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A Minimax Theorem for Linear Operators

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The aim of this note is to prove the following minimax theorem which generalizes a result by B. Ricceri: let E be an infinite-dimensional Banach space not containing ℓ^1, F be a Banach space, X be a convex subset of E whose interior is non-empty for the weak topology on bounded sets, S and T be linear and continuous operators from E to F, $\varphi : F \to \mathbb{R}$ be a continuous convex coercive map, $J \subset \mathbb{R}$ a compact interval and $\psi : J \to \mathbb{R}$ a convex continuous function. Assume moreover that $S \times T$ has a closed range in $F \times F$ and that S is not compact. Then

$$\sup_{x\in X} \inf_{\lambda\in J} \big(\varphi(Tx - \lambda Sx) + \psi(\lambda)\big) = \inf_{\lambda\in J} \sup_{x\in X} \big(\varphi(Tx - \lambda Sx) + \psi(\lambda)\big) .$$

In particular, if φ is the norm of F and $\psi = 0$, we get

$$\sup_{x\in X} \inf_{\lambda\in J} \|Tx - \lambda Sx\| = \inf_{\lambda\in J} \sup_{x\in X} \|Tx - \lambda Sx\| .$$

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1. Introduction

It was shown by E. Asplund and V. Pták in [AP] that for two normed spaces E and F of dimension ≥ 2 and any two A and B in $\mathcal{L}(E, F)$ the following inequality holds

$$\sup_{\|x\|\leq 1} \inf_{\lambda\in\mathbb{R}} \|Ax + \lambda Bx\| \leq \inf_{\lambda\in\mathbb{R}} \|A + \lambda B\| = \inf_{\lambda\in\mathbb{R}} \sup_{\|x\|\leq 1} \|Ax + \lambda Bx\|$$

and that the equality in the above relation is attained for every pair A, B in $\mathcal{L}(E, F)$ if and only if both E and F are inner product spaces. In the sequel the unit ball of E is replaced by a convex set whose interior is non-empty for the weak topology $\sigma(E, E^*)$ or more generally the 'weak topology on bounded sets' and we are interested in proving such a minimax equality.

For a Banach space E, we will denote by $\beta(E, E^*)$ the finest locally convex topology on E which agrees with the weak topology $\sigma(E, E^*)$ on the bounded

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subsets of E. This topology is finer than $\sigma(E, E^*)$ and coarser than the norm topology : so it is compatible with the duality betweeen E and E^*. Converging sequences for $\beta(E, E^*)$ are all weakly converging sequences, and a subset U of E is open for $\beta(E, E^*)$ if and only if every weakly converging sequence in E whose weak limit is in U has all but finitely many terms inside U. It can be noticed that a closed convex subset C of E is a neighborhood of 0 for $\beta(E, E^*)$ iff its polar set C^0 is norm-compact in E^*.

2. The minimax theorem

The following theorem has been proved by B. Ricceri in [3] (Theorem 3).

Theorem 2.1. *Let E be a infinite-dimensional reflexive Banach space, $T : E \to E$ a non-zero linear compact operator, $\varphi : E \to \mathbb{R}$ a convex continuous and coercive functional, $J \subset \mathbb{R}$ a compact interval with $0 \in J$, $\psi : J \to \mathbb{R}$ be a continuous convex function. Then, for each $r > \varphi(0)$, one has*

$$\sup_{x \in X} \inf_{\lambda \in J} (\varphi(Tx - \lambda x) + \psi(\lambda)) = r + \psi(0)$$

where $X = \{x \in E : \varphi(Tx) \leq r\}$.

Our main theorem extends this latter statement : indeed $\sup_{x\in X} \varphi(Tx - \lambda x) = +\infty$ for all $\lambda \in J \setminus \{0\}$ and so $\inf_{\lambda\in J} \sup_{x\in X} \varphi(Tx - \lambda x) + \psi(\lambda)$ is clearly equal to $r + \psi(0)$.

In order to prove it, we need some preliminary results.

Lemma 2.2. *Let E be a separable Banach space not containing ℓ^1, N be a closed linear subspace of E, $q : E \to E/N$ be the quotient mapping and (y_n) be a sequence in the quotient space E/N which converges weakly to 0. Then there is a subsequence (y_{n_k}) and a sequence (z_k) weakly converging to 0 in E such that $y_{n_k} = qz_k$.*

Proof. The weakly converging sequence (y_n) is bounded. So there is $M \in \mathbb{R}^+$ such that $\sup_n \|y_n\| < M$, and for all n we can find $x_n \in q^{-1}(y_n)$ such that $\|x_n\| < M$. Let θ be an accumulation point of $(\frac{x_n}{M})$ in the compact set $B_{E^{**}}$ equipped with the topology $\sigma(E^{**}, E^*)$. By Rosenthal's and Bourgain-Fremlin-Talagrand's theorems [2], θ is a Baire-1 function on the compact space B_{E^*} equipped with $\sigma(E^*, E)$, and there is a subsequence (x_{n_k}) extracted from (x_n) which converges to $M.\theta$ for $\sigma(E^{**}, E^*)$. Then $y_{n_k} = q^{**}(x_{n_k}) \to q^{**}(M.\theta)$ for $\sigma(E^{**}, E^*)$, hence $q^{**}(M.\theta) = 0$ and since q^{**} is the projection of E^{**} onto E^{**}/N^{**} it follows that $\theta \in N^{**}$. Since $N \subset E$ cannot contain ℓ^1, θ is a Baire-1 function on the compact space B_{N^*} equipped with $\sigma(N^*, N)$ and there is a sequence (w_k) in B_N such that $w_k \to \theta$ for $\sigma(N^{**}, N^*)$ which agrees on $B_{N^{**}}$ with $\sigma(E^{**}, E^*)$. It follows that $\langle \xi, w_k \rangle \to \langle \theta, \xi \rangle$ for all $\xi \in E^*$ and that $z_k = x_{n_k} - M.w_k$ converges to 0 for $\sigma(E^{**}, E^*)$ what means that $q(z_k) = q(x_{n_k}) - M.q(w_k) = q(x_{n_k}) = y_{n_k}$ and $z_k \to 0$ weakly. $\square$

Theorem 2.3. *Let E be a separable Banach space not containing ℓ^1, V be a Banach space and $T \in \mathcal{L}(E, V)$. If for every sequence (v_n) in E weakly converging to 0, the sequence (Tv_n) is relatively compact in V then the operator T is compact.*

Proof. It should be first noticed that a Banach space which contains an isomorphic copy of ℓ^1 cannot be reflexive. So the present theorem applies in particular to reflexive spaces. One can also remark that the hypothesis that E does not have a subspace isomorphic to ℓ^1 cannot be removed: indeed by Schur property, if $E = V = \ell^1$ and T is the identity mapping of E, the operator T is not compact but every weakly converging sequence (x_n) in E converges in norm, and that implies that (Tx_n) is relatively compact in V.

Let (v_n) be a sequence in the unit ball B_E of E. We want to show that there exists a subsequence (v_{n_k}) such that (Tv_{n_k}) converges in V. Let θ an accumulation point of (v_n) in the compact set $B_{E^{**}}$ equipped with the topology $\sigma(E^{**}, E^*)$. Again by Rosenthal's and Bourgain-Fremlin-Talagrand's theorems, θ is a Baire-1 function on the compact space B_{E^*} equipped with $\sigma(E^*, E)$, and there is a subsequence (v_{n_k}) extracted from (v_n) which converges to θ for $\sigma(E^{**}, E^*)$. If the set $\{Tv_{n_k} : k \in \mathbb{N}\}$ is relatively compact in V, there is a subsequence which converges in V.

