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Fractional Elliptic Problem With Two Non-Local Terms Involving $r(\xi)$-Laplacian

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In this present paper, we are interested in investigating the existence of weak solutions for a new class of fractional partial differential equations with $r(\xi)$-Laplacian involving the Kirchhoff-type equation in the space ψ-fractional $\mathsf{N}^{\alpha,\beta;\psi}_{r(\xi)}(\Omega)$ via Mountain Pass Theorem and Ekeland Variational Principle. The idea is to discuss the existence of a weak solution u_κ for $\kappa > 0$ and a positive solution u_κ for $\kappa \in (0, \kappa^*)$ for $\kappa^* > 0$.

Keywords: Fractional Elliptic Equation, ψ-Hilfer Fractional Derivative, Existence of Solutions, Mountain Pass Theorem, Ekeland Variational Principle.

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1. Introduction and Motivation

In this present paper, we consider the fractional elliptic problem with two nonlocal terms given by

$$\begin{cases} \square(\zeta)\,\mathsf{D}^{\alpha,\beta;\psi}_{T}\left(\left|\mathsf{D}^{\alpha,\beta;\psi}_{0+}u\right|^{r(\xi)-2}\mathsf{D}^{\alpha,\beta;\psi}_{0+}u\right) = \kappa\,|u|^{s(\xi)-2}\,u\left[\displaystyle\int_\Omega \frac{1}{s(\xi)}|u|^{s(\xi)}\,d\xi\right]^{\tau} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

with

$$\zeta := \int_\Omega \frac{1}{r(\xi)}\left|\mathsf{D}^{\alpha,\beta;\psi}_{0+}u\right|^{r(\xi)} d\xi$$

where $\Omega \subset \mathsf{R}^N$ is a smooth bounded domain, $\mathrm{M} : \mathsf{R}^+ \to \mathsf{R}^+$ is a continuous function, κ is a positive real parameter, $\tau > 0$, $r, s \in C(\Omega)$ are functions whose properties will be given later and $\mathsf{D}^{\alpha,\beta;\psi}_{T}(\cdot)$ and $\mathsf{D}^{\alpha,\beta;\psi}_{0+}(\cdot)$ are the ψ-Hilfer partial derivatives fractional on the right

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and on the left of order $0 < \alpha < 1$ and type $0 \leq \beta \leq 1$, respectively and $\alpha r(\xi) < N$. An interesting observation regarding problems involving fractional operators is that the negative sign does not need to be imposed in front of the Laplacian, as this sign is included in the definition of the left fractional derivative.

Problems of the form presented in Eq.(1.1) are associated with the energy functional

$$\mathfrak{a}_{\kappa}^{\alpha,\beta;\psi}(u) = \widehat{\square}\left(\int_\Omega \frac{1}{r(\xi)}\left|\mathbf{D}_{0+}^{\alpha,\beta;\psi}u\right|^{r(\xi)}d\xi\right) - \frac{\kappa}{r+1}\left[\int_\Omega \frac{1}{r(\xi)}\left|u\right|^{r(\xi)}d\xi\right]^{\tau+1}, \tag{1.2}$$

for all $u \in \mathcal{N}_{r(\xi)}^{\alpha,\beta;\psi}(\Omega)$ and $\widehat{\square}(t) = \int_0^t \square(s)ds$
Depending on the function $r(\cdot)$ the functional $\mathfrak{a}_{\kappa}^{\alpha,\beta;\psi}(\cdot)$ is differentiable and its Fréchet derivative is given by

$$\left\langle \left(\mathfrak{a}_{\kappa}^{\alpha,\beta;\psi}\right)'(u), v \right\rangle = \square\left(\int_\Omega \frac{1}{r(\xi)}\left|\mathbf{D}_{0+}^{\alpha,\beta;\psi}u\right|^{r(\xi)}d\xi\right)\int_\Omega \frac{1}{r(\xi)}\left|\mathbf{D}_{0+}^{\alpha,\beta;\psi}u\right|^{r(\xi)-2}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u\,\mathbf{D}_{0+}^{\alpha,\beta;\psi}v\,d\xi$$
$$-\kappa\left[\int_\Omega \frac{1}{s(\xi)}\left|u\right|^{s(\xi)}u\,d\xi\right]^{r}\int_\Omega \left|u\right|^{s(\xi)-2}\mathbf{D}_{0+}^{\alpha,\beta;\psi}uv\,d\xi$$

for all $u, v \in \mathcal{N}_{r(\xi)}^{\alpha,\beta;\psi}(\Omega)$. Thus, we can observe that the critical points of the functional $\mathfrak{a}_{\kappa}^{\alpha,\beta;\psi}(\cdot)$ are the weak solutions to the problem (1.1).

Consider the $p(x)$-Laplacian operator with variable exponent [46]

$$\Delta_{p(x)} = \operatorname{div}\left(\left|\nabla u\right|^{p(x)}\nabla u\right). \tag{1.3}$$

Note that for $p(x) = p$, we have the classic case of the p-Laplacian theory [3, 4, 9, 27]. Over the years, operators of the type (1.3) have drawn the attention of researchers to discuss issues of existence and multiplicity, uniqueness and regularity of solutions using variational techniques, in particular, via the Mountain Pass theorem and Nehari manifold [1, 10, 16, 32].

It is also worth highlighting the importance of these types of operators in problems involving electrorheological fluids, image processing, non-Newtonian mechanics, medicine, economics, ecology and other areas involved [17, 33, 34]. A class of problems that deserves special attention are those of the diffusive or diffusion type. Movement of fluids and study of porous media (liquids and gases) are widely investigated in the study of diffusive and convection problems, respectively [19, 35, 45, 47].

Lebesgue spaces with variable exponent appeared in the literature for the first time in 1931, through an article by Orlicz [31], who demonstrated several results, including Holder's inequality, in a discreet presentation in which he considered the spaces $L^{p(x)}$ on the real line. Problems involving the $p(x)$-Laplacian operator are widely used in modeling problems in elasticity theory, viscous flows of non-Newtonian fluids and fluid mechanics, more precisely, electrorheological type fluids (called fluids intelligent). Another application is related to image processing. For more details, see Mihailescu and Radulescu [29], Ruzicka [36] and their references. We emphasize that the first major discovery about electrorheological fluids is attributed to Willis Winslow, in 1949. Such fluids have the important property that their viscosity depends on the electric field in the fluid. He discovered that the Viscosity of such fluids (e.g. lithium polymethacrylate) in an electric field is inversely proportional to field strength. The field induces the formation of streamlines in the fluid, which are parallel to the field. They can increase viscosity by as much as 5 orders of magnitude. This phenomenon is known as the Winslow effect. We can also highlight the interesting work carried out by Ambrosio and Isernia on parametric Schrödinger equations driven by the fractional *p*-Laplacian operator [5].

On the other hand, Kirchhoff-type problems were motivated by physics problems, that is, related to diffusion processes. In other words, in 1883 G. Kirchhoff, motivated by D'Alembert's wave equation, considered the following equation

$$\sigma \frac{\partial^2}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^t \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2}{\partial x^2} = 0$$

where σ, P_0, h, E, L they are physical constants and can be consulted in [25, 28]. Kirchhoff 's original model was used to study problems in one-dimensional cases. In this sense, over the years, Kirchhoff-type problems have attracted attention. On the other hand, we highlight the combination of Kirchhoff-type problems with the $p(x)$-Laplacian operator, in other cases, it is also possible to discuss the stationary version of the wave-type Kirchhoff equation [2, 3, 8, 9, 12–14, 18, 48].

In the literature there are several differential equation problems involving Kirchhoff problems, for example, one of the best known differential equations is the stationary equation [15]

$$\begin{cases} -\square\left(\|t\|^2\right)\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ and $\square : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions that meet certain conditions and Δ is the Laplacian operator.

In 2013, Correa and Costa [15], considered the following problem

$$\begin{cases} -\square\left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right) \Delta_{p(x)} u = \kappa |u|^{s(x)-2} u \left[\int_\Omega \frac{1}{q(x)} |u|^{q(x)} dx \right]^r & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathsf{R}^N$ is a bounded smooth domain, κ, $r > 0$, are real parameters and $M : \mathsf{R}_+ \to \mathsf{R}_+$ is a continuous function, $p, s \in C(\bar{\Omega})$.

On the other hand, we can also highlight the problem addressed by Gomes and Sanchez [24] using

$$\begin{cases} -\Delta u = \kappa f(x,u) \left[\int_\Omega F(x,u) dx \right]^r & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $F(t) = \int_\Omega f(s) ds$ and f is a regular function.

On the other hand, non-local fractional problems with infinitely many solutions have attracted much attention in recent years. Fractional operators have applications in several areas, such as financial mathematics, quantum mechanics, water waves, phase transition, minimal surfaces, population dynamics, optimal control, game theory, Lévy processes in probability theory, among others. For more details on these matters, see [6, 7, 11, 26] and its references. There are some types of fractional operators that are used to discuss $p(x)$-Laplacian problems with Kirchhoff-type equations. First, we highlight the fractional operators introduced through the Fourier transform and its similar versions, that is, via the Riesz operator. On the other hand, we have fractional derivatives which are also used to discuss $p(x)$-Laplacian problems [4, 9, 30]. The vast majority of such fractional derivatives are well established. However, it is not a simple and trivial task to choose a given derivative to formulate and discuss properties of solutions of equations of the $p(x)$-Laplacian type. One way to overcome this problem is to propose more general operators where those existing in the literature are particular cases. In this sense, Sousa and Oliveira proposed the ψ-Hilfer fractional operator [37] and discussed some properties. The ψ-Hilfer fractional operator is well established with numerous properties, works and applications. In this sense, since 2018, the ψ-Hilfer fractional operator has attracted attention in helping to discuss theoretical problems involving problems of the type p, $p(x)$ Laplacian and Kirchhoff [38–40].

Recently Sousa et al. [38], considered a new class of equations of the $p(x)$-Kirchhoff type of the form

$$\begin{cases} \square\left(\displaystyle\int_\Omega \frac{1}{p(x)} \left|\mathrm{D}_{0+}^{\alpha,\beta;\psi} u\right|^{p(x)} dx\right) \mathbf{L}_{p(x)}^{\alpha,\psi} u = g(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \qquad (1.7)$$

where $\Omega=[0,T]\times[0,T]$ and $\mathbf{L}_{p(x)}^{\alpha,\beta;\psi} u := \mathrm{D}_{T}^{\alpha,\beta;\psi}\left(\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi} u\right|^{p(x)-2} \mathrm{D}_{0+}^{\alpha,\beta;\psi} u\right)$. In addition to the above, there are interesting works whose authors use fractional operators to formulate problems of elliptic differential equations, whether of the $p(x)$-Laplacian, p-Laplacian and Kirchhoff type, and discuss issues of existence and multiplicity of solutions to such formulated problems [41–43] and the references therein.

Motivated by the problems (1.4)–(1.7), in this paper, we will consider the existence of solutions for a new class of fractional elliptic problems involving Kirchhoff-type equations (see problem (1.1)). In other words, we are interested in discussing the following results:

Theorem 1.1. *Suppose that* $1<\alpha r(\xi)<N$ *with* $1<s(\xi)<r_\alpha^*(\xi)=\dfrac{Nr(\xi)}{N-\alpha r(\xi)}$ *for all* $\xi\in\overline{\Omega}$ *and* $0<\alpha<1$. *Furthermore, assume that there exists positive* ρ_0 *and* v_1 *such that* $\rho_0 \leq \mathrm{M}(t) \leq v_1$, *with* $\dfrac{\upsilon_1 r^+}{\rho_0} < \dfrac{(s^-)^{\tau+1}(\tau+1)}{(s^+)^r}$ *and* $(s^-)(\tau+1)>r^+$. *Then the problem* (*1.1*) *has a weak solution for all* $\kappa > 0$.

Theorem 1.2. *Suppose that* $1 < \alpha r(\xi) < N$ *with* $1<s(\xi)<r_\alpha^*(\xi)=\dfrac{Nr(\xi)}{N-\alpha r(\xi)}$ *for all* $\xi\in\Omega$ *and* $0 < \alpha < 1$. *Furthermore, assume that there exists* $0 < \rho_0$ *such that* $\rho_0 < \mathrm{M}(t) < v_1$.

If $(\tau+1)s^- < r^-$, *then there exists* $\kappa^* > 0$ *such that the problem* (1.1) *has a positive solution* u_κ *for each* $\kappa \in (0, \kappa^*)$.

Theorem 1.3. *Suppose that* $1 < \alpha r(\xi) < N$ *with* $1 < s(\xi) < r^*_\alpha(\xi) = \frac{Nr(\xi)}{N-\alpha r(\xi)}$ *for all* $\xi \in \Omega$ *and* $0 < \alpha < 1$. *Furthermore, assume that* $M(t) = a + bt$, *where* $a > 0$, $b > 0$ *and* $t \geq 0$. *If* $\frac{2(r^+)^2}{r^-} < \frac{(\tau+1)(s^-)^{\tau+1}}{(s^+)^r}$ *and* $(\tau+1)s^- > 2r^+$, *then the problem* (1.1) *has a weak positive solution for all* $\kappa > 0$.

Theorem 1.4. *Suppose that* $1 < \alpha r(\xi) < N$ *with* $1 < s(\xi) < r^*_\alpha(\xi) = \frac{Nr(\xi)}{N-\alpha r(\xi)}$ *for all* $\xi \in \overline{\Omega}$ *and* $0 < \alpha < 1$ *and* $M(t) = a + bt$, *where* $a > 0$, $b > 0$ *and* $t \geq 0$. *If* $(\tau+1)s^- < r^-$, *then there exists* $\kappa^* > 0$ *such that the problem* (1.1) *has a positive solution* u_κ *for each* $\kappa \in (0, \kappa^*)$.

Theorem 1.5. *Suppose that* $1 < \alpha r(\xi) < N$ *with* $1 < s(\xi) < r^*_\alpha(\xi) = \frac{Nr(\xi)}{N-\alpha r(\xi)}$ *for all* $\xi \in \overline{\Omega}$ *and* $0 < \alpha < 1$. *Let* $M(t) = t^{\mu-1}$ *such that* $\frac{\mu(r^+)^\mu}{(r^-)^{\mu-1}} < \frac{(s^-)^{\tau+1}(\tau+1)}{(s^+)^r}$ *and* $(\tau+1)s^- > \mu r^+ \geq \mu r^- > 1$. *Then the problem* (1.1) *has a weak positive solution for all* $\kappa > 0$.

Theorem 1.6. *Suppose that* $1 < \alpha r(\xi) < N$ *with* $1 < s(\xi) < r^*_\alpha(\xi) = \frac{Nr(\xi)}{N-\alpha r(\xi)}$ *for all* $\xi \in \overline{\Omega}$ *and* $0 < \alpha < 1$. *Let* $M(t) = t^{\mu-1}$. *If* $(\tau+1)s^- < \mu r^-$, *then there exists* $\kappa^* > 0$ *such that the problem* (1.1) *has a positive solution* u_κ *for each* $\kappa \in (0, \kappa^*)$.

A natural consequence of the study of fractional operators is, in the limit of the operator fractional order, to obtain the classical case, in this particular case, when $\alpha \to 1$. This procedure is repeated when the operator order is $1 < \alpha < 2$, within the limit of $\alpha \to 2$, the entire case is obtained, carrying out this procedure for the operator order $n - 1 < \alpha < n$ in the limit of $\alpha \to n$**,** we obtain the entire case. Furthermore, there are other more general operators that preserve all the properties of their respective particular cases. As highlighted in the introduction, the ψ-Hilfer fractional operator has a wide class of fractional operators based on the choice of parameters α, β and the function $\psi(\cdot)$. Furthermore, the problem addressed (1.1) also allows the choice of the function $M(\zeta)$. On the other hand, it is natural to expect that the particular cases of the results investigated here still continue to be valid, and in fact they are, since all the properties of the ψ-Hilfer fractional operator are preserved for its wide class of particular cases.

Otherwise, the paper is organized as follows: In Sec. 2, we present some classic concepts about fractional operators and properties related to them. In addition to the above, the ψ-fractional space with its respective norm and some results involving the functional $A^{\alpha,\beta;\psi}$ are investigated. In Sec. 3, it is intended to discuss the main contributions of the present article, i.e., the existence of solutions through the proof of Theorem 1.1–Theorem 1.2.

2. A Brief Framework and Preliminary Results

Let $\theta \in \{\theta_1, \theta_2\}$, $T \in \{T_1, T_2\}$ and $\alpha \in \{\alpha_1, \alpha_2\}$ where $0 < \alpha_j < 1$ with $\theta_j < T_j$, for all $j \in \{1, 2\}$. Also take $\Lambda = I_1 \times I_2 = [\theta_1, T_1] \times [\theta_2, T_2]$, where T_1, T_2 and θ_1, θ_2 are positive constants. Consider also $\psi(\cdot)$ an increasing, positive, monotone function on (θ_1, T_1), (θ_2, T_2), with continuous derivative $\psi'(\cdot)$ in $(\theta_1, T_1]$, $(\theta_2, T_2]$.

