Minimax Theory and its Applications
Volume 10 (2025), No. 1, 34-42

Existence and Uniqueness of Solution to the *p*-Laplacian Equations Involving Discontinuous Kirchhoff Functions Via A Global Minimum Principle of Ricceri

Yun-Ho Kim

Yun-Ho Kim, Department of Mathematics Education, Sangmyung University, Seoul 03016, Republic of Korea
kyh1213@smu.ac.kr

Received: August 28, 2024
Accepted: September 20, 2024

The aim of this paper is to establish the existence and uniqueness of solutions to the p-Laplacian problems with discontinuous Kirchhoff coefficients using a global minimum principle as its main tool. In particular, we apply the Dìaz-Saa inequality to observe the uniqueness of a positive weak solution to Brézis-Oswald type problem involving the p-Laplacian.

Keywords: Discontinuous Kirchhoff function; Weak solution; Uniqueness; Global minima.

2020 Mathematics Subject Classification: 35B09, 35D30, 35J25, 35J60, 49J35.

1. Introduction

The present paper is devoted to the existence and uniqueness of positive solutions to a Kirchhoff type problem driven by the *p*-Laplacian as follows:

$$\begin{cases} -M\left(\int_\Omega |\nabla v|^p \, dy\right)\Delta_p v = a(y)h(v) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \\ \int_\Omega |\nabla v|^p \, dy \in J, \end{cases} \qquad (P)$$

where $\Delta_p v := \operatorname{div}(|\nabla v|^{p-2}\nabla v)$, $1 < p < +\infty$, $\Omega \subset \mathbb{R}^N$ is a bounded open set with Lipschitz boundary $\partial\Omega$, $J \subseteq (0, +\infty)$ is an open interval, M is a discontinuous Kirchhoff type function and functions a and h are nonnegative which will be specified later.

As is well known, the problem (P) is a non-local problem because the function M appears in the equation. The study of Kirchhoff-type problem, initially suggested by Kirchhoff [10], has a strong background in various applications in physics and biology. For this reason, there has recently been a great deal of interest in the study of elliptic equations involving Kirchhoff coefficients, for instance [2,4-7,11-14,17,19] and the references therein.

ISSN 2199-1421 (electronic)

The authors in [5, 13, 14] have considered the existence of nontrivial solutions when the Kirchhoff function $M:\mathbb{R}_0^+ \to \mathbb{R}^+$ is an increasing and continuous function with the nondegenerate condition $\inf_{t\in\mathbb{R}^+_0} M(t) \ge m_0 > 0$, where m_0 is a constant. However, since the increasing condition excludes the non-monotonic case, the existence of multiple solutions to a class of fractional p-Laplacian equations of Schrödinger-Kirchhoff type has been obtained in [17], where the nondegenerate Kirchhoff function M is continuous and satisfies the assumption:

(K1) For $0 < s < 1$, there is $\vartheta \in [1, \frac{N}{N-sp})$ such that $\vartheta \mathcal{M}(t) \ge M(t)t$ for any $t \ge 0$, where $\mathcal{M}(t) := \int_0^t M(\tau)d\tau$.

Pursuantly, the condition (K1) includes the typical example $M(t) = 1 + at^{\vartheta}(a \ge 0, t \ge 0)$ as well as the non-monotonic case. In this perspective, nonlinear elliptic equations with Kirchhoff coefficient satisfying (K1) have been studied extensively by many researchers in recent years; see [2, 6–9, 17, 19]. In view of these related papers, the functional $\Phi : W_0^{1,p}(\Omega) \to \mathbb{R}$ corresponding to the primitive part in (P) is defined by

$$\Phi(u) = \frac{1}{p}\mathcal{M}\left(\int_\Omega |\nabla u|^p \, dy\right)$$

for any $u \in W_0^{1,p}(\Omega)$, where M is given in (K1). Then it follows from the fact $M \in C(\mathbb{R}_0^+)$ that $\Phi \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$ and its Frechet derivative is given by

$$\langle \Phi'(u), \omega\rangle = M\left(\int_\Omega |\nabla u|^p \, dy\right)\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \omega dy$$

for any $u, \omega \in W_0^{1,p}(\Omega)$. In particular, conditions $M \in C(\mathbb{R}_0^+)$ and (K1) play a fundamental role in obtaining some topological properties of functionals Φ, Φ' and the Palais-Smale type compactness condition for an energy functional corresponding to (P), which are important for applying variational methods such as the mountain pass theorem, Ekeland variational principle, fountain theorem and dual fountain theorem. However, the continuity of the nondegenerate Kirchhoff function M in $[0, \infty)$ excludes many examples, for instance, let us consider the Kirchhoff functions

$$M(t) = \tan t \quad \text{for} \quad 0 < t < \frac{\pi}{2}$$

and

$$M(t) = (c-t)^{-s} \quad \text{for} \quad t \in (-\infty, c), \quad \text{where} \quad c > 0, s \in (0,1).$$

