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Semi-algebraic Semicontinuity Of Vector-valued Maps And Applications

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This paper proposes concepts of semicontinuity of vector-valued maps in the sense of “semi-algebra”, that is, the image spaces are not endowed with any particular topology. Properties as well as characterizations of these concepts are discussed. We use these results together with new concepts of generalized quasiconvexity of vector-valued map to study existence conditions and the (semi)continuity of solution maps to vector equilibrium problems. Wellposedness under perturbations for vector equilibrium problems with equilibrium constraints is also presented.

Keywords: Semicontinuity, algebraic interior, vector equilibrium problem, existence of solution, generalized quasiconvexity.

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1. Introduction

In the wide range of applications of vector optimization, it is recognized that one has to face with many models where the image spaces of objective maps are the linear ones not having any particular topology (see references quoted in [2]). Therefore, there have been in the literature a lot of works devoted to optimization-related problems without using topological tools (see, e.g., [1–3, 11, 19, 21, 23] and the references therein). The main mathematical tools in the mentioned works are convex separation theorems, alternative theorems, and some algebraic counterparts of topological notions such as the algebraic interior, the relative algebraic interior and the vector closure.

The vector equilibrium problems introduced in [9, 12] have been extensively studied since they provide an unified framework for several important problems in optimization such as vector optimization problems, vector variational inequality problems, vector Nash equilibrium problems, vector complementarity problems, vector saddle point problems

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(see, e.g., [14, 15, 18, 27]). Recently, in [23], many kinds of solution to vector equilibrium problems in the non-topological settings have been characterized via scalarization. Also via scalarization, results on the existence of solutions to such problems were obtained in [22].

The main goal of this paper is to introduce and study properties and characterizations of the semicontinuity of vector-valued maps when they take values just in linear spaces. Also, we propose new concepts of generalized quasiconvexity of vector-valued maps with values in a linear space. The properties and characterizations are utilized to formulate existence conditions and to establish sufficient conditions for the continuity of solution maps to vector equilibrium problems whose image spaces are not endowed with any topology. Moreover, sufficient conditions for the wellposedness under perturbations for vector equilibrium problems with equilibrium constraints in such spaces are investigated.

The rest of the paper is organized as follows. Sect. 2 recalls some notions along with preliminaries in a linear space. In Sect. 3, we introduce concepts of semicontinuity and generalized quasiconvexity of vector-valued maps taking values in linear spaces and study their properties and characterizations. Sufficient conditions for the nonemptiness of solution sets and the continuity of solution maps to vector equilibrium problems are established in Sect. 4. In Sect. 5, these results are applied to study wellposedness under perturbations for vector equilibrium problems with equilibrium constraints. Finally, Sect. 6 gives some concluding remarks.

2. Preliminaries

Let Y be a linear space. A nonempty subset $A \subset Y$ is said to be convex if for all $y_1, y_2 \in A$ and $0 \leq \lambda \leq 1$, one has $\lambda y_1 + (1-\lambda) y_2 \in A$ and proper if $0 \neq A \neq Y$. A subset K of Y is called a cone if for all $\lambda \geq 0$ and $k \in K$, we have $\lambda k \in K$. The cone K is said to be pointed if $K \cap (-K) = \{\mathbf{0}_Y\}$ (0_Y is the zero vector in Y). The cone K is said to be nontrivial if $k \neq 0$, $K \neq \{\mathbf{0}_Y\}$ and $k \neq Y$ hold. For $u, v \in Y$, we use the notations

$$[u,v] := \{tu + (1-t)v \mid t \in [0,1]\},$$
$$[u,v) := [u,v] \setminus \{v\}.$$

Here, in order to avoid topological concepts, we use algebraic counterparts. The following concepts can be found in [1, 23, 26].

Definition 2.1. The set

$$\text{core}(A) := \left\{a \in A \mid \forall v \in Y, \exists \lambda > 0 \text{ s.t. } a + [0,\lambda]v \subset A\right\},$$

is called the algebraic interior of A. If $A = \text{core}(A)$, then A is called algebraically open. It is said that A is solid if core(A) is nonempty.

Definition 2.2. An element $\bar{x} \in Y$ is said to be linearly accessible from A, if there is $x \in A$, $x \neq \bar{x}$ such that $\lambda x + (1-\lambda)\bar{x} \in A$ for all $\lambda \in (0,1]$. The union of A and the set of all linearly accessible elements from A is called the algebraic closure of A and it is denoted by lin(A). If $A = \text{lin}(A)$, then the set A is called algebraically closed.

We note that when Y is endowed with a topology, we have the following inclusions

$$\mathrm{int}(A) \subset \mathrm{core}(A) \subset A \subset \mathrm{lin}(A) \subset \mathrm{cl}(A),$$

where int(A), cl(A) are the topological interior and closure of A, respectively.

We now discuss some properties of the algebraic interior and the algebraic closure of sets in linear spaces. Let us start with a relation between the algebraic interior and the algebraic closure of a subset.

Lemma 2.1. (*See* [7]) *If A is algebraically closed and convex in Y, then its complement is algebraically open in Y.*

The following example shows that the conclusion of Lemma 2.1 is not true if A is nonconvex.

Example 2.1. Let $Y = \mathbb{R}$, $A = \left\{\frac{1}{n} | n \in \mathbb{N} \setminus \{0\}\right\}$. Then, A is algebraically closed. Indeed, it is easily verified the set of all linearly accessible elements from A is empty. Therefore, lin(A) = A, implying that A is algebraically closed. However, the set $Y \setminus A$ is not algebraically open. To see this, consider $a = 0 \in Y \setminus A$ and $v = 1 \in Y$. Then, for all λ>0, we have $a + [0,\lambda]v = [0,\lambda]$, which is not a subset of $Y \setminus A$. Thus, $a \notin \mathrm{core}(Y \setminus A)$. Consequently, $Y \setminus A$ is not equal to $Y \setminus \mathrm{core}(A)$.

