

Minimax and Lipschitz Operators

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The aim of this short note is to show that the minimax equality proved in [3] cannot be extended to the case where linear operators are replaced by Lipschitz operators.

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In his paper [3], extending the previous [2], the author established the following minimax theorem:

Theorem. *Let E be a infinite-dimensional Banach space, F be a Banach space, X be a convex subset of E whose interior is non-empty for the weak topology on bounded sets, Δ a finite-dimensional convex compact subset of $\mathcal{L}(E, F)$, $\varphi : F \rightarrow \mathbb{R}$ be a continuous convex coercive map, and $\psi : \Delta \rightarrow \mathbb{R}$ a convex continuous function. Assume moreover that Δ contains at most one compact operator. Then*

$$\sup_{x \in X} \inf_{T \in \Delta} \varphi(Tx) + \psi(T) = \inf_{T \in \Delta} \sup_{x \in X} \varphi(Tx) + \psi(T)$$

In particular, taking $\varphi(x) = \|x\|$ and $\psi = 0$, this gets the minimax equality :

$$\sup_{x \in X} \inf_{T \in \Delta} \|Tx\| = \inf_{T \in \Delta} \sup_{x \in X} \|Tx\|.$$

The aim of this note is to replace the finite-dimensional convex compact subset Δ of $\mathcal{L}(E, F)$ by a finite-dimensional convex compact set of Lipschitz mappings from E to F , and more precisely of mappings of the type $x \mapsto \Phi(x) + \psi(x)$ where Φ is a fixed continuous surjective linear mapping from E to F , and $\psi : E \rightarrow F$ is a Lipschitz mapping.

If E and F are Banach spaces and $\Phi : E \rightarrow F$ a continuous linear mapping with closed range (in particular if Φ is onto) we will denote by $\nu(\Phi)$ the quantity

$$\nu(\Phi) = \sup_{\substack{y \in \Phi(E) \\ \|y\| \leq 1}} d(0, \Phi^{-1}(y)).$$

Of course, if moreover Φ is one-to-one, $\nu(\Phi)$ is the norm of the continuous linear mapping $\Phi^{-1} : \Phi(F) \rightarrow E$. And if $\ker(\Phi) \neq \{0\}$, Φ factors through the quotient $\hat{E} = E/\ker(\Phi)$ and a one-to-one linear mapping $\hat{\Phi} = \hat{E} \rightarrow \Phi(E)$, and then $\nu(\Phi)$ is the norm of the linear mapping $\hat{\Phi}^{-1}$.

The following result which addresses the case where Δ is an interval, is due to B. Ricceri and follows from Theorem 2.1 in [1].

Theorem. *Let X and Y be two Banach spaces, with $\dim(Y) \geq 2$. Let $\Phi : X \rightarrow Y$ be a continuous surjective linear operator and let $\Psi_1, \Psi_2 : X \rightarrow Y$ be two β -Lipschitz operators, where $1/\beta = \nu(\Phi)$. Then, one has*

$$\sup_{x \in X} \inf_{\lambda \in [0,1]} \|\Phi(x) + \lambda\Psi_1(x) + (1-\lambda)\Psi_2(x)\| = \min \left\{ \sup_{x \in X} \|\Phi(x) + \Psi_1(x)\|, \sup_{x \in X} \|\Phi(x) + \Psi_2(x)\| \right\}$$

Consider now the case where the finite-dimensional compact convex set Δ has dimension at least 2 and its elements have small Lipschitz constants.

Theorem 1. *Let X and Y be Banach spaces, Φ be a linear continuous operator from X onto Y , $\beta = 1/\nu(\Phi)$ and Δ be a finite-dimensional compact convex set of Lipschitz mappings from X to Y . If each element of Δ has a Lipschitz constant $< \beta$, then*

$$\inf_{\psi \in \Delta} \sup_{x \in X} \|\Phi(x) + \psi(x)\| = \sup_{x \in X} \inf_{\psi \in \Delta} \|\Phi(x) + \psi(x)\| = +\infty$$

Proof. On the compact space Δ , the function which assigns to each ψ its Lipschitz constant $L_\psi = \sup_{\substack{x,z \in X \\ x \neq z}} \frac{\|\psi(x) - \psi(z)\|}{\|x - z\|}$ is continuous and attains its supremum $\lambda < \beta$. Also by compactness $R = \sup_{\psi \in \Delta} \|\psi(0)\| < +\infty$.