These examples cannot be addressed by any of the results known up to now. Very recently, in order to obtain the existence and uniqueness of positive solutions to elliptic equations with discontinuous Kirchhoff coefficients, Ricceri [18] has provided new approach that is different from previous related studies [2, 4–7, 11–14, 17, 19]. Based on this work, the main purpose of this paper is to demonstrate the existence and uniqueness of positive solutions to the p-Laplacian problems with discontinuous Kirchhoff type function. To this end, the main tools are the abstract global minimum principle in [18] and the uniqueness result of Brézis-Oswald type problem based on [1]. As in [18], the novelties of this paper are the lack of the continuity of the Kirchhoff function M in $[0, \infty)$ and the localization of the solution. Unlike in the paper [18], we apply the Dìaz-Saa inequality [3] to observe the uniqueness of a positive weak solution to Brézis-Oswald type problem involving the p-Laplacian.

The outline of this paper is as follows. We first present a global minimum principle provided by B. Ricceri [18], and then we provide the variational framework and a very useful auxiliary related to problem (P). Finally, we illustrate the existence and uniqueness of positive solutions under suitable assumptions.

2. Preliminaries and Main result

In this section, we introduce the variational setting corresponding to the problem (P), and then we demonstrate the existence result of a unique positive solution to the p-Laplacian problem with discontinuous Kirchhoff coefficients. To do this, we give the abstract theorem provided by B. Ricceri [18].

Let us define the basic function space

$$X = \left\{ v \in W^{1,p}(\Omega) : \int_\Omega |\nabla v|^p \, dy < +\infty \right\}$$

with the norm

$$\|v\|_X = \left(\int_\Omega |\nabla v|^p \, dy \right)^{\frac{1}{p}},$$

which is equivalent to the norm $\|v\|_{W^{1,p}(\Omega)}$.

Definition 2.1. *We say that $v \in X$ is a weak solution of (P) if*

$$M\left(\int_\Omega |\nabla v|^p \, dy \right) \int_\Omega |\nabla v|^{p-2} \nabla v \cdot \nabla \omega \, dy = \int_\Omega a(y) h(v) \omega \, dy$$

for any $\omega \in X$.

Let us define the functional $\Phi : X \to \mathbb{R}$ by

$$\Phi(v) := \frac{1}{p} \int_\Omega |\nabla v|^p \, dy$$

for any $v \in X$.

Definition 2.2. *Let X be a topological space, a function $g : X \to \mathbb{R}$ is said to be inf-compact (resp. sup-compact) provided that, for each $r \in \mathbb{R}$, the set $g^{-1}((-\infty, r])$ (resp. $g^{-1}([r, +\infty))$ is compact.*

The following abstract global minimum principle in [18] is crucial to obtain our main result.

Theorem 2.3. *Let X be a topological space, and let $\Phi : X \to \mathbb{R}$, with $\Phi^{-1}(0) \neq \emptyset$ and $\psi : X \to \mathbb{R}$ be two functions such that, for each $\eta > 0$, the function $\eta\Phi - \Psi$ is lower semicontinuous, inf-compact and has a unique global minimum. Moreover, assume that ψ has no global* *maxima in X. Furthermore, let* $J \subseteq (0, +\infty)$ *be an open interval and* $M : J \to \mathbb{R}$ be an *increasing function such that* $M(J) = (0, +\infty)$. *Then, there exists a unique* $\tilde{u} \in X$ *such that* $\Phi(\tilde{u}) \in J$ *and*

$$M(\Phi(\tilde{u}))\Phi(\tilde{u}) - \Psi(\tilde{u}) = \inf_{u \in X}(M(\Phi(\tilde{u}))\Phi(u) - \Psi(u)).$$

Now we introduce the Dìaz-Saa inequality [3] which is a useful tool in the study of uniqueness of positive solutions to boundary value problems with the p-Laplacian.