Let us assume further that K is an algebraically closed, pointed, convex and solid cone in the linear space Y. It is known that K induces preorder relations $\square_K$, $▪_K$ defined by, for all $x, y \in Y$,

$$x \,\square_K\, y :\to\to y - x \in K,$$
$$x \,▪_K\, y :\to\to y - x \in \mathit{core}(K).$$

Lemma 2.2. (*See* [26, p. 8-10]) *The following statements are true.*

(i) *The set* core(K) *is algebraically open.*

(iii) *For all b* ∈ Y, *b* + core(K) *is algebraically open.*

(iii) *If* K *is convex, then* core(K) + K = core(K).

Lemma 2.3. *A nonempty subset A of Y is algebraically closed if and only if v + A is algebraically closed for all v ∈ Y.*

Proof. The proof can be easily obtained from a simple observation that for every nonempty subset A of Y and $v \in Y$, one has that $\mathrm{lin}(v + A) = v + \mathrm{lin}(A)$.

Lemma 2.4. (*See* [7]) *The subset* $A \subset Y$ *is algebraically open if and only if for each* $y_0 \in A$, *there exists an algebraically open set U such that* $y_0 \in U \subset A$.

3. Semi-algebraic semicontinuity concepts

In the literature, there exist notions of semicontinuity of a vector-valued map where the image space of the map is given in a topological space. In what follows, we introduce notions of the upper, lower semicontinuity and continuity of a vector-valued map only taking values in a linear space. Subsequently, we discuss their properties and characterizations. With the help of these notions, we will investigate the existence conditions and the (semi)continuity of solution maps to optimization-related problems in the nontopological settings.

Definition 3.1. Let X be a topological linear space, Y be a linear space and $h: X \to Y$ be a vector-valued map. It is said that

(a) h is K-upper semicontinuous at $x_0 \in X$ if for each algebraically open set V with $h(x_0) \in V$, there exists a neighborhood U of x_0 such that $h(x) \in V - \mathrm{K}$ for all $x \in U$.

(b) h is K-lower semicontinuous at $x_0 \in X$ if for each algebraically open set V with $h(x_0) \in V$, there exists a neighborhood U of x_0 such that $h(x) \in V + \mathrm{K}$ for all $x \in U$.

(c) h is K-continuous at $x_0 \in X$ if it is K-upper semicontinuous as well as K-lower semicontinuous at x_0.

As usual, we say that a map has a certain property on a subset $D \subset X$ if such property is satisfied at every point of D.

We observe that verifying the upper or lower semicontinuity of a vector-valued map solely through its definition can be challenging. Hence, in the subsequent discussion, we provide characterizations for both upper and lower semicontinuity, which will help us in addressing this issue.

Theorem 3.1. *Given a vector-valued map $h: X \to Y$, then we have that*

(i) *h is* K-*upper semicontinuous on X if and only if for each $b \in Y$, the set $h^{-1}(b - core(\mathrm{K}))$ is an open subset of X.*

(ii) *h is* K-*lower semicontinuous on X if and only if for each $b \in Y$, the set $h^{-1}(b + core(\mathrm{K}))$ is an open subset of X.*

Proof. (i) Assume that h is K-upper semicontinuous on X. For each $b \in Y$, let $x_0 \in h^{-1}(b - \mathrm{core}(\mathrm{K}))$ be arbitrary. Then, $h(x_0) \in b - core(\mathrm{K})$. By Lemma 2.2(ii), b–core(K) is algebraically open, which together with the K-upper semicontinuity of h leads to the existence of a neighborhood U of x_0 such that $h(x) \in b - core(\mathrm{K}) - \mathrm{K}$ for all $x \in U$. It follows from Lemma 2.2(iii) that $b - core(\mathrm{K}) - \mathrm{K} = b - core(\mathrm{K})$. Therefore, for each $x \in U$, $x \in h^{-1}(b - core(\mathrm{K}))$, which implies that $x_0 \in U \subset h^{-1}(b - core(\mathrm{K}))$. Consequently, $h^{-1}(b - core(\mathrm{K}))$ is an open subset of X.

Conversely, assume that $h^{-1}(b - core(\mathrm{K}))$ is an open subset of X for all $b \in Y$, we prove that h is K-upper semicontinuous at every point $x_0 \in X$. Let V is an algebraically open subset of Y satisfying $h(x_0) \in V$ and e be an arbitrary element in core(K). Then, there is $\lambda > 0$ such that

$$h(x_0) + \frac{\lambda}{2} e \in V.$$

We note that

$$x_0 \in h^{-1}(h(x_0) + \frac{\lambda}{2} e - core(\mathrm{K})),$$

and $h^{-1}(h(x_0) + \frac{\lambda}{2} e - core(\mathrm{K}))$ is an open set, and hence there is a neighborhood U_0 of x_0 such that

$$x_0 \in U_0 \subset h^{-1}\left(h(x_0) + \frac{L}{2} e - core(\mathrm{K})\right),$$

or equivalently, for all $x \in U_0$, one has

$$x \in h^{-1}\left(h(x_0) + \frac{L}{2} e - core(\mathrm{K})\right).$$

Therefore,

$$h(x) \in h(x_0) + \frac{L}{2} e - core(\mathrm{K}),$$

which yields that $h(x) \in V - \mathrm{K}$ for all $x \in U_0$.

(ii) The proof can be proceeded in the same way as that of (i).

Theorem 3.2. *The following statements are equivalent.*

(i) *The vector-valued map* $h : X \to Y$ *is* K-*upper semicontinuous on* X.

(ii) *For each* $b \in Y$, *the upper level set*

$$\mathrm{lev}_{\square_{\mathrm{K}} b} h := \{x \in X \mid h(x) \square_{\mathrm{K}} b\}$$

is closed in X.

(iii) *For every* $x_0 \in X$ *and* $e \in core(\mathrm{K})$, *there exists an open neighborhood* U *of* x_0 *such that*

$$h(x) \in h(x_0) + e - core(\mathrm{K}),$$

for all $x \in U$.

