Minimax Theory and its Application
Volume 10 (2025), No. 1, 109-134

On Nonnegative Solutions of Quasilinear Equations with Oscillating Nonlinearity in $\mathbb{R}^N$

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Received: February 7, 2024
Accepted: June 28, 2024

In this paper, it is obtained the existence and multiplicity of solutions for a class of quasilinear problems with an oscillating nonlinearity satisfying an area condition. The approach consists in introducing an auxiliary semilinear problem through a suitable change of variables which is equivalent to the original one, and solve it by means of the Schaefer's Fixed Point Theorem, sub-supersolutions and minimax results.

Keywords: Comparison principles, Variational methods, Nonlinear elliptic equations

2020 Mathematics Subject Classification: 35B51, 35J20, 35J60

1. Introduction and main results

The purpose of this manuscript is to consider the existence and multiplicity of non-negative solutions for the quasilinear problem with oscilating nonlinearity given by

$$\begin{cases} -\Delta u - \Delta(u^2)u = \lambda P(x) f(u) \text{in} \mathbb{R}^N \\ \lim\limits_{|x|\to\infty} u(x) = 0, u \in D^{1,2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \end{cases} \tag{P}$$

where $N \geq 3$, $f : [0, +\infty) \to \mathbb{R}$ is a continuous function and

$$P \in C^+_{rad}(\mathbb{R}^N, \mathbb{R}) := \{u \in C(\mathbb{R}^N, \mathbb{R}); u(x) = u(|x|) \text{ and } u(x) > 0, \forall x \in \mathbb{R}^N\}$$

satisfies

(f_1) $f(0) \geq 0$,

(f_2) There are $2m - 1$ zeros of f, $0 < a_1 < b_1 < a_2 < b_2 < \cdots < b_{m-1} < a_m$ such that for $k = 1, \cdots, m-1$

$$\begin{cases} f(t) \geq 0, \ t \in (b_k, a_{k+1}), \\ f(t) \leq 0, \ t \in (a_k, b_k), \end{cases}$$

(f_3) $\int_{a_k}^{a_{k+1}} f(s)ds > 0$, for all $k \in \{1, 2, \cdots, m-1\}$,

(P_1) $\int_{\mathbb{R}^N} |x|^{2-N} P(x)dx < +\infty$,

ISSN 2199-1421 (electronic)

(P_2) $P \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$
and

(P_3) $\displaystyle\int_{\mathbb{R}^N} \frac{P(y)}{|x-y|^{N-2}}\,dy \le \frac{C}{|x|^{N-2}}$, for all $x \in \mathbb{R}^N \setminus \{0\}$ and some constant $C > 0$.

Problem (P) arises, for instance, in the study of solutions for the SchrÖdinger equation given by

$$i\partial_t \phi = -\Delta\phi + V(x)\phi - \Delta(\rho(|\phi|^2))\rho'(|\phi|^2)\phi - \tilde{h}(|\phi|^2)\phi \text{ in } \Omega \subseteq \mathbb{R}^N \text{ and } t > 0, \qquad (1)$$

where $\varphi : \mathbb{R}_+ \Omega \to \mathbb{C}$, the function $V(x)$ is a given potential and ρ and $\tilde{h}$ are real functions. Depending on the function ρ in (1) one can consider relevant facts regarding the superfluid film equation, theoretical and numerical aspects related to high-power ultra short laser in matter and self-trapped electrons, see for instance, the references [4–6,9,29,30,34]. For others applications see [3,24,28,32,33,35]. The first manuscript that considered the quasilinear operator in (P) was [12], where it was considered the problem

$$-\Delta u - \Delta(u^2)u = g(x, u) \text{ in } \mathbb{R}^N.$$

By performing a change of variables, the above problem is transformed in a equivalent semilinear one given by

$$-\Delta v = \frac{1}{\sqrt{1+f^2(v)}}\, g(x, f(v)) \text{ in } \mathbb{R}^N,$$

where f is suitably chosen. Under certain conditions it was proved that if v solves the previous problem, then $u = f(v)$ solves the original one. Through a variational approach, the authors obtained several existence results.

We also quote [40] in which it was considered the problem

$$-\Delta_p u - \Delta_p(u^2)u + V(x)|u|^{p-2}u = h(u) \text{ in } \mathbb{R}^N,$$

where $1 < p \le N$, $-\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, V is a positive continuous potential bounded away from zero and $h(u)$ is a subcritical nonlinearity. By means of minimax results, it was obtained the existence of a $C^{1,\alpha}(\mathbb{R}^N)$ solution and that such solution decays to zero at infinity when $1 < p < N$.

In the recent reference [20], it was considered the SchrÖdinger problem with zero-mass

$$\begin{cases} -\Delta u - \Delta(u^2)u = h(x)u^q \text{ in } \mathbb{R}^N, \\ u \ge 0, u \in D^{1,2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \end{cases}$$

where $0 \le q < 2.2^* - 1$ with $2^* = 2N/(N-2)$; $N \ge 3$ and h is a locally bounded function which can change its sign. By applying the change of variable introduced in [12], the Mountain Pass Theorem and the Ekeland Variational Principle, it was obtained the existence of solutions.

The consideration of oscillating terms in partial differential equations arises, for instance, in physical phenomena governed by the Sine-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -\sin u,$$

which describes several wave phenomena. These include phase differences across Josephson junctions, the propagation of dislocations in crystals, waves along lipid membranes, torsion waves in strings and pendula. Moreover, such equation has wide applications in the Theory of Special Relativity. We also point ou that the Sine-Gordon equation was intensely studied in the nineteenth century due to its connection with the Theory of Pseudospherical Surfaces. For more details on the mentioned applications, see for instance [38,39,41] and its references.

On the other hand, problems with oscillating behavior like $(f_2) – (f_3)$ has been attracting the attention of several researchers in the last decades. We point out for example the classical references [7,8] where it was considered the equation given by

$$\begin{cases} Lu = \lambda f(x,u) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, L is a uniformly elliptic operator, with $f(x, 0) > 0$ and $f(x, t)$ satisfies (f_2) and (f_3). Brown and Budin [8], by means of variational arguments, proved that for $\lambda > 0$ large the above equation has at least n nonnegative solutions $\psi_1, \psi_2, \cdots, \psi_n$ with $\psi_1 \le \psi_2 \le \cdots \le \psi_n$. An improvement of such result can be found for example in [25], where it was applied the Leray-Schauder degree theory and minimization arguments to obtain multiple solutions for the equation

$$\begin{cases} -\Delta u = \lambda f(u) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded with smooth boundary, $\lambda \ge 0$ a parameter, $f : \mathbb{R}_+ \to \mathbb{R}$ is a C^1−function with $f(0) > 0$ and $f(a_k) = 0$ for $k = 1, \cdots, m$ for constants $0 < a_1 < \cdots < a_m$ and

$$\max\{F(s); 0 \le s \le a_{k-1}\} < F(a_k), \quad k = 2, \cdots, m,$$

where $F(s) = \int_0^s f(t)dt$.