Lemma 2.4. *Assume that* $\Omega \subset \mathbb{R}^N$ *is a open bounded set with Lipschitz boundary* $\partial\Omega$. *Then the following inequality*

$$\int_\Omega \left(-\frac{\Delta_p v_1}{v_1^{p-1}} + \frac{\Delta_p v_2}{v_2^{p-1}} \right)(v_1^p - v_2^p)dy \geq 0$$

holds (in the sense of distributions) for all $v_1, v_2 \in X$, *such that* $v_1 > 0$, $v_2 > 0$ *for almost all in* Ω *and both* $\frac{v_1}{v_2}, \frac{v_2}{v_1} \in L^\infty(\Omega)$.

As showing each assumption of Theorem 2.3 is satisfied, we demonstrate our main result.

Theorem 2.5. *Assume that there exists an open interval* $J \subseteq (0, +\infty)$ *such that the restriction of M to J is increasing and* $M(J) = (0, +\infty)$. *Let* $h : [0, +\infty) \to [0, +\infty)$ *be continuous function such that the following assumptions hold:*

(B1) *The function* $\xi \mapsto \frac{h(\xi)}{\xi^{p-1}}$ *is decreasing in* $(0, +\infty)$;

(B2) $\lim_{\xi \to +\infty} \frac{h(\xi)}{\xi^{p-1}} = 0$ *and* $\lim_{\xi \to 0^+} \frac{h(\xi)}{\xi^{p-1}} = +\infty$.

Then, for each $a \in L^\infty(\Omega)$, *with* $a > 0$, *problem (P) has a unique weak solution* $\tilde{u}$, *which is the unique global minimum in X of the functional*

$$v \mapsto \frac{1}{p} M\left(\int_\Omega |\nabla u|^p dy\right)\int_\Omega |\nabla v(y)|^p dy - \int_\Omega \left(a(y)\int_0^{v^+(y)} h(\xi)d\xi\right)dy,$$

where $v^+ = \max\{0, v\}$.

Proof. First of all, extend h to $\mathbb{R}$ putting $h(\xi) = 0$ for all $\xi < 0$. To apply Theorem 2.3, defining Ψ by

$$\Psi(v) := p\int_\Omega a(y)H(v^+(y))dy$$

for all $v \in X$, where $H(\xi) = \int_0^\xi h(t)dt$. The functional Ψ is C^1 with derivatives given by

$$\langle \Psi'(v), w \rangle = p\int_\Omega h(v)w(y)dy$$

for all $v, w \in X$. Moreover, due to the subcritical growth of h, the functional Ψ is sequentially weakly continuous. Fix $\eta > 0$. Also, we infer that the functional $\eta\Phi - \Psi$ is sequentially weakly lower semicontinuous on X. Choose $\varepsilon \in \left(0, (pC_{p,imb})^{-1}\eta\right)$, where $C_{p,imb}$ is an imbedding constant of $X \hookrightarrow L^p(\Omega)$. Since $\lim_{\xi \to +\infty} \frac{H(\xi)}{\xi^p} = 0$, there is $C(\varepsilon) > 0$ such that

$$H(\xi) \leq \frac{\varepsilon}{p\mathrm{C}} |\xi|^p + \frac{C(\varepsilon)}{p} \tag{2.1}$$

for any $\xi \in \mathbb{R}$, where $\mathrm{C} := \operatorname{ess\,sup}_\Omega a$. Thus, one has

$$\Psi(v) \leq \varepsilon \int_\Omega |v(y)|^p \, dy + C(\varepsilon) \int_\Omega a(y) dy.$$

This together with (2.1) yields that for any $v \in X$,

$$\begin{aligned} \eta\Phi(v) - \Psi(v) &\geq \frac{\eta}{p} \int_\Omega |\nabla v(y)|^p \, dy - \varepsilon \int_\Omega |v(y)|^p \, dy - C(\varepsilon) \int_\Omega a(y) dy \\ &= \frac{\eta}{p} \|v\|_X^p - \varepsilon \int_\Omega |v(y)|^p \, dy - C(\varepsilon) \int_\Omega a(y) dy \\ &\geq \left(\frac{\eta}{p} - \varepsilon C_{p,imb} \right) \|v\|_X^p - C(\varepsilon) \int_\Omega a(y) dy. \end{aligned}$$