Proof. [(i) ⇒ (ii)] It follows directly from Theorem 3.1(i).

Another proof: Assume that h is K-upper semicontinuous on X. Let $\{x_\alpha\}$ be a net of $\mathrm{lev}_{\square b} h$ with $x_\alpha \to x_0$, we need to show that $x_0 \in \mathrm{lev}_{\square b} h$. Suppose that $x_0 \notin \mathrm{lev}_{\square b} h$, then $h(x_0) \in b$ – core(K). Since b – core(K) is algebraically open and h is K-upper semicontinuous at x_0, we can find an open neighborhood U of x_0 satisfying

$$h(x) \in b - core(\mathrm{K}) - \mathrm{K} = b - core(\mathrm{K}),$$

for all $x \in U$. By the fact that $\{x_\alpha\}$ converges to x_0, there exists α_0 such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$. This implies that $h(x_\alpha) \in b - core(\mathrm{K})$, which is impossible as $x_\alpha \in \mathrm{lev}_{\square b} h$ for all α.

[(ii) ⇒(iii)] For each $x_0 \in X$ and $e \in$ core(K), it is easy to see that

$$h(x_0) \in h(x_0) + e - core(\mathrm{K}),$$

or equivalently

$$x_0 \in h^{-1}(h(x_0) + e - core(\mathrm{K})),$$

which is open in X due to (ii). Thus, there is an open neighborhood U of x_0 such that

$$x \in U \subset h^{-1}(h(x_0) + e - \mathrm{core}(\mathrm{K})),$$

that is,

$$h(x) \in h(x_0) + e - core(\mathrm{K}).$$

[(iii) ⇒ (i)] Let $x_0 \in X$ and V is an algebraically open subset of Y with $h(x_0) \in V$. For $e \in$ core(K), there is $\lambda > 0$ such that

$$h(x_0) + \frac{\lambda}{2} e \in V.$$

Let $e_0 := \frac{\lambda}{2} e \in core(\mathrm{K})$, it follows from (iii) that there exists an open neighborhood U of x_0 such that for all $x \in U$,

$$h(x) \in h(x_0) + e_0 - core(\mathrm{K}) \subset V - core(\mathrm{K}) \subset V - \mathrm{K}.$$

The K-upper semicontinuity of h is proved.
Passing to the K-lower semicontinuity, we have the following result.

Theorem 3.3. *The following statements are equivalent.*

(i) *The vector-valued map* $h: X \to Y$ *is* K-*lower semicontinuous on* X.

(ii) *For each* $b \in Y$, *the lower level set*

$$lev_{\leq_\mathrm{K} b} h := \{x \in X \mid h(x) \leq_\mathrm{K} b\}$$

is closed in X.

(iii) *For every* $x_0 \in X$ *and* $e \in$ core(K), *there exists an open neighborhood* U *of* x_0 *such that*

$$h(x) \in h(x_0) - e + core(\mathrm{K}),$$

for all $x \in U$.

Proof. Since the proofs are similar to those of Theorem 3.2, we omit them.

From Theorems 3.2 and 3.3, we suggest the following natural properties which are weaker than K-semicontinuities by imposing requirements only at a given level.

Definition 3.2. Let $h: X \to Y$ be a vector-valued map and $b \in Y$ be given.

(a) h is said to be (b, K)-upper semicontinuous at x_0 if the upper level set $\mathrm{lev}_{\geq_\mathrm{K} b} h$ is closed at x_0.

(b) h is said to be (b, K)-lower semicontinuous at x_0 if the lower level set $\mathrm{lev}_{\leq_\mathrm{K} b} h$ is closed at x_0.

(c) h is said to be (b, K)-continuous at x_0 if it is both (b, K)-upper semicontinuous and (b, K)-lower semicontinuous at x_0.

Let Z be a topological space and $h_1 : X \to Y$, $h_2 : Z \to Y$ be vector-valued maps. Then, we consider the map $h_1 + h_2 : X \times Z \to Y$ defined by

$$(h_1 + h_2)(x, z) := h_1(x) + h_2(z), \ \ \forall (x, z) \in X \times Z.$$

Theorem 3.4. *The following assertions hold.*

(i) *If* h_1 *and* h_2 *are* K-*upper semicontinuous at* x_0 *and* z_0, *respectively, then* $h_1 + h_2$ *is* K-*upper semicontinuous at* (x_0, z_0).

(ii) *If* h_1 *and* h_2 *are* K-*upper semicontinuous at* x_0 *and* z_0, *respectively, then* $h_1 + h_2$ *is* K-*lower semicontinuous at* (x_0, z_0).

Proof. Since the technique used for proving (i) and (ii) is similar, we will only present the proof for (ii).

Assume that V is an algebraically open set with $h_1(x_0) + h_2(z_0) \in V$. Then, $-h_1(x_0) - h_2(z_0) + V$ is an algebraically open set containing 0. Therefore, there are algebraically open sets V_1, V_2 containing 0 such that

$$V_1 + V_2 \subset -h_1(x_0) - h_2(z_0) + V.$$

Since $h_1(x_0) + V_1$ is an algebraically open set containing $h_1(x_0)$ and h_1 is K-lower semicontinuous at x_0, there is a neighborhood U_1 of x_0 such that, for all $x \in U_1$,

$$h_1(x) \in h_1(x_0) + V_1 + \mathrm{K}, \tag{1}$$

Similarly, we can find a neighborhood U_2 of z_0 such that, for all $z \in U_2$,

$$h_2(z) \in h_2(z_0) + V_2 + \mathrm{K}, \tag{2}$$

It follows from (1) and (2) that, for all $(x,z) \in U_1 \times U_2$,

$$h_1(x) + h_2(z) \in h_1(x_0) + h_2(z_0) + V_1 + V_2 + \mathrm{K} + \mathrm{K} \subset V + \mathrm{K}.$$

Therefore, $h_1 + h_2$ is K-lower semicontinuous at (x_0, z_0).