Let $u, \psi \in C^n(\Lambda)$ be two functions such that ψ is increasing and $\psi'(\xi_j) \neq 0$ with $\xi_j \in [\theta_j, T_j]$, $j \in \{1, 2\}$. The ψ-Hilfer partial fractional derivatives at two variables of $u, \psi \in AC^n(\Lambda)$ of order $\alpha \in \{\alpha_1, \alpha_2\}$ $(0 < \alpha_j < 1)$ and of type $\beta \in \{\beta_1, \beta_2\}$ where $0 \leq \beta_j \leq 1$, are defined by (see [37])

$$\mathsf{D}_{\theta}^{\alpha,\beta;\psi} u(\xi_1,\xi_2) = \mathbf{I}_{\theta}^{\beta(1-\alpha),\psi}\left(\frac{1}{\psi'(\xi_1)\psi'(\xi_2)}\left(\frac{\partial^2}{\partial\xi_1\partial\xi_2}\right)\right)\mathbf{I}_{\theta}^{(1-\beta)(1-\alpha),\psi} u(\xi_1,\xi_2)$$

and

$$\mathsf{D}_{T}^{\alpha,\beta;\psi} u(\xi_1,\xi_2) = \mathbf{I}_{T}^{\beta(1-\alpha),\psi}\left(-\frac{1}{\psi'(\xi_1)\psi'(\xi_2)}\left(\frac{\partial^2}{\partial\xi_1\partial\xi_2}\right)\right)\mathbf{I}_{T}^{(1-\beta)(1-\alpha),\psi} u(\xi_1,\xi_2),$$

where $\mathbf{I}_{\theta}^{\alpha,\psi} u(\xi_1,\xi_2)$ and $\mathbf{I}_{T}^{\alpha,\psi} u(\xi_1,\xi_2)$ are the ψ-Riemann-Liouville fractional integrals of $u \in L^1(\Lambda)$ of order α $(0 < \alpha < 1)$ given by

$$\mathbf{I}_{\theta}^{\alpha,\psi} u(\xi_1,\xi_2) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_{\theta_1}^{\xi_1}\int_{\theta_2}^{\xi_2}\psi'(s_1)\psi'(s_2)(\psi(\xi_1)-\psi(s_1))^{\alpha_1-1}(\psi(\xi_2)-\psi(s_2))^{\alpha_2-1}u(s_1,s_2)\,ds_2\,ds_1,$$

for $\theta_1 < s_1 < \xi_1$, $\theta_2 < s_2 < \xi_2$ and

$$\mathbf{I}_{T}^{\alpha,\psi} u(\xi_1,\xi_2) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_{\xi_1}^{T_1}\int_{\xi_2}^{T_2}\psi'(s_1)\psi'(s_2)(\psi(s_1)-\psi(\xi_1))^{\alpha_1-1}(\psi(s_2)-\psi(\xi_2))^{\alpha_2-1}u(s_1,s_2)\,ds_2\,ds_1,$$

with $\xi_1 < s_1 < T_1$, $\xi_2 < s_2 < T_2$, $\xi_1 \in [\theta_1, T_1]$ and $\xi_2 \in [\theta_2, T_2]$. In this sense, it extends to N variables and can be consulted in the work [44].

Let $\theta \in \{\theta_1, \theta_2\}$, $T \in \{T_1, T_2\}$ and $\alpha \in \{\alpha_1, \alpha_2\}$. Then, we have the integration by parts for the fractional integral given by (see [38])

$$\int_{\theta_1}^{T_1}\int_{\theta_2}^{T_2}\left(\mathbf{I}_{\theta}^{\alpha;\psi}\varphi(\xi_1,\xi_2)\right)\phi(\xi_1,\xi_2)\,d\xi_2\,d\xi_1 = \int_{\theta_1}^{T_1}\int_{\theta_2}^{T_2}\varphi(\xi_1,\xi_2)\psi'(\xi_1)\psi'(\xi_2)\mathbf{I}_{T}^{\alpha;\psi}\left(\frac{\phi(\xi_1,\xi_2)}{\psi'(\xi_1)\psi'(\xi_2)}\right)d\xi_2\,d\xi_1 \quad (2.1)$$

is holds.

Theorem 2.1. [38] *Let* $\psi(\cdot)$ be *an increasing, monotone and positive function in* $[\theta_1, T_1] \times [\theta_2, T_2]$, with *continuous derivative* $\psi'(\cdot) \neq 0$ *on* $(\theta_1, T_1) \times (\theta_2, T_2)$. *If* $\alpha \in \{\alpha_1, \alpha_2\}$ $(0 < \alpha_j < 1)$ *and* $\beta \in \{\beta_1, \beta_2\}$, $0 \leq \beta_j \leq 1$, *then*

$$\begin{aligned}\int_{\theta_1}^{T_1}\int_{\theta_2}^{T_2}&\left(\mathsf{D}_{\theta}^{\alpha,\beta;\psi}\varphi(\xi_1,\xi_2)\right)\phi(\xi_1,\xi_2)\,d\xi_2\,d\xi_1\\ &= \int_{\theta_1}^{T_1}\int_{\theta_2}^{T_2}\varphi(\xi_1,\xi_2)\psi'(\xi_1)\psi'(\xi_2)\,\mathsf{D}_{T}^{\alpha,\beta;\psi}\left(\frac{\phi(\xi_1,\xi_2)}{\psi'(\xi_1)\psi'(\xi_2)}\right)d\xi_2\,d\xi_1\end{aligned}$$

for any $\varphi \in C^1$ and $\phi \in C^1$ *satisfying the boundary conditions* $\varphi(\theta_1, \theta_2) = 0 = \varphi(T_1, T_2)$.

To discuss the Problem (1.1) we need some knowledge about the spaces $L^{r(\xi)}(\Omega)$ and $N^{\alpha,\beta;\psi}_{r(\xi)}(\Omega)$ which we call generalized ψ-fractional space.

Let Ω be a bounded domain of $\mathbb{R}^2$, denote [1,32]

$$C_+(\overline{\Omega}) = \{ r(\xi); r(\xi) \in C(\overline{\Omega}), r(\xi) > 1 \text{ for all } \xi \in \overline{\Omega}\}$$

and

$$r^- = \inf_{\Omega} r(\xi) \le r(\xi) \le r^+ = \sup_{\Omega} r(\xi) < 2.$$

For any $m \in C_+(\overline{\Omega})$, we introduce the Lebesgue space with variable exponent

$$L^{r(\cdot)}(\Omega) = \{ u; u \text{ is a real measurable function such that } \int_\Omega |u(\xi)|^{r(\xi)}\, d\xi < \infty\}$$

endowed with the so-called Luxembourg standard

$$\|u\|_{L^{r(\xi)}} = |u|_{r(\cdot)} = \inf\left\{ \alpha > 0 : \int_\Omega \left|\frac{u(\xi)}{\alpha}\right|^{r(\xi)} d\xi \le 1 \right\},$$

which is a separable Banach space and reflexive.

Proposition 2.2. [1,32] The *space* $(L^{r(\xi)}(\Omega), |\cdot|_{r(\xi)})$ is *separable, uniformly convex, reflexive and its conjugate space is* $(L^{r(\xi)}(\Omega), |\cdot|_{s(\xi)})$ *where* $s(\xi)$ *is the conjugate function of* $r(\xi)$, *i.e.*

$$\frac{1}{r(\xi)} + \frac{1}{s(\xi)} = 1, \text{ for all } \xi \in \Omega.$$

For all $u \in L^{r(\xi)}(\Omega)$, $v \in L^{s(\xi)}(\Omega)$, *the Hölder type inequality*

$$\left|\int_\Omega uv\, d\xi\right| \le \left(\frac{1}{r^-} + \frac{1}{s^-}\right) |u|_{r(\xi)} |v|_{s(\xi)}$$

is valid.

An important role in work with generalized Lebesgue-Sobolev spaces is played by the $r(\cdot)$- modular of the space $L^{r(\cdot)}(\Omega)$ which is the modular $\sigma_r(\cdot)$ of space $L^{r(\cdot)}(\Omega)$

$$\sigma_{r(\cdot)}(u) := \int_\Omega |u|^{r(\xi)}\, d\xi.$$

Lemma 2.3. [20-22] *Suppose* u_n, $u \in L^{r(\cdot)}$ *and p- < +∞. So, the following properties apply:*

1. For $u \neq 0$, $|u|_{r(\xi)} = \kappa \Leftrightarrow \sigma\left(\frac{u}{\kappa}\right) = 1$;
2. $|u|_{r(\cdot)} < 1 (= 1; > 1) \Leftrightarrow \sigma(u) < 1 (= 1; > 1)$;
3. $|u|_{r(\cdot)} > 1 \Rightarrow |u|^{r^-}_{r(\cdot)} \le \sigma_{r(\cdot)}(u) \le |u|^{r^+}_{r(\cdot)}$;
4. $|u|_{r(\cdot)} < 1 \Rightarrow |u|^{r^+}_{r(\cdot)} \le \sigma_{r(\cdot)}(u) \le |u|^{r^-}_{r(\cdot)}$;
5. $|u_n|_{r(\cdot)} \to 0$ (respectively $\to +\infty$) $\Leftrightarrow \sigma_{r(\cdot)}(u_n) \to 0$; (respectively $\to +\infty$);

6. $\lim\limits_{n\to\infty}|u_n - u|_{r(\cdot)} = 0 \Leftrightarrow \lim\limits_{n\to\infty}\sigma_{r(\cdot)}(u_n - u) = 0$;

7. $\lim\limits_{k\to+\infty}|u_k|_{r(\xi)} = +\infty \Leftrightarrow \lim\limits_{k\to+\infty}\sigma(u_k) = +\infty.$

Corollary 2.4. [23] *Let E be a Banach space and let $I : E \to \mathbb{R}$ be a functional of class C^1 that is lower bound. If I satisfies condition $(PS)_c$ with $c = \inf_{u\in E} I(u)$, then c is reached at a point $u_0 \in E$ and u_0 is the critical point of I.*

Theorem 2.5. [23](Vainberg) Let (f_n) *be a sequence of functions in $L^r(\Omega)$ such that $f_n \to f$ in $L^r(\Omega)$. Then there exists a subsequence $(f_{n_j}) \subset (f_n)$ such that:*

(A) *$f_{n_j}(\xi) \to f_n(\xi)$, a.e. in Ω.*
(B) *There exists $h \in L^r(\Omega)$ such that $|f_{n_j}(\xi)| \le h(\xi)$ for $j \in \mathbb{N}$.*

The ψ-fractional space with variable exponent $\mathcal{N}^{\alpha,\beta;\psi}_{r(\xi)}(\Omega)$ is defined by [44]

$$\mathcal{N}^{\alpha,\beta;\psi}_{r(\xi)}(\Omega) = \left\{ u \in L^{r(\cdot)}(\Omega), \left|\mathbf{D}^{\alpha,\beta;\psi}_{0+} u\right| \in L^{r(\cdot)}(\Omega) \right\}$$

and is equipped with norm

$$\|u\|_{\mathcal{N}^{\alpha,\beta;\psi}_{r(\xi)}} = \|u\|_{r(\xi)} + \left\|\mathbf{D}^{\alpha,\beta;\psi}_{0+} u\right\|_{r(\xi)}$$

Then $\mathcal{N}^{\alpha,\beta;\psi}_{r(\xi),0}(\Omega)$ is defined as the closure of $\mathcal{N}^{\alpha,\beta;\psi}_{r(\xi)}(\Omega)$ in $C^{\infty}_0(\Omega)$ with respect to the norm $\|u\|_{\mathcal{N}^{\alpha,\beta;\psi}_{r(\xi)}}\cdot$ In this sense, $L^{r(\xi)}(\Omega)$, $\mathcal{N}^{\alpha,\beta;\psi}_{r(\xi),0}(\Omega)$ and $\mathcal{N}^{\alpha,\beta;\psi}_{r(\xi)}(\Omega)$ become separable and reflexive Banach spaces.

Proposition 2.6. *The functional* $\mathbf{a}^{\alpha,\beta;\psi}_{\kappa,1}(u) = \square\left(\int_\Omega \frac{1}{r(\xi)}\left|\mathbf{D}^{\alpha,\beta;\psi}_{0+} u\right|^{r(\xi)} d\xi\right)$ *is of class C^1 in* $\mathcal{N}^{\alpha,\beta;\psi}_{r(\xi)}(\Omega)$.

Proof. The proof of this result will be discussed in two steps.

Step 1: $\mathbf{a}^{\alpha,\beta;\psi}_{\kappa,1}$ is differentiable.
Consider the function $G : \mathbb{R} \to \mathbb{R}$ given by $G(s) = \left|\mathbf{D}^{\alpha,\beta;\psi}_{0+} u + st\, \mathbf{D}^{\alpha,\beta;\psi}_{0+} v\right|^{r(\xi)}$, where u,v $\in \mathcal{N}^{\alpha,\beta;\psi}_{r(\xi)}(\Omega)$. Given $\xi \in \Omega$ and $0 < |t| < 1$, there exists $\gamma(\xi, t) = \gamma \in (0, 1)$ such that

$$\frac{G(1) - G(0)}{1} = G'(\gamma).$$

In this sense, yields that

$$\frac{\left|\mathbf{D}^{\alpha,\beta;\psi}_{0+} u + t\, \mathbf{D}^{\alpha,\beta;\psi}_{0+} v\right|^{r(\xi)} - \left|\mathbf{D}^{\alpha,\beta;\psi}_{0+} u\right|^{r(\xi)}}{t\, \mathbf{D}^{\alpha,\beta;\psi}_{0+} v} = r(\xi)\left|\mathbf{D}^{\alpha,\beta;\psi}_{0+} u + \gamma t\, \mathbf{D}^{\alpha,\beta;\psi}_{0+} v\right|^{r(\xi)-1}.$$

What does it imply

$$\frac{\left|\mathbf{D}^{\alpha,\beta;\psi}_{0+} u + t\, \mathbf{D}^{\alpha,\beta;\psi}_{0+} v\right|^{r(\xi)} - \left|\mathbf{D}^{\alpha,\beta;\psi}_{0+} u\right|^{r(\xi)}}{r(\xi) t}$$
$$= \left|\mathbf{D}^{\alpha,\beta;\psi}_{0+} u + \gamma t\, \mathbf{D}^{\alpha,\beta;\psi}_{0+} v\right|^{r(\xi)-2}\left(\mathbf{D}^{\alpha,\beta;\psi}_{0+} u + \gamma t\, \mathbf{D}^{\alpha,\beta;\psi}_{0+} v\right)\mathbf{D}^{\alpha,\beta;\psi}_{0+} v.$$

Note that

$$\phi=\left|\mathsf{D}^{\alpha,\beta;\psi}_{0+}u+\gamma t\,\mathsf{D}^{\alpha,\beta;\psi}_{0+}v\right|^{r(\xi)-2}\left(\mathsf{D}^{\alpha,\beta;\psi}_{0+}u+\gamma t\,\mathsf{D}^{\alpha,\beta;\psi}_{0+}v\right)\mathsf{D}^{\alpha,\beta;\psi}_{0+}v$$

$$\to\left|\mathsf{D}^{\alpha,\beta;\psi}_{0+}u\right|^{r(\xi)-2}\mathsf{D}^{\alpha,\beta;\psi}_{0+}u\,\mathsf{D}^{\alpha,\beta;\psi}_{0+}v\ \text{ a.e.in }\ \Omega \tag{2.2}$$

when $t\to 0$. In addition to the above, we have

$$\phi\le\left|\mathsf{D}^{\alpha,\beta;\psi}_{0+}u+\gamma t\,\mathsf{D}^{\alpha,\beta;\psi}_{0+}v\right|^{r(\xi)-1}\left|\mathsf{D}^{\alpha,\beta;\psi}_{0+}v\right|\le\left(\left|\mathsf{D}^{\alpha,\beta;\psi}_{0+}u\right|+\left|\mathsf{D}^{\alpha,\beta;\psi}_{0+}v\right|\right)^{r(\xi)-1}\left|\mathsf{D}^{\alpha,\beta;\psi}_{0+}v\right|. \tag{2.3}$$

Using the fact that

$$\left|\mathsf{D}^{\alpha,\beta;\psi}_{0+}u\right|,\left|\mathsf{D}^{\alpha,\beta;\psi}_{0+}v\right|\in L^{r(\xi)}(\Omega)$$

we have

$$\left(\left|\mathsf{D}^{\alpha,\beta;\psi}_{0+}u\right|+\left|\mathsf{D}^{\alpha,\beta;\psi}_{0+}v\right|\right)^{r(\xi)-1\ \frac{r(\xi)}{\ }}\in L^{r(\xi)-1}(\Omega).$$

Finally, using Hölder's inequality, it's follows that

$$\left(\left|\mathsf{D}^{\alpha,\beta;\psi}_{0+}u\right|+\left|\mathsf{D}^{\alpha,\beta;\psi}_{0+}v\right|\right)^{r(\xi)-1}\left|\mathsf{D}^{\alpha,\beta;\psi}_{0+}v\right|\in L^{1}(\Omega). \tag{2.4}$$

Using Eq.(2.2)–Eq.(2.4) and applying Lebesgue's Dominated Convergence Theorem, we obtain

$$\lim_{t\to 0}\int_{\Omega}\frac{\left|\mathsf{D}^{\alpha,\beta;\psi}_{0+}(u+tv)\right|^{r(\xi)}-\left|\mathsf{D}^{\alpha,\beta;\psi}_{0+}u\right|^{r(\xi)}}{r(\xi)t}d\xi=\int_{\Omega}\left|\mathsf{D}^{\alpha,\beta;\psi}_{0+}u\right|^{r(\xi)-2}\mathsf{D}^{\alpha,\beta;\psi}_{0+}u\,\mathsf{D}^{\alpha,\beta;\psi}_{0+}v\,d\xi.$$

Using the Chain rule, we have

$$\left(\mathrm{a}^{\alpha,\beta;\psi}_{\kappa,1}\right)'(u)v=\left[\square\left(\int_{\Omega}\frac{1}{r(\xi)}\left|\mathsf{D}^{\alpha,\beta;\psi}_{0+}u\right|^{r(\xi)}d\xi\right)\right]'\lim_{t\to 0}\int_{\Omega}\frac{\left|\mathsf{D}^{\alpha,\beta;\psi}_{0+}(u+tv)\right|^{r(\xi)}-\left|\mathsf{D}^{\alpha,\beta;\psi}_{0+}u\right|^{r(\xi)}}{r(\xi)t}d\xi.$$

In this sense, it follows that

$$\left(\mathrm{a}^{\alpha,\beta;\psi}_{\kappa,1}\right)'(u)v=\square\left(\int_{\Omega}\frac{1}{r(\xi)}\left|\mathsf{D}^{\alpha,\beta;\psi}_{0+}u\right|^{r(\xi)}d\xi\right)\int_{\Omega}\left|\mathsf{D}^{\alpha,\beta;\psi}_{0+}u\right|^{r(\xi)-2}\mathsf{D}^{\alpha,\beta;\psi}_{0+}u\,\mathsf{D}^{\alpha,\beta;\psi}_{0+}v\,d\xi,$$

and hence $\mathrm{a}^{\alpha,\beta;\psi}_{\kappa,1}$ is Gateaux-differentiable.

