

Weak Solutions For Nonlinear Steklov Eigenvalue Problems

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Using variational approaches, sufficient conditions are established under which the nonlinear Steklov eigenvalue problem

$$\begin{cases} -\Delta_p u + |u|^{p-2} u = a(x)f(u) & \text{in } \Omega, \\ \nabla u^{p-2} \frac{\partial u}{\partial \nu} = g(u) & \text{on } \partial\Omega, \end{cases}$$

has a nontrivial weak solution when $\mu > 0$ is sufficiently small. Examples are provided to show the applicability of the results.

Keywords: Steklov problems, p -Laplacian operator, weak solutions, eigenvalues, critical points.

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1. Introduction

Steklov eigenvalue problems, characterized by the spectral parameter appearing in the boundary conditions, have been the focus of extensive mathematical research in recent years. These problems find applications in diverse areas such as fluid dynamics, elasticity, electrostatics, etc [3–5,12]. Here, we are concerned with the nonlinear Steklov eigenvalue problem

$$\begin{cases} -\Delta_p u + |u|^{p-2} u = a(x)f(u) & \text{in } \Omega, \\ \nabla u^{p-2} \frac{\partial u}{\partial \nu} = \mu g(u) & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$, $1 < p < N$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian operator, $\frac{\partial}{\partial \nu}$ is the outer normal derivative, $a \in L^\infty(\Omega) \setminus \{0\}$ with $a \geq 0$, $\mu \geq 0$ is a spectral parameter, and $f, g : \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions.

Various variations of problem (1.1) have been explored in the literature. A small sample of such work can be found in, for example, [1, 2, 6, 7, 10, 11, 13, 14, 18, 19] and the references therein. In particular, paper [14] established upper bounds for the variational eigenvalues of the problem

$$\begin{cases} -\Delta_p u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \mu |u|^{p-2} u & \text{on } \partial\Omega, \end{cases}$$

and paper [19] identified sufficient conditions for the existence and multiplicity of weak solutions to the problem

$$\begin{cases} \overline{\Delta} u = \lambda b u + f(x, u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda m u + g(x, u) & \text{on } \partial\Omega, \end{cases}$$

where $m \in L^r(\Omega)$, $b \in L^s(\partial\Omega)$ with $r > N/2$ and $s > N - 2$, and λ is a real parameter.

In this paper, we investigate the existence of weak solutions for problem (1.1) and establish criteria under which problem (1.1) has a nontrivial weak solutions when $\mu > 0$ is small. Our proof employs variational theorems derived by Ricceri in [15, 16] (see Lemmas 2.1 and 2.2 in Section 2). This study was inspired by the work in [15].

Throughout this paper, let the Sobolev space $W^{1,p}(\Omega)$ be equipped with the norm

$$\|u\| = \left(\int_{\Omega} (|\nabla u(x)|^p + |u(x)|^p) dx \right)^{1/p}, \quad (1.2)$$

and for any $k \geq 1$, let $\|\cdot\|_k$ be the norm of the space $L^k(\Omega)$ and $\|\cdot\|_{k,\partial\Omega}$ be the norm of the space $L^k(\partial\Omega)$, that is,

$$\|u\|_k = \left(\int_{\Omega} |u(x)|^k dx \right)^{1/k} \quad \text{and} \quad \|u\|_{k,\partial\Omega} = \left(\int_{\partial\Omega} |u(x)|^k ds \right)^{1/k},$$

where ds denotes the boundary measure on $\partial\Omega$.

Let

$$p^* = \frac{Np}{N-p}, \quad c_q = \sup_{u \in W^{1,p}(\Omega) \setminus \{0\}} \frac{\|u\|_q^q}{\|u\|_p^p} \quad \text{and} \quad d_p = \sup_{u \in W^{1,p}(\Omega) \setminus \{0\}} \frac{\|u\|_{p,\partial\Omega}^p}{\|u\|_p^p}. \quad (1.3)$$

where $q \in (1, p^*)$ is given below in assumption (H2). Then, in view of the respective compact embedding of $W^{1,p}(\Omega)$ into $L^q(\Omega)$ and $L^p(\partial\Omega)$, we see that $c_q, d_p \in (0, \infty)$. Moreover, we have

$$\|u\|_q \leq c \|u\|_q \quad \text{and} \quad \|u\|_{p,\partial\Omega} \leq d \|u\|_p. \quad (1.4)$$

In the next section, we present our theorem and its proof.