The following example prevents us to state similar results to Theorem 3.4 for a given level b.

Example 3.1. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $\mathrm{K} = \mathbb{R}^2_+$, $x_0 = 0$ and $h_1, h_2 : \mathbb{R} \to \mathbb{R}^2$ be defined by $h_2(x) = (2,-3)$ for all $x \in \mathbb{R}$ and

$$h_1(x) = \begin{cases} (-3,2), & x = 0, \\ (3,4), & x \neq 0. \end{cases}$$

Then, h_1 is (0, K)-upper semicontinous at 0 and h_2 is evidently (0, K)-upper semicontinuous at 0 since it is continuous at 0. But

$$h(x) = (h_1 + h_2)(x) = \begin{cases} (-1,-1), & x = 0, \\ (5,1), & x \neq 0, \end{cases}$$

which is not (0, K)-upper semicontinous at 0 since $\mathrm{lev}_{\square_K 0} h$ is not closed at 0 (to see this, take $x_n := \frac{1}{n}$).

By modifying h_1 and h_2 as $h_2(x) = (2,-1)$ and

$$h_1(x) = \begin{cases} (-1,2), & x = 0, \\ (-3,-3), & x \neq 0. \end{cases}$$

Then, we see that both h_1 and h_2 are (0, K)-lower semicontinous at 0, but $h_1 + h_2$ is not.

Theorem 3.5. *Let* $h : X \to Y$ *be a vector-valued map and* $r > 0$. *It holds that* rh *is* K-*upper semicontinuous (lower semicontinuous, respectively) at* x_0 *provided that* h *is* K-*upper semicontinuous (lower semicontinuous, respectively) at* x_0.

Proof. We only present the proof for the case of the upper semicontinuity. For each $e \in \text{core}(\text{K})$, by Theorem 3.2, there exists an open neighborhood U of x_0 such that

$$h(x) \in h(x_0) + \frac{e}{r} - core(\text{K}), \ \forall x \in U.$$

It results

$$(rh)(x) := rh(x) \in rh(x_0) + e - core(\text{K}).$$

The proof is complete.

4. Existence of solutions and continuity of solution maps to vector equilibrium problems

Let Ω be a nonempty closed convex subset of a Hausdorff topological vector space X and Y be a vector space. Let $f : \Omega \times \Omega \to Y$ be a vector-valued map with $f(x,x) \square_{\text{K}} \mathbf{0}_Y$ for all $x \in \Omega$. The vector equilibrium problem under our consideration is as follows.

(VEP) Find $\bar{x} \in \Omega$ such that for all $y \in \Omega$,

$$f(\bar{x}, y) \notin -\text{K} \setminus \{\mathbf{0}_Y\}.$$

Recall that a point $\bar{x} \in \Omega$ is called a weakly efficient solution of (VEP) if

$$f(\bar{x}, y) \square_{\text{K}} \mathbf{0}_Y, \ \forall y \in \Omega.$$

Denote the weakly efficient solution set of (VEP) by S, i.e.,

$$S := \left\{ x \in \Omega \mid f(\bar{x}, y) \square_{\text{K}} \mathbf{0}_Y, \forall y \in \Omega \right\}.$$

4.1. Existence of solutions

In this subsection, we formulate a result on the existence of weakly efficient solutions to (VEP). For this aim, we first recall the concept of KKM map.

Definition 4.1. (See [28, Definition 3.1] Let Φ be an arbitrary subset of a topological vector space X. A set-valued map $F : \Phi \rightrightarrows \Phi$ is called a KKM-map if, for any finite subset $\{y_1, y_2, \ldots, y_n\}$ of Φ, conv$\{y_1, y_2, \ldots, y_n\}$ is a subset of $\bigcup_{i=1}^{n} F(y_i)$, where "conv" stands for the convex hull.

The following result is usually known as the KKM-Fan lemma.

Lemma 4.1. (*See* [17]) *Assume that* $F : \Phi \rightrightarrows \Phi$ *is a KKM map with closed values. If there exists a nonempty compact convex subset* Φ_0 *of* Φ *such that* $\bigcap_{y \in \Phi_0} F(y)$ *is compact, then* $\bigcap_{y \in \Phi} F(y) \neq \emptyset$.

Definition 4.2. (See [8, 24]) Let E be a nonempty convex subset of X. A vector-valued map $h : X \to Y$ is said to be

(a) K-convex on E if for all $x_1, x_2 \in E$ and $t \in [0,1]$,

$$h(tx_1 + (1-t)x_2) \square_{\text{K}} th(x_1) + (1-t)h(x_2).$$

(b) strictly K-convex on E if for all $x_1, x_2 \in E, x_1 \neq x_2$ and $t \in (0,1)$,

$$h(tx_1 + (1-t)x_2) \prec_{\mathrm{K}} th(x_1) + (1-t)h(x_2).$$

(c) K-quasiconvex on E if for all $x_1, x_2 \in E$, and $t \in (0,1)$, either

$$h(tx_1 + (1-t)x_2) \square_{\mathrm{K}} h(x_1), \text{ or } h(tx_1 + (1-t)x_2) \square_{\mathrm{K}} h(x_2).$$

(d) strict properly K-quasiconvex on E if for all $x_1, x_2 \in E$, and $t \in (0,1)$, either

$$h(tx_1 + (1-t)x_2) \prec_{\mathrm{K}} h(x_1), \text{ or } h(tx_1 + (1-t)x_2) \prec_{\mathrm{K}} h(x_2).$$

The map h is said to be K-concave (respectively, strictly K-concave, K-quasiconcave, strict properly K-quasiconcave) if $-h$ is K-convex (respectively, strictly K-convex, K-quasiconvex, strict properly K-quasiconvex).

We now introduce a level concavity concept of a vector-valued map.