Step 2: $\left(\mathrm{a}^{\alpha,\beta;\psi}_{\kappa,1}\right)'(u)$ is continuous in $\mathrm{N}^{\alpha,\beta;\psi}_{r(\xi),0}(\Omega)$.

Consider the sequence $(u_j)\subset\mathrm{N}^{\alpha,\beta;\psi}_{r(\xi),0}(\Omega)$ such that $u_j\to u$ in $\mathrm{N}^{\alpha,\beta;\psi}_{r(\xi),0}(\Omega)$ and $v\in\mathrm{N}^{\alpha,\beta;\psi}_{r(\xi),0}(\Omega)$ with $||v||\le 1$. Thus,

$$\mathsf{D}^{\alpha,\beta;\psi}_{0+}u_j\to\mathsf{D}^{\alpha,\beta;\psi}_{0+}u\ \text{ in }\left(L^{r(\xi)}(\Omega)\right)^{N}.$$

Using Vainberg Theorem, it's follows that

$$\mathsf{D}^{\alpha,\beta;\psi}_{0+}u_j(\xi)\to\mathsf{D}^{\alpha,\beta;\psi}_{0+}u(\xi)\ \text{ a.e. in }\ \Omega \tag{2.5}$$

and

$$\left|\mathbf{D}^{\alpha,\beta;\psi}_{0+}u_j(\xi)\right| \le g(\xi) \text{ a.e. in } \Omega, \tag{2.6}$$

where $g(\xi)\in L^{r(\xi)}(\Omega)$. Using Eq.(2.5), yields

$$\left|\mathbf{D}^{\alpha,\beta;\psi}_{0+}u_j(\xi)\right| \to \left|\mathbf{D}^{\alpha,\beta;\psi}_{0+}u(\xi)\right| \text{ a.e. in } \Omega. \tag{2.7}$$

In order to facilitate the development of calculations, consider the following notation $\Theta^{\alpha,\beta}_{\psi}(u)=\int_{\Omega}\frac{1}{r(\xi)}\left|\mathbf{D}^{\alpha,\beta;\psi}_{0+}u\right|^{r(\xi)}d\xi$. So, we have

$$\begin{aligned}
&\left|\left\langle \left(\Theta^{\alpha,\beta}_{\psi}\right)'(u_j)-\left(\Theta^{\alpha,\beta}_{\psi}\right)'(u),\nu \right\rangle\right| \\
&=\left|\int_{\Omega}\left(\left|\mathbf{D}^{\alpha,\beta;\psi}_{0+}u_j\right|^{r(\xi)-2}\mathbf{D}^{\alpha,\beta;\psi}_{0+}u_j-\left|\mathbf{D}^{\alpha,\beta;\psi}_{0+}u\right|^{r(\xi)-2}\mathbf{D}^{\alpha,\beta;\psi}_{0+}u\right)\mathbf{D}^{\alpha,\beta;\psi}_{0+}\nu\, d\xi\right| \\
&\le \int_{\Omega}\left|\left|\mathbf{D}^{\alpha,\beta;\psi}_{0+}u_j\right|^{r(\xi)-2}\mathbf{D}^{\alpha,\beta;\psi}_{0+}u_j-\left|\mathbf{D}^{\alpha,\beta;\psi}_{0+}u\right|^{r(\xi)-2}\mathbf{D}^{\alpha,\beta;\psi}_{0+}u\right|\left|\mathbf{D}^{\alpha,\beta;\psi}_{0+}\nu\right|d\xi.
\end{aligned} \tag{2.8}$$

On the other hand, choosing

$$f_j=\left|\left|\mathbf{D}^{\alpha,\beta;\psi}_{0+}u_j\right|^{r(\xi)-2}\mathbf{D}^{\alpha,\beta;\psi}_{0+}u_j-\left|\mathbf{D}^{\alpha,\beta;\psi}_{0+}u\right|^{r(\xi)-2}\mathbf{D}^{\alpha,\beta;\psi}_{0+}u\right|,\ j\in\mathbb{N},$$

we obtain

$$f_j \le \left|\mathbf{D}^{\alpha,\beta;\psi}_{0+}u_j\right|^{r(\xi)-1}+\left|\mathbf{D}^{\alpha,\beta;\psi}_{0+}u\right|^{r(\xi)-1}\in L^{\frac{r(\xi)}{r(\xi)-1}}(\Omega). \tag{2.9}$$

Thus, we have that $f_j\in L^{\frac{r(\xi)}{r(\xi)-1}}(\Omega)$. Using Hölder's inequality in the inequality (2.8), we obtain

$$\left|\left\langle \left(\Theta^{\alpha,\beta}_{\psi}\right)'(u_j)-\left(\Theta^{\alpha,\beta}_{\psi}\right)'(u),\nu \right\rangle\right| \le C\left|f_j\right|_{\frac{r(\xi)}{r(\xi)-1}}\left|\mathbf{D}^{\alpha,\beta;\psi}_{0+}\nu\right| \le C\left|f_j\right|_{\frac{r(\xi)}{r(\xi)-1}}\|\nu\|.$$

Therefore, we get

$$\left\|\left(\Theta^{\alpha,\beta}_{\psi}\right)'(u_j)-\left(\Theta^{\alpha,\beta}_{\psi}\right)'(u)\right\| \le C\,|f_j|_{\frac{r(\xi)}{r(\xi)-1}}.$$

Using (2.5) and (2.7)

$$f_j\to 0 \text{ a.e. in } \Omega. \tag{2.10}$$

From (2.6) and (2.9), yields that

$$f_j\le g(\xi)^{r(\xi)-1}+\left|\mathbf{D}^{\alpha,\beta;\psi}_{0+}u(\xi)\right|^{r(\xi)-1} \text{ a.e. in } \Omega.$$

Thus, it's follows that

$$f_j(\xi)^{s(\xi)}\le 2^{s^+}\left(g(\xi)^{r(\xi)}+\left|\mathbf{D}^{\alpha,\beta;\psi}_{0+}u(\xi)\right|^{r(\xi)}\right) \text{ a.e. in } \Omega, \tag{2.11}$$

where $s(\xi)=\dfrac{r(\xi)}{r(\xi)-1}$. Using Eq.(2.10) and Eq.(2.11) and Lebesgue's Dominated Convergence Theorem, we obtain

$$\int_\Omega f_j^q(\xi)d\xi\to 0 \text{ when } j\to\infty.$$

Therefore, by Lemma 2.3 we have that $|f_j|_{s(\xi)}\to 0$ when $j\to\infty$. Therefore,

$$\left\|\left(\Theta_\psi^{\alpha,\beta}\right)'(u_j)-\left(\Theta_\psi^{\alpha,\beta}\right)'(u)\right\|\to 0 \text{ when } j\to\infty.$$

Therefore, we conclude that the Gâteaux derivative $\left(\Theta_\psi^{\alpha,\beta}\right)'$ is continuous, which implies that $a_{\kappa,1}^{\alpha,\beta;\psi}(\cdot)$ is of class $C^1\left(N_{p(\xi),0}^{\alpha,\beta;\psi}(\Omega),\mathbb{R}\right)$.

Proposition 2.7. The *functional* $a_{\kappa,2}^{\alpha,\beta;\psi}(u)=\dfrac{\kappa}{r+1}\left[\int_\Omega\dfrac{1}{s(\xi)}|u|^{s(\xi)}d\xi\right]^{\tau+1}$ *is of class* $C^1\left(N_{r(\xi),0}^{\alpha,\beta;\psi}(\Omega),\mathbb{R}\right)$.

Proof. Firstly, note that

$$\left(a_{\kappa,2}^{\alpha,\beta;\psi}\right)'(u)v=\kappa\frac{r+1}{r+1}\left[\int_\Omega\frac{1}{s(\xi)}|u|^{s(\xi)}d\xi\right]^r\left(\int_\Omega\frac{1}{s(\xi)}|u|^{s(\xi)}d\xi\right)'.$$

Analogously to the idea $a_{\kappa,1}^{\alpha,\beta;\psi}(\cdot)$, we have that

$$\left(\int_\Omega\frac{1}{s(\xi)}|u|^{s(\xi)}d\xi\right)'=\int_\Omega|u|^{s(\xi)-2}uvd\xi.$$

Like this,

$$\left(a_{\kappa,2}^{\alpha,\beta;\psi}\right)'(u)v=\kappa\left[\int_\Omega\frac{1}{s(\xi)}|u|^{s(\xi)}d\xi\right]^r\int_\Omega|u|^{s(\xi)-2}uvd\xi.$$

Let's show that $\left(a_{\kappa,2}^{\alpha,\beta;\psi}\right)'$ is continuous.

Let $u_n\to u$ in $N_{r(\xi),0}^{\alpha,\beta;\psi}(\Omega)$, i.e., $(u_n-u)\to 0$ in $N_{r(\xi),0}^{\alpha,\beta;\psi}(\Omega)$. For all $v\in N_{r(\xi),0}^{\alpha,\beta;\psi}(\Omega)$ with $\|v\|\le 1$, we get

$$\left|\left(a_{\kappa,2}^{\alpha,\beta;\psi}\right)'(u_n-u)\nu\right|=\left|\kappa\left[\int_\Omega\frac{1}{s(\xi)}|(u_n-u)|^{s(\xi)},d\xi\right]^r\int_\Omega|u_n-u|^{s(\xi)-2}(u_n-u)\nu\, d\xi\right|$$
$$\le\frac{\kappa}{(s^-)^r}\left[\int_\Omega|(u_n-u)|^{s(\xi)}d\xi\right]^r\int_\Omega|u_n-u|^{s(\xi)-1}\nu\, d\xi.$$

Like $(u_n-u)\in L^{s(\xi)}$, we have that $|u_n-u|^{s(\xi)-1}\in L^{\frac{s(\xi)}{s(\xi)-1}}$. Now using Hölder's inequality, we obtain

$$\frac{\kappa}{(s^-)^r}\left[\int_\Omega|(u_n-u)|^{s(\xi)}d\xi\right]^r\int_\Omega|u_n-u|^{s(\xi)-1}\nu\, d\xi\le\frac{\kappa}{(s^-)^r}[\sigma(u_n-u)]^r|u_n-u|_{r(\xi)}|\nu|_{s(\xi)},$$

where $r(\xi)=\frac{s(\xi)}{s(\xi)-1}$. In this sense, through continuous Sobolev embeddings, there exists $C>0$ such that

$$\frac{\kappa}{(s^-)^r}[\sigma(u_n-u)]^r|u_n-u|_{r(\xi)}|v|_{s(\xi)}\leq\frac{\kappa}{(s^-)^r}[\sigma(u_n-u)]^r C\|u_n-u\|\|v|_{s(\xi)}.$$

Using the fact that $(u_n-u)\to 0$ in $\mathcal{N}^{\alpha,\beta;\psi}_{r(\xi),0}(\Omega)$, we have that $\|u_n-u\|\to 0$, which is equivalent to $\left|\mathrm{D}^{\alpha,\beta;\psi}_{0+}u_n-\mathrm{D}^{\alpha,\beta;\psi}_{0+}u\right|_{r(\xi)}\to 0$. Using Poincaré's inequality, there exists $C'>0$ such that

$$|u_n-u|_{r(\xi)}\leq C'\left|\mathrm{D}^{\alpha,\beta;\psi}_{0+}u_n-\mathrm{D}^{\alpha,\beta;\psi}_{0+}u\right|_{r(\xi)}\to 0.$$

Therefore, $|u_n-u|_{r(\xi)}\to 0$ and then $\sigma|u_n-u|\to 0$. Therefore, we conclude that

$$\left\|\left(\mathrm{a}^{\alpha,\beta;\psi}_{\kappa,2}\right)'(u_n)-\left(\mathrm{a}^{\alpha,\beta;\psi}_{\kappa,2}\right)'(u)\right\|\to 0\ \text{ when }\ n\to\infty.$$

Next, we have the definition of a weak solution to the problem (1.1).

Definition 2.8. A *function* $u\in\mathcal{N}^{\alpha,\beta;\psi}_{r(\xi),0}(\Omega)$ *is a weak solution of problem* (1.1) *if*

$$\square\left(\int_\Omega\frac{1}{r(\xi)}\left|\mathrm{D}^{\alpha,\beta;\psi}_{0+}u\right|^{r(\xi)}d\xi\right)\int_\Omega\left|\mathrm{D}^{\alpha,\beta;\psi}_{0+}u\right|^{r(\xi)-2}\mathrm{D}^{\alpha,\beta;\psi}_{0+}u\,\mathrm{D}^{\alpha,\beta;\psi}_{0+}v\,d\xi$$
$$=\kappa\left[\int_\Omega\frac{1}{s(\xi)}(u^+)^{s(\xi)}d\xi\right]^r\int_\Omega(u^+)^{s(\xi)-1}v\,d\xi$$

for all $u\in\mathcal{N}^{\alpha,\beta;\psi}_{r(\xi),0}(\Omega)$.

Theorem 2.9. [38] *Let* $\Upsilon_{r(\xi)}:\mathcal{N}^{\alpha,\beta;\psi}_{r(\xi),0}(\Omega)\to\left(\mathcal{N}^{\alpha,\beta;\psi}_{r(\xi),0}(\Omega)\right)^*$ *the application given by*

$$\Upsilon_{r(\xi)}(u)v=\int_\Omega\left|\mathrm{D}^{\alpha,\beta;\psi}_{0+}u\right|^{r(\xi)-2}\mathrm{D}^{\alpha,\beta;\psi}_{0+}u\,\mathrm{D}^{\alpha,\beta;\psi}_{0+}v\,d\xi,\ \forall u,v\in\mathcal{N}^{\alpha,\beta;\psi}_{r(\xi),0}(\Omega).$$

So, we assert that

1. $\Upsilon_{r(\xi)}:\mathcal{N}^{\alpha,\beta;\psi}_{r(\xi),0}(\Omega)\to\left(\mathcal{N}^{\alpha,\beta;\psi}_{r(\xi),0}(\Omega)\right)^*$ *is a continuous, bounded and strictly monotonous operator;*
2. $\Upsilon_{r(\xi)}$ *is an application of type* S_+, *i.e., if* $u_n\rightharpoonup u$ *in* $\mathcal{N}^{\alpha,\beta;\psi}_{r(\xi),0}(\Omega)$ *and in addition* $\limsup\left\langle\Upsilon_{p(x)}(u_n)-\Upsilon_{p(x)}(u),u_n-u\right\rangle\leq 0$ *then* $u_n\to u$ *in* $\mathcal{N}^{\alpha,\beta;\psi}_{r(\xi),0}(\Omega)$;
3. $\Upsilon_{r(\xi)}:\mathcal{N}^{\alpha,\beta;\psi}_{r(\xi),0}(\Omega)\to\left(\mathcal{N}^{\alpha,\beta;\psi}_{r(\xi),0}(\Omega)\right)^*$ *is a homomorphism.*

3. Main Results

Theorem 3.1. *Suppose that* $1<\alpha r(\xi)<N$ $(0<\alpha<1)$ *with* $1<s(\xi)<r^*_\alpha(\xi)=\frac{Nr(\xi)}{N-\alpha r(\xi)}$ *for all* $\xi\in\overline{\Omega}$. *Furthermore, assume that there exist positive* ρ_0 *and* v_1 *such that*

$\rho_0 \le \mathcal{M}(t) \le v_1$, *with* $\dfrac{\upsilon_1 r^+}{\rho_0} < \dfrac{(s^-)^{\tau+1}(\tau+1)}{(s^+)^r}$ and $(s^-)(\tau+1) > r^+$. *Then the problem* (1.1) *has a weak solution for all* $\kappa > 0$.