2. Main results

Let

$$F(t) = \int_0^t f(s) ds \quad \text{and} \quad G(t) = \int_0^t g(s) ds \quad \text{for any } t \in \mathbb{R}.$$

We make the following assumptions:

(H1) F has no global maximum in \mathbf{R} ;

(H2) $f(0) \neq 0$ and there exists $q \in (1, p^*]$ such that the function $k(t) := \frac{f(t)}{|t|^{q-2}t}$ is strictly monotone in $(-\infty, 0)$ and $(0, \infty)$, respectively, and $\lim_{|t| \rightarrow \infty} k(t) = 0$;

(H3) there exists a constant $k > 0$ such that $|G(t)| < K|t|^p$ for all $t \in \mathbf{R}$;

(H4) for any $\mu \in (0, \mu^*)$, there exist $r = r(\mu) > 0$ such that

$$\max \left\{ F \left(r^{1/q} \right), F \left(-r^{1/q} \right) \right\} < \frac{p(c \operatorname{ess\,sup}_q a)^{p/q} \left(\frac{1 - pd_p \kappa \mu}{\int_{\Omega} a(x) dx} \right)^{(q-p)/p} r^{p/q}}{p(c \operatorname{ess\,sup}_q a)^{p/q} \left(\frac{1 - pd_p \kappa \mu}{\int_{\Omega} a(x) dx} \right)^{(q-p)/p} r^{p/q}},$$

where c_q and d_p are dened in (1.3) and

$$\mu^* = \frac{1}{pd_p \kappa}. \tag{2.1}$$

Remark 2.1. Regarding the assumption (H2), we have the following observations:

- (a) The function $k(t)$ in (H2) may exhibit different monotonic behavior in the intervals $(-\infty, 0)$ and $(0, \infty)$.
- (b) Assumption (H2) implies that $f(t)$ has a constant sign over \mathbf{R} , meaning that $f(t)$ is either always positive or always negative for all $t \in \mathbf{R}$.

Remark 2.2. It is easy to check that assumption (H4) is satisfied if

$$\lim_{r \rightarrow \infty} \frac{F \left(r^{1/q} \right)}{r^{p/q}} = \lim_{r \rightarrow \infty} \frac{F \left(-r^{1/q} \right)}{r^{p/q}} = 0. \tag{2.2}$$

Definition 2.1. By a weak solution of problem (1.1), we mean a function $u \in W^{1,p}(\Omega)$ such that

$$\int_{\Omega} \left(|\nabla u(x)|^{p-2} \nabla u(x) \nabla v(x) + |u(x)|^{p-2} u(x) v(x) \right) dx - \mu \int_{\partial\Omega} g(u(x)) dS - \int_{\Omega} a(x) f(u(x)) v(x) dx = 0 \quad \text{in } \Omega \text{ for all } v \in W^{1,p}(\Omega).$$

When u is nontrivial, the corresponding μ is called an eigenvalue of problem (1.1).

Define two functionals $I, J : W^{1,p}(\Omega) \rightarrow \mathbf{R}$ by

$$I(u) = \frac{1}{p} \int_{\Omega} \left(|\nabla u(x)|^p + |u(x)|^p \right) dx - \mu \int_{\partial\Omega} G(u(x)) dS \tag{2.3}$$

and

$$J(u) = \int_{\Omega} a(x) F(u(x)) dx.$$

Then $I, J \in C^1(W^{1,p}(\Omega), \mathbf{R})$ with derivatives given by

$$\langle I'(u), v \rangle = \int_{\Omega} \left(|\nabla u(x)|^{p-2} \nabla u(x) \nabla v(x) + |u(x)|^{p-2} u(x) v(x) \right) dx - \mu \int_{\partial\Omega} g(u(x)) v(x) dS \tag{2.4}$$

and

$$\langle J'(u), v \rangle = \int_{\Omega} a(x) f(u(x)) v(x) dx \quad (2.5)$$

for all $u, v \in W^{1,p}(\Omega)$. Obviously, a weak solution of problem (1.1) is exactly a critical point of the energy functional $I - J$ on $W^{1,p}(\Omega)$.

We now state the main theorem of the paper.