For $\varepsilon > 0$ let $X_\varepsilon = \{x \in X : \|\Phi(x)\| \geq \beta(1 - \varepsilon)\|x\|\}$. By definition of β , we have $\Phi(X_\varepsilon) = \Phi(X) = Y$, hence $\sup_{x \in X_\varepsilon} \|x\| = +\infty$. Indeed if X_ε was bounded so would be $Y = \Phi(X_\varepsilon)$. Thus for each $\psi \in \Delta$ and each $x \in X_\varepsilon$ we have

$$\begin{aligned} \|\Phi(x) + \psi(x)\| &\geq \|\Phi(x)\| - \|\psi(x)\| \geq \|\Phi(x)\| - (\|\psi(0)\| + \|\psi(x) - \psi(0)\|) \\ &\geq \beta(1 - \varepsilon)\|x\| - (R + \lambda\|x\|) = (\beta(1 - \varepsilon) - \lambda)\|x\| - R. \end{aligned}$$

Choosing $\varepsilon < \frac{1}{2}(1 - \frac{\lambda}{\beta})$, we get $\inf_{\psi \in \Delta} \|\Phi(x) + \psi(x)\| \geq \frac{\beta - \lambda}{2}\|x\| - R$ for $x \in X_\varepsilon$ hence

$$\sup_{x \in X} \inf_{\psi \in \Delta} \|\Phi(x) + \psi(x)\| \geq \sup_{x \in X_\varepsilon} \inf_{\psi \in \Delta} \|\Phi(x) + \psi(x)\| \geq \frac{\beta - \lambda}{2} \cdot \left(\sup_{x \in X_\varepsilon} \|x\| \right) - R = +\infty$$

and

$$\inf_{\psi \in \Delta} \sup_{x \in X} \|\Phi(x) + \psi(x)\| \geq \sup_{x \in X} \inf_{\psi \in \Delta} \|\Phi(x) + \psi(x)\| = +\infty.$$

So the proof is complete.

We now look at the case where Δ has dimension > 1 and each element of Δ is β -Lipschitz, with $\beta = 1/\nu(\Phi)$ and want to prove that the previous minimax equality does no longer hold by constructing a counterexample.

Theorem 2. *There exist X and Y Banach spaces, $\Phi : X \rightarrow Y$ linear and onto, and $(\Psi_j)_{1 \leq j \leq 3} : X \rightarrow Y$, β -Lipschitz, where $\beta = 1/\nu(\Phi)$ such that*

$$\sup_{x \in X} \inf_{\alpha \in \Delta} \left\| \Phi(x) + \sum \alpha_j \Psi_j(x) \right\| < \inf_{\alpha \in \Delta} \sup_{x \in X} \left\| \Phi(x) + \sum \alpha_j \Psi_j(x) \right\|$$

where Δ is the canonical 2-dimensional simplex : $\{\alpha \in \mathbb{R}_+^3 : \alpha_1 + \alpha_2 + \alpha_3 = 1\}$.

The following result is probably well known.

Lemma 3. *Let H be a real Hilbert space and C be a nonempty closed convex subset of H . Denote by p the orthogonal projection on C and by $\varpi : x \mapsto p(x) - x$ the projecting line. For all x and y in H we have*

$$\|p(x) - p(y)\|^2 + \|\varpi(x) - \varpi(y)\|^2 \leq \|x - y\|^2$$

In particular p and ϖ are 1-Lipschitz.

Proof. In the case where C is a closed linear subspace this inequality is in fact an equality and corresponds to Pythagoras' theorem.

Let x and y be two points of H , $\xi = p(x)$ and $\eta = p(y)$ be their projections on C . We have

$$\langle x - \xi, \eta - \xi \rangle \leq 0 \text{ and } \langle y - \eta, \xi - \eta \rangle \leq 0,$$

thus

$$\begin{aligned} \langle x - y, \xi - \eta \rangle &= \langle (x - \xi) + (\xi - \eta) + (\eta - y), \xi - \eta \rangle \\ &= \langle x - \xi, \xi - \eta \rangle + \langle \xi - \eta, \xi - \eta \rangle + \langle \eta - y, \xi - \eta \rangle \\ &= \|\xi - \eta\|^2 - \langle x - \xi, \eta - \xi \rangle - \langle y - \eta, \xi - \eta \rangle \\ &\geq \|\xi - \eta\|^2. \end{aligned}$$

It follows that

$$\begin{aligned} \|p(x) - p(y)\|^2 + \|\varpi(x) - \varpi(y)\|^2 &= \|\xi - \eta\|^2 + \|(\xi - x) - (\eta - y)\|^2 \\ &= \|\xi - \eta\|^2 + \|(\xi - \eta) - (x - y)\|^2 \\ &= 2\|\xi - \eta\|^2 + \|x - y\|^2 - 2\langle x - y, \xi - \eta \rangle \\ &\leq \|x - y\|^2 + 2\|\xi - \eta\|^2 - 2\|\xi - \eta\|^2 = \|x - y\|^2, \end{aligned}$$

whence the statement follows.