Proof. First, consider the following functional

$$\mathrm{a}_{\kappa}^{\alpha,\beta;\psi}(u) = \widehat{\mathcal{M}}\left(\int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u\right|^{r(\xi)}d\xi\right) - \frac{\kappa}{r+1}\left[\int_\Omega \frac{1}{s(\xi)}(u^+)^{s(\xi)}d\xi\right]^{\tau+1} \ \forall\, \kappa, r > 0. \quad (3.1)$$

Let us now verify that Eq.(3.1) satisfies the first and second geometry of the Mountain Pass Theorem. Note that, for $t \ge 0$

$$\rho_0 \le \mathcal{M}(s) \Rightarrow \int_0^t \rho_0\, ds \le \int_0^t \mathcal{M}(s)ds = \widehat{\mathcal{M}}(t) \Rightarrow \rho_0 t \le \widehat{\mathcal{M}}(t).$$

Therefore, writing $t = \int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u\right|^{r(\xi)}d\xi$, we get

$$\widehat{\mathcal{M}}\left(\int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u\right|^{r(\xi)}d\xi\right) \ge \rho_0 \int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u\right|^{r(\xi)}d\xi. \quad (3.2)$$

Since $r^+ = \max_{\xi\in\bar{\Omega}} r(\xi)$, we obtain

$$\rho_0 \int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u\right|^{r(\xi)}d\xi \ge \frac{\rho_0}{r^+}\int_\Omega \left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u\right|^{r(\xi)}d\xi. \quad (3.3)$$

Furthermore, since $s^- = \min_{\xi\in\bar{\Omega}} s(\xi)$, yields

$$\frac{\kappa}{r+1}\left[\int_\Omega \frac{1}{s(\xi)}(u^+)^{s(\xi)}d\xi\right]^{\tau+1} \le \frac{\kappa}{(\tau+1)(s^-)^{\tau+1}}\left[\int_\Omega |u|^{s(\xi)}d\xi\right]^{\tau+1}. \quad (3.4)$$

Using the inequalities (3.2)–(3.4), we have

$$\mathrm{a}_{\kappa}^{\alpha,\beta;\psi}(u) \ge \frac{\rho_0}{r^+}\int_\Omega \left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u\right|^{r(\xi)}d\xi - \frac{\kappa}{(\tau+1)(s^-)^{\tau+1}}\left[\int_\Omega |u|^{s(\xi)}d\xi\right]^{\tau+1}. \quad (3.5)$$

Taking $\|u\| < 1$ in Lemma 2.3 item 4, we obtain

$$\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u\right|_{r(\xi)} < 1 \Rightarrow \sigma\left(\mathrm{D}_{0+}^{\alpha,\beta;\psi}u\right) \ge \left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u\right|_{r(\xi)}^{r^+} \Rightarrow \int_\Omega \left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u\right|^{r(\xi)}d\xi \ge \left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u\right|_{r(\xi)}^{r^+},$$

that is,

$$\int_\Omega \left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u\right|^{r(\xi)}d\xi \ge \|u\|^{r^+}. \quad (3.6)$$

Therefore, from the inequalities (3.5) and (3.6), it follows that

$$\mathrm{a}_{\kappa}^{\alpha,\beta;\psi}(u) \ge \frac{\rho_0}{r^+}\|u\|^{r^+} - \frac{\kappa}{(\tau+1)(s^-)^{\tau+1}}\left[\int_\Omega |u|^{s(\xi)}d\xi\right]^{\tau+1}. \quad (3.7)$$

By embedding $\mathrm{N}_{r(\xi),0}^{\alpha,\beta;\psi}(\Omega) \hookrightarrow L^{s(\xi)}(\Omega)$, there exists $C > 0$ such that

$$|u|_{s(\xi)} \le C\|u\|, \quad (3.8)$$

and adopting $\sigma = \|u\|$ sufficiently small, we have

$$|u|_{s(\xi)} \le C\|u\| \le C\sigma < 1,$$

which implies

$$|u|_{s(\xi)}^{s^-} \le \left(C\|u\|\right)^{s^-}. \tag{3.9}$$

From Lemma 2.3 (4), it follows that

$$\int_\Omega |u|^{s(\xi)} d\xi \le |u|_{s(\xi)}^{s^-}. \tag{3.10}$$

Using the inequality and raising ($\tau + 1$) on both sides of inequalities (3.9)–(3.10), we obtain

$$\left(\int_\Omega |u|^{s(\xi)} d\xi\right)^{\tau+1} \le \left(C\|u\|\right)^{(\tau+1)s^-}. \tag{3.11}$$

Replacing $\|u\|$ with a in (3.7) and considering (3.11), we obtain

$$\mathsf{a}_\kappa^{\alpha,\beta;\psi}(u) \ge \frac{\rho_0}{r^+}\sigma^{r^+} - \frac{\kappa}{(\tau+1)(s^-)^{\tau+1}}(C\sigma)^{(\tau+1)s^-}.$$

In this sense, it follows that

$$\mathsf{a}_\kappa^{\alpha,\beta;\psi}(u) \ge \sigma^{r^+}\left[\frac{\rho_0}{r^+} - \frac{\kappa C_1}{(\tau+1)(s^-)^{\tau+1}}\sigma^{(\tau+1)s^- - r^+}\right],$$

where $C_1 = C^{(\tau+1)s-}$.

Since $s(\xi) > 1$ and $r > 0$, we have $r^+ < s^-(r+1) < (s^-)^{\tau+1}(r+1)$. Therefore, we can find a, σ (constants) such that $\mathsf{a}_\kappa^{\alpha,\beta;\psi}(u) \ge a > 0$, if $\|u\| = \sigma$ for all $\kappa > 0$. Therefore, $\mathsf{a}_\kappa^{\alpha,\beta;\psi}(\cdot)$ satisfies the first geometry of the Mountain Pass Theorem.

Now, take $0 < \omega \in N_{r(\xi),0}^{\alpha,\beta;\psi}(\Omega)$, so we have

$$\mathsf{a}_\kappa^{\alpha,\beta;\psi}(t\omega) = \hat{\square}\left(\int_\Omega \frac{1}{r(\xi)}\left|\mathsf{D}_{0+}^{\alpha,\beta;\psi} t\omega\right|^{r(\xi)} d\xi\right) - \frac{\kappa}{r+1}\left[\int_\Omega \frac{1}{s(\xi)}|t\omega|^{s(\xi)} d\xi\right]^{\tau+1}. \tag{3.12}$$

For $t > 1$, we have $t^{r(\xi)} \le t^{r^+}$ and $t^{s^-} \le t^{s(\xi)}$. Furthermore, it follows that

$$\square(s) \le \upsilon_1 \Rightarrow \int_0^t \square(s)ds \le \int_0^t \upsilon_1 ds \Rightarrow \hat{\square}(t) \le \upsilon_1 t.$$

Consequently,

$$\hat{\square}\left(\int_\Omega \frac{1}{r(\xi)}\left|\mathsf{D}_{0+}^{\alpha,\beta;\psi} t\omega\right|^{r(\xi)} d\xi\right) \le \upsilon_1 \int_\Omega \frac{1}{r(\xi)}\left|\mathsf{D}_{0+}^{\alpha,\beta;\psi} t\omega\right|^{r(\xi)} d\xi. \tag{3.13}$$

In addition to the above, note that

$$\upsilon_1 \int_\Omega \frac{1}{r(\xi)}\left|\mathsf{D}_{0+}^{\alpha,\beta;\psi} t\omega\right|^{r(\xi)} d\xi \le \frac{\upsilon_1 t^{r^+}}{r^-}\int_\Omega \left|\mathsf{D}_{0+}^{\alpha,\beta;\psi} \omega\right|^{r(\xi)} d\xi \tag{3.14}$$

and

$$\frac{\kappa}{r+1}\left[\int_\Omega \frac{1}{s(\xi)}|t\omega|^{s(\xi)}d\xi\right]^{\tau+1} \ge \frac{\kappa}{r+1}\frac{t^{(\tau+1)s^-}}{(s^+)^{\tau+1}}\left[\int_\Omega |\omega|^{s(\xi)}d\xi\right]^{\tau+1}. \tag{3.15}$$

From the inequalities (3.12)–(3.15), we obtain the following inequality

$$\mathrm{a}_\kappa^{\alpha,\beta;\psi}(t\omega) \le \upsilon\, t^{r^+}\frac{1}{r^-}\int_\Omega \left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}\omega\right|^{r(\xi)}d\xi - \frac{\kappa}{r+1}\frac{t^{(\tau+1)s^-}}{(s^+)^{\tau+1}}\left[\int_\Omega |\omega|^{s(\xi)}d\xi\right]^{\tau+1}.$$

Considering that $(\tau+1)s^- > r^+$ and doing $t \to \infty$, we have $\mathrm{a}_\kappa^{\alpha,\beta;\psi}(t\omega) \to -\infty$. With this, we guarantee that $\mathrm{a}_\kappa^{\alpha,\beta;\psi}(\cdot)$ fulfills the second geometry of the Mountain Pass Theorem.

Now, let's discuss $\mathrm{a}_\kappa^{\alpha,\beta;\psi}(\cdot)$ satisfies the Palais-Smale condition, i.e., that every sequence $(u_n) \subset \mathrm{N}_{r(\xi),0}^{\alpha,\beta;\psi}(\Omega)$ such that

$$\mathrm{a}_\kappa^{\alpha,\beta;\psi}(u_n) \to C_\kappa \quad \text{and} \quad \left(\mathrm{a}_\kappa^{\alpha,\beta;\psi}\right)'(u_n) \to 0 \tag{3.16}$$

contains a convergent subsequence under the norm of $\mathrm{N}_{r(\xi),0}^{\alpha,\beta;\psi}(\Omega)$. Then, consider the sequence $(u_n) \subset \mathrm{N}_{r(\xi),0}^{\alpha,\beta;\psi}(\Omega)$ that satisfies Eq.(3.16). We will show that (u_n) is bounded. Choosing θ such that $\frac{\upsilon_1 r^+}{\rho_0} < \theta < \frac{(s^-)^{\tau+1}(r+1)}{(s^+)^r}$, we have

$$\begin{aligned} C_\kappa + \|u_n\| \ge{} & \mathcal{M}\left(\int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{r(\xi)}d\xi\right) - \frac{\kappa}{r+1}\left[\int_\Omega \frac{1}{s(\xi)}|u_n|^{s(\xi)}d\xi\right]^{\tau+1} \\ & - \frac{1}{\theta}\mathcal{M}\left(\int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{r(\xi)}d\xi\right)\int_\Omega \left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{r(\xi)}d\xi \\ & + \frac{\kappa}{\theta}\left[\int_\Omega \frac{1}{s(\xi)}|u_n|^{s(\xi)}d\xi\right]^{r}\int_\Omega |u_n|^{s(\xi)}d\xi. \end{aligned}$$

Note, since $\mathcal{M}(t) \ge \rho_0(t)$ and $r^+ > 1$, we obtain that

$$\mathcal{M}\left(\int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{r(\xi)}d\xi\right) \ge \rho_0\int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{r(\xi)}d\xi \ge \frac{\rho_0}{r^+}\int_\Omega \left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{r(\xi)}d\xi. \tag{3.17}$$

As $\upsilon_1 \ge \mathrm{M}(t)$, we have

$$\mathcal{M}\left(\int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{r(\xi)}d\xi\right) \le \upsilon_1. \tag{3.18}$$

Furthermore, we can state that

$$\frac{\kappa}{r+1}\left[\int_\Omega \frac{1}{s(\xi)}|u_n|^{s(\xi)}d\xi\right]^{\tau+1} \le \frac{\kappa}{(\tau+1)(s^-)^{\tau+1}}\left[\int_\Omega |u_n|^{s(\xi)}d\xi\right]^{\tau+1} \tag{3.19}$$

and

$$\kappa\left[\int_\Omega \frac{1}{s(\xi)}|u_n|^{s(\xi)}d\xi\right]^{r}\left(\int_\Omega |u_n|^{s(\xi)}d\xi\right) \ge \frac{\kappa}{(s^+)^r}\left[\int_\Omega |u_n|^{s(\xi)}d\xi\right]^{\tau+1}. \tag{3.20}$$

Using the inequalities (3.17)–(3.20), we have

$$C_\kappa + \|u_n\| \geq \left(\frac{\rho_0}{r^+} - \frac{\upsilon_1}{\theta}\right)\int_\Omega \left|\mathsf{D}^{\alpha,\beta;\psi}_{0+} u_n\right|^{r(\xi)} d\xi + \kappa\left(\frac{1}{\theta(s^+)^r} - \frac{1}{(\tau+1)(s^-)^{\tau+1}}\right)\left[\int_\Omega \left|u_n\right|^{s(\xi)} d\xi\right]^{\tau+1}.$$

Note that

$$\frac{\upsilon_1 r^+}{\rho_0} < \theta \Rightarrow \upsilon_1 r^+ < \rho_0\theta \Rightarrow \frac{\upsilon_1}{\theta} < \frac{\rho_0}{r^+}$$

and

$$\theta(s^+)^r < (\tau+1)(s^-)^{\tau+1}$$

are positive values.

Therefore, from the inequality (3.21), we have

$$C_\kappa + \|u_n\| \geq \left(\frac{\rho_0}{r^+} - \frac{\upsilon_1}{\theta}\right)\int_\Omega \left|\mathsf{D}^{\alpha,\beta;\psi}_{0+} u_n\right|^{r(\xi)} d\xi. \tag{3.22}$$

For $||u|| \geq 1$ and using Lemma 2.3 (3), it follows that

$$\left|\mathsf{D}^{\alpha,\beta;\psi}_{0+} u_n\right|_{r(\xi)} > 1 \Rightarrow \sigma\left(\mathsf{D}^{\alpha,\beta;\psi}_{0+} u_n\right) \geq \left|\mathsf{D}^{\alpha,\beta;\psi}_{0+} u_n\right|_{r(\xi)}^{r^-} \Rightarrow \int_\Omega \left|\mathsf{D}^{\alpha,\beta;\psi}_{0+} u_n\right|^{r(\xi)} d\xi \geq \left|\mathsf{D}^{\alpha,\beta;\psi}_{0+} u_n\right|_{r(\xi)}^{r^-},$$

i.e., less than a constant

$$\int_\Omega \left|\mathsf{D}^{\alpha,\beta;\psi}_{0+} u_n\right|^{r(\xi)} d\xi \geq \|u_n\|^{r^-}. \tag{3.23}$$

If (u_n) is an unbounded sequence in $\mathrm{N}^{\alpha,\beta;\psi}_{r(\xi),0}(\Omega)$, and without loss of generality passing to a subsequence, if necessary, that $||u_n|| > 1$ and using (3.22) and (3.23), we have

$$C_\kappa + \|u_n\| \geq \left(\frac{\rho_0}{r^+} - \frac{\upsilon_1}{\theta}\right)\|u_n\|^{r^-}.$$

Thus, we obtain an absurd, since $r^- > 1$. Therefore, (u_n) is bounded. Since $\mathrm{N}^{\alpha,\beta;\psi}_{r(\xi),0}(\Omega)$ is a reflexive Banach space, there exist a subsequence (u_n) such that $u_n \rightharpoonup u$ in $\mathrm{N}^{\alpha,\beta;\psi}_{r(\xi),0}(\Omega)$. Thus, from

$$\left(\mathsf{a}^{\alpha,\beta;\psi}_\kappa\right)'(u_n) \to 0,$$

it's follows that

$$\begin{aligned}
&\left(\mathsf{a}^{\alpha,\beta;\psi}_\kappa\right)'(u_n)(u_n - u)\\
&= \square\left(\int_\Omega \frac{1}{r(\xi)}\left|\mathsf{D}^{\alpha,\beta;\psi}_{0+} u_n\right|^{r(\xi)} d\xi\right)\int_\Omega \left|\mathsf{D}^{\alpha,\beta;\psi}_{0+} u_n\right|^{r(\xi)-2} \mathsf{D}^{\alpha,\beta;\psi}_{0+} u_n\, \mathsf{D}^{\alpha,\beta;\psi}_{0+}(u_n - u)d\xi\\
&-\kappa\left[\int_\Omega \frac{1}{s(\xi)}|u_n|^{s(\xi)} d\xi\right]^r \int_\Omega |u_n|^{s(\xi)-2} u_n(u_n - u)d\xi \to 0.
\end{aligned} \tag{3.24}$$

Therefore, we have that $s(\xi) < r_\alpha^*(\xi)$, for all $\xi \in (\bar{\Omega})$. Thus, $\mathcal{N}^{\alpha,\beta;\psi}_{r(\xi),0}(\Omega)$ is compactly immersed in $L^{s(\xi)}(\Omega)$. Using this fact and Hölder's inequality, we have

$$\left|\int_\Omega |u_n|^{s(\xi)-2} u_n (u_n - u) d\xi\right| \leq \int_\Omega |u_n|^{s(\xi)-1} |u_n - u| d\xi \leq C \|u_n\|_{\frac{s(\xi)}{s(\xi)-1}} |u_n - u|_{s(\xi)}.$$

As a consequence of the embedding of $\mathcal{N}^{\alpha,\beta;\psi}_{r(\xi),0}(\Omega)$ in $L^{s(\xi)}(\Omega)$, $(u_n) \to u$ strongly in $L^{s(\xi)}(\Omega)$. Therefore $\int_\Omega |u_n|^{s(\xi)-2} u_n (u_n - u) d\xi \to 0$. On the other hand, there are c_1 and c_2 (positive constants) such that

$$c_1 \leq \int_\Omega \frac{1}{s(\xi)} |u_n|^{s(\xi)} d\xi \leq c_2.$$

This implies that

$$\kappa \left[\int_\Omega \frac{1}{s(\xi)} |u_n|^{s(\xi)} d\xi\right]^r \int_\Omega |u_n|^{s(\xi)-2} u_n (u_n - u) d\xi \to 0. \tag{3.25}$$

Using Eq.(3.24), we obtain

$$\Upsilon_{r(\xi)}(u)(u_n - u) = \int_\Omega \left|\mathbf{D}^{\alpha,\beta;\psi}_{0+} u_n\right|^{r(\xi)-2} \mathbf{D}^{\alpha,\beta;\psi}_{0+} u_n \, \mathbf{D}^{\alpha,\beta;\psi}_{0+} (u_n - u) d\xi \to 0, \tag{3.26}$$

since there are positive constants b_1, b_2 such that

$$b_1 \leq \square \left(\int_\Omega \frac{1}{r(\xi)} \left|\mathbf{D}^{\alpha,\beta;\psi}_{0+} u_n\right|^{r(\xi)} d\xi \right) \leq b_2.$$

Consequently,

$$\left\langle \Upsilon_{r(\xi)}(u_n) - \Upsilon_{r(\xi)}(u), u_n - u \right\rangle \to 0.$$

Using Theorem 2.9, we have that $u_n \to u$ in $\mathcal{N}^{\alpha,\beta;\psi}_{r(\xi),0}(\Omega)$. Thus, the functional $\mathbf{a}^{\alpha,\beta;\psi}_\kappa(\cdot)$ satisfies the Mountain Pass geometries and satisfies the condition (*PS*). Therefore, the problem (1.1) has a weak solution.