If not there is some $\varepsilon > 0$ and a sequence extracted from (v_{n_k}), still denoted by (v_{n_k}), such that $\|Tv_{n_k} - Tv_{n_\ell}\| \geq \varepsilon$ for $k \neq \ell$. Consider then the sequence $w_k = v_{n_{k+1}} - v_{n_k}$ which satisfies $\|Tw_k\| \geq \varepsilon$ for all k. Nevertheless we have for all $x^* \in E^*$:

$$\langle x^*, w_k\rangle = \langle x^*, v_{n_{k+1}}\rangle - \langle x^*, v_{n_k}\rangle \to \langle \theta, x^*\rangle - \langle \theta, x^*\rangle = 0$$

which shows that the sequence (w_k) converges weakly to 0 in E. Then the sequence (Tw_k) should be relatively compact in V and converge weakly to 0, hence converge in norm to 0. And this is in contradiction to

$$\|Tw_k\| = \left\|Tv_{n_{k+1}} - Tv_{n_k}\right\| \geq \varepsilon \text{ for all } k.$$

Moreover since the compactness of T follows from the compactness of T_{E_0} for all separable subspace E_0 of E, the conclusion of Theorem 2.3 holds even if E is not assumed to be separable. □

Lemma 2.4. *Let E be a infinite-dimensional Banach space not containing ℓ^1 and X be a convex subset of E whose interior for $\beta(E, E^*)$ is non-empty. Then there exists a Banach space V, a compact linear mapping $\pi : E \to V$ with dense range and a convex open subset Y of V such that $\pi^{-1}(Y) \subset X \subset \overline{\pi^{-1}(Y)}$.*

Proof. It is clear that if $\pi : E \to V$ is compact, it is continuous from (E, β) to $(V, \|.\|)$, hence that the set $\pi^{-1}(Y)$ has necessarily a non-empty $\beta(E, E^*)$-interior as soon as the interior of Y in V itself is non-empty.

Let X_0 be the interior of X for $\beta(E, E^*)$ and $a \in X_0$. Put $W = (X_0 - a) \cap (a - X_0)$ which is a symmetric convex subset of E. Then W is open for $\beta(E, E^*)$ and contains 0. Since $\beta(E,E^*)$ is coarser than the norm-topology, W is absorbing and the Minkowski functional $p_W : x \mapsto \inf\{r > 0 : r^{-1}x \in W\}$ is a semi-norm on E. Denote by V the separated completion of (E, p_W) and π the canonical mapping from E to V. By definition $\pi(E)$ is dense in the Banach space V, and $\pi(W) = \pi(E) \cap B(0,1)$. We will show that $\pi(X_0)$ is open in $\pi(E)$ and that if Y denotes the interior of $\overline{\pi(X_0)} = \overline{\pi(X)}$, we have $X_0 = \pi^{-1}(Y)$. Then since $X_0 \neq \emptyset$ we have $\pi^{-1}(Y) = X_0 \subset X \subset \overline{X_0} = \overline{\pi^{-1}(Y)}$.

Indeed let $b \in \pi(X_0)$ and $u \in X_0$ such that $\pi u = b$. The set $\{t \in \mathbb{R} : u+t(a-u) \in X_0\}$ is open and contains 0. Hence it contains also $-\varepsilon$ for some $\varepsilon > 0$ and we let $v = u - \varepsilon(a-u)$. The homothety with center $v \in X_0$ and ratio $\eta = \dfrac{\varepsilon}{1+\varepsilon} < 1$ transforms X_0 into itself and in particular a into u, hence $a+W \subset X_0$ into $u+\eta\cdot W$. Thus $\pi(X_0) \supset \pi(E) \cap B(b,\eta)$ and $\pi(X_0)$ is a neighborhood of b in $\pi(E)$. Finally if Y is the interior of the convex set $\overline{\pi(X_0)}$ then $Y \cap \pi(E)$ is a convex open subset of $\pi(E)$ contained in $\pi(E) \cap \overline{\pi(X_0)}$ and containing $\pi(X_0)$: indeed if $u \in \pi(X_0)$ we have shown the existence of a ball $B(u,r)$ such that $\pi(E) \cap B(u,r) \subset \pi(X_0)$, hence that $\overline{\pi(E) \cap B(u,r)} \subset B(u,r) \subset \overline{\pi(X_0)}$ and $u \in Y$.

It remains to show the compactness of π. For this by theorem 2.3 it is enough to show that whenever (w_n) is a sequence which converges weakly to 0 in E the sequence (πw_n) converges to 0 in V, hence show that $p_W(w_n) \to 0$. Let $R > 0$; the sequences $(a + R.w_n)$ and $(a - R.w_n)$ converge both weakly to a which is an interior point of X_0 for $\beta(E,E^*)$. So there is an integer N such that $a \pm R.w_n \in X_0$ for all $n \geq N$. This means that $R.w_n \in (X_0 - a) \cap (a - X_0) = W$, hence that $\|\pi w_n\| = p_W(w_n) \leq \dfrac{1}{R}$ if $n \geq N$ and that $\pi w_n \to 0$ in the normed space V. ☐

Lemma 2.5. *Let E be an infinite-dimensional Banach space not containing ℓ^1, F be a Banach space, $J \subset \mathbb{R}$ be a compact interval, S and T in $\mathcal{L}(E,F)$. Assume that $S \times T : E \to F \times F$ is one-to-one and that $(S \times T)(E)$ is closed in $F \times F$. If there is no $\lambda \in J$ such that $T - \lambda S$ is compact then there exists a sequence (w_n) in E converging weakly to 0 and $\varepsilon > 0$ such that $\|Tw_n - \lambda.Sw_n\| \geq \varepsilon$ for all $\lambda \in J$ and all $n \in \mathbb{N}$.*

Proof. Notice first that since $S \times T$ is one-to-one and $(S \times T)(E)$ closed in the space $F \times F$ equipped with the norm $(x,y) \mapsto \|x\| + \|y\|$, there exists by the open mapping Theorem some $\delta > 0$ such that $\|Sx\| + \|Tx\| \geq \delta.\,\|x\|$ for all $x \in E$.

Assume that for all sequence (w_n) weakly converging to 0 in the unit ball B_E of E we had $\lim_n \inf_{\lambda \in J} \|Tw_n - \lambda.Sw_n\| = 0$. Then for every sequence (w_n) weakly converging to 0 in B_E it would exist a sequence (λ_n) in J such that $\|Tw_n - \lambda_n.Sw_n\| \to 0$, hence an accumulation point $\lambda \in J$ of (λ_n), and since

$$\begin{aligned} \|Tw_n - \lambda.Sw_n\| &\leq \|Tw_n - \lambda_n.Sw_n\| + |\lambda_n - \lambda|\,.\,\|Sw_n\| \\ &\leq \|Tw_n - \lambda_n.Sw_n\| + |\lambda_n - \lambda|\,.\,\|S\| \end{aligned}$$

we would have a subsequence of (w_n) and a fixed $\lambda \in J$ with $\|Tw_n - \lambda.Sw_n\| \to 0$.