Theorem 2.1. *Assume that (H1)–(H4) hold. Then, for each $\mu \in (0, \mu^*)$, problem (1.1) has a nontrivial weak solution u which is a local minimum of the energy function $I - J$ and satisfies*

$$\int_{\Omega} (|\nabla u(x)|^p + |u(x)|^p) dx \leq \left(\frac{r \int_{\Omega} a(x) dx}{c_q \operatorname{ess\,sup}_{\Omega} a} \right)^{p/q}. \quad (2.6)$$

Moreover, if $f(t) < 0$ and $g(t) \leq 0$ in \mathbb{R} , u is nonnegative almost everywhere in Ω , and if $f(t) > 0$ and $g(t) \geq 0$ in \mathbb{R} , there exists a subset Ω_1 of Ω with positive Lebesgue measure such that u is positive in Ω_1 .

In view of Remark 2.2, the following result is a direct consequence of Theorem 2.1.

Corollary 2.1. *Assume that (H1)–(H3) and (2.2) hold. Then, for all $\mu > 0$ sufficiently small, the conclusion of Theorem 2.1 holds.*

Before proving Theorem 2.1, we provide two examples to illustrate its applicability.

Example 2.1. Consider the problem

$$\begin{cases} -\operatorname{div}(\sqrt{\nabla u}^2 \nabla u) + \mu^2 u = a(x) \tau_1 (3u^2 + 1) & \text{in } \Omega, \\ \sqrt{\nabla u}^2 \frac{\partial u}{\partial \nu} = \mu \tau_2 u^3 & \text{on } \partial\Omega, \end{cases} \quad (2.7)$$

where $\Omega \subseteq \mathbb{R}^6$ is a bounded domain with a smooth boundary $\partial\Omega$, $a \in L^{\infty}(\Omega) \setminus \{0\}$ with $a \geq 0$, τ_1 and τ_2 are nonzero real numbers, and $\mu > 0$ is a parameter.

We claim that, for all $\mu > 0$ sufficiently small, problem (2.7) has a nontrivial weak solution u . Moreover, if $\tau_1 < 0$ and $\tau_2 < 0$, u is nonnegative almost everywhere in Ω , and if $\tau_1 > 0$ and $\tau_2 > 0$, u is positive at least in a subset of Ω with positive Lebesgue measure.

In fact, with $N = 6$, $p = 4$, $f(t) = \tau_1(3t^2 + 1)$, and $g(t) = \tau_2|t|^3$, it is clear that problem (2.7) is of the form of problem (1.1), $p^* = 12$, $F(t) = \tau_1(t^3 + t)$, and

$$G(t) = \begin{cases} \frac{\tau_1}{4} t^4, & t \geq 0. \\ -\frac{\tau_2}{4} t^4, & t < 0. \end{cases}$$

Then, (H1) and (H3) are obviously satisfied. Moreover, choosing $q = 4$, it is easy to see that (H2) and (2.2) hold. Therefore, the claim follows from Corollary 2.1.

Example 2.2. Consider the problem

$$\begin{cases} \left. \begin{aligned} -\operatorname{div}(|\nabla u|^{p-2} \nabla u) + |u|^{p-2} u &= a(x) \left(1 + \frac{|u|^{p-1}}{1+u^{2p}} \right) & \text{in } \Omega, \\ \nabla u^{p-2} \frac{\partial u}{\partial \nu} &= \mu \sin^{p-1}(u) & \text{on } \partial\Omega, \end{aligned} \right\} \quad (2.8)$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$, $1 < p < N$, $a \in L^\infty(\Omega) \setminus \{0\}$ with $a \geq 0$, $\rho \in (1, p^*]$, and $\mu > 0$ is a parameter.

We claim that, for all $\mu > 0$ sufficiently small, problem (2.8) has a nontrivial weak solution.

In fact, with $f(t) = 1 + \frac{|t|^{p-1}}{1+t^{2p}}$ and $g(t) = \sin^{p-1}(t)$, we see that problem (2.8) is of the form of problem (1.1). Moreover, we have

$$F(t) = \int_0^t \left(1 + \frac{|s|^{p-1}}{1+s^{2p}} \right) ds = t + \frac{1}{\rho} \operatorname{sgn}(t) \arctan(|t|^\rho)$$

and

$$|G(t)| = \left| \int_0^t \sin^{p-1}(s) ds \right| \leq \left| \int_0^t |s|^{p-1} ds \right| = \frac{1}{p} |t|^p.$$

Thus, with $q = \rho$ and $\kappa = \frac{1}{p}$, we see that (H1), (H2), (H3), and (2.2) hold. Therefore, the claim follows from Corollary 2.1.