Theorem 3.2. Suppose that $1 < \alpha r(\xi) < N$ with $1 < s(\xi) < r_\alpha^*(\xi) = \dfrac{N r(\xi)}{N - \alpha r(\xi)}$ for all $\xi \in (\bar{\Omega})$ and $0 < \alpha < 1$. Furthermore, assume that there exists $0 < \rho_0$ such that $\rho_0 \leq M(t) \leq \upsilon_1$. If $(\tau + 1)s^- < r^-$, then there exists $\kappa^* > 0$ such that the problem (1.1) has a positive solution u_κ for each $\kappa \in (0, \kappa^*)$.

Proof. To prove this result, we will use Ekeland's Variational Principle. Using the inequality (3.7), we have

$$\mathbf{a}^{\alpha,\beta;\psi}_\kappa(u) \geq \frac{\rho_0}{r^+} \|u\|^{r^+} - \frac{\kappa}{(\tau+1)(s^-)^{\tau+1}} \left[\int_\Omega |u|^{s(\xi)} d\xi\right]^{\tau+1}.$$

Taking $||u|| = \sigma$, sufficiently small and considering (3.11), it follows that

$$\mathrm{a}_{\kappa}^{\alpha,\beta;\psi}(u) \geq \frac{\rho_0}{r^+}\sigma^{r^+} - \frac{\kappa}{(\tau+1)(s^-)^{\tau+1}}(C\sigma)^{(\tau+1)s^-}.$$

Choosing $c_1 = C^{(\tau+1)s^-}$, we have

$$\mathrm{a}_{\kappa}^{\alpha,\beta;\psi}(u) \geq \sigma^{(\tau+1)s^-}\left[\frac{\rho_0\sigma^{r^+-(\tau+1)s^-}}{r^+} - \frac{\kappa C_1}{(\tau+1)(s^-)^{\tau+1}}\right] \geq a \geq 0, \text{ if } 0 < \kappa < \kappa^*$$

for some $\kappa^* > 0$. In this sense, for all $\kappa \in (0, \kappa^*)$

$$\inf_{\partial B_\sigma(0)} \mathrm{a}_{\kappa}^{\alpha,\beta;\psi} > 0. \tag{3.27}$$

Here the κ parameter plays a crucial role. Using hypothesis, $(\tau + 1)s^- < r^-$. Therefore, there exists $\varepsilon_0 > 0$ such that $(\tau + 1)s^- + \varepsilon_0(\tau + 1) < r^-$. Since $q \in C(\overline{\Omega})$, there exists an open set $\Omega_0 \subset \Omega$ such that $|s(\xi) - s^-| < \varepsilon_0$ for all $\xi \in \Omega_0$, which implies that $(s(\xi) - s^-)(\tau + 1) < \varepsilon_0(\tau + 1)$. In this sense, yields that

$$s(\xi)(\tau+1) \leq (\tau+1)s^- + \square_0(\tau+1) < r^-, \forall \xi \in \Omega_0. \tag{3.28}$$

Consider $\phi \in C_0^\infty(\Omega)$ such that $\operatorname{supp}\phi \supset \Omega_0$, $\phi(\xi) = 1$ for all $\xi \in \Omega_0$ and $0 \leq \phi \leq 1$ in Ω. Taking a sufficiently small t, in particular $t \in (0,1)$, we have $t\phi \in B_\sigma(0)$. Additionally, note that

$$\mathrm{a}_{\kappa}^{\alpha,\beta;\psi}(t\phi) = \square\left(\int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi} t\phi\right|^{r(\xi)} d\xi\right) - \frac{\kappa}{r+1}\left[\int_\Omega \frac{1}{s(\xi)}|t\phi|^{s(\xi)} d\xi\right]^{\tau+1}. \tag{3.29}$$

Analogously to inequality (3.2), it follows that

$$\square\left(\int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi} t\phi\right|^{r(\xi)} d\xi\right) \leq \upsilon_1 \int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi} t\phi\right|^{r(\xi)} d\xi. \tag{3.30}$$

Since $t \in (0, 1)$, it follows that $t^{r(\xi)} \leq t^{r^-}$. Furthermore, since $\frac{1}{r(\xi)} \leq \frac{1}{r^-}$, we have

$$\upsilon_1 \int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi} t\phi\right|^{r(\xi)} d\xi \leq \frac{\upsilon_1 t^{r^-}}{r^-}\int_\Omega \left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}\phi\right|^{r(\xi)} d\xi. \tag{3.31}$$

Inthissense, wehavethat $\frac{1}{(s^+)^{\tau+1}} \leq \frac{1}{s(\xi)}$. Furthermore, if $s(\xi)(r+1) \leq s^-(r+1) + \square_0(r+1)$, then $t^{s^-(r+1)+\square_0(r+1)} \leq t^{s(\xi)(\tau+1)}$. Thus,

$$\frac{\kappa}{r+1}\left[\int_\Omega \frac{1}{s(\xi)}|t\phi|^{s(\xi)} d\xi\right]^{\tau+1} \geq \frac{\kappa t^{(\tau+1)s^-+\square_0(r+1)}}{(\tau+1)(s^+)^{\tau+1}}\left[\int_\Omega |\phi|^{s(\xi)} d\xi\right]^{\tau+1}. \tag{3.32}$$

Therefore, using Eq.(3.29) and the inequalities (3.30)–(3.32), we obtain

$$\mathrm{a}_{\kappa}^{\alpha,\beta;\psi}(t\phi) \leq \frac{\upsilon_1 t^{r^-}}{r^-}\int_\Omega \left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}\phi\right|^{r(\xi)} d\xi - \frac{\kappa t^{(\tau+1)s^-+\square_0(r+1)}}{(\tau+1)(s^+)^{\tau+1}}\left[\int_\Omega |\phi|^{s(\xi)} d\xi\right]^{\tau+1}.$$

Since $\int_\Omega \left|\mathrm{D}^{\alpha,\beta;\psi}_{0+}\phi\right|^{r(\xi)} d\xi > 0$ and $t^{r^-} < t^{(\tau+1)s^- + \square_0(r+1)}$ for all $t \in (0, 1)$, we obtain

$$\mathrm{a}^{\alpha,\beta;\psi}_{\kappa}(t\phi) < 0.$$

In this sense, for all $u \in B_\sigma(0)$, we have

$$\mathrm{a}^{\alpha,\beta;\psi}_{\kappa}(u) \geq \frac{\rho_0}{r^+}\|u\|^{r^+} - \frac{\kappa C}{(\tau+1)(s^-)^{\tau+1}}\|u\|^{(\tau+1)s^-}.$$

Thus, it follows that

$$-\infty < \underline{c} := \inf_{B_\sigma(0)} \mathrm{a}^{\alpha,\beta;\psi}_{\kappa} < 0. \tag{3.33}$$

Using (3.27) and (3.33), we know that $\inf_{\partial B_\sigma(0)} \mathrm{a}^{\alpha,\beta;\psi}_{\kappa} - \inf_{B_\sigma(0)} \mathrm{a}^{\alpha,\beta;\psi}_{\kappa} > 0$. So we choose $0 < \square < \inf_{\partial B_\sigma(0)} \mathrm{a}^{\alpha,\beta;\psi}_{\kappa} - \inf_{B_\sigma(0)} \mathrm{a}^{\alpha,\beta;\psi}_{\kappa}$. Using Ekeland's Variational Principle for the functional $\mathrm{a}^{\alpha,\beta;\psi}_{\kappa} : \overline{B_\sigma(0)} \to \mathbb{R}$, we find $u_\square \in \overline{B_\sigma(0)}$ such that

$$\mathrm{a}^{\alpha,\beta;\psi}_{\kappa}(u_\square) < \inf_{B_\sigma(0)} \mathrm{a}^{\alpha,\beta;\psi}_{\kappa} + \square \text{ and } \mathrm{a}^{\alpha,\beta;\psi}_{\kappa}(u_\square) < \mathrm{a}^{\alpha,\beta;\psi}_{\kappa}(u) + \square\|u - u_\square\|, u \neq u_\square.$$

Thus, we obtain a sequence $(w_n) \subset B_\sigma(0)$ such that

$$\mathrm{a}^{\alpha,\beta;\psi}_{\kappa}(w_n) \to \underline{c} \text{ and } \left(\mathrm{a}^{\alpha,\beta;\psi}_{\kappa}\right)'(w_n) \to 0.$$

Since $s(\xi) < r^*_\alpha(\xi)$ and $\mathrm{a}^{\alpha,\beta;\psi}_{\kappa}(\cdot)$ satisfies the condition (*PS*), we have a subsequence (w_n), and an element $w \in \mathrm{N}^{\alpha,\beta;\psi}_{r(\xi),0}(\Omega)$ such that $w_n \to w$ and $\mathrm{a}^{\alpha,\beta;\psi}_{\kappa}(w) = \underline{c} < 0$. By the Corollary 2.4, $\mathrm{a}^{\alpha,\beta;\psi}_{\kappa}(w) = 0$, and thus we conclude that w is a non-trivial weak solution to the problem (1.1).

Theorem 3.3. *Suppose that* $1 < \alpha r(\xi) < N$ with $1 < s(\xi) < r^*_\alpha(\xi) = \dfrac{Nr(\xi)}{N - \alpha r(\xi)}$ for *all* $\xi \in (\overline{\Omega})$ and $0 < \alpha < 1$. *Furthermore, assume that* $\mathrm{M}(t) = a + bt$, *where* $a > 0$, $b > 0$ and $t \geq 0$. *If* $\dfrac{2(r^+)^2}{r^-} < \dfrac{(\tau+1)(s^-)^{\tau+1}}{(s^+)^r}$ *and* $(\tau + 1)s^- > 2r^+$, *then the problem* (1.1) *has a weak positive solution for all* $\kappa > 0$.

Proof. Since $\mathrm{M}(t) = a + bt$, we obtain

$$\widehat{\square}(t) = \int_0^t \square(s)ds = at + \frac{bt^2}{2}.$$

So, in this sense, we get

$$\begin{aligned}\mathrm{a}^{\alpha,\beta;\psi}_{\kappa}(u) &= \widehat{\square}\left(\int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}^{\alpha,\beta;\psi}_{0+}u\right|^{r(\xi)} d\xi\right) - \frac{\kappa}{r+1}\left[\int_\Omega \frac{1}{s(\xi)}|u|^{s(\xi)} d\xi\right]^{\tau+1} \\ &= a\int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}^{\alpha,\beta;\psi}_{0+}u\right|^{r(\xi)} d\xi + \frac{b}{2}\left[\int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}^{\alpha,\beta;\psi}_{0+}u\right|^{r(\xi)} d\xi\right]^2 - \frac{\kappa}{r+1}\left[\int_\Omega \frac{1}{s(\xi)}|u|^{s(\xi)} d\xi\right]^{\tau+1}.\end{aligned} \tag{3.34}$$

Note that as $\frac{1}{r^+} \le \frac{1}{r(\xi)}$, we have

$$a\int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u\right|^{r(\xi)} d\xi \ge \frac{a}{r^+}\int_\Omega \left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u\right|^{r(\xi)} d\xi \tag{3.35}$$

and

$$\frac{b}{2}\left[\int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u\right|^{r(\xi)} d\xi\right]^2 \ge \frac{b}{2(r^+)^2}\left[\int_\Omega \left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u\right|^{r(\xi)} d\xi\right]^2. \tag{3.36}$$

Furthermore, since $\frac{1}{s(\xi)^{\tau+1}} \le \frac{1}{(s^-)^{\tau+1}}$, it follows that

$$\frac{\kappa}{r+1}\left[\int_\Omega \frac{1}{s(\xi)}|u|^{s(\xi)} d\xi\right]^{\tau+1} \le \frac{\kappa}{(\tau+1)(s^-)^{\tau+1}}\left[\int_\Omega |u|^{s(\xi)} d\xi\right]^{\tau+1}. \tag{3.37}$$

Using the inequalities (3.34)–(3.37), and taking $\|u\| \le 1$ in the inequality (3.6), we have

$$\mathrm{a}_\kappa^{\alpha,\beta;\psi}(u) \ge \frac{a}{r^+}\|u\|^{r^+} + \frac{b}{2(r^+)^2}\|u\|^{2r^+} - \frac{\kappa}{(\tau+1)(s^-)^{\tau+1}}\left[\int_\Omega |u|^{s(\xi)} d\xi\right]^{\tau+1}. \tag{3.38}$$

Using the inequality (3.11) and substituting $\|u\|$ and (3.38), it follows that

$$\mathrm{a}_\kappa^{\alpha,\beta;\psi}(u) \ge \frac{a}{r^+}\|u\|^{r^+} + \frac{b}{2(r^+)^2}\|u\|^{2r^+} - \frac{C_1}{(\tau+1)(s^-)^{\tau+1}}\left[\int_\Omega |u|^{s(\xi)} d\xi\right]^{\tau+1}. \tag{3.39}$$

Since $(\tau + 1)s^- > 2r^+$, we find δ, σ such that

$$\mathrm{a}_\kappa^{\alpha,\beta;\psi}(u) \ge \delta > 0 \text{ if } \|u\| = \sigma,\ \forall\kappa > 0,$$

and thus $\mathrm{a}_\kappa^{\alpha,\beta;\psi}(\cdot)$ satisfies the first geometry of the Mountain Pass Theorem.

Now take $0 < \omega \in \mathrm{N}_{r(\xi),0}^{\alpha,\beta;\psi}(\Omega)$. Note that

$$\mathrm{a}_\kappa^{\alpha,\beta;\psi}(t\omega) = a\int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}t\omega\right|^{r(\xi)} d\xi + \frac{b}{2}\left[\int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}t\omega\right|^{r(\xi)} d\xi\right]^2 - \frac{\kappa}{r+1}\left[\int_\Omega \frac{1}{s(\xi)}\left|t\omega\right|^{s(\xi)} d\xi\right]^{\tau+1}.$$

For $t > 1$, it follows that $t^{r(\xi)} < t^{r^+}$ and $t^s < t^{s(\xi)}$. Hence

$$\mathrm{a}_\kappa^{\alpha,\beta;\psi}(t\omega) \le \frac{at^{r^+}}{r^-}\int_\Omega \left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}\omega\right|^{r(\xi)} d\xi + \frac{bt^{2r^+}}{2(r^-)^2}\left[\int_\Omega \left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}\omega\right|^{r(\xi)} d\xi\right]^2 - \frac{\kappa t^{(\tau+1)s^+}}{(\tau+1)(s^-)^{\tau+1}}\left[\int_\Omega |\omega|^{s(\xi)} d\xi\right]^{\tau+1}.$$

Since $s^-(r + 1) > 2r^+$, we have $\mathrm{a}_\kappa^{\alpha,\beta;\psi}(t\omega) \to -\infty$ when $t \to +\infty$. We are done that $\mathrm{a}_\kappa^{\alpha,\beta;\psi}(\cdot)$ satisfies the second geometry of the Mountain Pass Theorem.