Then two cases could occur : — either this λ is the same for all sequences (w_n) converging to 0 for the weak topology but no in norm, in B_E — or there are two sequences (w_n) and (x_n) in B_E both converging to 0 weakly but not in norm, and $\lambda \neq \mu$ in J such that $Tw_n - \lambda.Sw_n \to 0$ and $Tx_n - \mu.Sx_n \to 0$.

We show that in the first case the operator $T - \lambda S$ is compact. Indeed for every sequence (w_n) weakly converging to 0 we have $Tw_n - \lambda Sw_n \to 0$ and the conclusion follows from Theorem 2.3.

In the second one, take subsequences of (w_n) and (x_n) satisfying $\inf_n \|w_n\| > 0$ and $\inf_n \|x_n\| > 0$ and choose for all n some $x_n^* \in F^*$ of norm 1 such that $\langle x_n^*, Sx_n\rangle = \|Sx_n\|$. Since (w_p) converges weakly to 0 we have $\lim_{p\to\infty}\langle x_n^*, Sw_p\rangle = \lim_{p\to\infty}\langle S^*x_n^*, w_p\rangle = 0$. So we can replace (w_n) by some subsequence still denoted by (w_n) such that $|\langle x_n^*, Sw_n\rangle| \leq 2^{-n}$ and for every $\zeta \in \mathbb{R}$ we get

$$\begin{aligned}\|Sx_n - \zeta.Sw_n\| \geq \langle x_n^*, Sx_n - \zeta.Sw_n\rangle &\geq \|Sx_n\| - |\zeta|\,.\,|\langle x_n^*, Sw_n\rangle| \\ &\geq \|Sx_n\| - |\zeta|\,.2^{-n}\,.\end{aligned}$$

Put $z_n = \dfrac{1}{2}(w_n + x_n)$; since (z_n) is a sequence in B_E weakly converging to 0 there must exist $\nu \in J$ such that $Tz_n - \nu.Sz_n \to 0$. So we have :

$$\begin{aligned}&Tw_n - \lambda.Sw_n \to 0 \quad ; \quad Tx_n - \mu.Sx_n \to 0 \\ &2(Tz_n - \nu.Sz_n) = Tw_n + Tx_n - \nu.S(x_n + z_n) \to 0\,,\end{aligned}$$

hence $\lambda.Sw_n + \mu.Sx_n - \nu(Sw_n + Sx_n) = (\lambda - \nu).Sw_n + (\mu - \nu).Sx_n \to 0$. If $\mu = \nu$ we get $(\lambda - \mu).Sw_n \to 0$, hence $\|Sw_n\| \to 0$ and $\|Tw_n\| \leq \|Tw_n - \lambda Sw_n\| + |\lambda|\,.\,\|Sw_n\| \to 0$. Then $\|w_n\| \leq \delta^{-1}.(\|Sw_n\| + \|Tw_n\|) \to 0$, a contradiction. And else, with $\zeta = \dfrac{\nu - \lambda}{\mu - \nu}$,

$$\begin{aligned}\|Sx_n\| &\leq \|S(x_n - \zeta w_n)\| + |\zeta|\,.2^{-n} \\ &\leq \frac{1}{|\mu - \nu|}\,\|(\lambda - \nu).Sw_n + (\mu - \nu).Sx_n\| + |\zeta|\,.2^{-n} \to 0\end{aligned}$$

and $\|Tx_n\| \leq \|Tx_n - \mu Sx_n\| + |\mu|\,.\,\|Sx_n\| \to 0$. So $\|x_n\| \leq \delta^{-1}.(\|Sx_n\| + \|Tx_n\|) \to 0$, what is again a contradiction. This second case cannot occur and the proof is complete. ☐

Lemma 2.6. *Let E be an infinite-dimensional Banach space not containing ℓ^1, F and V Banach spaces, $\lambda_0 \neq 0$ be a real number, $S \in \mathcal{L}(E, F)$, $H \in \mathcal{L}(E, F)$ be a compact operator and $T = \lambda_0.S + H$. Assume that $S \times T$ is a one-to-one operator with closed range, Y is a non-empty convex open subset of V, $\pi \in \mathcal{L}(E, V)$ a compact operator with dense range and $X = \pi^{-1}(Y)$. Then for $\lambda \neq \lambda_0$, $\sup_{x\in X} \|Tx - \lambda.Sx\| = +\infty$.*

Proof. Notice first that $S \times H$ is also one-to-one with closed range, since the mapping $(u, v) \mapsto (u, v - \lambda_0 u)$ is an isomorphism from $F \times F$ onto itself.

As above there is a $\delta > 0$ such that $\|Sx\| + \|Hx\| \geq \delta.\|x\|$ for all $x \in E$. Let $y_0 \in Y \cap \pi(E)$, $x_0 \in \pi^{-1}(y_0)$ and $r > 0$ such that $B(y_0, r) \subset Y$. One can find in the unit sphere S_E of E a vector w_0 and a sequence $(w_n)_{n\geq 1}$ such that for all $n \geq 0$ $d(w_{n+1}, \mathrm{span}(w_0, w_1, \dots, w_n)) \geq \dfrac{2}{3}$. As in the proof of Theorem 2.3 we can extract from (w_n) a sequence (w_{n_k}) which converges for $\sigma(E^{**}, E^*)$ to some $w^{**} \in E^{**}$. Then for $k \geq 1$, $z_k = w_{n_k} - w_{n_{k-1}}$ satisfies $\|z_k\| \geq \dfrac{2}{3}$ and (z_k) converges weakly to 0 in E. We deduce that $\|Hz_k\| \to 0$ and that $\|\pi z_k\| \to 0$, since H and π are compact. Then put $t := \dfrac{r}{2^{-k} + \|Hz_k\| + \|\pi z_k\|}$ and $x_k = x_0 + t \cdot z_k$. It follows that

$$\|\pi x_k - y_0\| = \|\pi x_k - \pi x_0\| = t.\|\pi z_k\| < r\ ,$$

hence that $\pi x_k \in B(y_0, r) \subset Y$. This means that $x_k \in X$ and that

$$\begin{aligned}\|Tx_k - \lambda S x_k\| &= \|(\lambda_0 - \lambda).Sx_k + Hx_k\| \\ &\geq |\lambda - \lambda_0|.(\|Sx_k\| + \|Hx_k\|) - (1 + |\lambda - \lambda_0|).\|Hx_k\| \\ &\geq \delta.|\lambda - \lambda_0|.\|x_k\| - (1 + |\lambda - \lambda_0|).\|Hx_k\|\ ,\end{aligned}$$

and at the same time

$$\|x_k\| \geq t.\|z_k\| - \|x_0\| \geq \frac{2}{3} \cdot t - \|x_0\| \to +\infty$$

and $\|Hx_k\| \leq \|Hx_0\| + t.\|Hz_k\| \leq \|Hx_0\| + r$.

Thus $\lim_k \|Tx_k - \lambda S x_k\| = +\infty$, hence $\sup_{x\in X} \|Tx - \lambda.Sx\| = +\infty$. □

Corollary 2.7. *Let E be an infinite-dimensional Banach space not containing ℓ^1, F a Banach space, $\lambda_0 \neq 0$ be a real number, $S \in \mathcal{L}(E, F)$, $H \in \mathcal{L}(E, F)$ be a compact operator, $T = \lambda_0.S + H$ and $\varphi : F \to \mathbb{R}$ be a convex continuous and coercive function. Assume that $S \times T$ is one-to-one with closed range, that Y is a non-empty convex open subset of the normed space V, $\pi \in \mathcal{L}(E, V)$ is a compact operator with dense range and $X = \pi^{-1}(Y)$.*

Then for $\lambda \neq \lambda_0$, $\sup_{x\in X} \varphi(Tx - \lambda.Sx) = +\infty$.