Next, we prove Theorem 2.1. Lemma 2.1 below is essential in the proof and is taken from [15, Theorem A]. For some other related results, see [8, 17].

Lemma 2.1. *Let X be a reflexive real Banach space and let $\Phi, \Psi : X \rightarrow \mathbb{R}$ be two sequentially weakly lower semicontinuous functionals with Ψ also coercive and $\Phi(0) = \Psi(0) = 0$. Then, for each $\sigma > \inf_X \Psi$ and each λ satisfying*

$$\lambda > - \frac{\inf_X \Phi}{\sigma},$$

the restriction of $\lambda\Psi + \Phi$ to $\Psi^{-1}((-\infty, \sigma])$ has a global minimum.

Let $0 \leq a \leq b < \infty$. For a generic pair of functions $\phi, \psi : \mathbb{R} \rightarrow \mathbb{R}$, if $\lambda \in [a, b]$, we denote by $M(\phi, \psi, \lambda)$ the set of all global minimum of the function $\lambda\psi - \phi$ or the empty set according to whether $\lambda < \infty$ or $\lambda = \infty$. We adopt the conventions $\sup 0 = -\infty$ and $\inf 0 = \infty$. Let

$$\alpha(\phi, \psi, b) = \max \left\{ \inf_{\mathbb{R}} \psi, \sup_{M(\phi, \psi, b)} \psi \right\}$$

and

$$\beta(\phi, \psi, a) = \min \left\{ \sup_{\mathbb{R}} \psi, \inf_{M(\phi, \psi, a)} \psi \right\}.$$

The following minimax result can be found in [15, Propostion A], which is a special case of [16, Theorem 1].

Lemma 2.2. Let $\phi, \psi : \mathbb{R} \rightarrow \mathbb{R}$ be two functions such that, for each $\lambda \in (a, b)$, the function $\lambda\psi - \phi$ is lower semicontinuous, coercive, and has a unique global minimum in \mathbb{R} . Assume that

$$\alpha(\phi, \psi, b) < \beta(\phi, \psi, a).$$

Then, for each $s \in (\alpha(\phi, \psi, b), \beta(\phi, \psi, a))$, there exists $\lambda_s \in (a, b)$ such that the global minimum of the function $\lambda_s \psi - \phi$ lies in $\psi^{-1}(s)$.

For $u \in W^{1,p}(\Omega)$, let $u_- = -\min\{u, 0\}$. Lemma 2.3 below follows from [9, Theorem 7.6].

Lemma 2.3. Let $u \in W^{1,p}(\Omega)$. Then $u_- \in W^{1,p}(\Omega)$ and

$$\nabla u_- = \begin{cases} \nabla u & \text{if } u < 0, \\ 0 & \text{if } u \geq 0. \end{cases}$$

We are now in a position to prove our main theorem.

Proof of Theorem 2.1. Using (1.2), the second inequality in (1.4), (H3), and (2.3), we have

$$\begin{aligned} I(u) &\geq \frac{1}{p} \|u\|^p - \kappa\mu \int_{\partial\Omega} |u(x)|^p dS \\ &= \frac{1}{p} \|u\|_p^p - \kappa\mu \|u\|_{p, \partial\Omega}^p \\ &\geq \frac{1}{p} \|u\|_p^p - d \kappa\mu \|u\|_p^p \\ &= \left(\frac{1}{p} - d \kappa\mu \right) \|u\|_p^p. \end{aligned} \quad (2.9)$$

Below, we always assume that $0 < \mu < \mu^*$. Let r be given in (H4) and define

$$\sigma = \frac{1 - pd \kappa\mu}{p} \left(\frac{r \int_{\Omega} a(x) dx}{c_{q\text{ess sup}\Omega a}} \right)^{p/q}. \quad (2.10)$$

Then, we have

$$c_{q\text{ess sup}\Omega a} \left(\frac{p\sigma}{1 - pd \kappa\mu} \right)^{q/p} = r \int_{\Omega} a(x) dx. \quad (2.11)$$

From the first inequality in (1.4), (2.1), and (2.9), we see that

$$\begin{aligned} \left\{ u \in W^{1,p}(\Omega) : I(u) \leq \sigma \right\} &\subseteq \left\{ u \in W^{1,p}(\Omega) : \|u\|^p \leq \frac{p\sigma}{1 - pd \kappa \mu} \right\} \\ &\subseteq \left\{ u \in L^q(\Omega) : \|u\|_q^q \leq c^q \left(\frac{p\sigma}{1 - pd \kappa \mu} \right)^{q/p} \right\}. \end{aligned}$$