Definition 4.3. Let $b \in Y$ and E is a convex subset of X. A vector-valued map $h : X \to Y$ is said to be (b, K)-concave on E if for all $x_1, x_2 \in E$, and $t \in (0,1)$, the following implication holds

$$h(x_1) \mathsf{A}_{\mathrm{K}} b, h(x_2) \mathsf{g}_{\mathrm{K}} b \to h(tx_1 + (1-t)x_2) \mathsf{g}_{\mathrm{K}} b.$$

Proposition 4.1. *If h is strict properly* K*-quasiconcave on E, then h is* (b, K)*-concave on E.*

Proof. Let $x_1, x_2 \in E$ and $t \in (0,1)$ with $h(x_1) \square_{\mathrm{K}} b$ and $h(x_2) \mathsf{g}_{\mathrm{K}} b$, we need to show that $h(x_t) \square_{\mathrm{K}} b$ where $x_t := tx_1 + (1-t)x_2$. Since h is strict properly K-quasiconcave, we have either $h(x_t) \prec_{\mathrm{K}} h(x_1)$ or $h(x_t) \prec_{\mathrm{K}} h(x_2)$. Suppose that $h(x_t) \square_{\mathrm{K}} b$, then there exists $k_1 \in K$ such that $h(x_t) - b = -k_1$. If $h(x_t) \prec_{\mathrm{K}} h(x_1)$, then there is $k_2 \in \mathrm{core(K)}$, $h(x_t) = h(x_1) + k_2$. Hence, $h(x_1) + k_2 - b = -k_1$, or equivalently

$$h(x_1) - b = -k_1 - k_2 \in -\mathrm{K} - \mathrm{core(K)} = -\mathrm{core(K)}.$$

This is impossible as $h(x_1) \square_{\mathrm{K}} b$. For the case of $h(x_t) \prec_{\mathrm{K}} h(x_2)$, we also have another contradiction.

Theorem 4.1. *Assume that all the following conditions are satisfied:*

(i) *for each $x \in \Omega$, $f(x, \cdot)$ is* K*-quasiconvex on* Ω;

(ii) *for every $y \in \Omega$, $f(\cdot, y)$ is* ($\mathbf{0}_Y$,K) *-upper semicontinuous on* Ω;

(iii) *there exist a nonempty compact subset G of Ω and a nonempty convex compact subset H of Ω such that for each $x \in \Omega \setminus G$, there exists $y \in H$ satisfying $f(x, y) \prec_{\mathrm{K}} \mathbf{0}_Y$.*

Then, the weakly efficient solution set S to (VEP) is nonempty.

Proof. Let $F : \Omega \rightrightarrows \Omega$ defined by

$$F(y) := \{x \in \Omega \mid f(x, y) \square_{\mathrm{K}} \mathbf{0}_Y\}, \ \forall y \in \Omega.$$

It is easy to see that $\bar{x} \in S$ if and only if $\bar{x} \in \bigcap_{y \in \Omega} F(y)$. Obviously, $y \in F(y)$, and hence $F(y)$ is a nonempty set.

We now show that F is a KKM map. Suppose to the contrary that there is a finite subset $\{y_1, y_2, ..., y_n\}$ of Ω, and $y \in \text{conv}\{y_1, y_2, ..., y_n\}$, but $y \notin F(y_i)$ for all i. Since $y \in \text{conv}$ $\{y_1, y_2, ..., y_n\}$, one has

$$y = \sum_{i=1}^{n} t_i y_i, \ \sum_{i=1}^{n} t_i = 1, \ t_i \in [0,1].$$

It follows from the fact $y \notin F(y_i)$ that $f(y, y_i) \cdot_K \mathbf{0}_Y$. By the K-quasiconvexity of f, one gets

$$\mathbf{0}_Y \lneq_K f(y,y) = f\left(y, \sum_{i=1}^{n} t_i y_i\right) \leq_K f(y, y_i) \cdot_K \mathbf{0}_Y,$$

which is a contradiction. Therefore, F is a KKM map.

We prove that $F(y)$ is closed subset of Ω for all $y \in \Omega$. Let $x_\alpha \in F(y)$ with $x_\alpha \to x_0$, then $x_\alpha \in \Omega$ and $f(x_\alpha, y) \square_K \mathbf{0}_Y$ for all α. It follows from the (0_Y, K)-upper semicontinuity of f that $f(x_0, y) \square_K \mathbf{0}_Y$, which implies $x_0 \in F(y)$. Hence, $F(y)$ is a closed set for all $y \in \Omega$. Using (iii), we deduce that $\cap_{y\in H} F(y)$ is a closed subset of G. Therefore, $\cap_{y\in H} F(y)$ is compact. Applying Lemma 4.1, we have

$$\bigcap_{y\in\Omega} F(y) \neq \emptyset.$$

This allows us to infer that the solution set S is nonempty.

Remark 4.1. If Ω is compact, the condition (iii) in Theorem 4.1 is satisfied automatically. In addition, in this case the solution set S is compact. Indeed, let $x_\alpha \in S$ with $x_\alpha \to x_0$. If $x_0 \notin S$, then there is $y_0 \in \Omega$ such that

$$f(x_0, y_0) \cdot_K \mathbf{0}_Y. \tag{3}$$

Since $x_\alpha \in S$ and $y_0 \in \Omega$, we have $f(x_\alpha, y_0) \square_K \mathbf{0}_Y$. Combining this with the (0_Y, K)-upper semicontinuity of f, we get

$$f(x_0, y_0) \square_K \mathbf{0}_Y. \tag{4}$$

Form (3) and (4), we get a contradiction. Therefore, $x_0 \in S$, which leads to the closedness of S in the compact set Ω. Therefore, S is also compact.

Remark 4.2. In [22], the authors obtain an existence result of weakly efficient solutions to (VEP) by using a scalarization function in the sense of Gerstewitz. Here, in contrast to the method employed in [22], we directly study (VEP) and achieve the same result as the aforementioned paper.