Hence, due to the choice of ϵ, we have

$$\lim_{\|v\|_X \to +\infty} (\eta\Phi(v) - \Psi(v)) = +\infty.$$

This together with the reflexivity of X and the Eberlein-Smulyan theorem yields that the sequentially weakly lower semicontinuous functional $\eta\Phi - \psi$ is weakly inf-compact. We now claim that it has a unique global minimum in X. Indeed, its critical points are exactly the weak solutions of the problem

$$\begin{cases} -\Delta_p v = \dfrac{a(y)}{\eta} h(v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where $v \in X$ is a weak solution of (2.2) if

$$\int_\Omega |\nabla v|^{p-2} \nabla v \cdot \nabla \omega \, dy = \frac{1}{\eta} \int_\Omega a(y) h(v) \omega \, dy$$

for any $\omega \in X$. Now, fix any nonnegative function $v \in C^2(\Omega) \cap C_0(\overline{\Omega})$, with $v = 0$ on $\partial\Omega$, such that

$$\eta_1 \int_\Omega |v(y)|^p \, dy = \int_\Omega |\nabla v(y)|^p \, dy,$$

where $\eta 1$ is a positive eigenvalue that can be characterized as

$$\eta_1 = \min_{v \in W^{1,p}(\Omega)} \left\{ \int_\Omega |\nabla v(y)|^p \, dy : \|v\|_{L^p(\Omega)} = 1 \right\}.$$

Also, fix $\tau \in (0, \|a\|_{L^\infty(\Omega)})$. Of course, the set

$$\Omega_\tau := \{ y \in \Omega : a(y) \geq \tau \}$$

has a positive measure. Furthermore, fix $M > 0$ so that

$$M > \frac{\eta\eta_1 \int_\Omega |v(y)|^p \, dy}{p\tau \int_{\Omega_\tau} |v(y)|^p \, dy}.$$

Since $\lim_{\xi \to 0^+} \frac{H(\xi)}{\xi^p} = +\infty$, there exists $\xi_0 > 0$ such that

$$H(\xi) \geq M\xi^p$$

for all $\xi \in [0, \xi_0]$. Now, set $u = \mu v$, where $\mu = \frac{\xi_0}{\sup_{\bar{\Omega}} v}$. Then, we have

$$\begin{aligned} \Psi(u) &= p\int_\Omega a(y)H(u(y))dy \\ &\geq pM\int_\Omega a(y)|u(y)|^p \, dy \geq \tau M \int_{\Omega_\tau} |u(y)|^p \, dy \\ &> \eta\eta_1 \int_\Omega |u(y)|^p \, dy = \eta \int_\Omega |\nabla u(y)|^p \, dy = \eta\Phi(u). \end{aligned}$$

This implies that 0 is not a global minimum for the functional $\eta\Phi - \Psi$.

Since the right-hand side of the equation (2.2) is non-negative, we know that the non-zero weak solutions of the problem are positive in Ω. Let $v_1, v_2 \in X$ be two positive solutions of problem (2.2). We write

$$-\frac{\Delta_p v_1}{v_1^{p-1}} = \frac{1}{\eta} \frac{a(y)h(v_1)}{v_1^{p-1}} \tag{2.3}$$

and

$$-\frac{\Delta_p v_2}{v_2^{p-1}} = \frac{1}{\eta} \frac{a(y)h(v_2)}{v_2^{p-1}}. \tag{2.4}$$

Subtracting the above two equations (2.3) and (2.4), we know

$$-\frac{\Delta_p v_1}{v_1^{p-1}} + \frac{\Delta_p v_2}{v_2^{p-1}} = \frac{1}{\eta}\left(\frac{a(y)h(v_1)}{v_1^{p-1}} - \frac{a(y)h(v_2)}{v_2^{p-1}} \right). \tag{2.5}$$