In De Figueiredo [19], a priori estimate for the oscillating problem

$$\begin{cases} -\Delta u = \lambda \sin u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

was considered, where $\Omega \subset \mathbb{R}^N$ $(N \ge 2)$ is a bounded domain and $\lambda > \lambda_1$ is a parameter with λ_1 being the first eigenvalue of the Laplacian operator. By assuming a hypothesis on the geometry of the domain and on the level sets of the solutions of the above problem, an a priori L^∞ estimate was derived. Moreover, it was proved that the result implies the uniqueness of the solution, and examples showing the applicability of the result were presented.

We also point out the recent reference [21], which applied truncation arguments, minimization methods, comparison techniques, the topological degree and sub-supersolutions method to obtain the multiplicity of positive solutions for the quasilinear problem

$$\begin{cases} -\Delta u - \kappa\Delta(u^2)u + \mu |u|^{q-2} u = \lambda f(u) + h(u) \text{ in } \Omega \\ u = 0, \text{ on } \partial\Omega, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded with smooth boundary, $\kappa, \mu, \lambda > 0$, $q \geq 1$, and $f, g : \mathbb{R} \to \mathbb{R}$ are functions satisfying certain conditions with $f(0) > 0$, $(f_2) - (f_3)$, and $h(t) \geq 0$, for all $t \geq 0$.

With respect to the study of oscillating problems with area conditions in unbounded domains there is only the reference [15], whose authors obtained, through variational methods and sub-supersolutions, existence and multiplicity of positive solutions for the semilinear problem

$$\begin{cases} -\Delta u = \lambda P(x) f(u) \text{ in } \mathbb{R}^N \\ \lim\limits_{|x| \to \infty} u(x) = 0, \end{cases}$$

where f, P and Q satisfy $(f_1) - (f_3)$ and $(P_1) - (P_3)$.

Thus, based on the previous comments and mentioned papers, we propose studying the quasilinear equation (P).

We say that $u : \mathbb{R}^N \to \mathbb{R}$ is a distributional solution for (P) if $u \in D^{1,2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $u \neq 0$, $u \geq 0$ a.e. in $\mathbb{R}^N$ and

$$\int_{\mathbb{R}^N} (1 + 2|u|^2) \nabla u \nabla \varphi + 2 \int_{\mathbb{R}^N} |\nabla u|^2 u\varphi = \int_{\mathbb{R}^N} \lambda P(x) f(u) \varphi, \forall \varphi \in C_0^\infty(\mathbb{R}^N). \tag{3}$$

As described in [12,20,40], variational methods cannot be directly applied to elliptic problems involving the operator $L(u) := -\Delta u - \Delta(u^2)u$ due to the fact that is not possible to guarantee that the formal energy functional is well defined. In order to avoid such difficulty, it will be performed the dual approach of [12], which provides an equivalent semilinear problem. After this, through nontrivial modifications, we adapt the arguments of [15,25] to prove Theorems 1.1 and 1.2. Below we describe the main results of this manuscript.

Theorem 1.1. *Consider that the function f satisfies $(f_1) - (f_3)$ and P verifies $(P_1) - (P_3)$. There exists a $\lambda_0 > 0$ such that if $\lambda > \lambda_0$, the problem (P) has at least $m - 1$ non-negative weak solutions $\{u_1, \ldots, u_{m-1}\} \subset L^\infty(\mathbb{R}^N)$ with $a_{k-1} < |u_k|_\infty a_k$, for $k = 2, \ldots, m$.*

In the infinitely many zeros case we consider the condition below on f.

$(f_2)_\infty$ There is a sequence of infinitely many zeros of f, $0 < a_1 < b_1 < a_2 < b_2 < \ldots < b_{m-1} < a_m < \ldots$ with

$$\begin{cases} f(t) \geq 0, \ t \in (b_k, a_{k+1}), \\ f(t) \leq 0, \ t \in (a_k, b_k), \end{cases}$$

$(f_3)_\infty$ There is $\beta > 0$ such that

$$\frac{F(a_k) - \max\{F(s); 0 \leq s \leq a_{k-1}\}}{\max\{|F(s)|; 0 \leq s \leq a_k\}} > \beta, \forall k \geq 2,$$

where $F(s) = \int_0^s f(t)dt$, and

$(f_4)_\infty$

$$\max\{|F(s)|;\ 0 \le s \le a_k\} \ge d a_k,\ \forall k \ge 2,$$

where d is a positive constant with $a_k \to a$, for some $a \in \mathbb{R}$ or $a_k \to +\infty$.

Theorem 1.2. *Consider that the function f satisfies (f_1) and $(f_2)_\infty - (f_4)_\infty$. There exists a $\tilde{\lambda}_0 > 0$ such that for all $\lambda > \tilde{\lambda}_0$, the problem (P) has infinitely many non-negative weak solutions $\{u_1, \ldots, u_{m-1}, \ldots\} \subset L^\infty(\mathbb{R}^N)$ such that $a_{k-1} < |u_k|_\infty \le a_k$, for $k = 2, \cdots, m$.*

Theorem 1.3. *Assume that $f(0) > 0$ and (f_2). If the problem*

$$\begin{cases} -\Delta u - \Delta(u^2)u = P(x) f(u) \text{ in } \mathbb{R}^N \\ \lim_{|x| \to \infty} u(x) = 0, \end{cases}$$

has a nonnegative weak solution u with $|u|_\infty \in (a_k, a_{k+1}]$, for some $k \in 1, \ldots, m-1$, then for such k it holds that

$$\int_{A_K}^{a_{k+1}} f(s)ds > 0. \tag{4}$$

In what follows we describe the notations that will be used in this manuscript.