Notations

Consider a triangle T in the euclidean plane \mathbb{R}^2 , with edges a_1, a_2, a_3 , and suppose all of its angles are acute. Let Γ be the circumscribed circle to T , ω be the center and ρ the radius of Γ . Then ω is interior to T .

To simplify the notations, for $j \in \mathbb{Z}$, a_j will denote the point a_i with $i \in \{1, 2, 3\}$ such that $i = j \pmod{3}$. Denote $m_j = \frac{a_{j+1} + a_{j+2}}{2}$ the midpoint of the side $J_j = [a_{j+1}, a_{j+2}]$ opposite to a_j ; in particular $\omega - m_j$ is orthogonal to J_j . We have

$$\begin{aligned} (a_j - a_{j+1}) \wedge (a_{j+1} - a_{j+2}) &= (a_j - a_{j+2}) \wedge (a_{j+1} - a_{j+2}) - (a_{j+1} - a_{j+2}) \wedge (a_{j+1} - a_{j+2}) \\ &= (a_j - a_{j+2}) \wedge (a_{j+1} - a_{j+2}) = (a_{j+3} - a_{j+2}) \wedge (a_{j+1} - a_{j+2}) \\ &= -(a_{j+2} - a_{j+3}) \wedge (a_{j+1} - a_{j+2}) \\ &= (a_{j+1} - a_{j+2}) \wedge (a_{j+2} - a_{j+3}) \end{aligned}$$

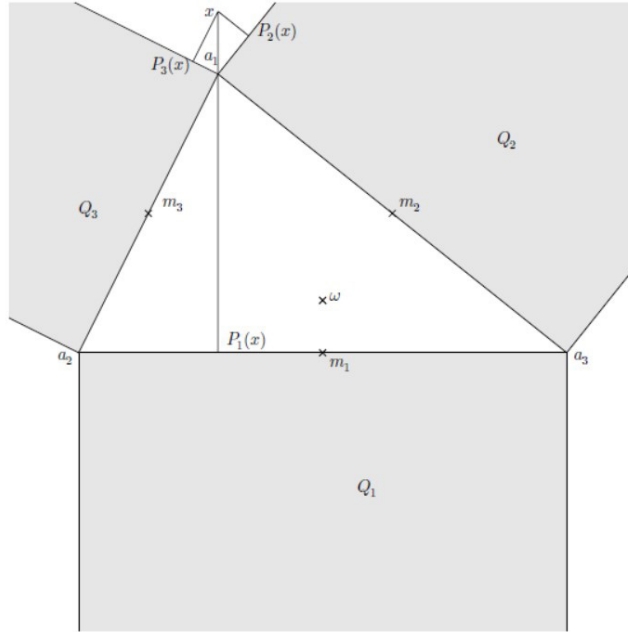
so this outer product is independent from j . Up to swapping a_2 and a_3 if necessary, we will assume that $(a_j - a_{j+1}) \wedge (a_{j+1} - a_{j+2}) > 0$ for all j . The linear functionals $u \mapsto \langle \omega - m_j, u \rangle$ and $u \mapsto u \wedge (a_{j+1} - a_{j+2})$ have same kernel hence are proportional; so there exists $\gamma_j \in \mathbb{R}$ such that $u \mapsto u \wedge (a_{j+1} - a_{j+2}) = \gamma_j \langle \omega - m_j, u \rangle$. Applying to $u = a_j - a_{j+1}$ we check that $\gamma_j \langle \omega - m_j, a_j - a_{j+1} \rangle = \gamma_j \langle \omega - m_j, a_j - m_j \rangle > 0$, hence that $\gamma_j > 0$.