Proof. Since φ is coercive, for each $M \in \mathbb{R}^+$, there is $R > 0$ such that $\varphi(u) < M \implies \|u\| < R$ for all $u \in F$. Following Lemma 2.6, there exists $x \in X$ such that $\|Tx - \lambda.Sx\| \geq R$; then we have for this vector x : $\varphi(Tx - \lambda.Sx) \geq M$. □

Lemma 2.8. *Let E be an infinite-dimensional Banach space not containing ℓ^1, F a Banach space, $S \in \mathcal{L}(E, F)$, $J \subset \mathbb{R}$ be a compact interval, $T \in \mathcal{L}(E, F)$ such that $T - \lambda.S$ be compact for none $\lambda \in J$. Assume that $S \times T$ is a one-to-one operator with closed range, that Y is a non-empty open subset of the normed space V, $\pi \in \mathcal{L}(E, V)$ is a compact operator with dense range and $X = \pi^{-1}(Y)$.*

Then $\sup_{x\in X} \inf_{\lambda\in J} \|Tx - \lambda.Sx\| = +\infty$.

Proof. Let $y_0 \in Y \cap \pi(E)$, $x_0 \in \pi^{-1}(y_0)$ and $r > 0$ such that the closed ball $\tilde{B}(y_0, r)$ is contained in Y. It follows from Lemma 2.5 that one can find in B_E a sequence (w_n) converging weakly to 0 and $\varepsilon > 0$ such that for all $n \in \mathbb{N}$ and all $\lambda \in J$ the inequality $\|Tw_n - \lambda.Sw_n\| \geq \varepsilon$ holds. Then the sequence (πw_n) converges to 0 in V and we put for $n \geq 1$: $x_n = x_0 + \dfrac{r}{2^{-n} + \|\pi w_n\|} \cdot w_n$. We then have

$$\|\pi x_n - y_0\| = \|\pi x_n - \pi x_0\| = \frac{r.\,\|\pi w_n\|}{2^{-n} + \|\pi w_n\|} < r$$

hence $\pi x_n \in B(y_0, r) \subset Y$, it is $x_n \in X$, and for $\lambda \in J$:

$$\begin{aligned} \|Tx_n - \lambda.Sx_n\| &\geq \frac{r}{2^{-n} + \|\pi w_n\|} \cdot \|(T - \lambda.S)w_n\| - \|Tx_0 - \lambda Sx_0\| \\ &\geq \frac{r.\varepsilon}{2^{-n} + \|\pi w_n\|} - \|Tx_0 - \lambda Sx_0\| \ , \end{aligned}$$

hence $\inf_{\lambda\in J} \|Tx_n - \lambda.Sx_n\| \geq \dfrac{r.\varepsilon}{2^{-n} + \|\pi w_n\|} - \|Tx_0 - \lambda Sx_0\|$ and

$$\sup_{x\in X} \inf_{\lambda\in J} \|Tx - \lambda.Sx\| \geq \sup_n \frac{r.\varepsilon}{2^{-n} + \|\pi w_n\|} - \|Tx_0 - \lambda Sx_0\| = +\infty \ ,$$

which completes the proof of the lemma. □

Corollary 2.9. *Let E be an infinite-dimensional Banach space not containing ℓ^1, F a Banach space, $S \in \mathcal{L}(E, F)$, $J \subset \mathbb{R}$ be a compact interval, $T \in \mathcal{L}(E, F)$ such that $T - \lambda.S$ be compact for none $\lambda \in J$ and $\varphi : F \to \mathbb{R}$ a convex continuous and coercive function. Assume that $S \times T$ is a one-to-one operator with closed range, Y is a non-empty convex open subset of the normed space V, $\pi \in \mathcal{L}(E, V)$ a compact operator with dense range and $X = \pi^{-1}(Y)$.*

Then $\sup_{x\in X} \inf_{\lambda\in J} \varphi(Tx - \lambda.Sx) = +\infty$.

Proof. Since φ is coercive, for each $M \in \mathbb{R}^+$, there is $R > 0$ such that $\varphi(u) < M \implies \|u\| < R$ for all $u \in F$. Following Lemma 2.8, there exists $x \in X$ such that $\|Tx - \lambda.x\| \geq R$ for all $\lambda \in J$; then we have for this vector x : $\inf_{\lambda\in J} \varphi(Tx - \lambda.Sx) \geq M$. □

Lemma 2.10. *Let φ be a convex continuous and coercive function on the Banach space E. Then the dual function φ^* has a proper domain D which is a convex neighborhood of 0 in E^*. And if Z is any dense subset of E^*, we have for all $x \in E$: $\varphi(x) = \sup_{\xi\in Z}\langle \xi, x\rangle - \varphi^*(\xi)$.*

Proof. Since φ^* is convex and the proper domain of φ^* is

$$D = \{\xi : \varphi^*(\xi) < +\infty\} = \bigcup_n \{\xi : \varphi^*(\xi) \leq n\}$$

it is clear that D is convex. Since φ is coercive the set $\{x : \varphi(x) \le 1 + \varphi(0)\,\}$ is bounded in E, and there exists $R > 0$ such that $\varphi(x) < 1+\varphi(0) \Longrightarrow \|x\| < R$. By convexity we deduce that $\varphi(x) \ge \varphi(0) + \dfrac{\|x\|}{R}$ if $\|x\| \ge R$ hence that $\varphi(x) - \langle \xi, x\rangle$ is bounded from below outside of the ball of radius R if $\xi \in E^*$ and $\|\xi\| < \dfrac{1}{R}$.

The convex continuous function $x \mapsto \varphi(x) - \langle \xi, x\rangle$ is necessarily bounded from below on the bounded convex complete set $\tilde{B}(0, R)$; it follows that $D \supset B(0, \dfrac{1}{R})$ hence that the interior D^o of D is non-empty. The convex l.s.c. function φ^* is finite on D. Hence, if L_m denotes the closed subset of D^o defined by $\{\xi \in D^o : \varphi^*(\xi) \le m\}$ we have $D^o = \bigcup_{m\in\mathbb{N}} L_m$ and some L_m has non-empty interior by Baire's Category Theorem. So φ^* is bounded from above on a neighborhood of some point of D^o , hence is continuous on D^o. Since $\langle \xi, x\rangle \le \varphi(x) + \varphi^*(\xi)$ for all $x \in E$ and all $\xi \in E^*$, we have $\varphi(x) \ge \sup_{\xi\in Z} [\langle \xi, x\rangle - \varphi^*(\xi)]$ for any $Z \subset E^*$.