Basic Notation

- $B_r(x)$ denotes the open ball in $\mathbb{R}^N$ centered at x with radius $r > 0$.
- $C^+_{rad}(\mathbb{R}^N, \mathbb{R}) = \{u \in C(\mathbb{R}^N, \mathbb{R});\ u(x) = u(|x|) \text{ and } u(x) > 0, \forall x \in \mathbb{R}^N\}$
- $L^s(\mathbb{R}^N)$, for $1 \le s \le \infty$, denotes the Lebesgue space with the usual norm $|\cdot|_s$.
- If $H : \mathbb{R}^N \to \mathbb{R}^+$ is a measurable function, we define

$$L^2_H(\mathbb{R}^N) := \{u : \mathbb{R}^N \to \mathbb{R} \text{ measurable}; \int_{\mathbb{R}^N} H(x)|u(x)|^2\, dx < \infty\}.$$

  $L^2_H(\mathbb{R}^N)$ is a Hilbert space with the inner product

$$(u,v)_{2,H} = \int_{\mathbb{R}^N} H(x)u(x)v(x)dx, \qquad \forall u,v \in L^2_H(\mathbb{R}^N),$$

  whose associated norm will be denoted by $|\cdot|_{2,H}$.
- $D^{1,2}(\mathbb{R}^N)$ denotes the Sobolev space

$$D^{1,2}(\mathbb{R}^N) := \{u \in L^{2^*}(\mathbb{R}^N);\ |\nabla u| \in L^2(\mathbb{R}^N)\},$$

  where 2^* is the Sobolev critical exponent: $2^* = \frac{2N}{N-2}$, if $N \ge 3$, and $2^* = \infty$, if $N = 1, 2$, which will be endowed with inner product

$$(u,v)_{1,2} = \int_{\mathbb{R}^N} \nabla u \nabla v dx, \qquad u,v \in D^{1,2}(\mathbb{R}^N).$$

Its associated inner product will be denoted by $|\cdot|_{1,2}$.

- $H^1(\mathbb{R}^N)$ denotes the Sobolev space with the inner product

$$(u,v)_{H^1} = \int_{\mathbb{R}^N} \nabla u \nabla v dx + \int_{\mathbb{R}^N} uv dx, \quad u,v \in H^1(\mathbb{R}^N).$$

The associated norm will be denoted by $|\cdot|_{H^1}$.

- $D(\mathbb{R}^N)$ denotes the space of C^∞−functions with compact support in $\mathbb{R}^N$.
- $D(\mathbb{R}^N)$ denotes the set of distributions on $\mathbb{R}^N$.
- We denote by E the Banach space given by

$$E := \left\{ u \in C(\mathbb{R}^N); \quad \sup_{x \in \mathbb{R}^N} |u(x)| < \infty \right\}$$

endowed with the norm $|\cdot|_\infty$.

- $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^N$.
- If u is a mensurable function, we denote by u^+ and u^- the positive and negative part of u respectively, which are given by

$$u^+ = \max\{u, 0\} \qquad \text{and} \qquad u^- = \min\{u, 0\}$$

and so $u = u^+ + u^-$.

- $C, C_1, \cdots, C_n$ are positive constants.

2. Examples

In what follows, we present a class of weights that verifies the conditions $(P_1) - (P_3)$. The example and the justification that it satisfies $(P_1) - (P_3)$ can be found in [1]. It will also be presented examples of nonlinearities that satisfy $(f_1) - (f_3)$, and the case where (f_1) and $(f_2)_\infty - (f_4)_\infty$ occurs simultaneously.

We will begin with an example regarding the weights. Consider $P \in C^+_{rad}(\mathbb{R}^N, \mathbb{R})$ of the form $P(x) = Q(x)R(x)$, with $Q(x) = Q(|x|)$, $R(x) = R(|x|)$ being decreasing radial functions of $C^+_{rad}(\mathbb{R}^N, \mathbb{R})$ that satisfy

$$\sup_{x \in \mathbb{R}^N}(|x|^{N-2} Q(x)) < +\infty \text{ and } \frac{R(x)}{|x|^{N-2}} \in L^1(\mathbb{R}^N). \tag{5}$$

Note that P satisfy (P_1). In fact, the continuity and the mononicity of Q, provides that such function is bounded and

$$\begin{aligned} \int_{\mathbb{R}^R} |x|^{2-N} P(x)dx &= \int_{\mathbb{R}^R} x|^{2-N} P(x)dx \\ &= \int_{\mathbb{R}^R} |x|^{2-N} Q(x)R(x)dx \\ &\leq |Q|_\infty \int_{\mathbb{R}^R} \frac{R(x)}{|x|^{N-2}} dx \\ &< +\infty, \end{aligned}$$

where we used (5) in the last inequality. Thus, (P_1) is proved.

Now it will be proved (P_2). Note that the monotonicity and the continuity of Q and R provides that P is bounded. We also have

$$\begin{aligned}\int_{\mathbb{R}^N} P(x)dx &= \int_{\mathbb{R}^N} Q(x)R(x)dx \\ &= \int_{\mathbb{R}^N} \frac{R(x)}{|x|^{N-2}} |x|^{N-2} Q(x)dx \\ &\leq \sup_{x\in\mathbb{R}^N}(|x|^{N-2} Q(x)) \int_{\mathbb{R}^N} \frac{R(x)}{|x|^{N-2}} dx \\ &< +\infty nn\end{aligned}$$

where we used (5) in the last inequality, which concludes the proof of (P_2).

In order to prove (P_3) note that for a fixed $x \in \mathbb{R}^N$ if $A_x := \{y \in \mathbb{R}^N ; |x-y| \leq |x|/2\}$ and $B_x := \{y \in \mathbb{R}^N ; |x-y| > |x|/2\}$, then

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{P(y)}{|x-y|^{N-2}}dy &= \int_{A_x} \frac{P(y)}{|x-y|^{N-2}}dy + \int_{B_x} \frac{P(y)}{|x-y|^{N-2}}dy \\ &= \int_{A_x} \frac{P(y)}{|x-y|^{N-2}}dy + \frac{2^{N-2}}{|x|^{N-2}} \int_{\mathbb{R}^N} P(y)dy.\end{aligned} \tag{6}$$

Let $y \in \mathbb{R}^N$ be such that $|x-y| \leq |x|/2$. For $z := x-y$ we have

$$|x-z| \geq |x| - |z| \geq |x| - \frac{|x|}{2} = \frac{|x|}{2},$$

and

$$|x-z| \geq |x| - |z| \geq 2|z| - |z| = |z|.$$

Since R, and Q are decreasing we have $Q(|x-z|) \leq Q\left(\frac{|x|}{2}\right)$, and $R(|x-z|) \leq R(|z|)$.

Therefore

$$\begin{aligned}\int_{A_x} \frac{P(y)}{|x-y|}dy &= \int_{|z|\leq|x|/2} \frac{P(|x-z|)}{|z|^{N-2}} dz \\ &\leq \int_{|z|\leq|x|/2} \frac{Q(|x|/2)R(|z|)}{|z|^{N-2}} dz \\ &= Q\left(\frac{|x|}{2}\right) \int_{\mathbb{R}^N} \frac{R(z)}{|z|^{N-2}} dz,\end{aligned}$$

which combined with (5) and (6) provides (P_3).