Finally, we will prove that $\mathrm{a}_\kappa^{\alpha,\beta;\psi}(\cdot)$ satisfies the condition (*PS*). Let $(u_n) \subset \mathrm{N}_{r(\xi),0}^{\alpha,\beta;\psi}(\Omega)$ be a sequence such that

$$\mathrm{a}_\kappa^{\alpha,\beta;\psi}(u_n) \to C_\kappa \text{ and } \mathrm{a}_\kappa^{\alpha,\beta;\psi}(u_n) \to 0.$$

Therefore, taking

$$\max\left\{r^+, \frac{2(r^+)^2}{r^-}\right\} < \theta < \frac{(\tau+1)(s^-)^{\tau+1}}{(s^+)^r},$$

it follows that

$$\begin{aligned}
C_\kappa + \|u_n\| &\geq \mathrm{a}_\kappa^{\alpha,\beta;\psi}(u_n) - \frac{1}{\theta}\left(\mathrm{a}_\kappa^{\alpha,\beta;\psi}\right)'(u_n)u_n \\
&= a\int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{r(\xi)}d\xi \\
&\quad + \frac{b}{2}\left[\int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{r(\xi)}d\xi\right]^2 - \frac{\kappa}{r+1}\left[\int_\Omega \frac{1}{s(\xi)}\left|u_n\right|^{s(\xi)}d\xi\right]^{\tau+1} \\
&\quad - \frac{a}{\theta}\int_\Omega\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{r(\xi)}d\xi \\
&\quad - \frac{b}{\theta}\int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{r(\xi)}d\xi\int_\Omega\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{r(\xi)}d\xi + \frac{\kappa}{\theta}\left[\int_\Omega \frac{1}{s(\xi)}\left|u_n\right|^{s(\xi)}d\xi\right]^r\int_\Omega\left|u_n\right|^{s(\xi)}d\xi.
\end{aligned} \tag{3.40}$$

As $\frac{1}{r^+} \leq \frac{1}{r(\xi)} \leq \frac{1}{r^-}$, we have the following inequalities

$$a\int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{r(\xi)}d\xi \geq \frac{a}{r^+}\int_\Omega\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{r(\xi)}d\xi, \tag{3.41}$$

$$\frac{b}{2}\left[\int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{r(\xi)}d\xi\right]^2 \geq \frac{b}{2(r^+)^2}\left[\int_\Omega\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{r(\xi)}d\xi\right]^2 \tag{3.42}$$

and

$$\frac{b}{\theta}\int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{r(\xi)}d\xi\int_\Omega\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{r(\xi)}d\xi \leq \frac{b}{\theta r^-}\left[\int_\Omega\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{r(\xi)}d\xi\right]^2. \tag{3.43}$$

On the other hand, as $\frac{1}{s^+} \leq \frac{1}{s(\xi)} \leq \frac{1}{s^-}$, therefore

$$\frac{\kappa}{r+1}\left[\int_\Omega \frac{1}{s(\xi)}\left|u_n\right|^{s(\xi)}d\xi\right]^{\tau+1} \leq \frac{\kappa}{(\tau+1)(s^-)^{\tau+1}}\left[\int_\Omega\left|u_n\right|^{s(\xi)}d\xi\right]^{\tau+1} \tag{3.44}$$

and

$$\frac{\kappa}{\theta}\left[\int_\Omega \frac{1}{s(\xi)}\left|u_n\right|^{s(\xi)}d\xi\right]^r\int_\Omega\left|u_n\right|^{s(\xi)}d\xi \geq \frac{\kappa}{\theta(s^+)^r}\left[\int_\Omega\left|u_n\right|^{s(\xi)}d\xi\right]^{\tau+1}. \tag{3.45}$$

Using (3.40)–(3.45), we have

$$\begin{aligned}
C + \|u_n\| &\geq \left(\frac{a}{r^+} - \frac{a}{\theta}\right)\int_\Omega\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{r(\xi)}d\xi + \left(\frac{b}{2(r^+)^2} - \frac{b}{\theta r^-}\right)\left[\int_\Omega\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{r(\xi)}d\xi\right]^2 \\
&\quad + \kappa\left(\frac{1}{\theta(s^+)^r} - \frac{1}{(\tau+1)(s^-)^{\tau+1}}\right)\left[\int_\Omega\left|u_n\right|^{s(\xi)}d\xi\right]^{\tau+1}.
\end{aligned} \tag{3.46}$$

Note that $r^+ < \theta$,

$$\frac{2(r^+)^2}{r^-} < \theta \Rightarrow 2(r^+)^2 < \theta r^-$$

and

$$\theta(s^+)^r < (\tau+1)(s^-)^{\tau+1}.$$

Therefore, from the inequality (3.46), one has

$$C_\kappa + \|u_n\| \geq \left(\frac{a}{r^+} - \frac{a}{\theta}\right)\int_\Omega \left|\mathrm{D}_{0+}^{\alpha,\beta;\psi} u_n\right|^{r(\xi)} d\xi + \left(\frac{b}{2(r^+)^2} - \frac{b}{\theta r^-}\right)\left[\int_\Omega \left|\mathrm{D}_{0+}^{\alpha,\beta;\psi} u_n\right|^{r(\xi)} d\xi\right]^2. \quad (3.47)$$

Suppose that the sequence (u_n) is unbounded in $\mathcal{N}_{r(\xi),0}^{\alpha,\beta;\psi}(\Omega)$. Taking to a subsequence, if necessary, we have $\|u_n\| > 1$ and considering (3.23) and (3.47), we obtain

$$C_\kappa + \|u_n\| \geq \left(\frac{a}{r^+} - \frac{a}{\theta}\right)\|u_n\|^{r^-} + \left(\frac{b}{2(r^+)^2} - \frac{b}{\theta(r^-)^2}\right)\|u_n\|^{2r^-},$$

obtaining an absurd, since $r^- > 1$. Consequently, the sequence (u_n) is bounded in $\mathcal{N}_{r(\xi),0}^{\alpha,\beta;\psi}(\Omega)$, and so there exists a subsequence, also denoted by (u_n), such that $(u_n) \rightharpoonup u$ in $\mathcal{N}_{r(\xi),0}^{\alpha,\beta;\psi}(\Omega)$. From $\left(\mathfrak{a}_\kappa^{\alpha,\beta;\psi}\right)'(u_n) \to 0$, we have

$$\left(\mathfrak{a}_\kappa^{\alpha,\beta;\psi}\right)'(u_n)(u_n - u) = \left(a + b\int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi} u_n\right|^{r(\xi)} d\xi\right)\Upsilon_{r(\xi)}(u_n)(u_n - u)$$

$$-\kappa\left[\int_\Omega \frac{1}{s(\xi)}|u_n|^{s(\xi)} d\xi\right]^r \int_\Omega |u_n|^{s(\xi)-2} u_n(u_n - u) d\xi \to 0.$$

Using Theorem 3.1, we have

$$\kappa\left[\int_\Omega \frac{1}{s(\xi)}|u_n|^{s(\xi)} d\xi\right]^r \int_\Omega |u_n|^{s(\xi)-2} u_n(u_n - u) d\xi \to 0.$$

Furthermore, there are positive constants c_1 and c_2 so that $c_1 \leq \int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi} u_n\right|^{r(\xi)} d\xi \leq c_2$ and

$$\Upsilon_{r(\xi)}(u_n)(u_n - u) = \int_\Omega \left|\mathrm{D}_{0+}^{\alpha,\beta;\psi} u_n\right|^{r(\xi)-2} \mathrm{D}_{0+}^{\alpha,\beta;\psi} u_n \, \mathrm{D}_{0+}^{\alpha,\beta;\psi}(u_n - u) d\xi \to 0.$$

In addition to the above, yields that

$$\Upsilon_{r(\xi)}(u)(u_n - u) = \int_\Omega \left|\mathrm{D}_{0+}^{\alpha,\beta;\psi} u\right|^{r(\xi)-2} \mathrm{D}_{0+}^{\alpha,\beta;\psi} u \, \mathrm{D}_{0+}^{\alpha,\beta;\psi}(u_n - u) d\xi \to 0.$$

Consequently,

$$\left\langle \Upsilon_{r(\xi)}(u_n) - \Upsilon_{r(\xi)}(u), u_n - u \right\rangle \to 0.$$

Therefore, by Theorem 2.9, that $u_n \to u$ in $\mathrm{N}^{\alpha,\beta;\psi}_{r(\xi),0}(\Omega)$. Thus, $\mathrm{a}^{\alpha,\beta;\psi}_{\kappa}(\cdot)$ satisfies the condition (*PS*) and, thus, by the Mountain Pass Theorem, the problem (1.1) has a weak positive solution for all $\kappa > 0$.

Theorem 3.4. *Suppose that* $1 < \alpha r(\xi) < N$, $0 < \alpha < 1$ with $1 < s(\xi) < r^*_\alpha(\xi) = \frac{Nr(\xi)}{N - \alpha r(\xi)}$ for *all* $\xi \in (\overline{\Omega})$ *and* $\mathrm{M}(t) = a + bt$, *where* $a > 0$, $b > 0$ *and* $t \geq 0$. *If* $(\tau + 1)s^- < r^-$, *then there exists* $\kappa^* > 0$ *such that the problem* (1.1) *has a positive solution* u_κ *for each* $\kappa \in (0, \kappa^*)$.

Proof. Using the inequality (3.39), when $\|u\| < 1$, it follows that

$$\mathrm{a}^{\alpha,\beta;\psi}_{\kappa}(u) \geq \frac{a}{r^+}\|u\|^{r^+} + \frac{b}{2(r^+)^2}\|u\|^{2r^+} - \frac{\kappa C_1}{(\tau+1)(s^-)^{\tau+1}}\|u\|^{(\tau+1)s^-}.$$

We will show that, for sufficiently small $\|u\| = \sigma$,

$$\mathrm{a}^{\alpha,\beta;\psi}_{\kappa}(u) \geq \delta > 0, \; if \; 0 < \kappa < \kappa^*,$$

where

$$\kappa^* = \frac{1}{3}\min\left\{ \frac{(\tau+1)(s^-)^{\tau+1}a(\sigma)^{r^+-(\tau+1)s^-}}{C_1 r^+}, \frac{(\tau+1)(s^-)^{\tau+1}b(\sigma)^{2r^+-(\tau+1)s^-}}{2C_1(r^+)^2} \right\}.$$

In fact, if $\kappa^* = \frac{1}{3}\frac{(\tau+1)(s^-)^{\tau+1}a(\sigma)^{p^+-(\tau+1)s^-}}{C_1 r^+}$, and assuming that $\kappa = \kappa^*$, then

$$\begin{aligned}
\mathrm{a}^{\alpha,\beta;\psi}_{\kappa}(u) &\geq \frac{a}{r^+}\sigma^{r^+} + \frac{b}{2(r^+)^2}\sigma^{2r^+} - \frac{1}{3}\frac{(\tau+1)(s^-)^{\tau+1}a(\sigma)^{r^+-(\tau+1)s^-}C_1(\sigma)^{(\tau+1)s^-}}{(\tau+1)(s^-)^{\tau+1}C_1 r^+} \\
&= \frac{b}{2(r^+)^2}\sigma^{2r^+} + \frac{2}{3}\frac{a}{r^+}\sigma^{r^+} > 0.
\end{aligned} \tag{3.48}$$

On the other hand, if $\kappa^* = \frac{1}{3}\frac{(\tau+1)(s^-)^{\tau+1}b(\sigma)^{2p^+-(\tau+1)s^-}}{2C_1(r^+)^2}$, and suppose that $\kappa = \kappa^*$, we have

$$\begin{aligned}
\mathrm{a}^{\alpha,\beta;\psi}_{\kappa}(u) &\geq \frac{a}{r^+}\|u\|^{r^+} + \frac{b}{2(r^+)^2}\|u\|^{2r^+} - \frac{\kappa}{(\tau+1)(s^-)^{\tau+1}}(C\|u\|)^{(\tau+1)s^-} \\
&= \frac{a}{r^+}\sigma^{r^+} + \frac{1}{3}\frac{b}{(r^+)^2}\sigma^{2r^+} > 0.
\end{aligned} \tag{3.49}$$

Furthermore, returning to (3.39), note that $(\tau+1)s^- < r^- \leq 2r^+$ and $\|u\| < 1$. Therefore, from the inequalities (3.48) and (3.49), it follows that

$$\mathrm{a}^{\alpha,\beta;\psi}_{\kappa}(u) \geq \delta > 0, \text{if } 0 < \kappa < \kappa^*.$$

In this sense, we obtain that

$$\inf_{\partial B_\sigma(0)} \mathrm{a}^{\alpha,\beta;\psi}_{\kappa}(u) > 0.$$

Let $\phi \in C_0^\infty(\Omega)$ be such that $\operatorname{supp}\phi \supset \Omega_0$, $\phi(\xi) = 1$ for all $\xi \in \Omega_0$ and $0 \le \phi \le 1$ in Ω. Note that for $t \in (0,1)$, sufficiently small, we have $t\phi \in B_\sigma(0)$ and $\mathrm{a}_\kappa^{\alpha,\beta;\psi}(t\phi) < 0$. In fact, it is immediate that

$$\begin{aligned}\mathrm{a}_\kappa^{\alpha,\beta;\psi}(t\phi) = {} & a\int_\Omega \frac{1}{r(\xi)}\left|\mathbf{D}_{0+}^{\alpha,\beta;\psi} t\phi\right|^{r(\xi)} d\xi + \frac{b}{2}\left[\int_\Omega \frac{1}{r(\xi)}\left|\mathbf{D}_{0+}^{\alpha,\beta;\psi} t\phi\right|^{r(\xi)} d\xi\right]^2 \\ & - \frac{\kappa}{r+1}\left[\int_\Omega \frac{1}{s(\xi)}\left|\mathbf{D}_{0+}^{\alpha,\beta;\psi} t\phi\right|^{s(\xi)} d\xi\right]^{\tau+1}.\end{aligned} \tag{3.50}$$

Since $t \in (0, 1)$, we have $t^{r^-} \ge t^{r(\xi)}$ and $\dfrac{1}{r^-} \ge \dfrac{1}{r(\xi)}$. Hence, it follows that

$$a\int_\Omega \frac{1}{r(\xi)}\left|\mathbf{D}_{0+}^{\alpha,\beta;\psi} t\phi\right|^{r(\xi)} d\xi \le \frac{at^{r^-}}{r^-}\int_\Omega \left|\mathbf{D}_{0+}^{\alpha,\beta;\psi}\phi\right|^{r(\xi)} d\xi \tag{3.51}$$

and

$$\frac{b}{2}\left[\int_\Omega \frac{1}{r(\xi)}\left|\mathbf{D}_{0+}^{\alpha,\beta;\psi} t\phi\right|^{r(\xi)} d\xi\right]^2 \le \frac{bt^{r^-}}{2(r^-)^2}\left[\int_\Omega \left|\mathbf{D}_{0+}^{\alpha,\beta;\psi}\phi\right|^{r(\xi)} d\xi\right]^2. \tag{3.52}$$

Using the inequality (3.28) (see Theorem 3.2), there exists $\varepsilon_0 > 0$ such that $s(\xi)(\tau + 1) \le s^-(\tau + 1) + \varepsilon_0(\tau + 1) < r^-$, $\forall \xi \in \Omega_0 \subset \Omega$. Since $\dfrac{1}{s^+} \le \dfrac{1}{s(\xi)}$, yields that

$$\frac{\kappa}{r+1}\left[\int_\Omega \frac{1}{s(\xi)}|t\phi|^{s(\xi)} d\xi\right]^{\tau+1} \ge \frac{\kappa}{r+1}\,\frac{t^{(\tau+1)s^- + \varepsilon_0(\tau+1)}}{(s^+)^{\tau+1}}\left[\int_\Omega |\phi|^{s(\xi)} d\xi\right]^{\tau+1}. \tag{3.53}$$

Using (3.50)–(3.53), it follows that

$$\begin{aligned}\mathrm{a}_\kappa^{\alpha,\beta;\psi}(t\phi) \le {} & \frac{at^{r^-}}{r^-}\int_\Omega \left|\mathbf{D}_{0+}^{\alpha,\beta;\psi}\phi\right|^{r(\xi)} d\xi + \frac{bt^{r^-}}{2(r^-)^2}\left[\int_\Omega \left|\mathbf{D}_{0+}^{\alpha,\beta;\psi}\phi\right|^{r(\xi)} d\xi\right]^2 \\ & - \frac{\kappa}{r+1}\,\frac{t^{(\tau+1)s^- + \varepsilon_0(\tau+1)}}{(s^+)^{\tau+1}}\left[\int_\Omega |\phi|^{s(\xi)} d\xi\right]^{\tau+1}.\end{aligned}$$

Therefore, since $\displaystyle\int_\Omega \left|\mathbf{D}_{0+}^{\alpha,\beta;\psi}\phi\right|^{r(\xi)} d\xi > 0$, we obtain that $\mathrm{a}_\kappa^{\alpha,\beta;\psi}(t\phi) < 0$.

As for all $u \in B_\sigma(0)$, we have

$$\mathrm{a}_\kappa^{\alpha,\beta;\psi}(u) \ge \frac{a}{r^+}\|u\|^{r^+} + \frac{b}{2(r^+)^2}\|u\|^{2r^+} - \frac{\kappa C_1}{(\tau+1)(s^-)^{\tau+1}}\|u\|^{(\tau+1)s^-},$$

thus, it follows that

$$-\infty < \underline{c} := \inf_{B_\sigma(0)} \mathrm{a}_\kappa^{\alpha,\beta;\psi} < 0.$$

Applying Ekeland's Variational Principle to the functional $\mathrm{a}_\kappa^{\alpha,\beta;\psi} : \overline{B_\sigma}(0) \to \mathbb{R}$, we have $(w_n) \in B_\sigma(0)$ such that

$$\mathsf{a}_{\kappa}^{\alpha,\beta;\psi}(w_n) \to \underline{c} \text{ and } \left(\mathsf{a}_{\kappa}^{\alpha,\beta;\psi}\right)'(w_n) \to 0.$$

Therefore, by the same idea as the Theorem 3.2, we conclude that $w = \lim_{n\to\infty} w_n$ is a nontrivial weak solution to the problem (1.1).