Since D^o is open, $D^o \cap Z$ is dense in D^o. By continuity of φ^* on D^o we necessarily have

$$\varphi(x) \ge \sup_{\xi\in D^o\cap Z} [\langle \xi, x\rangle - \varphi^*(\xi)] = \sup_{\xi\in D^o} [\langle \xi, x\rangle - \varphi^*(\xi)] \ .$$

If $\alpha < \varphi(x)$, the point (x, α) does not belong to the closed convex set $G = \{(u, t) : t \ge \varphi(u)\}$. Then it follows from Hahn-Banach's theorem that exists $\xi \in E^*$ such that

$$\langle \xi, x\rangle - \alpha > \sup_{(u,s)\in G} [\langle \xi, u\rangle - s] = \sup_{u} [\langle \xi, u\rangle - \varphi(u)] = \varphi^*(\xi)$$

what implies $\xi \in D$ and $\langle \xi, x\rangle - \varphi^*(\xi) > \alpha$, whence $\varphi(x) = \sup_{\xi\in D} [\langle \xi, x\rangle - \varphi^*(\xi)]$. For $\xi \in D$, we have $t\xi \in D^o$ for all $t \in [0, 1[$ since $0 \in D^o$. The function $t \mapsto \varphi^*(t\xi)$ is convex and l.s.c. on $[0, 1]$, hence continuous, and it follows that

$$\langle \xi, x\rangle - \varphi^*(\xi) = \lim_{t\to 1, t<1} \langle t\xi, x\rangle - \varphi^*(t\xi) \le \sup_{\xi\in D^o} [\langle \xi, x\rangle - \varphi^*(\xi)] \ ,$$

hence that

$$\varphi(x) = \sup_{\xi\in D} [\langle \xi, x\rangle - \varphi^*(\xi)] \le \sup_{\xi\in D^o} [\langle \xi, x\rangle - \varphi^*(\xi)] \le \sup_{\xi\in Z} [\langle \xi, x\rangle - \varphi^*(\xi)] \le \varphi(x) \ .$$

□

Lemma 2.11. *Let E be an infinite-dimensional Banach space , V be a normed space and $K : E \to V$ a compact linear operator. Then there exists a σ-compact set $Z_0 \subset E^*$ and for all $\xi \notin Z_0$ a sequence (w_n) in V such that $\langle \xi, w_n\rangle = 1$ for all n and $\|Kw_n\| \to 0$.*

Proof. Since K is compact, the operator $K^* : V^* \to E^*$ is compact too and for all $m \in \mathbb{N}$ the set $T_m = \overline{K^*(mB_{V^*})}$ is a compact subset of the space E^*. Thus $Z_0 = \bigcup_m T_m$ is σ-compact and if $\xi \notin Z_0$ there cannot exist any continuous linear functional $\hat{\eta}$ on V such that $\xi = K^*(\hat{\eta})$.

If it existed $\gamma > 0$ such that $\{x : \langle\xi, x\rangle = 1\}$ be disjoint from $\{x : \|Kx\| \leq \gamma\}$ the set $\{\langle\xi, x\rangle : \|Kx\| \leq \gamma\}$ would be convex and symmetric hence an interval centered at 0 and non containing 1. So it would exist some $r \leq 1$ such that $|\langle\xi, x\rangle| \leq \dfrac{r}{\gamma} \cdot \|Kx\|$ for all $x \in E$ and it would exist a linear functional η on $K(E) \subset V$ whose norm would be at most $\dfrac{r}{\gamma}$ such that $\xi = \eta \circ K$: indeed if $y \in K(E)$ satisfies $y = Kx = Kx'$, we have $|\langle\xi, x - x'\rangle| \leq \dfrac{r}{\gamma}\,\|Kx - Kx'\| = 0$; hence $\eta(y) = \langle\xi, x\rangle$ depends only on $y = Kx$ and satisfies $|\eta(y)| \leq \dfrac{r}{\gamma}\,\|Kx\| = \dfrac{r}{\gamma}\,\|y\|$.

By Hahn-Banach's theorem it would exist $\hat{\eta} \in V^*$ extending η and we would have $\xi = \hat{\eta} \circ K = K^*(\hat{\eta})$, a contradiction with the choice of Z_0. We conclude that if $\xi \notin Z_0$ we can find for all $n \in \mathbb{N}$ some w_n such that $\|Kw_n\| \leq 2^{-n}$ and $\langle\xi, w_n\rangle = 1$. ☐

Lemma 2.12. *Let E be an infinite-dimensional Banach space, F be a normed space, $S : E \to F$ be a continuous non-compact linear operator, $H \in \mathcal{L}(E, F)$ be a compact operator, Y be a non-empty convex open subset of the normed space V and $\pi : E \to V$ be a compact operator with dense range, $X_0 = \pi^{-1}(Y)$, $J \subset \mathbb{R}$ be a compact interval, $\lambda_0 \in J$ and $\psi : J \to \mathbb{R}$ be a convex continuous function. Then there exists a dense subset Z of F^* such that*

$$\sup_{x\in X_0}\Big(\lambda_0\langle\xi, Sx\rangle + \langle\xi, Hx\rangle - \psi^*(\langle\xi, Sx\rangle)\Big) \geq \psi(\lambda_0) + \sup_{x\in X_0}\langle\xi, Hx\rangle$$

for all $\xi \in Z$.

Proof. Let K be the operator $x \mapsto (Hx, \pi x)$ from E to the product space $W = F \times V$ normed by $\|(y, u)\| = \|y\| + \|u\|$. Since H and π are compact, K is compact too and by Lemma 2.11 one can find an σ-compact set $Z_0 \subset E^*$ such that for $\xi_0 \notin Z_0$ there exists a sequence (w_n) in E such that $\|Kw_n\| = \|Hw_n\| + \|\pi w_n\| \to 0$ and that $\langle\xi, w_n\rangle = 1$.

Let (T_m) be a sequence of compact subsets of E^* such that $Z_0 = \bigcup_m T_m$. If the closed set ${S^*}^{-1}(T_m)$ had non-empty interior in F^* there would be some ball $B(\xi, r)$ in F^* such that $S^*(B(\xi, r)) \subset T_m$ and S^* would be a compact operator which in turn would imply that S is compact. So ${S^*}^{-1}(T_m)$ is nowhere dense and $M = \bigcup_m {S^*}^{-1}(T_m)$ is meager in F^*. Therefore $Z = F^* \setminus M$ is dense in F^*. Moreover, if $\xi \in Z$, then $\xi_0 = S^*\xi \notin Z_0$ and there exists a sequence $(w_n) \in E$ such that $Kw_n \to 0$ and $\langle\xi, Sw_n\rangle = \langle\xi_0, w_n\rangle = 1$.