In order to apply Lemma 2.4, we also need to check that

$$\frac{v_1}{v_2}, \frac{v_2}{v_1} \in L^\infty(\Omega).$$

To do this, we claim that

$$\max\left\{ \limsup_{y \to y_0} \frac{v_1(y)}{v_2(y)}, \limsup_{y \to y_0} \frac{v_2(y)}{v_1(y)} \right\} < +\infty, \tag{2.6}$$

whenever $y \to y_0 \in \partial\Omega$ with $y \in \Omega$. In fact, it follows from the Hopf boundary point lemma in the strong maximum principle for problem (2.2) that solutions v_1 and v_2 satisfy the following properties

$$\frac{\partial v_1}{\partial \nu} < 0, \quad \frac{\partial v_2}{\partial \nu} < 0 \qquad \text{for all } y_0 \in \partial\Omega, \tag{2.7}$$

where ν denotes the exterior normal unit vector to $\partial\Omega$; see [16, Theorem 5.5.1]. Recalling that $v_1, v_2 \in C^1(\overline{\Omega})$, it is clear from l'Hôpital's theorem and (2.7) that (2.6) is satisfied. Consequently, we arrive $v_1/v_2, v_2/v_1 \in L^\infty(\Omega)$.

Multiply (2.5) through by $v_1^p - v_2^p$ and integrate over Ω. Applying Lemma 2.4, one has

$$\int_\Omega \left(-\frac{\Delta_p v_1}{v_1^{p-1}} + \frac{\Delta_p v_2}{v_2^{p-1}} \right)(v_1^p - v_2^p)\,dy \ge 0.$$

Then we infer

$$\frac{1}{\eta}\int_\Omega \left(\frac{a(y)h(v_1)}{v_1^{p-1}} - \frac{a(y)h(v_2)}{v_2^{p-1}} \right)(v_1^p - v_2^p)\,dy \ge 0.$$

Hence, since the function $\xi \mapsto \dfrac{a(y)h(\xi)}{\xi^{p-1}}$ is decreasing in $(0, +\infty)$, we obtain that $v_1 = v_2$. Therefore, we ensure that the problem has at most one positive solution. As a consequence, we deduce that the functional $\eta\Phi - \Psi$ has a unique global minimum in X, since otherwise, in view of [15, Corollary 1], it would have at least three critical points. Consequently, the global minimum of this functional agrees with its only non-zero critical point.

Finally, let us show that Ψ has no global maxima. Arguing by contradiction, suppose that $\hat{v} \in X$ is a global maximum of Ψ. Clearly, $\Psi(\hat{v}) > 0$. Consequently, the set

$$\Omega_+ := \{ y \in \Omega : a(y)h(\hat{v}(y)) > 0 \}$$

has a positive measure. Fix a closed set $C \subset \Omega_+$ of positive measure. Let $\omega \in X$ be such that $\omega > 0$ and $\omega(y) = 1$ for all $y \in C$. Then, we have

$$\int_\Omega a(y)h(\hat{v}(y))\omega(y)\,dy \ge \int_C a(y)h(\hat{v}(y))\,dy > 0,$$

and thus $\Psi'(\hat{v}) \ne 0$, which is absurd.

Therefore, each assumption of Theorem 2.3 is satisfied. As a result, there exists a unique $\tilde{u} \in X$, with $\|\tilde{u}\|_X \in J$, such that

$$M\left(\int_\Omega |\nabla \tilde{u}|^p dy\right)\int_\Omega |\nabla \tilde{u}|^p dy - p\int_\Omega a(y)H(\tilde{u}^+(y))dy$$
$$= \inf_{v\in X}\left\{ M\left(\int_\Omega |\nabla \tilde{u}|^p dy\right)\int_\Omega |\nabla v|^p dy - p\int_\Omega a(y)H(v^+(y))dy \right\}.$$

Obviously, as seen above, the function $\tilde{u}$ is the unique positive weak solution to problem (P). This completes the proof.

As an application of Theorem 2.5, we give a simple example as follows:

Example 2.6. *Let us consider functions* $M : (0,1) \to \mathbb{R}$ and $h : [0, +\infty) \to [0, +\infty)$ *defined by*

$$M(\xi) = \tanh^{-1}\xi \quad \text{and} \quad h(\xi) = (1 + \xi^{p-1})e^{-\xi},$$

respectively. Then, for each $a \in L^\infty(\Omega)$, with $a > 0$, *the problem*

$$\begin{cases} -M\left(\int_\Omega |\nabla v|^p\,dy\right)\Delta_p v = a(y)h(v) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \\ \|v\|_X^p \in (0,1) \end{cases}$$

has a unique weak positive solution $\tilde{u}$, which is the unique global minimum in X of the functional

$$v \mapsto \frac{1}{p}M\left(\int_\Omega |\nabla \tilde{u}|^p dy\right)\int_\Omega |\nabla v(y)|^p dy - \int_\Omega \left(a(y)\int_0^{v^+(y)} h(\xi)\,d\xi\right)dy.$$