Theorem 3.5. *Suppose that* $1 < \alpha r(\xi) < N, 0 < \alpha < 1$ with $1 < s(\xi) < r_{\alpha}^{*}(\xi) = \frac{Nr(\xi)}{N - \alpha r(\xi)}$ for *all* $\xi \in (\bar{\Omega})$. Let $\mathrm{M}(t) = t^{\mu-1}$ *such that* $\frac{\mu(r^+)^{\mu}}{(r^-)^{\mu-1}} < \frac{(s^-)^{\tau+1}(\tau+1)}{(s^+)^{r}}$ *and* $(\tau+1)s^- > \mu r^+ \geq \mu r^- > 1.$
Then the *problem* (1.1) *has a weak positive solution for all* $\kappa > 0$.
Proof. Firstly, note that

$$\widehat{\square}(t) = \int_0^t s^{\mu-1}\, ds = \frac{t^{\mu}}{\mu}.$$

Thus, taking $t = \int_{\Omega} \frac{1}{r(\xi)} \left| \mathsf{D}_{0+}^{\alpha,\beta;\psi} u \right|^{r(\xi)} d\xi$, we obtain that

$$\begin{aligned}\mathsf{a}_{\kappa}^{\alpha,\beta;\psi}(u) &= \widehat{\square}\left(\int_{\Omega} \frac{1}{r(\xi)} \left| \mathsf{D}_{0+}^{\alpha,\beta;\psi} u \right|^{r(\xi)} d\xi \right) - \frac{\kappa}{r+1} \left[\int_{\Omega} \frac{1}{s(\xi)} \left| u \right|^{s(\xi)} d\xi \right]^{\tau+1} \\ &= \frac{1}{\mu}\left[\int_{\Omega} \frac{1}{r(\xi)} \left| \mathsf{D}_{0+}^{\alpha,\beta;\psi} u \right|^{r(\xi)} d\xi \right]^{\mu} - \frac{\kappa}{r+1} \left[\int_{\Omega} \frac{1}{s(\xi)} \left| u \right|^{s(\xi)} d\xi \right]^{\tau+1},\end{aligned} \tag{3.54}$$

for all $\kappa > 0$.

Note that $\frac{1}{r^+} \leq \frac{1}{r(\xi)}$ and $\frac{1}{s(\xi)} \leq \frac{1}{s^+}$. So, we have

$$\frac{1}{\mu}\left[\int_{\Omega} \frac{1}{r(\xi)} \left| \mathsf{D}_{0+}^{\alpha,\beta;\psi} u \right|^{r(\xi)} d\xi \right]^{\mu} \geq \frac{1}{\mu(r^+)^{\mu}} \left[\int_{\Omega} \left| \mathsf{D}_{0+}^{\alpha,\beta;\psi} u \right|^{r(\xi)} d\xi \right]^{\mu} \tag{3.55}$$

and

$$\frac{\kappa}{r+1}\left[\int_{\Omega} \frac{1}{s(\xi)} \left| u \right|^{s(\xi)} d\xi \right]^{\tau+1} \leq \frac{\kappa}{(\tau+1)(s^-)^{\tau+1}} \left[\int_{\Omega} \left| u \right|^{s(\xi)} d\xi \right]^{\tau+1}. \tag{3.56}$$

Using (3.54)–(3.56), it follows that

$$\mathsf{a}_{\kappa}^{\alpha,\beta;\psi}(u) \geq \frac{1}{\mu(r^+)^{\mu}} \left[\int_{\Omega} \left| \mathsf{D}_{0+}^{\alpha,\beta;\psi} u \right|^{r(\xi)} d\xi \right]^{\mu} - \frac{\kappa}{(\tau+1)(s^-)^{\tau+1}} \left[\int_{\Omega} \left| u \right|^{s(\xi)} d\xi \right]^{\tau+1}. \tag{3.57}$$

Since $\|u\| < 1$ by (3.57) and the inequalities (3.6) and (3.11), we obtain

$$\mathsf{a}_{\kappa}^{\alpha,\beta;\psi}(u) \geq \frac{1}{\mu(r^+)^{\mu}} \|u\|^{\mu r^+} - \frac{\kappa}{(\tau+1)(s^-)^{\tau+1}} \left(C\|u\| \right)^{(\tau+1)s^-}. \tag{3.58}$$

Taking $\|u\| = \sigma$ in (3.58), yields that

$$\mathsf{a}_{\kappa}^{\alpha,\beta;\psi}(u) \geq \sigma^{\mu r^{+}} \left[\frac{1}{\mu (r^{+})^{\mu}} - \frac{\kappa C_1}{(\tau+1)(s^{-})^{\tau+1}} \sigma^{(\tau+1)s^{-}-\mu r^{+}} \right].$$

Since $s^{-}(r+1) > \mu r^{+}$, we find positive numbers a, σ such that

$$\mathsf{a}_{\kappa}^{\alpha,\beta;\psi}(u) \geq a > 0 \text{ if } \|u\| = \sigma, \forall \kappa > 0.$$

Let $0 < \omega \in \mathrm{N}_{r(\xi),0}^{\alpha,\beta;\psi}(\Omega)$. Note that

$$\begin{aligned} \mathsf{a}_{\kappa}^{\alpha,\beta;\psi}(t\omega) &= \square \left(\int_{\Omega} \frac{1}{r(\xi)} \left| \mathsf{D}_{0+}^{\alpha,\beta;\psi} t\omega \right|^{r(\xi)} d\xi \right) - \frac{\kappa}{r+1} \left[\int_{\Omega} \frac{1}{s(\xi)} \left| t\omega \right|^{s(\xi)} d\xi \right]^{\tau+1} \\ &= \frac{1}{\mu} \left[\int_{\Omega} \frac{1}{r(\xi)} \left| \mathsf{D}_{0+}^{\alpha,\beta;\psi} t\omega \right|^{r(\xi)} d\xi \right]^{\mu} - \frac{\kappa}{r+1} \left[\int_{\Omega} \frac{1}{s(\xi)} \left| t\omega \right|^{s(\xi)} d\xi \right]^{\tau+1}. \end{aligned} \tag{3.59}$$

Assuming $t > 1$, note that $t^{r(\xi)} \leq t^{r^{+}}$ and $\frac{1}{r(\xi)} \leq \frac{1}{r^{-}}$, we have

$$\frac{1}{\mu} \left[\int_{\Omega} \frac{1}{r(\xi)} \left| \mathsf{D}_{0+}^{\alpha,\beta;\psi} t\omega \right|^{r(\xi)} d\xi \right]^{\mu} \leq \frac{t^{\mu r^{+}}}{\mu (r^{-})^{\mu}} \int_{\Omega} \left| \mathsf{D}_{0+}^{\alpha,\beta;\psi} \omega \right|^{r(\xi)} d\xi. \tag{3.60}$$

In addition to the above, for $t^{s^{-}} \leq t^{s(\xi)}$ and $\frac{1}{s^{+}} \leq \frac{1}{s(\xi)}$, one has

$$\frac{\kappa}{r+1} \left[\int_{\Omega} \frac{1}{s(\xi)} \left| t\omega \right|^{s(\xi)} d\xi \right]^{\tau+1} \geq \frac{\kappa}{r+1} \frac{t^{(\tau+1)s^{-}}}{(s^{+})^{\tau+1}} \left[\int_{\Omega} |\omega|^{s(\xi)} d\xi \right]^{\tau+1}. \tag{3.61}$$

Using the inequalities (3.59)–(3.61), and since $(\tau+1)s^{-} > \mu r^{+}$, yields

$$\mathsf{a}_{\kappa}^{\alpha,\beta;\psi}(t\omega) \leq \frac{t^{\mu r^{+}}}{\mu (r^{-})^{\mu}} \int_{\Omega} \left| \mathsf{D}_{0+}^{\alpha,\beta;\psi} \omega \right|^{r(\xi)} d\xi - \frac{\kappa}{r+1} \frac{t^{(\tau+1)s^{-}}}{(s^{+})^{\tau+1}} \left[\int_{\Omega} \left| \omega \right|^{s(\xi)} d\xi \right]^{\tau+1} \to -\infty$$

when $t \to +\infty$ from which we conclude that $\mathsf{a}_{\kappa}^{\alpha,\beta;\psi}(\cdot)$ satisfies the geometries of the Mountain Pass Theorem.

We assert that $\mathsf{a}_{\kappa}^{\alpha,\beta;\psi}(\cdot)$ satisfies the condition (*PS*). Indeed, let $(u_n) \subset \mathrm{N}_{r(\xi),0}^{\alpha,\beta;\psi}(\Omega)$, a sequence such that

$$\mathsf{a}_{\kappa}^{\alpha,\beta;\psi}(u_n) \to C_{\kappa} \text{ and } \left(\mathsf{a}_{\kappa}^{\alpha,\beta;\psi} \right)'(u_n) \to 0.$$

Taking $\frac{\mu (r^{+})^{\mu}}{(r^{-})^{\mu-1}} < \theta < \frac{(s^{-})^{\tau+1}(\tau+1)}{(s^{+})^{r}}$, and note that

$$
\begin{aligned}
C_\kappa + \|u_n\| &\geq \mathrm{a}_\kappa^{\alpha,\beta;\psi}(u_n) - \frac{1}{\theta}\left(\mathrm{a}_\kappa^{\alpha,\beta;\psi}\right)'(u_n)u_n \\
&= \frac{1}{\mu}\left[\int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{r(\xi)}d\xi\right]^\mu \\
&\quad - \frac{\kappa}{r+1}\left[\int_\Omega \frac{1}{s(\xi)}|u_n|^{s(\xi)}d\xi\right]^{\tau+1} + \frac{\kappa}{\theta}\left[\int_\Omega \frac{1}{s(\xi)}|u_n|^{s(\xi)}d\xi\right]^r \int_\Omega |u_n|^{s(\xi)}d\xi \\
&\quad - \frac{1}{\theta}\left[\int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{r(\xi)}d\xi\right]^{\mu-1}\int_\Omega\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{r(\xi)}d\xi.
\end{aligned}
\tag{3.62}
$$

Since $\frac{1}{r^+} \leq \frac{1}{r(\xi)} \leq \frac{1}{r^-}$, one has

$$
\frac{1}{\mu}\left[\int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{r(\xi)}d\xi\right]^\mu \geq \frac{1}{\mu(r^+)^\mu}\left[\int_\Omega\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{r(\xi)}d\xi\right]^\mu
\tag{3.63}
$$

and

$$
\frac{1}{\theta}\left[\int_\Omega \frac{1}{r(\xi)}\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{r(\xi)}d\xi\right]^{\mu-1}\int_\Omega\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{r(\xi)}d\xi \leq \frac{1}{\theta(r^-)^{\mu-1}}\left[\int_\Omega\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{s(\xi)}d\xi\right]^\mu.
\tag{3.64}
$$

On the other hand, $\frac{1}{s^+} \leq \frac{1}{s(\xi)} \leq \frac{1}{s^-}$, it,s follows that

$$
\frac{\kappa}{r+1}\left[\int_\Omega \frac{1}{s(\xi)}(u_n)^{s(\xi)}d\xi\right]^{\tau+1} \leq \frac{\kappa}{(s^-)^{\tau+1}(\tau+1)}\left[\int_\Omega |u_n|^{s(\xi)}d\xi\right]^{\tau+1}
\tag{3.65}
$$

and

$$
\frac{\kappa}{\theta}\left[\int_\Omega \frac{1}{s(\xi)}|u_n|^{s(\xi)}d\xi\right]^r\int_\Omega|u_n|^{s(\xi)}d\xi \geq \frac{\kappa}{\theta(s^+)^r}\left[\int_\Omega|u_n|^{s(\xi)}d\xi\right]^{\tau+1}.
\tag{3.66}
$$

Using the inequalities (3.62)–(3.66), we obtain

$$
\begin{aligned}
C + \|u_n\| &\geq \left(\frac{1}{\mu(r^+)^\mu} - \frac{1}{\theta(r^-)^{\mu-1}}\right)\left[\int_\Omega\left|\mathrm{D}_{0+}^{\alpha,\beta;\psi}u_n\right|^{r(\xi)}d\xi\right]^\mu \\
&\quad + \kappa\left(\frac{1}{\theta(s^+)^r} - \frac{1}{(s^-)^{\tau+1}(\tau+1)}\right)\left[\int_\Omega |u_n|^{s(\xi)}d\xi\right]^{\tau+1}.
\end{aligned}
\tag{3.67}
$$

Note that

$$
\frac{\mu(r^+)^\mu}{(r^-)^{\mu-1}} < \theta \Rightarrow \mu(r^+)^\mu < \theta(r^-)^{\mu-1}
$$

and

$$\theta(s^+)^r < (s^-)^{\tau+1}(\tau+1),$$

are positive terms.

Therefore, from (3.67), we have

$$C+\|u_n\| \geq \left(\frac{1}{\mu(r^+)^{\mu}}-\frac{1}{\theta(r^-)^{\mu-1}}\right)\left[\int_\Omega \left|\mathbf{D}^{\alpha,\beta;\psi}_{0+}u_n\right|^{r(\xi)}d\xi\right]^{\mu}. \tag{3.68}$$

Now, suppose that the sequence (u_n) is unbound. Moving on to a subsequence, if necessary, we take $\|u_n\| > 1$ and considering (3.23) and (3.68), we obtain

$$C+\|u_n\| \geq \left(\frac{1}{\mu(r^-)^{\mu}}-\frac{1}{\theta(r^-)^{\mu-1}}\right)\|u_n\|^{\mu r^-},$$

which is absurd, since $\mu r^- > 1$. Consequently, (u_n) is bounded in $\mathcal{N}^{\alpha,\beta;\psi}_{p(\xi),0}(\Omega)$, which is a reflexive Banach space. Thus, there exists a subsequence (u_n), such that $(u_n) \rightharpoonup u$ in $\mathcal{N}^{\alpha,\beta;\psi}_{r(\xi),0}(\Omega)$. Using $\left(\mathfrak{a}^{\alpha,\beta;\psi}_{\kappa}\right)'(u_n)\to 0$, we have

$$\begin{aligned}\left(\mathfrak{a}^{\alpha,\beta;\psi}_{\kappa}\right)'(u_n)(u_n-u) &= \left[\int_\Omega \frac{1}{r(\xi)}\left|\mathbf{D}^{\alpha,\beta;\psi}_{0+}u_n\right|^{r(\xi)}d\xi\right]^{\mu-1}\Upsilon_{r(\xi)}(u_n)(u_n-u)\\ &\quad -\kappa\left[\int_\Omega \frac{1}{s(\xi)}|u_n|^{s(\xi)}d\xi\right]^{r}\int_\Omega |u_n|^{s(\xi)-2}u_n(u_n-u)d\xi \to 0.\end{aligned}$$

Using the idea of Eq.(3.25) (see Theorem 3.1), yields

$$\kappa\left[\int_\Omega \frac{1}{s(\xi)}|u_n|^{s(\xi)}d\xi\right]^{r}\int_\Omega |u_n|^{s(\xi)-2}u_n(u_n-u)d\xi \to 0.$$

Furthermore, there are positive constants b_1 and b_2 so that

$$b_1 \leq \int_\Omega \frac{1}{r(\xi)}\left|\mathbf{D}^{\alpha,\beta;\psi}_{0+}u_n\right|^{r(\xi)}d\xi \leq b_2$$

and

$$\Upsilon_{r(\xi)}(u_n)(u_n-u) = \int_\Omega \left|\mathbf{D}^{\alpha,\beta;\psi}_{0+}u_n\right|^{r(\xi)-2}\mathbf{D}^{\alpha,\beta;\psi}_{0+}u_n\,\mathbf{D}^{\alpha,\beta;\psi}_{0+}(u_n-u)d\xi \to 0.$$

Also note

$$\Upsilon_{r(\xi)}(u)(u_n-u) = \int_\Omega \left|\mathbf{D}^{\alpha,\beta;\psi}_{0+}u\right|^{r(\xi)-2}\mathbf{D}^{\alpha,\beta;\psi}_{0+}u\,\mathbf{D}^{\alpha,\beta;\psi}_{0+}(u_n-u)d\xi \to 0.$$

Consequently,

$$\langle \Upsilon_{r(\xi)}(u_n)-\Upsilon_{r(\xi)}(u),u_n-u\rangle \to 0.$$

We conclude by Theorem 2.9, that $(u_n) \to u$ in $\mathrm{N}^{\alpha,\beta;\psi}_{r(\xi),0}(\Omega)$. Therefore $\mathrm{a}^{\alpha,\beta;\psi}_{\kappa}(\cdot)$ satisfies the condition (*PS*) and, by the Mountain Pass Theorem, the problem 1.1 has a weak positive solution for $\kappa > 0$.

Theorem 3.6. *Suppose that* $1 < \alpha r(\xi) < N$, $0 < \alpha < 1$ with $1 < s(\xi) < r^*_\alpha(\xi) = \dfrac{Nr(\xi)}{N-\alpha r(\xi)}$ *for all* $\xi \in (\bar{\Omega})$. Let $\mathrm{M}(t) = t^{\mu-1}$. *If* $(\tau + 1)s^- < \mu r^-$, *then there exists* $\kappa^* > 0$ *such that the problem* (1.1) *has a positive solution* u_κ *for each* $\kappa \in (0, \kappa^*)$.

