent*, Lecture Notes in Math. Springer, 2017.

[13] X. L. Fan, *Solutions for p(x)-Laplacian Dirichlet problems with singular coefficients*, J. Math. Anal. Appl. 312 (2005), 464–477.

[14] X. L. Fan, *Eigenvalues of the p(x)-Laplacian Neumann problem*, Nonlinear Anal. 67 (2007), 2982–2992.

[15] X. L. Fan, *Boundary trace embedding theorems for variable exponent Sobolev spaces*, J. Math. Anal. Appl. 339 (2008), 1395–1412.

[16] X. L. Fan and Q. H Zhang, *Existence of solutions for p(x)-Laplacian Dirichlet problem*, Nonlinear Anal. 52 (2003), 1843–1852.

[17] X. L. Fan, Q. Zhang and D. Zhao, *Eigenvalues of p(x)-Laplacian Dirichlet problem*, J. Math. Anal. Appl. 302 (2015), 306–317.

[18] X. L. Fan and D. Zhao, *On the spaces* $L^{p(x)}(\Omega)$ *and* $W^{m,p(x)}(\Omega)$, J. Math. Anal. Appl. 263 (2001), 424–446.

[19] L. Friedlander, *Asymptotic behavior of the eigenvalues of the p-Laplacian*, Comm. Partial Differential Equations 14 (1989), 1059–1069.

[20] T. C. Halsey, *Electrorheological fluids, Science* 258 (1992), 761–766.

[21] O. Kovăčik and J. Rákosník, *On spaces* $L^{p(x)}(\Omega)$ *and* $W^{k,p(x)}(\Omega)$, Czechoslovak Math. J. 41(116) (1991), 592–618.

[22] A. Lê, *Eigenvalue problems for the p-Laplacian*, Nonlinear Anal. 64 (2006), 1057–1099.

[23] L. Ljusternik and L. *Schnirelmann, Méthodes topologiques dans les problémes variationels*, Hermann, Paris, 1934.

[24] R. A. Mashiyev, B. Cekic, M. Avci, and Z. Yucedag, *Existence and multiplicity of weak solutions for nonuniformly elliptic equations with nonstandard growth condition*, Complex Variables Elliptic Equa. 57(5) (2012), 579–595.

[25] M. Mihăilescu, G. Moroşanu and D. Stancu-Dumitru, *Equations involving a variable exponent Grushin-type operator*, Nonlinearity 24 (2011), 2663–2680.

[26] M. Mihăailescu and V. Rădulescu, *A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids*, Proc. Royal Soc. A. 462 (2006), 2625–2641.

[27] M. Mihăilescu and V. Rădulescu, *On a nonhomogenuous quasilinear eigenvalue problem in Sobolev cpaces with variable exponent*, Proc. Amer. Math. Soc. 135 (2007), 2929–2937.

[28] M. Mihăilescu, V. Rădulescu and D. Stancu-Dumitru, *A Caffarelli-Kohn-Nirenberg-type inequality with variable exponent and applications to PDEs*, Complex Var. Elliptic Equa. 56 (2011), 659–669.

[29] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Application to Differential Equations*, CBMS Reg. Conf. Ser. in Math., Vol. 65, Am. Math. Soc., Providence, 1986.

[30] M. Růžička, *Electrotheological fluids: Modeling and Mathematical Theory, Lecture Notes in Mathematics*, Vol. 1784, Berlin, Springer, 2000.

[31] A. Szulkin, *Ljusternik-Schnirelmann theory on C^1-manifolds*, Ann. Inst. Henri Poincaré 5(2) (1988), 119–139.

[32] E. Zeidler, *Nonlinear Functional Analysis and its Applications I: Fixed-Point Theorems, III: Variational Methods and Optimization*, Springer-Verlag, Now York, Berlin, Heidelberg, London, Paris, Tokyo, 1990.

[33] D. Zhao, WJ. Qing and X.L. Fan, *On generalized Orlicz space $L^{p(x)}(\Omega)$*, J. Gansu Sci. 9(2) (1996), 1–7. (in Chinese).

[34] VV. Zhikov, *Averaging of functionals of the calculus of variation and elasticity theory*, Math. USSR, Izv. 29 (1987), 33–66.