Note that the functions $R(x) = \dfrac{1}{|x|^{\alpha} + |x|^{\beta}}, 0 < \alpha < 2 < \beta$ and $Q(x) = \dfrac{1}{e^{|x|}}$ are decreasing radial functions of $C^{+}_{rad}(\mathbb{R}^N, \mathbb{R})$ and satisfy (5).

Concerning a function f satisfying $(f_1)-(f_3)$, note that a direct computation shows that the function

$$f(t)=\begin{cases} t\sin t, & t\in[0,6\pi] \\ 0, & t>6\pi, \end{cases}$$

where $a_1=\pi$, $a_2=3\pi$, $a_3=5\pi$, $b_1=2\pi$, $b_2=4\pi$, and $b_3=6\pi$ verifies $(f_1)-(f_3)$.

Now, it will be presented an example where (f_1), $(f_2)_\infty-(f_4)_\infty$ hold true. Consider $f(t)=e^t\sin^+ t$, and $a_k =$r$(2k-1)\pi$, $b_k=2k\pi$, $k\geq 1$. Note that (f_1) and $(f_2)_\infty$ occurs.

We now set $F(s)=\int_0^s f(t)dt,\ s\geq 0.$ In order to prove $(f_3)_\infty$, note that for $k\geq 1$ we have

$$F(a_k)=\sum_{i=1}^{k}\int_{(2i-2)\pi}^{(2i-1)\pi} e^t\sin t dt = \frac{1+\cdots+e^{(2k-1)\pi}}{2}=\frac{e^{2k\pi}-1}{2(e^\pi-1)}. \tag{7}$$

Since F is a nondecreasing and nonnegative function, it follows from (7) that

$$\begin{aligned}\frac{F(a_k)-\max\{|F(s)|;0\leq s\leq a_{k-1}\}}{\max\{|F(s)|;0\leq s\leq a_k\}} &= \frac{F(a_k)-\max\{F(s);0\leq s\leq a_{k-1}\}}{\max\{F(s);0\leq s\leq a_k\}} \\ &= \frac{F(a_k)-F(a_{k-1})}{F(a_k)} \\ &= \frac{e^{(2k-1)\pi}}{2F(a_k)} \\ &= \left(\frac{e^{(2k-1)\pi}}{e^{2k\pi}-1}\right)(e^\pi-1) \\ &= \left(\frac{e^{2k\pi}}{e^{2k\pi}-1}\right)\left(\frac{e^\pi-1}{e^\pi}\right) \\ &\geq \left(\frac{e^\pi-1}{e^\pi}\right),\end{aligned}$$

which proves $(f_3)_\infty$.

With respect to the proof of (f_4), note that from (7), and the inequality $e^x>1+x$, $x>0$, we obtain that

$$\begin{aligned}\max\{|F(s)|;0\leq s\leq a_k\} &= \max\{F(s);0\leq s\leq a_k\} \\ &= F(a_k) \\ &= \frac{e^{2k\pi}-1}{2(e^\pi-1)} \\ &> \frac{2k\pi+1-1}{2(e^\pi-1)} \\ &= \frac{a_k}{2(e^\pi-1)},\end{aligned}$$

which imply $(f_4)_\infty$.

3. Existence and multiplicity of solutions

In this section, we will obtain the existence and multiplicity of solutions to problem (P). We start by stating a technical result, which can be found in [15, Lemma 2.1], that will be important to ensure the positivity of the solutions and that allows us to distinguish the obtained multiple solutions. The proof of this lemma is based on the arguments of [37] and we invite the reader to see the proof on [15].

Lemma 3.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

$$\begin{cases} g(t) \geq 0, & t \in (-\infty, 0) \\ g(t) \leq 0, & t \in (s_0, +\infty). \end{cases}$$

for some $s_0 \geq 0$. If $u \in D^{1,2}(\mathbb{R}^N)$ is a weak solution of the problem

$$-\Delta u = P(x)g(u) \text{ in } \mathbb{R}^N, \tag{Q}$$

then $u \geq 0$ a.e. in Ω and $|u|_\infty \leq s_0$.

We will apply the dual approach developed in the papers [12,36] to deal with the quasilinear equation (P). The quasilinear problem (P) will be transformed into an equivalent semilinear one by performing a suitable change of variables. More precisely, we make the change of variables $v = h^{-1}(u)$, where h satifies

$$\begin{aligned} h'(t) &= \frac{1}{[1+2h^2(t)]^{1/2}} \quad \text{on} \quad [0,+\infty), \\ h(t) &= -h(-t) \quad \text{on} \quad (-\infty, 0]. \end{aligned} \tag{8}$$

The classical ODE theory provides that h is uniquely determined, invertible and that belongs to $C^2(\mathbb{R}, \mathbb{R})$. Therefore, by this change of variables, the original equation (P) is transformed in the following auxiliary semilinear problem:

$$\begin{cases} -\Delta v = \lambda P(x) h'(v) f(h(v)) \text{ in } \mathbb{R}^N \\ \lim\limits_{|x|\to\infty} v(x) = 0. \end{cases} \tag{P'}$$

We say that that $v \in D^{1,2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $v \neq 0$ is a distributional solution for (P') if

$$\int_{\mathbb{R}^N} \nabla v \nabla \varphi = \int_{\mathbb{R}^N} \lambda P(x) h'(v) f(h(v)) \varphi, \forall\, \varphi \in C_0^\infty(\Omega). \tag{9}$$

Note that from [20, Lemma 1], it follows that if a function $v \in D^{1,2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ satisfy the identity (9), where h is given in (8), then the function $u = h(v)$ solves the problem (P), i.e., satisfies (3).

In what follows, we present the main properties of the function $h : \mathbb{R} \to \mathbb{R}$ given in (8), which will be often applied in our study of the problem (P').