References

[1] H. Brezis and L. Oswald, Remarks on sublinear elliptic equations, *Nonlinear Anal.,* **10** (1986), 55–64. 2

[2] G.W. Dai and R.F. Hao, Existence of solutions of a *p*(*x*)-Kirchhoff-type equation, *J. Math. Anal. Appl.*, **359** (2009), 275–284. 1, 2

[3] J.I. Diaz and J.E. Saa, Existence et unicité de solutions positives pour certaines équations elliptiques quasilinéaires, *Comptes Rendus Acad. Sc. Paris. Série I,* **305**, (1987), 521–524. 2, 4

[4] A. Fiscella, Schrödinger-Kirchhoff-Hardy *p*-fractional equations without the Ambrosetti-Rabinowitz condition, *Discrete Contin. Dyn. Syst. Ser. S*, **13** (2020), 1993–2007. 1, 2

[5] A. Fiscella and E. Valdinoci, A critical Kirchhoff type problem involving a nonlocal operator, *Nonlinear Anal.*, **94** (2014), 156–170. 1, 2

[6] S. Gupta and G. Dwivedi, Kirchhoff type elliptic equations with double criticality in Musielak-Sobolev spaces, *Math. Methods Appl. Sci.*, **46** (2023), 8463–8477. 1, 2

[7] T. Huang and S. Deng, Existence of ground state solutions for Kirchhoff type problem without the Ambrosetti-Rabinowitz condition, *Appl. Math. Lett.*, **113** (2021), 106866. 1, 2

[8] I.H. Kim and Y.-H. Kim, Infinitely many small energy solutions to nonlinear Kirchhoff-Schrödinger equations with the *p*-Laplacian, *Bull. Malays. Math. Sci. Soc.*, **47**, (2024). 2

[9] I.H. Kim, Y.-H. Kim and K. Park, Multiple solutions to a non-local problem of Schrödinger-Kirchhoff type in $\mathbb{R}^N$, *Fractal Fract.*, **7** (2023), 627. 2

[10] G.R. Kirchhoff, *Vorlesungen über Mathematische Physik*, *Mechanik,* Teubner, Leipzig, 1876. 1

[11] J.L. Lions, On some questions in boundary value problems of mathematical physics, *North-Holland Math. Stud.*, **30** (1978), 284–346. 1, 2

[12] D.C. Liu, On a *p*-Kirchhoff-type equation via fountain theorem and dual fountain theorem, *Nonlinear Anal.*, **72** (2010), 302–308. 1, 2

[13] N. Nyamoradi, Existence of three solutions for Kirchhoff nonlocal operators of elliptic type, *Math. Commun.*, **18** (2013) 489–502. 1, 2

[14] P. Pucci and S. Saldi, Critical stationary Kirchhoff equations in $\mathbb{R}^N$ involving nonlocal operators, *Rev. Mat. Iberoam.*, **32** (2016), 1–22. 1, 2

[15] P. Pucci and J. Serrin, A mountain pass theorem, *J. Differential Equations*, **60** (1985), 142–149. 7

[16] P. Pucci and J. Serrin, The maximum principle, Birkhäuser, Basel, 2007. 6

[17] P. Pucci, M.Q. Xiang and B.L. Zhang, Multiple solutions for nonhomogeneous Schrödinger-Kirchhoff type equations involving the fractional p-Laplacian in $\mathbb{R}^N$, *Calc. Var. Partial Differ. Equ.*, **54** (2015), 2785–2806. 1, 2

[18] B. Ricceri, Existence, uniqueness, localization and minimization property of positive solutions for nonlocal problems involving discontinuous Kirchhoff functions. *Adv. Nonlinear Anal.*, **13** (2024), 20230104. https://doi.org/10.1515/anona-2023-0104 2, 3

[19] M.Q. Xiang, B.L. Zhang and M. Ferrara, Existence of solutions for Kirchhoff type problem involving the non-local fractional p-Laplacian, *J. Math. Anal. Appl.*, **424** (2015), 1021–1041. 1, 2