4.2 Continuity of solution maps to vector equilibrium problems

In this subsection, let P be a Hausdorff topological space. When the set Ω and the mapping f are perturbed by a parameter λ which varies over the subset $\Lambda \subset P$, we consider the following parametric vector equilibrium problem:

(PVEP) Find $\bar{x} \in \Omega(\Lambda)$ such that for all $y \in \Omega(\Lambda)$,

$$f(\bar{x}, y, \lambda) \in -K\setminus\{0_Y\}.$$

Herein, $\Omega : \Lambda \rightrightarrows X$ is a set-valued map and $f : X \times X \times \Lambda \to Y$ is a vector-valued map.

For each $\lambda \in \Lambda$, we denote the weakly efficient solution set of (PVEP) at λ by $S(\lambda)$, i.e.,

$$S(\lambda) := \{x \in \Omega(\lambda) \mid f(x, y, \lambda) \not\leq_K 0_Y,\ \forall y \in \Omega(\lambda)\}.$$

We will investigate the stability in the sense of continuity of S as a set-valued map from Λ into X.

Let us recall concepts of upper and lower semicontinuity of set-valued maps and their properties.

Definition 4.4. (See [10, Definitions 1.4.1, 1.4.2, p. 38] Let X_1 and X_2 be Hausdorff topological vector spaces and $Q : X_1 \rightrightarrows X_2$ be a set-valued map.

(a) Q is upper semicontinuous at x_0 if for any neighborhood U of $Q(x_0)$, there is a neighborhood N of x_0 such that $Q(N) \subset U$.

(b) Q is lower semicontinuous at x_0 if for all nets $\{x_\alpha\}$ converging to x_0 and $y_0 \in Q(x_0)$, then there exist $y_\alpha \in Q(x_\alpha)$ such that $y_\alpha \to y_0$.

(c) Q is continuous at x_0 if it is both upper semicontinuous and lower semicontinuous at x_0.

Lemma 4.2. (See [25, Proposition 2.6, p. 37] *The set-valued mapping Q is lower semicontinuous at x_0 if for all $\{x_\alpha\}$ converging to x_0, then one has*

$$Q(x_0) \subset \liminf Q(x_\alpha) := \{y_0 \in X_2 \mid \exists y_\alpha \in Q(x_\alpha), y_\alpha \to y_0\}.$$

Lemma 4.3. (See [25, Proposition 2.19, p. 41] *If $Q(x_0)$ is compact, then Q is upper semicontinuous at x_0 if and only if for any net $\{x_\alpha\}$ converging to x_0 and $y_\alpha \in Q(x_\alpha)$, there is a subnet $\{y_\beta\}$ of the net $\{y_\alpha\}$ converging to some $y_0 \in Q(x_0)$.*

The following theorem gives sufficient conditions for the map S to be continuous on Λ.

Theorem 4.2. *Assume that the following conditions hold:*

(i) Ω *is continuous and compact-convex-valued on* Λ *and* $\Omega(\Lambda)$ *is convex;*

(ii) *for each* $\lambda \in \Lambda$ *and* $x \in \Omega(\lambda)$, $f(x, \cdot, \lambda)$ *is* K-*quasiconvex on* $\Omega(\Lambda)$;

(iii) *for every* $\lambda \in \Lambda$ *and* $y \in \Omega(\lambda)$, $f(\cdot, y, \lambda)$ *is* $(0_Y, K)$-*concave on* $\Omega(\Lambda)$.

(iv) *f is* K-*continuous on* $X \times X \times \Lambda$.

Then, the solution map S is nonempty compact-valued and continuous on Λ.

Proof. Obviously, all conditions of Theorem 4.1 are satisfied, and hence $S(\lambda)$ is nonempty for all $\lambda \in \Lambda$.

In order to prove the continuity of S, we divide the proof into two steps.

Step 1: We prove the upper semicontinuity of S on Λ. Suppose to the contrary that there is $\lambda_0 \in \Lambda$ with S is not upper semicontinuous at this point. Then there exists a neighborhood U of $S(\lambda_0)$ such that for all neighborhood N of λ_0, there is $\lambda' \in N$ with $S(\lambda') \not\subset U$.

Therefore, there exists a net $\{\lambda_\alpha\}$ converging to λ_0 satisfying $S(\lambda_\alpha) \not\subseteq U$ for all α. This derives that there are $x_\alpha \in S(\lambda_\alpha)$ with $x_\alpha \notin U$ for all α. Since Ω is upper semicontinuous at λ_0, $\Omega(\lambda_0)$ is compact and $x_\alpha \in \Omega(\lambda_\alpha)$, by Lemma 4.3, without loss of generality, we can assume that x_α tends to some $x_0 \in \Omega(\Lambda_0)$. We claim that $x_0 \in S(\Lambda_0)$. Indeed, if $x_0 \notin S(\Lambda_0)$, then there is $y_0 \in \Omega(\Lambda_0)$ satisfying

$$f(x_0, y_0, \lambda_0) \not\geq_K 0_Y.$$

It follows from the lower semicontinuity of Ω that there exist $y_\alpha \in \Omega(\lambda_\alpha)$ with $y_\alpha \to y_0$. Since $x_\alpha \in S(\lambda_\alpha)$ and $y_\alpha \in \Omega(\lambda_\alpha)$, one has

$$f(x_\alpha, y_\alpha, \lambda_\alpha) \geq_K 0_Y.$$

This together with the K-upper semicontinuity of f yields that

$$f(x_0, y_0, \lambda_0) \geq_K 0_Y,$$

which is a contradiction. Therefore, we get $x_0 \in S(\Lambda_0)$. This is again a contradiction as $x_\alpha \notin U$ for all α. Consequently, we obtain the upper semicontinuity of S on Λ.