Then if $x_0 \in X_0$, $y_0 = \pi(x_0) \in Y$ and $t \in \mathbb{R}$, $x_n = x_0 + (t - \langle\xi_0, x_0\rangle).w_n$ satisfies $\langle S^*\xi, x_n\rangle = \langle\xi_0, x_n\rangle = t$, $\pi x_n = y_0 + (t - \langle S^*\xi, x_0\rangle).\pi w_n \to y_0$. Because Y is open in V we have $\pi x_n \in Y$ for n large enough, hence $x_n \in X_0$. Moreover

$$\langle\xi, Hx_n\rangle = \langle\xi, Hx_0\rangle + (t - \langle S^*\xi, x_0\rangle).\langle\xi, Hw_n\rangle \to \langle\xi, Hx_0\rangle\ ,$$

from what it follows that

$$\begin{aligned}\sup_{x\in X_0}\Big(\lambda_0\langle\xi,Sx\rangle+\langle\xi,Hx\rangle-\psi^*(\langle\xi,Sx\rangle)\Big)&\\ &\geq \limsup_n\Big(\lambda_0\langle\xi,Sx_n\rangle+\langle\xi,Hx_n\rangle-\psi^*(\langle\xi,Sx_n\rangle)\Big)\\ &=\limsup_n\Big(\lambda_0\langle\xi_0,x_n\rangle+\langle\xi,Hx_n\rangle-\psi^*(\langle\xi_0,x_n\rangle)\Big)\\ &=\lambda_0 t+\langle\xi,Hx_0\rangle-\psi^*(t)\ ,\end{aligned}$$

and since this holds for all $t\in\mathbb{R}$ and all $x_0\in X_0$ we get

$$\begin{aligned}\sup_{x\in X_0}\Big(\lambda_0\langle\xi,Sx\rangle+\langle\xi,Hx\rangle-\psi^*(\langle\xi,Sx\rangle)\Big)&\geq\sup_t\big(\lambda_0 t-\psi^*(t)\big)+\sup_{x\in X_0}\langle\xi,Hx\rangle\\ &=\psi(\lambda_0)+\sup_{x\in X_0}\langle\xi,Hx\rangle\ ,\end{aligned}$$

what is the wanted inequality. □

Lemma 2.13. *Let E and F be infinite-dimensional Banach spaces, $S:E\to F$ be a continuous non-compact linear operator, π be a compact linear mapping with dense range from E to a normed space V, Y be a convex subset of V with non-empty interior, $H\in\mathcal{L}(E,F)$ be a compact operator, $J\subset\mathbb{R}$ be a compact interval, $\lambda_0\in J$, $\varphi:F\to\mathbb{R}$ be a convex continuous and coercive function and $\psi:J\to\mathbb{R}$ be a continuous convex function. Assume $S\times T$ is one-to-one with closed range. Then*

$$\sup_{\pi(x)\in Y}\inf_{\lambda\in J}\varphi\big((\lambda_0-\lambda).Sx+Hx\big)+\psi(\lambda)\geq\psi(\lambda_0)+\sup_{\pi(x)\in Y}\varphi(Hx)\ .$$

Proof. Put $\Phi(\lambda,x)=\varphi\big((\lambda_0-\lambda).Sx+Hx\big)$. It follows from Lemma 2.10 that, for a dense subset Z of F^*, $\Phi(\lambda,x)=\sup_{\xi\in Z}\langle\xi,(\lambda_0-\lambda).Sx+Hx\rangle-\varphi^*(\xi)$, hence

$$\begin{aligned}\inf_{\lambda\in J}\big(\Phi(\lambda,x)+\psi(\lambda)\big)&=\inf_{\lambda\in J}\sup_{\xi\in Z}\Big(\langle\xi,(\lambda_0-\lambda).Sx+Hx\rangle-\varphi^*(\xi)+\psi(\lambda)\Big)\\ &\geq\sup_{\xi\in Z}\inf_{\lambda\in J}\Big(\langle\xi,(\lambda_0-\lambda).Sx+Hx\rangle-\varphi^*(\xi)+\psi(\lambda)\Big)\\ &\geq\sup_{\xi\in Z}\Big(\lambda_0\langle\xi,Sx\rangle+\langle\xi,Hx\rangle-\varphi^*(\xi)+\inf_{\lambda\in J}\big(-\lambda\langle\xi,Sx\rangle+\psi(\lambda)\big)\Big)\\ &\geq\sup_{\xi\in Z}\Big(\lambda_0\langle\xi,Sx\rangle+\langle\xi,Hx\rangle-\varphi^*(\xi)-\psi^*(\langle\xi,Sx\rangle)\Big)\ ,\end{aligned}$$

and after Lemma 2.12,

$$\begin{aligned}
\sup_{\pi(x)\in Y}\inf_{\lambda\in J}\big(\Phi(\lambda,x)+\psi(\lambda)\big)&\\
&\geq \sup_{\pi(x)\in Y}\sup_{\xi\in Z}\Big(\lambda_0\langle\xi,Sx\rangle+\langle\xi,Hx\rangle-\varphi^*(\xi)-\psi^*(\langle\xi,Sx\rangle)\Big)\\
&\geq \sup_{\xi\in Z}\Big(-\varphi^*(\xi)+\sup_{\pi(x)\in Y}\big(\lambda_0\langle\xi,Sx\rangle+\langle\xi,Hx\rangle-\psi^*(\langle\xi,Sx\rangle)\big)\Big)\\
&\geq \sup_{\xi\in Z}\big(-\varphi^*(\xi)+\psi(\lambda_0)+\sup_{\pi(x)\in Y}\langle\xi,Hx\rangle\big)\\
&\geq \psi(\lambda_0)+\sup_{\pi(x)\in Y}\big(\sup_{\xi\in Z}\langle\xi,Hx\rangle-\varphi^*(\xi)\big)\\
&= \psi(\lambda_0)+\sup_{\pi(x)\in Y}\varphi(Hx)
\end{aligned}$$

what is the wanted inequality. □

Theorem 2.14. *Let E be an infinite-dimensional Banach space not containing ℓ^1, F a Banach space, $S,T\in\mathcal{L}(E,F)$, $J\subset\mathbb{R}$ be a compact interval, $\varphi:F\to\mathbb{R}$ be a convex continuous and coercive function, X be a convex subset of E whose interior is non-empty for the topology $\beta(E,E^*)$, $J\subset\mathbb{R}$ be a compact interval and $\psi:J\to\mathbb{R}$ be a convex continuous function. Assume $S\times T$ has a closed range and S is not compact. Then the following holds :*

$$\sup_{x\in X}\inf_{\lambda\in J}\big(\varphi(Tx-\lambda Sx)+\psi(\lambda)\big)=\inf_{\lambda\in J}\sup_{x\in X}\big(\varphi(Tx-\lambda Sx)+\psi(\lambda)\big)\ .$$

Lemma 2.15. *We can reduce the problem to the case where $S\times T:x\mapsto(Sx,Tx)$ is one-to-one from E to $F\times F$.*

Proof. Denote by $N=\ker S\cap\ker T$ the kernel of $S\times T$ and by $q:E\to E/N$ the quotient mapping. It is clear that S and T factor through q : $S=\tilde{S}\circ q$ and $T=\tilde{T}\circ q$; so for all $\lambda\in\mathbb{R}$, $Tx-\lambda Sx$ depends only on qx. So we can replace E by $\tilde{E}=E/N$, S by $\tilde{S}$, T by $\tilde{T}$, and X by $\tilde{X}=q(X)$. Clearly if $\tilde{S}$ was compact, S would be compact too. Finally it is enough to check that the convex subset $\tilde{X}$ of $\tilde{E}$ has non-empty interior for the topology $\beta(\tilde{E},\tilde{E}^*)$: let a be an interior point of X for $\beta(E,E^*)$ and $\tilde{a}=qa$. We want to prove that $\tilde{a}$ is an interior point of $\tilde{X}$ for $\beta(\tilde{E},\tilde{E}^*)$. If not it would exist a sequence (y_n) in $\tilde{E}\setminus\tilde{X}$ weakly converging to $\tilde{a}$ and we could find by Lemma 2.2 a subsequence (y_{n_k}) and a sequence (z_k) in E weakly converging to 0 such that $q(z_k)=y_{n_k}-\tilde{a}$. Then (z_k+a) converges weakly to a thus satisfies $z_k+a\in X$ hence $q(z_k+a)=y_{n_k}\in\tilde{X}$ for k large enough, a contradiction. □

Proof. By Lemma 2.15, we can and do assume that $S\times T$ is one-to-one. By Lemma 2.4, we know that there exists a normed space V, a compact linear mapping $\pi:E\to V$ and a convex open subset Y of V such that $V=\overline{\pi(E)}$ and $\pi^{-1}(Y)\subset X\subset\pi^{-1}(\overline{Y})$. It is then easy to see that the supremum over X is equal to the supremum on $\pi^{-1}(Y)$, and we will only prove the statement when $X = \pi^{-1}(Y)$. The inequality

$$\sup_{x\in X}\inf_{\lambda\in J}\big(\varphi(Tx-\lambda Sx)+\psi(\lambda)\big) \leq \inf_{\lambda\in J}\sup_{x\in X}\big(\varphi(Tx-\lambda Sx)+\psi(\lambda)\big)$$

is standard. So we have only to prove the converse inequality.