Lemma 3.2 (see [12,40])**.** *The function h and its derivative enjoy the following properties:*

(i) *h is uniquely defined, C^2 and invertible;*
(ii) $|h'(t)| \leq 1$ *for all $t \in \mathbb{R}$;*

(iii) $|h(t)| \le |t|$ *for all* $t \in \mathbb{R}$*;*
(iv) $h(t)/t \to 1$ *as* $t \to 0$*;*
(v) $|h(t)| \le 2^{1/4}|t|$ *for all* $t \in \mathbb{R}$*;*
(vi) $h(t)/2 < th'(t) < h(t)$ *for all* $t > 0$*, and the reverse inequalities hold for* $t < 0$*;*
(vii) $h(t)/\sqrt{t} \to 2^{\frac{1}{4}}$ *as* $t \to +\infty$*;*
(viii) *there exists a constant* $c > 0$ *such that*

$$|h(t)| \ge \begin{cases} c|t|, & |t| \le 1 \\ c|t|^{1/2}, & |t| \ge 1; \end{cases}$$

(ix) $h^2(ts) \ge th^2(s)$ *for all* $t \ge 1$ *and* $s \ge 0$*.*

To study (P') we will consider a truncation of the nonlinearity. Consider for each $k = 1, \cdots, m$ the function f_k defined as

$$f_k(s) = \begin{cases} f(0) \text{ if } s \in (-\infty, 0], \\ f(s) \text{ if } s \in [0, a_k], \\ 0 \text{ if } s \in [a_k, +\infty). \end{cases}$$

It will be considered the truncated problem

$$\begin{cases} -\Delta v = \lambda P(x) h'(v) f_k(h(v)) \text{ in } \mathbb{R}^N, \\ \lim_{|x|\to\infty} v(x) = 0, \end{cases} \qquad (P_k')$$

and the energy functional associated to (P_k'), say $I_{k,\lambda} : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$, which is defined by

$$I_{k,\lambda}(v) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\, dx - \lambda \int_{\mathbb{R}^N} P(x) F_k(v) dx,$$

where

$$F_k(s) = \int_0^s h'(t) f_k(h(t)) dt.$$

Note that $I_{k,\lambda} \in C^1(D^{1,2}(\mathbb{R}^N), \mathbb{R})$ with the derivative $I'_{k,\lambda}$ given by

$$I'_{k,\lambda}(v)w = \int_{\mathbb{R}^N} \nabla v \nabla w dx - \lambda \int_{\mathbb{R}^N} P(x) h'(v) f_k(h(v)) w dx, \quad \forall w \in D^{1,2}(\mathbb{R}^N).$$

The proof of Theorem 1.1 will be splitted in some lemmas.

Lemma 3.3. $I_{k,\lambda}$ *is coercive, that is,* $I_{k,\lambda}(v) \to +\infty$ *as* $|v|_{1,2} \to +\infty$*.*

Proof. The boundedness of f_k and (ii)−Lemma 3.2 implies that there exists c_k with

$$|F_k(v)| \le c_k |v|,$$

which provides

$$I_{k,\lambda}(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\, dx - \lambda \int_{\mathbb{R}^N} P(x) F_k(v) dx \ge \frac{1}{2}|v|^2_{1,2} - \lambda c_k \int_{\mathbb{R}^N} P(x)|v|\, dx. \qquad (10)$$

Since

$$\int_{\mathbb{R}^N} P(x)|v|dx = \int_{\mathbb{R}^N} P^{1/2}(x)P^{1/2}(x)|v|dx,$$

it follows from Hölder's inequality that

$$\int_{\mathbb{R}^N} P(x)|v|dx \le \left(\int_{\mathbb{R}^N} P(x)dx\right)^{1/2}\left(\int_{\mathbb{R}^N} P(x)|v|^2\,dx\right)^{1/2}. \tag{11}$$

By applying (10), (11) and the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^2_p(\mathbb{R}^N)$ (see [22]) we obtain that

$$I_{k,\lambda}(v) \ge \frac{1}{2}|v|^2_{1,2} - \lambda \tilde{c}_k |v|_{1,2}.$$

Therefore, $I_{k,\lambda}(v) \to +\infty$ as $|v|_{1,2} \to +\infty$.

Lemma 3.4. *$I_{k,\lambda}$ is bounded below.*

Proof. Since $I_{k,\lambda}$ is coercive, given $M > 0$ there is $R > 0$ such that,

$$I_{k,\lambda}(v) > M, \text{ when } |v|_{1,2} > R. \tag{12}$$

On the other hand we have

$$|I_{k,\lambda}(v)| \le \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,dx + \lambda \int_{\mathbb{R}^N} |P(x)F_k(v)|dx \le \frac{1}{2}|v|^2_{1,2} + \lambda \tilde{c}_k |v|_{1,2}.$$

By fixing $|v|_{1,2} < R$, we get

$$|I_{k,\lambda}(v)| \le \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,dx + \lambda \int_{\mathbb{R}^N} |P(x)F_k(v)|dx \le \frac{1}{2}R^2 + \lambda \tilde{c}_k R := M_1.$$

Therefore, $I_{k,\lambda}(v) \ge -M_1$, for all $v \in D^{1,2}(\mathbb{R}^N)$.

It follows, by Lemmas 3.3 and 3.4, that there is a global minimum v_k of $I_{k,\lambda}$. The function v_k $D^{1,2}(\mathbb{R}^N)$ is a critical point of $I_{k,\lambda}$ and it follows from the Riesz Potencial Theory (see [31, p. 43]) that

$$v_k(x) = C\int_{\mathbb{R}^N} \frac{P(x)h'(v_k)f_k(h(v_k))w}{|x-\xi|^{N-2}}d\xi, \quad \forall x \in \mathbb{R}^N.$$

From the boundedness of f_k, (*ii*) of Lemma 3.2 and the hypothesis (P_3) we have

$$|v_k(x)| \le \frac{C}{|x|^{N-2}},$$

and therefore $\lim_{|x|\to\infty} u_k(x) = 0$. Note that

$$h'(t)f_k(h(t)) = \begin{cases} h'(0)f(0) \ge 0, \text{if } t \le 0,\\ 0, \text{if } t \ge h^{-1}(a_k).\end{cases}$$

Thus, it follows from Lemma 3.1 that $0 \leq v_k \leq h^{-1}(a_k)$. From Lemma 3.2 we conclude that for each $k = 2, \cdots, m$, the problem (P) has a solution $u_k = h^{-1}(v_k)$ with $0 \leq u_k\ a_k$.

In the next lemma it will be obtained that

$$a_{k-1} < |u_k|_\infty \leq a_k,\ k = 2, \cdots, m,$$

and so, (P) has at least $m - 1$ solutions when $\lambda > 0$ is large enough.

Lemma 3.5. *For each $k = 2, \cdots, m$, there is $\lambda_k > 0$, such that for all $\lambda > \lambda_k$ it holds*

$$a_{k-1} < |u_k|_\infty \leq a_k.$$

Proof. It suffices to prove that $h^{-1}(a_{k-1}) < |v_k|_\infty$. Suppose that $|v_k|_\infty \leq h^{-1}(a_{k-1})$.