Proof. Note that if $\|u_n\| < 1$, then we have through the inequalities (3.6) and (3.11), we have that

$$\left[\int_\Omega \left|\mathbf{D}^{\alpha,\beta;\psi}_{0+} u\right|^{r(\xi)} d\xi\right]^{\mu} \geq \|u\|^{\mu r^+} \quad \text{and} \quad \left[\int_\Omega |u|^{s(\xi)} d\xi\right]^{\tau+1} \leq C\|u\|^{(\tau+1)s^-}.$$

Furthermore, since $\dfrac{1}{r^+} \leq \dfrac{1}{r(\xi)}$ and $\dfrac{1}{s(\xi)} \leq \dfrac{1}{s^-}$, we obtain

$$\frac{1}{\mu}\left[\int_\Omega \frac{1}{r(\xi)}\left|\mathbf{D}^{\alpha,\beta;\psi}_{0+} u\right|^{r(\xi)} d\xi\right]^{\mu} \geq \frac{1}{\mu(r^+)^{\mu}}\|u\|^{\mu r^+} \tag{3.69}$$

and

$$\frac{\kappa}{r+1}\left[\int_\Omega \frac{1}{s(\xi)}|u|^{s(\xi)} d\xi\right]^{\tau+1} \leq \frac{\kappa}{(\tau+1)(s^-)^{\tau+1}}\left(C\|u_n\|\right)^{(\tau+1)s^-}. \tag{3.70}$$

By (3.69) and (3.70), it follows that

$$\begin{aligned}\mathrm{a}^{\alpha,\beta;\psi}_{\kappa}(u) &= \frac{1}{\mu}\left[\int_\Omega \frac{1}{r(\xi)}\left|\mathbf{D}^{\alpha,\beta;\psi}_{0+} u\right|^{r(\xi)} d\xi\right]^{\mu} - \frac{\kappa}{r+1}\left[\int_\Omega \frac{1}{s(\xi)}|u|^{s(\xi)} d\xi\right]^{\tau+1} \\ &\geq \frac{1}{\mu(r^+)^{\mu}}\|u\|^{\mu r^+} - \frac{\kappa C_1}{(\tau+1)(s^-)^{\tau+1}}\|u_n\|^{(\tau+1)s^-}.\end{aligned}$$

Replacing ||u|| with σ and highlighting $\sigma^{(\tau+1)s^-}$, we obtain

$$\mathrm{a}^{\alpha,\beta;\psi}_{\kappa}(u) \geq \sigma^{(\tau+1)s^-}\left[\frac{1}{\mu(r^+)^{\mu}}\sigma^{\mu r^+ - (\tau+1)s^-} - \frac{\kappa C_1}{(\tau+1)(s^-)^{\tau+1}}\right] \geq \delta > 0, \text{ if } 0 < \kappa < \kappa^*,$$

for some $\kappa^* > 0$. Thus, for $\kappa \in (0, \kappa^*)$, we have

$$\inf_{\partial B_\sigma(0)} \mathrm{a}^{\alpha,\beta;\psi}_{\kappa} > 0.$$

Given $\varepsilon_0 > 0$, there exists $\Omega_0 \subset \Omega$ such that $|s(\xi) - s^-| < \varepsilon_0$ for all $\xi \in \Omega_0$ (see Theorem 3.2). Analogously to (3.28), $s(\xi)(\tau + 1) \leq (\tau + 1)s^- + \varepsilon_0(\tau + 1) < \mu r^-$ or $s(\xi) < s^- + \varepsilon_0$, $\forall \xi \in \Omega_0$.

Let $\phi \in C_0^\infty(\Omega)$ be such that supp $\phi \supset \Omega_0$, $\phi(\xi) = 1$ for all $\xi \in \Omega_0$ and $0 \leq \phi \leq 1$ in Ω. For sufficiently small t, $t \in (0, 1)$, we have $t\phi \in B_\sigma(0)$. Note that

$$\mathrm{a}^{\alpha,\beta;\psi}_{\kappa}(t\phi) = \frac{1}{\mu}\left[\int_\Omega \frac{1}{r(\xi)}\left|\mathbf{D}^{\alpha,\beta;\psi}_{0+} t\phi\right|^{r(\xi)} d\xi\right]^{\mu} - \frac{\kappa}{r+1}\left[\int_\Omega \frac{1}{s(\xi)}|t\phi|^{s(\xi)} d\xi\right]^{\tau+1}. \tag{3.71}$$

Since $t \in (0,1)$, we have $t^{r(\xi)} \le t^{r^-}$. Furthermore $\frac{1}{r(\xi)} \le \frac{1}{r^-}$. In this sense, from the first part of Eq.(3.71), we have

$$\frac{1}{\mu}\left[\int_\Omega \frac{1}{r(\xi)}\left|\mathsf{D}^{\alpha,\beta;\psi}_{0+} t\phi\right|^{r(\xi)} d\xi\right]^{\mu} \le \frac{t^{\mu r^-}}{\mu(r^-)^{\mu}}\left[\int_\Omega \left|\mathsf{D}^{\alpha,\beta;\psi}_{0+}\phi\right|^{r(\xi)} d\xi\right]^{\mu}. \tag{3.72}$$

Since $t^{s^-(r+1)+\square_0(r+1)} \le t^{s(\xi)(\tau+1)}$ and $\frac{1}{s^+} \le \frac{1}{s(\xi)}$, it follows that

$$\frac{\kappa}{r+1}\left[\int_\Omega \frac{1}{s(\xi)}|t\phi|^{s(\xi)} d\xi\right]^{\tau+1} \ge \frac{\kappa}{r+1}\,\frac{t^{(\tau+1)s^-+\square_0(\tau+1)}}{(s^+)^{\tau+1}}\left[\int_\Omega |\phi|^{s(\xi)} d\xi\right]^{\tau+1}. \tag{3.73}$$

Using Eq.(3.71) and the inequalities (3.72)–(3.73), we have

$$\mathsf{a}_\kappa^{\alpha,\beta;\psi}(t\phi) \le \frac{t^{\mu r^-}}{\mu(r^-)^{\mu}}\left[\int_\Omega \left|\mathsf{D}^{\alpha,\beta;\psi}_{0+}\phi\right|^{r(\xi)} d\xi\right]^{\mu} - \frac{\kappa}{r+1}\,\frac{t^{(\tau+1)s^-+\square_0(\tau+1)}}{(s^+)^{\tau+1}}\left[\int_\Omega |\phi|^{s(\xi)} d\xi\right]^{\tau+1}.$$

Like $\int_\Omega \left|\mathsf{D}^{\alpha,\beta;\psi}_{0+}\phi\right|^{r(\xi)} d\xi > 0$, and $s(\xi)(\tau+1) \le (\tau+1)s^- + \square_0(\tau+1) < \mu r^-$, follows that $\mathsf{a}_\kappa^{\alpha,\beta;\psi}(t\phi) \to -\infty$ when $t \to +\infty$. Therefore, $\mathsf{a}_\kappa^{\alpha,\beta;\psi}(t\phi) < 0$

Since for all $u \in B_\sigma(0)$ we have

$$\mathsf{a}_\kappa^{\alpha,\beta;\psi}(u) \ge \frac{1}{\mu(r^+)^{\mu}}\|u\|^{\mu r^+} - \frac{\kappa C_1}{(\tau+1)(s^-)^{\tau+1}}\|u\|^{(\tau+1)s^-}.$$

In this sense, we have

$$-\infty < \underline{c} := \inf_{B_\sigma(0)} \mathsf{a}_\kappa^{\alpha,\beta;\psi} < 0.$$

Applying Ekeland's Variational Principle to the functional $\mathsf{a}_\kappa^{\alpha,\beta;\psi} : \bar{B}_\sigma(0) \to \mathbb{R}$, there is a sequence (w_n) such that

$$\mathsf{a}_\kappa^{\alpha,\beta;\psi}(w_n) \to \underline{c} \text{ and } \left(\mathsf{a}_\kappa^{\alpha,\beta;\psi}\right)'(w_n) \to 0.$$

Therefore, in this way, we have, in a similar way to what was demonstrated in Theorem 3.2 that $w = \lim_{n\to\infty} w_n$ is a non-trivial solution for the problem (1.1).

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Data availability statement

Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

Conflict of interests

There is no conflict of interest.

References

[1] Afrouzi, G. A., and M. Mirzapour., Eigenvalue problems for $p(x)$-Kirchhoff type equations. Electron. J. Differ. Equ. 253: 1–10, 2013. 2, 6

[2] Alves, C. O., Correa, F. J. S. A. e Ma, T. F., Positive solutions for a quasilinear elliptic equation of Kirchhoff type, Comput. Math. Appl. 49, 2005, 85–93. 3

[3] Ambrosio, V., T. Isernia, and V. D. Radulescu. Concentration of positive solutions for a class of fractional p-Kirchhoff type equations. Proc. Royal Soc. Edinburgh Section A: Math. 151.2: 601–651, 2021. 2, 3

[4] Ambrosio, V., A Kirchhoff Type Equation in R^N involving the fractional (p, q)-Laplacian. The J. Geometric Anal. 32(4): 135, 2022. 2, 3

[5] Ambrosio, V., and T. Isernia. Multiplicity and concentration results for some nonlinear Schrodinger equations with the fractional p-Laplacian." *Disc. Cont. Dyn. Syst.* 38.11 (2018): 5835–5881. 2

[6] Ambrosio, V., 'Nonlinear fractional schrödinger equations in R^N Front. Elliptic Parabol. Probl. Birkhauser/Springer, Cham, 2021, xvii+662 pp. 3

[7] Applebaum, D., Levy processes and stochastic calculus 2nd edn. Cambridge Studies.in Advanced Mathematics, vol. 116. Cambridge University Press, Cambridge, 2009 3

[8] Azroul, E., A. Benkirane, M. Shimi and M. Srati. On a class of fractional $p(x)$-Kirchhoff type problems. Applicable Anal. 100.2: 383–402, 2021. 3

[9] Binlin, Z., V. D. Radulescu, and L. Wang., Existence results for Kirchhoff-type superlinear problems involving the fractional Laplacian. Proc. Royal Soc. Edinburgh Section A: Math. 149(40): 1061–1081, 2019. 2, 3

[10] Boudjeriou, T., On the diffusion $p(x)$-Laplacian with logarithmic nonlinearity. J. Ellip. Parabolic Equ. 6(2): 773–794, 2020. 2

[11] Caffarelli, L., Nonlocal equations, drifts and games. Nonlinear Partial Diff. Equ. 7(2), 37–52, 2012. 3

[12] Cammaroto, F., and L. Vilasi. Multiple solutions for a Kirchhoff-type problem involving the $p(x)$-Laplacian operator. Nonlinear Analysis: Theory, Methods & Applications 74.5: 1841–1852, 2011. 3

[13] Cen, J., C. Vetro, and S. Zeng. A multiplicity theorem for double phase degenerate Kirchhoff problems. Appl. Math. Lett. 146: 108803, 2023. 3

[14] Colasuonno, F., and P. Pucci. Multiplicity of solutions for $p(x)$-polyharmonic elliptic Kirchhoff equations. Nonlinear Analysis: Theory, Methods & Applications 74.17: 5962–5974, 2011. 3

[15] Correa, F. J. S. A., and A. C. R. Costa. A variational approach for a bi-non-local elliptic problem involving the $p(x)$-Laplacian and non-linearity with non-standard growth. Glasgow Math. J. 56(2): 317–333, 2014. 3

[16] Chabrowski, J., and Y. Fu., Existence of solutions for $p(x)$-Laplacian problems on a bounded domain. J. Math. Anal. Appl. 306(2): 604–618, 2005. 2

[17] Chen, Y., S. Levine, and M. Rao. Variable exponent, linear growth functionals in image restoration. SIAM J. Appl. Math. 66(4): 1383–1406, 2006. 2

[18] Dai, G., and R. Ma. Solutions for a $p(x)$-Kirchhoff type equation with Neumann boundary data. Nonlinear Analysis: Real World Applications 12.5: 2666–2680, 2011. 3

[19] Ebmeyer, C., and W. B. Liu., Finite element approximation of the fast diffusion and the porous medium equations. SIAM J. Numer. Anal. 46(5): 2393–2410, 2008. 2

[20] Fan, X. L., Shen, J. S. e Zhao, D., Sobolev embedding theorems for spaces $W^{k,p(x)}$ (Q), J. Math Anal. Appl. 262: 749–760, 2001. 6

[21] Fan, X. L., Zhang, Q. H., Existence of solutions for $p(x)$-Laplacian Dirichlet problems, Nonlinear Anal. 52: 1843v1852, 2003. 6

[22] Fan, X. L. e Zhao, D., On the spaces $L^{p(x)}$ and $W^{m,p(x)}$, J. Math. Anal. Appl. 263: 424–446, 2001. 6

[23] Figueiredo, D. G., The Ekeland variational Principle with applications and detours. Springer - Tata Institute of Fundamental Research Bombay, 1989. 6, 7

[24] Gomes, J. M. e Sanchez, L., On a variational approach to some non-local boundary value problems, Appl. Anal., 84(9): 909–925, 2005. 3

[25] Kirchhoff, G., Mechanik, Teubner, Leipzig, 1883. 2

[26] Laskin, N., Fractional quantum mechanics and Lévy path integrals. Phys. Lett. A. 268, 298–305, 2000. 3

[27] Liu, Z., D. Motreanu, and S. Zeng., Multiple Solutions for a Kirchhoff-type Problem with Vanishing Nonlocal Term and Fractional p-Laplacian. Frontiers Math. 18(5): 1067–1082, 2023. 2

[28] Ma, T. F. Remarks on an elliptic equation of Kirchhoff type. Nonlinear Analysis: Theory, Methods & Applications 63.5-7 (2005): e1967–e1977. 2

[29] Mihailescu, M. e Radulescu, V., A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, Proc. R. Soc. A, 462: 2625–2641, 2006. 2

[30] Nezza, D., G. P. Eleonora, and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. Bull. Sci. Math. 136(5): 521–573, 2012. 3

[31] Orlicz, W., Uber konjugierte Exponentenfolgen, Studia Math. 3: 200–212, 1931. 2

[32] Pan, N., B. Zhang, and J. Cao. Weak solutions for parabolic equations with $p(x)$-growth. Electron. J. Diff. Equ. 209: 15, 2016. 2, 6

[33] Rajagopal, K. R., and M. Ruzicka., On the modeling of electrorheological materials. Mech. Research Commun. 23(4): 401–407, 1996. 2

[34] Rajagopal, K. R., and M. Ruzicka., Mathematical modeling of electrorheological materials. Continuum. Mech. Thermodyn. 13(1): 59–78, 2001. 2

[35] Rionero, S., Triple diffusive convection in porous media. Acta Mechanica 224(2): 447–458, 2013. 2

[36] Ruzicka, M., Electrorheological fluids: modelling and mathematical theory, Springer-Verlag, Berlin, Alemanha, 2000. 2

[37] Sousa , J. Vanterler da C. and Capelas de Oliveira, E., On the 0—Hilfer fractional derivative, Commun. Nonlinear Sci. Numer. Simul., 60: 72–91, 2018 3, 5

[38] Sousa, J. Vanterler da C., Kishor D. Kucche, and Juan J. Nieto. Existence and Multiplicity of Solutions for Fractional K(0)-Kirchhoff-Type Equation. Qual. Theory Dyn. Sys. 23(1): 27, 2024. 3, 5, 11

[39] Sousa, J. Vanterler da C., Mbarki Lamine, and Leandro S. Tavares. Generalized Telegraph equation with fractional $p(x)$-Laplacian. Minimax Theory Appl. 08(2): 423–441, 2023. 3

[40] Sousa, J. Vanterler da C., D. S. Oliveira, and Ravi P. Agarwal. Existence and multiplicity for fractional Dirichlet problem with $\Upsilon(\xi)$-Laplacian equation and Nehari manifold. Appl. Anal. Disc. Math. 17.2 (2023): 480–495. 3

[41] Sousa, J. Vanterler da C., Gabriela L. Araujo, Maria V. S. Sousa and Amalia R. E. Pereira. Multiplicity of solutions for fractional $K(x)$-Laplacian equations. J. Appl. Anal. Comput. 14(3): 1543–1578, 2024. 4

[42] Sousa, J. Vanterler da C., Karla B. Lima, and Leandro S. Tavares. Existence of solutions for a singular double phase problem involving a ψ-Hilfer fractional operator via Nehari manifold. Qual. Theory Dyn. Sys. 22.3: 94, 2023. 4

[43] Sousa, J. Vanterler da C., Jiabin Zuo, and Donal O'Regan. The Nehari manifold for a ψ-Hilfer fractional p-Laplacian. Applicable Anal. 101(14): 5076–5106, 2022. 4

[44] Srivastava, H. M., and J. Vanterler da C. Sousa. Multiplicity of solutions for fractional-order differential equations via the $K(x)$-Laplacian operator and the Genus theory. Fractal and Fractional 6.9: 481, 2022. 5, 7

[45] Straughan, B. Bidispersive double diffusive convection. Inter. J. Heat Mass Transfer 126: 504–508, 2018. 2

[46] Tavares, Leandro S., and J. Vanterler C. Sousa. Multiplicity results for a system involving the $p(x)$-Laplacian operator. Applicable Anal. 102(5): 1271–1280, 2023. 2

[47] Vazquez, J. Luis., Smoothing and decay estimates for nonlinear diffusion equations: equations of porous medium type. Vol. 33. OUP Oxford, 2006. 2

[48] Vetro, C., Variable exponent $p(x)$-Kirchhoff type problem with convection. J. Math. Anal. Appl. 506(2): 125721, 2022. 3