Step 2: We show the lower semicontinuity of S on Λ. Let us put

$$S_1(\lambda) := \{x \in \Omega(\lambda) \mid f(x, y, \lambda) \not\leq_K 0_Y,\ \forall y \in \Omega(\lambda)\}.$$

First, we show the lower semicontinuity of the map S_1. Suppose to the contrary that S_1 is not lower semicontinuous at any $\lambda_0 \in \Lambda$. By Definition 4.4(b), there are a net $\{\lambda_\alpha\}$ converging to λ_0 and a point $x_0 \in S_1(\Lambda_0)$ satisfying that any net $\{x_\alpha\}$ with $x_\alpha \in S_1(\lambda_\alpha)$ does not converge to x_0. By the lower semicontinuity of Ω at λ_0, there are $\bar{x}_\alpha \in \Omega(\lambda_\alpha)$ with $\bar{x}_\alpha \to x_0$. Due to the contradiction assumption, there must be a subnet $\{\bar{x}_\beta\}$ of $\{\bar{x}_\alpha\}$ such that $\bar{x}_\beta \notin S_1(\Lambda_\beta)$ for all β. This means that there exist $y_\beta \in \Omega(\Lambda_\beta)$ such that

$$f(\bar{x}_\beta, y_\beta, \lambda_\beta) \leq_K 0_Y,\ \forall \beta.$$

Since Ω is upper semicontinuous and $\Omega(\lambda_0)$ is compact, one has $y_0 \in \Omega(\lambda_0)$ with $y_\beta \to y_0$ (we can take a subnet if necessary). If $f(x_0, y_0, \lambda_0) \not\leq_K 0_Y$, then $f(x_0, y_0, \Lambda_0) \in Y \setminus (-K)$ which is algebraically open due to Lemma 2.1. This together with the K-lower semiconti-nuity of f derives that

$$f(\bar{x}_\beta, y_\beta, \Lambda_\beta) \in Y \setminus (-K) + K = Y \setminus (-K),$$

for β sufficient large, which is a contradiction. Thus,

$$f(x_0, y_0, \lambda_0) \leq_K 0_Y.$$

This is impossible as $x_0 \in S_1(\Lambda_0)$. Therefore, S_1 is lower semicontinuous at λ_0.

We now prove that $S(\Lambda_0) \subseteq \mathrm{cl}(S_1(\Lambda_0))$. Let $\bar{x} \in S(\Lambda_0)$, $x_1 \in S_1(\Lambda_0)$, we set $x_t := (1-t)\bar{x} + tx_1$. Then, for all $y \in \Omega(\Lambda_0)$, we have

$$f(\bar{x}, y, \lambda_0) \geq_K 0_Y$$

and

$$f(x_1, y, \lambda_0) \not\leq_K 0_Y.$$

It follows from (iii) that

$$f(x_t, y, \lambda_0)\ g_K\ 0_Y.$$

This means that $x_t \in S_1(\lambda_0, \mu_0)$. We see that $x_t \to \bar{x}$ when $t \to 0$, and thus $\bar{x} \in \mathrm{cl}(S_1(\lambda_0))$, which yields that $S(\lambda_0) \subset \mathrm{cl}(S_1(\lambda_0))$. Combining this with the lower semicontinuity of S_1 and the closedness of $S_1(\lambda_\alpha)$, we arrive at

$$S(\lambda_0) \subset \mathrm{cl}(S_1(\lambda_0)) \subset \liminf S_1(\lambda_\alpha) \subset \liminf S(\lambda_\alpha).$$

By Lemma 4.2, the lower semicontinuity of S is obtained.

From Step 1 and Step 2, we get the continuity of the solution map S, and hence the proof is complete.

5. Wellposedness under perturbations for vector equilibrium problems with equilibrium constraints

It is known that wellposedness for optimization-related problems is a topic received much attention of several researchers from optimization community (see, e.g., [4–6, 13, 16, 20, 29–31] and the references therein). In this section, we present a result on the wellposedness under perturbations in the sense of Tykhonov for vector equilibrium problems with equilibrium constraints.

Let X, P be normed spaces and Y, Ω, Λ, f be as in the previous sections. Assume that Z is a linear space equipped with an algebraically closed, pointed, convex and solid cone $\mathbf{K}$ and $g : \Omega(\lambda) \times \Omega(\lambda) \times \Lambda \to Z$ with $g(x, x, \lambda) \square_K 0_Z$ for all $x \in \Omega(\lambda)$ and $\lambda \in \Lambda$. We consider the following parametric vector equilibrium problem with equilibrium constraints:

(PVEPEC) Find $\bar{x} \in S(\lambda)$ such that for all $y \in S(\lambda)$,

$$g(\bar{x}, y, \lambda) \square_{\check{K}} 0_Z,$$

where $S(\lambda)$ is the weakly efficient solution set to (PVEP), i.e.,

$$S(\lambda) := \{x \in \Omega(\lambda) \mid f(x, y, \lambda) \square_K 0_Y,\ \forall y \in \Omega(\lambda)\}.$$

We assume further that the original problem is corresponding to $\lambda = \bar{\lambda} \in \Lambda$. For each $\lambda \in \Lambda$, $e \in core(\mathbf{K})$, and $e \in \mathbb{R}_+$, let us denote the s-approximate solution set with respect to (shortly, wrt) e of (PVEPEC) by

$$\Pi(\lambda, s) := \{x \in S(\lambda) \mid g(x, y, \lambda) + se \square_{\mathbf{K}} 0_Z,\ \forall y \in S(\lambda)\}.$$

Definition 5.1. Let $\{\lambda_n\} \subset \Lambda$ be a sequence converging to $\bar{\lambda}$. A sequence $\{x_n\}$ with $x_n \in S(\lambda_n)$ is said to be an asymptotically solving sequence for (PVEPEC) corresponding to $\{\lambda_n\}$ if there exists a sequence $\{e_n\} \subset \mathbb{R}_+$ converging to 0 such that

$$g(x_n, y, \lambda_n) + s_n e \square_{\mathbf{K}} 0_Z,\ \forall y \in S(\lambda_n).$$

Definition 5.2. It is said that the problem (PVEPEC) is wellposed under perturbations if:

(a) the solution set $\Pi(\bar{\lambda}, 0)$ to (PVEPEC) is nonempty;

(b) for any sequence $\{\lambda_n\} \subset \Lambda$ converging to $\bar{\lambda}$, every asymptotically solving sequence for (PVEPEC) corresponding to $\{\lambda_n\}$ has a subnet converging to some point in $\Pi(\bar{\lambda},0)$.