Following Corollary 2.9, it is enough to consider the case where $T = \lambda_0.S + H$, $\lambda_0 \in J$ and $H \in \mathcal{L}(E,F)$ is compact. Then by Corollary 2.7, we have

$$\inf_{\lambda\in J}\sup_{x\in X}\big(\varphi(Tx-\lambda Sx)+\psi(\lambda)\big) = \sup_{x\in X}\big(\varphi(Tx-\lambda_0 Sx)+\psi(\lambda_0)\big) = \sup_{x\in X}\varphi(Hx)+\psi(\lambda_0)\ ,$$

and it follows from Lemma 2.13 that

$$\begin{aligned}\sup_{\pi(x)\in Y}\varphi(Hx)+\psi(\lambda_0) &\leq \sup_{\pi(x)\in Y}\inf_{\lambda\in J}\varphi\big((\lambda_0-\lambda).Sx+Hx\big)+\psi(\lambda)\\ &= \sup_{\pi(x)\in Y}\inf_{\lambda\in J}\varphi(Tx-\lambda Sx)+\psi(\lambda),\end{aligned}$$

and this completes the proof of the theorem. □

The hypothesis made on the operator $S \times T$ to have closed range and on S to not be compact could seem quite artificial. In fact without any hypothesis on S the statement of Theorem 2.14 becomes false, as shown by the following.

Example 2.16. There exist two continuous linear operators S and T from the Hilbert space $\mathcal{E} = \ell^2$ to a Banach space $\mathcal{F}$, X a convex subset of $\mathcal{E}$ whose interior is non-empty for $\beta(\mathcal{E},\mathcal{E}^*)$, a compact interval $J \subset \mathbb{R}$ and a convex continuous and coercive function $\varphi : \mathcal{F} \to \mathbb{R}$ such that

$$\sup_{x\in X}\inf_{\lambda\in J}\big(\varphi(Tx-\lambda Sx)\big) < \inf_{\lambda\in J}\sup_{x\in X}\big(\varphi(Tx-\lambda Sx)\big)$$

Proof. We begin by exhibiting a concrete example of a pair A, B of operators between normed spaces which do not satisfy the minimax equality of Asplund and Pták.

Define the two-dimensional normed spaces E and F as the linear space $\mathbb{R}^2$ equipped respectively with the norms $\|.\|_1 : (x,y) \mapsto |x|+|y|$ and $\|.\|_\infty : (x,y) \mapsto \max(|x|,|y|)$. Denote by $A \in \mathcal{L}(E,F)$ the operator whose matrix in the canonical bases is $\begin{pmatrix} 3 & 1\\ -1 & 0\end{pmatrix}$ and by I the identity mapping.

Lemma 2.17. *We have :* $\inf_{\lambda\in\mathbb{R}}\sup_{\|z\|_1\leq 1}\|Az-\lambda z\|_\infty = \inf_{\lambda\in\mathbb{R}}\|A-\lambda I\| = \dfrac{3}{2}$.

Proof. It is easily noticed that the norm in $\mathcal{L}(E,F)$ of the operator T having matrix $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ is $\max(|\alpha|, |\beta|, |\gamma|, |\delta|)$. Indeed, denoting by (e_1, e_2) the canonical basis of $E \simeq F^*$, we have $\|e_1\| = \|e_2\| = 1$ and $|\alpha| = |\langle e_1, Te_1\rangle| \le \|e_1\| . \|T\| . \|e_1\|$ hence $|\alpha| \le \|T\|$; and similarly for $|\beta|, |\gamma|$ and $|\delta|$. Moreover, if $z = (x,y)$,

$$\begin{aligned}\|Tz\|_\infty &= \max(|\alpha x + \beta y|, |\gamma x + \delta y|)\\ &\le \max(\max(|\alpha|, |\beta|).(|x| + |y|), \max(|\gamma|, |\delta|).(|x| + |y|))\\ &= \max(|\alpha|, |\beta|, |\gamma|, |\delta|). \|z\|_1\end{aligned}$$

Thus $\|A - \lambda I\| = \max(|3-\lambda|, 1, 1, |\lambda|)$. And since

$$\max(|3-\lambda|, 1, 1, |\lambda|) \ge \max(|3-\lambda|, |\lambda|) \ge \frac{1}{2}(|3-\lambda| + |\lambda|) \ge \frac{1}{2}|3-\lambda+\lambda| = \frac{3}{2}$$

and $\left\|A - \frac{3}{2}I\right\| = \max(\frac{3}{2}, 1, 1, \frac{3}{2}) = \frac{3}{2}$, we conclude that $\inf_{\lambda\in\mathbb{R}} \|A - \lambda I\| = \frac{3}{2}$. □

Lemma 2.18. *We have:* $\sup_{\|z\|_1\le 1} \inf_{\lambda\in\mathbb{R}} \|Az - \lambda z\|_\infty \le \frac{5}{4} < \frac{3}{2}$. *More precisely, for each* z *in the unit ball of* E *there is a* $\lambda^* \in [0,2]$ *such that* $\|Az - \lambda^* z\|_\infty \le \frac{5}{4}$.

Proof. Let $x \in [0,1]$ and $|y| \le 1-x$. Then for $z = (x,y)$ we have

$$\begin{aligned}\|Az - \lambda z\|_\infty &= \max(|(3-\lambda)x + y|, |-x - \lambda y|) = \max(|(3-\lambda)x + y|, |x + \lambda y|)\\ &\le \max\big(|3-\lambda| .x + 1 - x, x + |\lambda| .(1-x)\big)\end{aligned}$$

Then for $\lambda^* = x + 1 \in [0,2]$ we get

$$\begin{aligned}\|Az - \lambda^* z\|_F &\le \max\big((2-x).x + 1 - x, x + (x+1)(1-x)\big)\\ &= \max(x + 1 - x^2, x + 1 - x^2)\\ &= x + 1 - x^2 \le \frac{5}{4}\end{aligned}$$

If $z = (x,y)$ satisfies $\|z\|_1 \le 1$, we have $|x| \le 1$ and up to replacing z by $-z$, which does not change $\|Az - \lambda z\|_\infty$, we can assume $0 \le x \le 1$. It follows from the previous computation that there exists some λ^* such that $\|Az - \lambda^* z\|_\infty \le \frac{5}{4}$.

Hence for any z such that $\|z\|_1 \le 1$ we get $\inf_{\lambda\in\mathbb{R}} \|Az - \lambda z\|_\infty \le \frac{5}{4}$.