Thus, by (2.11), it follows that

$$\begin{aligned} &\left\{ u \in W^{1,p}(\Omega) : I(u) \leq \sigma \right\} \\ &\subseteq \left\{ u \in L^q(\Omega) : \int_{\Omega} a(x) |u(x)|^q dx \leq c_q \operatorname{ess\,sup}_{\Omega} a \left(\frac{p\sigma}{1 - pd \kappa \mu} \right)^{q/p} \right\} \\ &= \left\{ u \in L^q(\Omega) : \int_{\Omega} a(x) |u(x)|^q dx \leq r \int_{\Omega} a(x) dx \right\}. \end{aligned} \tag{2.12}$$

For any $\lambda > 0$, in view of (H2), it is obvious that the function $l_{\lambda}(t) := \lambda|t|^q - F(t)$ is lower semicontinuous and coercive (i.e., $l_{\lambda}(t) \rightarrow \infty$ as $|t| \rightarrow \infty$). Now, we show that $l_{\lambda}(t)$ has a unique global minimum in \mathbf{R} . Assume, to the contrary, that $l_{\lambda}(t)$ has two distinct global minima $t_1, t_2 \in \mathbf{R}$ with $t_1 < t_2$. Then there are 3 cases to consider:

- (a) $0 < t_1 < t_2$, (b) $t_1 < t_2 < 0$, and (c) $t_1 < 0 < t_2$.

Case (a): $0 < t_1 < t_2$. Since $l'_{\lambda}(t_1) = l'_{\lambda}(t_2) = 0$, we have

$$\lambda q t_1^{q-1} - f(t_1) = \lambda q t_2^{q-1} - f(t_2) = 0.$$

Thus,

$$\frac{f(t_1)}{t_1^{q-1}} = \frac{f(t_2)}{t_2^{q-1}} = \lambda q,$$

contradicting to the assumption in (H2) that $k(t)$ is strictly monotone in $(0, \infty)$.

Case (b): $t_1 < t_2 < 0$. Arguing as in case (a), a similar contradiction can be derived in view of the assumption in (H2) that $k(t)$ is strictly monotone in $(-\infty, 0)$.

Case (c): $t_1 < 0 < t_2$. By Rolle's theorem, there exists $t_3 \in (t_1, t_2)$ such that $\lambda q t_3^{q-1} = f(t_3)$.

Since $f(0) \neq 0$ by (H2), we have $t_3 \neq 0$. Then, either $t_3 \in (t_1, 0)$ or $t_3 \in (0, t_2)$. Moreover, we have

$$\frac{f(t_1)}{t_1^{q-1}} = \frac{f(t_2)}{t_2^{q-1}} = \frac{f(t_3)}{t_3^{q-1}}. \tag{2.13}$$

Then, for both cases $t_3 \in (t_1, 0)$ and $t_3 \in (0, t_2)$, (2.13) yields a contradiction due to the monotonicity assumption on $k(t)$ in (H2). Now, we have proved that, for each $\lambda \in (0, \infty)$, $l_{\lambda}(t)$ has a unique global minimum in \mathbf{R} . Below, let θ_{λ} denote the unique global minimum in \mathbf{R} of $l_{\lambda}(t)$.

Obviously, $\alpha(F, |\cdot|^q, \infty) = 0$, and by (H1), $\beta(F, |\cdot|^q, 0) = \infty$. Thus, with $\phi(t) = F(t)$, $\psi(t) = |t|^q$, $a = 0$, $b = \infty$, and $\lambda \in (a, b)$, all the assumptions of Lemma 2.2 are satisfied. Then,

in view of $r \in (\alpha(F, |\cdot|^q, \infty), \beta(F, |\cdot|^q, 0))$, Lemma 2.2 implies that there exists $\lambda_r \in (0, \infty)$ such that

$$\Psi(\theta_{\lambda_r}) = |\theta_{\lambda_r}|^q = r$$

and

$$\lambda_r r - F(\theta_{\lambda_r}) \leq \lambda_r \Psi(t) - \Phi(t) = \lambda_r |t|^q - F(t) \quad \text{for all } t \in \mathbb{R}.$$