Denote by Q_j the half strip:

$$Q_j = \left\{ x \in \mathbb{R}^2 : (x - m_j) \wedge (a_{j+1} - a_{j+2}) \leq 0 \leq \left\langle x - m_j, a_{j+1} - a_{j+2} \right\rangle \leq \frac{1}{2} \|a_{j+1} - a_{j+2}\|^2 \right\},$$

whose boundary contains the side J_j , by P_j the orthogonal projection on the closed convex set Q_j and by $\varpi_j : x \mapsto P_j(x) - x$ the corresponding projecting line. It is easily seen that the intersection of Q_j and the line $(a_{j+1}, a_{j+2}) = \{x : (x - m_j) \wedge (a_{j+2} - a_{j+1}) = 0\}$ is the segment J_j .

Lemma 4. If $(x - m_j) \wedge (a_{j+1} - a_{j+2}) \geq 0$, then $P_j(x)$ belongs to J_j .



Proof. If not the point $y = P_j(x)$ should satisfy $(y - m_j) \wedge (a_{j+1} - a_{j+2}) < 0$ since $P_j(x) \in Q_j$ and the inequality $\langle y - m_j, a_{j+1} - a_{j+2} \rangle \leq 0$ holds. Then $y' = y + \varepsilon(\omega - m_j)$ would belong to Q_j for $\varepsilon > 0$ small enough, and we would have $y - x = \lambda(a_{j+2} - a_{j+1}) - \mu(\omega - m_j)$ for some convenient λ and μ , with $\mu \geq 0$ since $(\omega - m_j) \wedge (a_{j+1} - a_{j+2}) > 0$ and

$$\mu(\omega - m_j) \wedge (a_{j+1} - a_{j+2}) = (x - m_j) \wedge (a_{j+1} - a_{j+2}) - (y - m_j) \wedge (a_{j+1} - a_{j+2}) \geq 0,$$

hence also $y' - x = \lambda(a_{j+2} - a_{j+1}) - (y - \varepsilon)(\omega - m_j)$ thus

$$\begin{aligned} \|y' - x\|^2 &= \lambda^2 \|a_{j+1} - a_{j+2}\|^2 + (\mu - \varepsilon)^2 \|\omega - m_j\|^2 \\ &< \lambda^2 \|a_{j+1} - a_{j+2}\|^2 + \mu^2 \|\omega - m_j\|^2 = \|x - y\|^2, \end{aligned}$$

so $d(x, Q_j) \leq \|x - y'\| < \|x - y\| = d(x, Q_j)$, a contradiction.

Lemma 5. For all $x \in \mathbb{R}^2$ at least one $P_j(x)$ belongs to T .

Proof. Indeed

$$\begin{aligned} \sum_{j=1}^3 (x - m_j) \wedge (a_{j+1} - a_{j+2}) &= (x - \omega) \wedge \sum_{j=1}^3 (a_{j+1} - a_{j+2}) + \sum_{j=1}^3 (\omega - m_j) \wedge (a_{j+1} - a_{j+2}) \\ &= (x - \omega) \wedge 0 + \sum_{j=1}^3 (\omega - m_j) \wedge (a_{j+1} - a_{j+2}) = \sum_{j=1}^3 \gamma_j \langle \omega - m_j, \omega - m_j \rangle \\ &= \sum_j \gamma_j \|\omega - m_j\|^2 > 0. \end{aligned}$$

Then at least one of the terms $(x - m_j) \wedge (a_{j+1} - a_{j+2})$ is positive and the conclusion follows from lemma 4.

Define $X = Y = \mathbb{R}^2$, $\Phi(x) = x$ and $\Psi_j(x) = \overline{p}_j(x) - \omega$, hence $\Phi(x) + \Psi_j(x) = P_j(x) - \omega$. It is clear from what precedes that the mappings P_j and Ψ_j are 1-Lipschitz, and that Φ^{-1} is the identity mapping, so $v(\Phi) = 1$.

Recall that Δ is the simplex $\{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}_+^3 : \alpha_1 + \alpha_2 + \alpha_3 = 1\}$.

Lemma 6. For whichever $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \Delta$ the function $\varphi_\alpha = \sum_{j=1}^3 \alpha_j (P_j(x) - \omega)$ satisfies

$$\sup_{x \in \mathbb{R}^2} \|\varphi_\alpha(x)\| = +\infty.$$

Thus $\inf_{\alpha \in \Delta} \sup_{x \in \mathbb{R}^2} \|\varphi_\alpha(x)\| = +\infty$.