We have

$$I_{k,\lambda}(v_k) \leq I_{k,\lambda}(w),\ \forall w \in D^{1,2}(\mathbb{R}^N). \tag{13}$$

In order to prove the result, it will be obtained the existence of both $\lambda_k > 0$ and $w_k \in D^{1,2}(\mathbb{R}^N)$ with $w_k \geq 0$ and $|w_k|_\infty \leq a_k$, such that

$$I_{k,\lambda}(w_k) < I_{k-1,\lambda}(v_k),\ \forall \lambda > \lambda_k.$$

which contradicts (13).

We have

$$F_k(h^{-1}(a_k)) - F_k(h^{-1}(a_{k-1})) = \int_{h^{-1}(a_{k-1})}^{h^{-1}(a_k)} f(h(t))h'(t)dt.$$

Applying the change of variables $s = h(t)$ we obtain

$$F(h^{-1}(a_k)) - F(h^{-1}(a_{k-1})) = \int_{a_{k-1}}^{a_k} f(s)ds.$$

Since we supposed that f verifies (f_3), we have

$$\alpha := F_k(h^{-1}(a_k)) - \max_{0 \leq t \leq a_{k-1}} \left\{ F_k(h^{-1}(t)) \right\} > 0.$$

Then for all v in $D^{1,2}(\mathbb{R}^N)$ with $0 \leq v \leq h^{-1}(a_{k-1})$ a.e. in $\mathbb{R}^N$ we obtain that

$$\int_{\mathbb{R}^N} P(x)F_k(h^{-1}(a_k))dx \geq \int_{\mathbb{R}^N} P(x)F_k(v)dx + \alpha\, |P|_1.$$

Since $P \in L^1(\mathbb{R}^N)$, given $\varepsilon > 0$, there is $R = R(\varepsilon) > 0$ large enough, with

$$\int_{\mathbb{R}^N \setminus B_R(0)} |P|\, dx < \varepsilon.$$

Consider $w_R \in D^{1,2}(\mathbb{R}^N)$ with $0 \le w_R \le h^{-1}(a_k)$ and $w_R = h^{-1}(a_k)$ for all $x \in B_R(0)$. We have

$$\int_{\mathbb{R}^N} P(x)F_k(w_R)dx = \int_{B_R(0)} P(x)F_k(h^{-1}(a_k))dx + \int_{\mathbb{R}^N \setminus B_R(0)} P(x)F_k(w_R)dx$$
$$= \int_{\mathbb{R}^N} P(x)F_k(h^{-1}(a_k))dx - \int_{\mathbb{R}^N \setminus B_R(0)} P(x)[F_k(h^{-1}(a_k)) - F_k(w_R)]dx$$
$$\ge \int_{\mathbb{R}^N} P(x)F_k(h^{-1}(a_k))dx - 2C \int_{\mathbb{R}^N \setminus B_R(0)} P(x)dx,$$

where $C = \max F_k(t);\ 0 \le t \le h^{-1}(a_k)$. Thus, for all $v \in D^{1,2}(\mathbb{R}^N)$ with $0 \le v\ h^{-1}(a_{k-1})$ we have

$$\int_{\mathbb{R}^N} P(x)F_k(w_R)dx \ge \int_{\mathbb{R}^N} P(x)F_k(v)dx + \alpha\,|P|_1 - \varepsilon.$$

Fix $\varepsilon > 0$ such that $\eta = \alpha|P|_1 - \varepsilon > 0$, and define $w_k := w_R$. For all $v \in D^{1,2}(\mathbb{R}^N)$, $0 \le v \le h^{-1}(a_{k-1})$ we get

$$I_{k,\lambda}(w) - I_{k-1,\lambda}(v_{k-1}) = \frac{1}{2}(|w|^2_{1,2} - |v_{k-1}|^2_{1,2}) - \lambda\int_{\mathbb{R}^N} P(x)(F_k(w) - F_k(v_{k-1}))dx$$
$$\le \frac{1}{2}|w|_{1,2} - \lambda\eta < 0,$$

provided that $\lambda > \frac{|w_k|_{1,2}}{2\eta} := \lambda_k$. Therefore $h^{-1}(a_{k-1}) \le h^{-1}(a_{k-1}) < |v_k|_\infty \le h^{-1}(a_k)$, which provides that $a_{k-1} < |u_k|_\infty \le a_k$.

In view of this result, it follows that problem (P) has at least $m - 1$ solutions $u_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1, , m\ 1$ for $\lambda > \lambda_0$, where $\lambda_0 := \max\{\lambda_1, \cdots, \lambda_{m-1}\}$ with λ_k define in the proof of Lemma 3.5.

3.1. The infinite zeros case

Based in the ideas of [14] it will be considered the case of infinite zeros.

As in the case of Theorem 1.1, by Lemmas 3.3 and 3.4, there is a global minimum v_k of $I_{k,\lambda}$. Moreover, $0 \le v_k \le h^{-1}(a_k)$. The next result provides that for $k \in \{2, 3, 4, \cdots\}$

$$h^{-1}(a_{k-1}) < |v_k|_\infty \le h^{-1}(a_k),$$

in the infinite zeros case, for all $\lambda > 0$ large enough, and as in the first part we obtain infinitely many solutions u_j with

$$a_{k-1} < |u_k|_\infty \le a_k,$$

that completes the proof of Theorem 1.2.

Lemma 3.6. *Consider* (f_1), $(f_2)_\infty$ *and* $(f_3)_\infty$. *There is a positive constant* $\tilde{\lambda}_0$, *such that for all* $\lambda > \tilde{\lambda}_0$, *it holds that*

$$a_{k-1} < |u_k|_\infty \le a_k.$$

Proof. It is enough to show that $h^{-1}(a_{k-1}) < |v_k|_\infty$. Recall that

$$I_{k,\lambda}(v_k) \le I_{k,\lambda}(w),\ \forall w \in D^{1,2}(\mathbb{R}^N). \tag{14}$$

Suppose that $|v_k|_\infty \le h^{-1}(a_{k-1})$. The idea of the proof will consist in obtain a contradiction by showing the existence of both $\tilde{\lambda}_0 > 0$, $w \in D^{1,2}(\mathbb{R}^N)$ with $w \ge 0$, $|w|_\infty \le h^{-1}(a_k)$ and

$$I_{k,\lambda}(w) < I_{k-1,\lambda}(v_k),\ \forall\lambda > \tilde{\lambda}_0, \tag{15}$$

where $\lambda_0 > 0$ is a constant that will be described before.