Thus, in view of Remark 2.1 (b), we obtain that

$$F(\theta_{\lambda_r}) = \sup_{|s|^q = r} F(s) = \max \left\{ F(r^{1/q}), F(-r^{1/q}) \right\}, \quad (2.14)$$

and for each $u \in L^q(\Omega)$,

$$(\lambda_r r - F(\theta_{\lambda_r})) \int_{\Omega} a(x) dx \leq \int_{\Omega} a(x) (\lambda_r |u(x)|^q - F(u(x))) dx. \quad (2.15)$$

It is trivial to see that (2.14) and (2.15) further imply that, for each $u \in L^q(\Omega)$ satisfying

$$\int_{\Omega} a(x) |u(x)| dx < r \int_{\Omega} a(x) dx,$$

we have

$$\int_{\Omega} a(x) F(u(x)) dx \leq \max \left\{ F(r^{1/q}), F(-r^{1/q}) \right\} \int_{\Omega} a(x) dx.$$

Then, from (2.12), we have

$$\sup_{I^{-1}((-\infty, \sigma])} \int_{\Omega} a(x) F(u(x)) dx \leq \max \left\{ F(r^{1/q}), F(-r^{1/q}) \right\} \int_{\Omega} a(x) dx. \quad (2.16)$$

From (H4), (2.10), and (2.16), we see that

$$\sup_{I^{-1}((-\infty, \sigma])} \int_{\Omega} a(x) F(u(x)) dx < \sigma,$$

from which it follows that

$$\frac{\inf_{I^{-1}((-\infty, \sigma])} (-J(u))}{\sigma} < 1.$$

Then, with $X = W^{1,p}(\Omega)$, $\Phi = -J$, $\Psi = I$, and $\lambda = 1$, we can apply Lemma 2.1 to obtain the existence of a weak solution $u \in W^{1,p}(\Omega)$ of problem (1.1) which is a local minimum of $I - J$ and also satisfies (2.6) by virtue of (2.9) and (2.10). Since $f(0) \neq 0$, u is nontrivial.

In what follows, we show the ‘‘moreover’’ part of the theorem. We first assume that $f(t) < 0$ and $g(t) \leq 0$ in \mathbb{R} . Since u is a critical point of $I - J$, we obtain that

$$\langle I'(u), u_- \rangle - \langle J'(u), u_- \rangle = 0. \quad (2.17)$$

Then, from (2.4) and (2.5), it follows that

$$\begin{aligned} & \int_{\Omega} \left(|\nabla u(x)|^{p-2} \nabla u(x) \nabla u_-(x) + |u(x)|^{p-2} u(x) u_-(x) \right) dx \\ & = \mu \int_{\partial\Omega} g(u(x)) u_-(x) dS + \int_{\Omega} a(x) f(u(x)) u_-(x) dx. \end{aligned} \quad (2.18)$$

Assume, to the contrary, that u is not nonnegative almost everywhere in Ω . Let $\Omega_- = \{x \in \bar{\Omega} : u(x) < 0\}$. Then Ω_- has a positive Lebesgue measure. By Lemma 2.3 and (2.18), we have

$$\begin{aligned} 0 &< \int_{\Omega} (|\nabla u_-(x)|^p + |u_-(x)|^p) dx = \mu \int_{\partial\Omega} g(u(x))u_-(x) dS + \int_{\Omega} a(x)f(u(x))u_-(x) dx \\ &\leq \int_{\Omega} a(x)f(u(x))u_-(x) dx \leq 0, \end{aligned}$$

which is a contradiction. Thus, u is nonnegative almost everywhere in Ω .

Now, we assume that $f(t) > 0$ and $g(t) \geq 0$ in \mathbb{R} . Suppose by the contradiction that there does not exist a subset Ω_1 of $\bar{\Omega}$ with positive Lebesgue measure such that u is positive in Ω_1 . Then, u is nontrivial and nonpositive in Ω . As in (2.17), we have

$$\langle I'(u), u \rangle - \langle J'(u), u \rangle = 0,$$

Then, from (2.4) and (2.5), we derive that

$$0 < \int_{\Omega} (|\nabla u(x)|^p + |u(x)|^p) dx = \mu \int_{\partial\Omega} g(u(x))u(x) dS + \int_{\Omega} a(x)f(u(x))u(x) dx \leq 0.$$

We again reach a contradiction. Hence, there exists a subset Ω_1 of $\bar{\Omega}$ with positive Lebesgue measure such that u is positive in Ω_1 . This completes the proof of the theorem.

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