Proof. Consider first the case $\alpha_j = 1$, $\alpha_{j+1} = \alpha_{j+2} = 0$, hence $\varphi_\alpha(x) = P_j(x) - \omega$. Then, for $t \geq 0$, consider the point $x_t = m_j - t(\omega - m_j)$ which belongs to the strip Q_j . We have

$$\|\varphi_\alpha(x_t)\| = \|x_t - \omega\| = \|m_j - \omega - t(\omega - m_j)\| = (1+t)\|\omega - m_j\|$$

and $\sup_{x \in \mathbb{R}^2} \|\varphi_\alpha(x)\| \geq \sup_{t \geq 0} \|\varphi_\alpha(x_t)\| = +\infty$ since $m_j \neq \omega$.

Otherwise, if both α_{j+1} and α_{j+2} are not 0 (e.g. if $\sup_i \alpha_i < 1$ and $\alpha_j = \min(\alpha_1, \alpha_2, \alpha_3)$), consider for $t \geq 0$, the point $x_t = a_j + t(\omega - m_j)$. We have $P_j(x_t) = P_j(x_0) \in J_j$ for all t . And for $k \neq j$, ($k = j+1$ or $k = j+2$), we have $P_k(x_t) = a_j + \theta(m_k - \omega)$ for some $\theta \geq 0$. Then since $x_t - P_k(x_t)$ is orthogonal to $m_k - \omega$, we get :

$$\begin{aligned} 0 &= \langle x_t - P_k(x_t), m_k - \omega \rangle = \langle (a_j + t(\omega - m_j)) - (a_j + \theta(m_k - \omega)), m_k - \omega \rangle \\ &= t \langle \omega - m_j, m_k - \omega \rangle - \theta \langle m_k - \omega, m_k - \omega \rangle \end{aligned}$$

hence $\theta = t \frac{\langle \omega - m_j, m_k - \omega \rangle}{\|m_k - \omega\|^2}$ and

$$\langle \omega - m_j, P_k(x_t) - \omega \rangle = \langle \omega - m_j, a_j - \omega \rangle + t \frac{(\langle \omega - m_j, \omega - m_k \rangle)^2}{\|m_k - \omega\|^2}.$$

Finally we get

$$\begin{aligned} \langle \omega - m_j, \varphi_\alpha(x_t) \rangle &= \alpha_j \langle \omega - m_j, P_j(x_0) - \omega \rangle + \alpha_{j+1} \langle \omega - m_j, a_j - \omega \rangle + \alpha_{j+1} t \frac{(\langle \omega - m_j, \omega - m_{j+1} \rangle)^2}{\|m_{j+1} - \omega\|^2} \\ &\quad + \alpha_{j+2} \langle \omega - m_j, a_j - \omega \rangle + \alpha_{j+2} t \frac{(\langle \omega - m_j, \omega - m_{j+2} \rangle)^2}{\|m_{j+2} - \omega\|^2} \\ &= A + Bt \end{aligned}$$

where $B = \alpha_{j+1} \frac{(\langle \omega - m_j, \omega - m_{j+1} \rangle)^2}{\|\omega - m_{j+1}\|^2} + \alpha_{j+2} \frac{(\langle \omega - m_j, \omega - m_{j+2} \rangle)^2}{\|\omega - m_{j+2}\|^2} > 0$, since by the

hypothesis on the angles of T : $\langle \omega - m_j, \omega - m_k \rangle \neq 0$ for $j \neq k$.

So we get

$$\|\varphi_\alpha(x_t)\| \geq \frac{1}{\|\omega - m_j\|} \left| \langle \omega - m_j, \varphi_\alpha(x_t) \rangle \right| \geq \frac{Bt - |A|}{\|\omega - m_j\|}$$

whence $\lim_{t \rightarrow +\infty} \|\varphi_\alpha(x_t)\| = +\infty$ and $\sup_{x \in \mathbb{R}^2} \|\varphi_\alpha(t)\| = +\infty$.

Lemma 7. For whichever $x \in \mathbb{R}^2$ there exists some $\alpha \in \Delta$ such that $\|\varphi_\alpha(x)\| \leq \rho$. In particular

$$\sup_{x \in \mathbb{R}^2} \inf_{\alpha \in \Delta} \|\varphi_\alpha(x)\| \leq \rho < +\infty.$$

Proof. In virtue of lemma 5, for all $x \in \mathbb{R}^2$ there is at least one $P_j(x)$ in T and for such a j we have

$$\inf_{\alpha \in \Delta} \|\varphi_\alpha(x)\| \leq \|P_j(x) - \omega\| \leq \rho$$

since T is included in the disk of center ω and radius ρ .

Proof of theorem 2. Lemmas 6 and 7 show that the previous construction yields the desired counterexample.

References

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