From $(f_3)_\infty$ and $(f_4)_\infty$ we obtain that

$$0 < \alpha_k := F_k(h^{-1}(a_k)) - \max\{F_k(s);\ 0 \le s < h^{-1}(a_{k-1})\}.$$

Thus, it follows that for all $v \in D^{1,2}(\mathbb{R}^N)$ with $0 \le v \le h^{-1}(a_{k-1})$ a. e. in $\mathbb{R}^N$ we obtain that

$$\int_{\mathbb{R}^N} P(x)F_k(h^{-1}(a_k))dx \ge \int_{\mathbb{R}^N} P(x)F_k(v)dx + \alpha_k\, |P|_1\, .$$

Since $P \in L^1(\mathbb{R}^N)$, given $\varepsilon > 0$, we may find $R = R(\varepsilon) > 0$ large enough, with

$$\int_{\mathbb{R}^N\setminus B_R(0)} |P|\,dx < \varepsilon\, .$$

Consider $w_R \in D^{1,2}(\mathbb{R}^N)$ with $0 \le w_R \le h^{-1}(a_k)$, and $w_R = h^{-1}(a_k)$ for all $x \in B_R(0)$. Therefore

$$\begin{aligned}\int_{\mathbb{R}^N} P(x)F_k(w_R)dx &= \int_{B_R(0)} P(x)F_k(h^{-1}(a_k))dx + \int_{\mathbb{R}^N\setminus B_R(0)} P(x)F_k(w_R)dx\\ &= \int_{\mathbb{R}^N} P(x)F(h^{-1}(a_k))dx - \int_{\mathbb{R}^N\setminus B_R(0)} P(x)[F(h^{-1}(a_k)) - F(w_R)]dx\\ &\ge \int_{\mathbb{R}^N} P(x)F(h^{-1}(a_k))dx - 2C_k\int_{\mathbb{R}^N\setminus B_R(0)} P(x)dx,\end{aligned}$$

where $C_k = \max\{|F(t)|;\ 0 \le t \le h^{-1}(a_k)\}$. Thus, we have for all $v \in D^{1,2}(\mathbb{R}^N)$ with $0 \le v \le h^{-1}(a_{k-1})$ that

$$\int_{\mathbb{R}^N} P(x)F(w_R)dx \ge \int_{\mathbb{R}^N} P(x)F(v)dx + C_k\left(\frac{\alpha_k}{C_k}|P|_1 - 2\int_{\mathbb{R}^N\setminus B_R(0)} P(x)dx\right).$$

Fix $R > 0$ such that $\left(\frac{\alpha_k}{C_k}|P|_1 - 2\int_{\mathbb{R}^N\setminus B_R(0)} P(x)dx\right) > 0$ and define $w = w_R$. From $(f_3)_\infty$ and $(f_4)_\infty$ we get

$$\begin{aligned}I_{k,\lambda}(w) - I_{k-1,\lambda}(v_k) &= \frac{1}{2}(|w|^2_{1,2} - |v_k|^2_{1,2}) - \lambda\int_{\mathbb{R}^N} P(x)(F_k(w) - F_k(v_k))dx\\ &\le \frac{1}{2}|w|^2_{1,2} - \lambda C_k\left(\frac{\alpha_k}{C_k}|P|_1 - 2\int_{\mathbb{R}^N\setminus B_R(0)} P(x)dx\right).\end{aligned}\tag{16}$$

Note that from $(f_3)_\infty$, and $(f_4)_\infty$ we have

$$\frac{|w|^2_{1,2}}{2C_k\left(\frac{\alpha_k}{C_k}|P|_1 - 2\int_{\mathbb{R}^N\setminus B_R(0)} P(x)dx\right)} \le \frac{|w|^2_{1,2}}{2da_k\left(\beta|P|_1 - 2\int_{\mathbb{R}^N\setminus B_R(0)} P(x)dx\right)},\ \forall k \ge 2.$$

Since $a_k \to a$ or $a_k \to +\infty$, it follows that the right-hand side given in the last inequality is a bounded sequence. By defining

$$\tilde{\lambda}_0 := \sup_{k\geq 2} \frac{|w|_{1,2}^2}{2da_k\left(\beta |P|_1 - 2\int_{\mathbb{R}^N \setminus B_R(0)} P(x)dx\right)},$$

and considering $\lambda > \tilde{\lambda}_0$, it follows from (16) and (17) that

$$\begin{aligned} I_{k,\lambda}(w) - I_{k,\lambda}(v_k) &= I_{k,\lambda}(w) - I_{k-1,\lambda}(v_k) \\ &< 0, \end{aligned}$$

which contradicts (14).

From the previous result, it follows that problem (P) has has infinitely many non-negative weak solutions $u_i : \mathbb{R}_+ \to \mathbb{R}$, $i \geq 1$ for $\lambda > \tilde{\lambda}_0$, where $\tilde{\lambda}_0$ was defined in (18).

4. Asymptotic behaviour

The main goal of this section consists in analyze the behavior of u_λ, the "first solution" of problem (P), as $\lambda \to \infty$. As mentioned before, the ideas of this section were inspired by [10] and [18].

In this section it will be considered that f satisfies: $f(0) = f(a_1) = 0$, $f(t) > 0$ for all $0 < t < a_1$, the right derivative $f_+'(0)$ exists and satisfies $f_+'(0) > \lambda_1$, where λ_1 is the first eigenvalue of the problem

$$\begin{cases} -\Delta u = \lambda P(x)u, \text{ in } \mathbb{R}^N \\ \lim_{|x|\to+\infty} u(x) = 0 \end{cases} \tag{19}$$

Since in this section we are interested in f on the interval $[0, a_1]$ we consider an extension of f, still denoted by f, such that $f(t) = 0$ for $t \in \mathbb{R} \setminus [0, a_1]$.

In what follows it will be considered a pair of sub-supersolutions for (P'). We say that $0 < \underline{v} \in D^{1,2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is a subsolution for (P') if

$$\begin{cases} -\Delta \underline{v} \leq \lambda P(x)h'(\underline{v})f(h(\underline{v})), \text{ in } \mathbb{R}^N \\ \lim_{|x|\to\infty} \underline{v}(x) = 0, \end{cases} \tag{SB}$$

and $0 < \bar{v} \in D^{1,2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is a supersolution for (P') if

$$\begin{cases} -\Delta \bar{v} \geq \lambda P(x)h'(\bar{v})f(h(\bar{v})), \text{ in } \mathbb{R}^N \\ \lim_{|x|\to\infty} \bar{v}(x) = 0, \end{cases} \tag{SP}$$

in the weak sense.