And this achieves the proof that $\sup_{\|z\|_1\le 1} \inf_{\lambda\in\mathbb{R}} \|Az - \lambda z\|_\infty \le \frac{5}{4}$. □

Lemma 2.19. *There exist two Banach spaces* E_2 *and* F_2 *which are isomorphic to the Hilbert space* ℓ^2 *and operators* A_2 *and* B_2 *in* $\mathcal{L}(E_2, F_2)$ *such that*

$$\sup_{\|z\|_{E_2}\le 1} \inf_{\lambda\in[0,2]} \|A_2 z - \lambda B_2 z\|_{F_2} \le \frac{5}{4} < \frac{3}{2} \le \inf_{\lambda\in\mathbb{R}} \sup_{\|z\|_{E_2}\le 1} \|A_2 z - \lambda B_2 z\|_{F_2} .$$

Proof. Take $E_2 = E \times \ell^2$ with the norm : $(z,\xi) \mapsto \sqrt{\|z\|_1^2 + \|\xi\|^2}$, $F_2 = F \times \ell^2$ with the norm : $(z,\xi) \mapsto \sqrt{\|z\|_\infty^2 + \|\xi\|^2}$. These spaces are clearly isomorphic to the Hilbert space ℓ^2.

For $(z,\xi) \in E_2$ define $A_2(z,\xi) = (Az,\xi)$ and $B_2(z,\xi) = (z,\xi)$. Then A_2 and B_2 are clearly continuous linear operators from E_2 to F_2. It is easily checked that $\|A_2 - \lambda B_2\| \geq \frac{3}{2}$ for all real λ. Indeed

$$\begin{aligned}\sup_{\|(z,\xi)\|\leq 1} \|(A_2 - \lambda B_2)(z,\xi)\| &\geq \sup_{\|z\|_1\leq 1} \|(A_2 - \lambda B_2).(z,0)\| \\ &= \sup_{\|z\|_1\leq 1} \|(A - \lambda I)z\|_\infty = \|A - \lambda I\| \geq \frac{3}{2}\end{aligned}$$

Thus $\inf_{\lambda\in\mathbb{R}} \sup_{\|(z,\xi)\|\leq 1} \|(A_2 - \lambda B_2)(z,\xi)\| = \inf_{\lambda\in\mathbb{R}} \|A_2 - \lambda B_2\| \geq \frac{3}{2}$.

Conversely by Lemma 2.18 and by homogeneity, if $\|(z,\xi)\|_{E_2} \leq 1$ there is a λ^* in $[0,2]$ such that $\|Az - \lambda^* z\|_\infty \leq \frac{5}{4}\|z\|_1$. Then

$$\|(A_2 - \lambda^* B_2)(z,\xi)\|^2 = \|Az - \lambda^* z\|_\infty^2 + \|\xi - \lambda^*\xi\|^2 \leq (\frac{5}{4})^2 \|z\|_1^2 + (1-\lambda^*)^2 \|\xi\|^2 \ ,$$

and since $|1 - \lambda^*| \leq 1 \leq \frac{5}{4}$ we get

$$\|(A_2 - \lambda^* B_2)(z,\xi)\|^2 \leq (\frac{5}{4})^2\big(\|z\|_1^2 + \|\xi\|^2\big) = (\frac{5}{4})^2 \|(z,\xi)\|_{E_2}^2 \leq (\frac{5}{4})^2 \ ,$$

hence

$$\sup_{\|(z,\xi)\|\leq 1} \inf_{\lambda\in[0,2]} \|A_2(z,\xi) - \lambda B_2(z,\xi)\|_{F_2} \leq \frac{5}{4} < \frac{3}{2} \leq \inf_{\lambda\in\mathbb{R}} \sup_{\|(z,\xi)\|\leq 1} \|(A_2 - \lambda B_2)(z,\xi)\| \, ,$$

and this completes the proof. □

Since E_2 is isomorphic to ℓ^2, we can find a one-to-one compact operator $\pi : \mathcal{E} = \ell^2 \to E_2$ with dense range, define $\mathcal{F} = F_2$, $T = A_2 \circ \pi$ and $S = B_2 \circ \pi$ and $\varphi = \|.\|_{F_2}$. Choose $J = [0,2]$ and Y as the unit ball of E_2. Then define $X = \pi^{-1}(Y)$. So, for any $\lambda \in J$, we have $\sup_{y\in Y} \|(A_2 - \lambda B_2)y\| = \sup_{y\in Y\cap\pi(\ell^2)} \|(A_2 - \lambda B_2)y\|$, thus

$$\begin{aligned}\sup_{x\in X} \varphi(Tx - \lambda Sx) &= \sup_{x\in X} \|Tx - \lambda Sx\| = \sup_{\pi(x)\in Y} \|Tx - \lambda Sx\| \\ &= \sup_{\pi(x)\in Y} \|A_2\pi x - \lambda B_2\pi x\| = \sup_{y\in Y} \|A_2 y - \lambda B_2 y\| \\ &= \|A_2 - \lambda B_2\| \geq \frac{3}{2} \ ,\end{aligned}$$

whence $\inf_{\lambda\in J} \sup_{x\in X} \varphi(Tx - \lambda Sx) \geq \frac{3}{2}$.

Similarly, because the function $\nu : y \mapsto \inf_{\lambda\in J} \|(A_2 - \lambda B_2)y\|_{F_2}$ is Lipschitz, hence $\sup_{y\in Y} \nu(y) = \sup_{y\in Y\cap\pi(\ell^2)} \nu(y)$, and using Lemma 2.19,

$$\begin{aligned}
\sup_{x\in X}\Big(\inf_{\lambda\in J} \varphi(Tx-\lambda Sx)\Big) &= \sup_{x\in X}\Big(\inf_{\lambda\in J} \|Tx-\lambda Sx\|_{F_2}\Big)\\
&= \sup_{x\in X}\Big(\inf_{\lambda\in J} \|(A_2-\lambda B_2)(\pi x)\|_{F_2}\Big)\\
&= \sup_{\pi(x)\in Y}\Big(\inf_{\lambda\in J} \|(A_2-\lambda B_2)(\pi x)\|_{F_2}\Big)\\
&= \sup_{y\in Y}\Big(\inf_{\lambda\in J} \|(A_2-\lambda B_2)(y)\|_{F_2}\Big)\\
&= \sup_{\|y\|\le 1}\Big(\inf_{\lambda\in[0,2]} \|(A_2-\lambda B_2)(y)\|_{F_2}\Big) \le \frac{5}{4}\ ,
\end{aligned}$$

so $\sup_{x\in X}\Big(\inf_{\lambda\in J}\varphi(Tx-\lambda Sx)\Big) < \inf_{\lambda\in J}\sup_{x\in X}\varphi(Tx-\lambda Sx)$, as announced. It can be noticed that in this example B_2 is an isomorphism, so S is one-to-one. A fortiori $S\times T$ is one-to-one. But it is compact, and its range is not closed. ☐

It would be interesting to know whether the interior of X for $\beta(E,E^*)$ has to be non-empty for guaranteeing the validity of Theorem 2.14. Suppose E is a Banach space not containing ℓ^1, $X\subset E$ is a non-empty convex set and the conclusion of Theorem 2.14 holds for all F,S,T,J,φ,ψ as in the hypotheses: should the interior of X for the topology $\beta(E,E^*)$ be non-empty?

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