Also, we consider the following stronger property with the uniqueness requirement as in classical definitions.

Definition 5.3. The problem (PVEPEC) is said to be uniquely well-posed under perturbations if:

(a) there exists a unique solution $\bar{x}$ to (PVEPEC);
(b) for any sequence $\{\lambda_n\} \subset \Lambda$ converging to $\bar{\lambda}$, every asymptotically solving sequence for (PVEPEC) corresponding to $\{\lambda_n\}$ converges to $\bar{x}$.

The following result gives sufficient conditions for the wellposedness under perturbations of the problem (PVEPEC).

Theorem 5.1. *Assume that all assumptions of Theorem* 4.2 *are satisfied. Assume further that*

(i) *for each* $x \in S(\Lambda)$ *and* $\lambda \in \Lambda$, $g(x,\cdot,\lambda)$ *is* $\mathcal{K}$ *-quasiconvex on* $S(\Lambda)$;

(ii) g *is* $\mathcal{K}$ *-upper semicontinuous on* $S(\Lambda) \times S(\Lambda) \times \Lambda$.

Then, (PVEPEC) is well-posed under perturbations. Furthermore, if the solution set of (PVEPEC) is a singleton, then (PVEPEC) is uniquely well-posed under perturbations.

Proof. We see that according to Theorem 4.2, we get that the map S is nonempty compact valued and continuous on Λ. Applying Theorem 4.1 with $f = g$ and $\Omega = S$, we get the nonemptiness of the solution set $\Pi(\bar{\lambda},0)$.

We now show that Π is upper semicontinuous at $(\bar{\lambda},0)$. Suppose that there exist an open superset U of $\Pi(\bar{\lambda},0)$, a sequence $\{(\lambda_n,\varepsilon_n)\} \subset \Lambda \times \mathbb{R}_+$ converging $(\bar{\lambda},0)$, $x_n \in \Pi(\lambda_n,\varepsilon_n)$ such that $x_n \notin U$ for all n. By the upper semicontinuity of S at $\bar{\lambda}$ and the compactness of $S(\bar{\lambda})$, we can assume that $x_n \to x_0$ for some $x_0 \in S(\bar{\lambda})$. If $x_0 \notin \Pi(\bar{\lambda},0)$, then we can find $y_0 \in S(\bar{\lambda})$ such that

$$g(x_0,y_0,\bar{\lambda}) \prec_{\mathcal{K}} 0_Z.$$

The lower semicontinuity of S in turn shows the existence of $y_n \in S(\lambda_n)$ satisfying $y_n \to y_0$. It follows from $x_n \in \Pi(\lambda_n,\varepsilon_n)$ and $y_n \in S(\lambda_n)$ that

$$g(x_n,y_n,\lambda_n) + \varepsilon_n e \,\square_{\mathcal{K}}\, 0_Z.$$

Let us consider the identity map $i : \mathcal{K} \to \mathcal{K}$. Then, Theorems 3.4 and 3.5 derive the $\mathcal{K}$-upper semicontinuity of the map $(x,y,\lambda,\varepsilon) \mapsto g(x,y,\lambda) + \varepsilon e$ at $(x_0,y_0,\bar{\lambda},0)$. Applying Theorem 3.2, we obtain

$$g(x_0,y_0,\bar{\lambda}) \,\square_{\mathcal{K}}\, 0_Z,$$

which is impossible. Hence, $x_0 \in \Pi(\bar{\lambda},0)$, which is again a contradiction since $x_n \notin U$ for all n. Thus, we obtain the upper semicontinuity of Π at $(\bar{\lambda},0)$.

Next, we will show that $\Pi(\bar{\lambda},0)$ is compact by simply proving its closedness in $S(\bar{\lambda})$. Let $x_n \in \Pi(\bar{\lambda},0)$ be such that $x_n \to x_0$. Then, for every $y_0 \in S(\bar{\lambda})$ we have

$$g(x_n,y_0,\bar{\lambda}) \,\square_{\mathcal{K}}\, 0_Z.$$

Using the upper semicontinuity of g, we get

$$g(x_0, y_0, \bar{\lambda}) \,\square_K\, \mathbf{0}_Z.$$

Therefore, $x_0 \in$ Õ$(\bar{\lambda},0)$, which leads to the closedness of Õ$(\bar{\lambda},0)$.

We are ready complete the proof. Let $\{\lambda_n\} \subset \Lambda$ be a sequence converging to $\bar{\lambda}$ and $\{x_n\}$ be an asymptotically solving sequence for (PVEPEC) corresponding to $\{\lambda_n\}$. Since P is upper semicontiuous and compact-valued at $(\bar{\lambda},0)$, taking Lemma 4.3 into account, $\{x_n\}$ contains a subsequence converging to a point in Õ$(\bar{\lambda},0)$. Hence, we get the well-posedness under perturbations for (PVEPEC). In the particular case where Õ$(\bar{\lambda},0)$ is a singleton, the unique well-posedness under perturbations follows.

6. Conclusions

In this paper, we have studied concepts of semi-algebraic semicontinuity of vector-valued maps taking values in a linear space. Several characterizations as well as properties of these concepts are discussed. Such properties together with famous KKM lemma and generalized quasiconvexity of vector-valued maps are utilized to investigate new results on the existence of solutions. Moreover, we establish sufficient conditions for the continuity of solution maps when the data of the reference problems is perturbed by a parameter. Using obtained results, we provide sufficient conditions to wellposedness under perturbations for the vector equilibrium problems with equilibrium constraints. Considering optimization-related problems in the algebraic frameworks is an interesting direction and has received a growing attention from many researchers. Therefore, there would be several works devoted to topics such as existence conditions, stability analysis, wellposedness,... to other models in optimization.

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