First it will be obtained the subsolution. Since $f_+'(0) > \lambda_1$, by considering $\tilde{f}(t) = (f \circ h)(t)$, we have from (8) that

$$\tilde{f}_+'(0) = h'(0)f'(h(0)) = f_+'(0) > \lambda_1.$$

Hence, there is $t_0 > 0$ such that if $t \in (0, t_0)$, we have $\frac{h'(t)f(h(t))}{t} > \lambda_1$, and so

$$h'(t)f(h(t)) > \lambda_1 t,\ t \in (0, t_0).$$

On the other hand, we have $0 < \varepsilon\phi_1(x) < t_0$ for all $x \in \mathbb{R}^N$ if $0 < \varepsilon$ is small enough. Therefore,

$$-\Delta(\varepsilon\phi_1(x)) = \lambda_1 P(x)(\varepsilon\phi_1(x)) \le P(x)h'(\varepsilon\phi_1(x))f(h(\varepsilon\phi_1(x))), \text{ in } \mathbb{R}^N.$$

Thus, it follows for $\lambda >> 1$ that

$$-\Delta(\varepsilon\phi_1) \le \lambda P(x)h(\varepsilon\phi_1)f(h(\varepsilon\phi_1)) \text{ in } \mathbb{R}^N. \tag{20}$$

We quote that $\lambda \ge 1$ does not depends on $\varepsilon > 0$ and, consequently, $0 < \underline{v}(x) = \varepsilon\phi_1(x)$ is a subsolution of (P'), for λ large enough.

Now it will be considered the supersolution. For a fixed λ large enough as in the construction of the subsolution, consider $e \in D^{1,2}(\mathbb{R}^N)\, L^\infty(\mathbb{R}^N)$ be the unique solution of the problem

$$\begin{cases} -\Delta e = P(x), \text{ in } \mathbb{R}^N \\ \lim_{|x|\to\infty} e(x) = 0. \end{cases} \tag{21}$$

We will show that there is $M > 0$ large enough such that $\bar{u} = Me$ is a supersolution for the problem (P). Consider

$$\begin{aligned} \hat{f}: \mathbb{R} &\to \mathbb{R} \\ s &\mapsto \hat{f}(s) = \max_{0\le t\le s} h'(t)f(h(t)) \end{aligned}$$

Note that, $\hat{f}$ is nondecreasing with

$$\lim_{s\to\infty} \frac{\hat{f}(s)}{s} = 0$$

and so, it is possible to choose $M = M(\lambda) > 0$ such that

$$\frac{1}{\lambda |e|_\infty} \ge \frac{\hat{f}(M|e|_\infty)}{M|e|_\infty}.$$

Then, for $\lambda > 0$ large enough, we have that $\bar{v} = Me$ satisfies

$$\begin{aligned} -\Delta\bar{v} &= P(x)M \ge \lambda P(x)\hat{f}(M|e|_\infty) \\ &\ge \lambda P(x)\hat{f}(Me) \\ &\ge \lambda P(x)h'(\bar{v})f(h(\bar{v})), \text{ in } \mathbb{R}^N, \end{aligned}$$

which provides that $\bar{v} = Me$ is a supersolution for (P').

Now it will be proved that for $\varepsilon > 0$ small enough that

$$Me \ge \varepsilon\phi_1 \text{ in } \mathbb{R}^N.$$

We have

$$
\begin{aligned}
0 \le |(Me - \varepsilon\phi_1)^-|_{1,2}^2 &= \int_{\mathbb{R}^N} \nabla(Me - \varepsilon\phi_1)\nabla(Me - \varepsilon\phi_1)^- dx \\
&= M\int_{\mathbb{R}^N} \nabla e \nabla(Me - \varepsilon\phi_1)^- dx - \varepsilon\int_{\mathbb{R}^N} \nabla\phi_1\nabla(Me - \varepsilon\phi_1)^- dx \\
&= M\int_{\mathbb{R}^N} P(x)(Me - \varepsilon\phi_1)^- dx - \varepsilon\int_{\mathbb{R}^N} P(x)\lambda_1\phi_1(Me - \varepsilon\phi_1)^- dx \\
&= \int_{\mathbb{R}^N} P(x)(M - \varepsilon\lambda_1\phi_1)(Me - \varepsilon\phi_1)^- dx \le 0,
\end{aligned}
$$

which provides that $|(Me - \varepsilon\phi_1)^-|_{1,2}^2 = 0$ and, consequently, $\bar{v} = Me \ge \varepsilon\phi_1 = \underline{v}$ in $D^{1,2}(\mathbb{R}^N)$.

In the proof of the next lemma it will be used the following fixed point result whose proof may be found in Evans [23, p. 504]:

Theorem 4.1. (Schaefer's Fixed Point Theorem). *Suppose that $A : X \to X$ is a continuous and compact mapping in the Banach space X. Consider that the set*

$$\{u \in X;\ u = \mu A(u),\ \text{for some } 0 \le \mu \le 1\}$$

is bounded. Then A has a fixed point in X.

Lemma 4.2. *There exists a solution v_λ of (P') with $v_\lambda \in \mathrm{M}$, where M is the set defined by*

$$\mathrm{M} = \{v \in D^{1,2}(\mathbb{R}^N);\ \underline{v} \le v \le \bar{v} \text{ a.e. in } \mathbb{R}^N\}.$$

Proof. Initially, note that $\underline{v} \le \bar{v}$ and $\underline{v}, \bar{v} \in D^{1,2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Consider the operator $T : D^{1,2}(\mathbb{R}^N) \to D^{1,2}(\mathbb{R}^N)$ given by

$$
Tv_\lambda(x) = \begin{cases} \bar{v}(x), & \text{if } v(x) \ge \bar{v}(x), \\ v(x), & \text{if } \underline{v}(x) \le v(x) \le \bar{v}(x), \\ \underline{v}(x), & \text{if } v(x) \le \underline{v}(x). \end{cases}
$$

Hence, we can consider the auxiliary problem

$$
\begin{cases} -\Delta w = \lambda P(x)\tilde{f}(T_\lambda v) \text{ in } \mathbb{R}^N, \\ \lim_{|x|\to\infty} w(x) = 0, \end{cases} \tag{22}
$$

where $\tilde{f}$ was defined previously and the fixed $\lambda > 0$ is as in the construction of the sub-supersolutions. We have that for each $v \in L^2_P(\mathbb{R}^N)$ there is a unique solution $w \in D^{1,2}(\mathbb{R}^N)$ for (22), because $f \in L^\infty(\mathbb{R}^N)$ and $P \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. So, it is well defined the solution operator for (22) defined by $S : L^2_P(\mathbb{R}^N) \to D^{1,2}(\mathbb{R}^N)$, where $w := Sv$, $v \in L^2_P(\mathbb{R}^N)$ is the unique solution of (22). Since the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^2_P(\mathbb{R}^N)$ is compact (see [22]), we can consider that $S : L^2_P(\mathbb{R}^N) \to L^2_P(\mathbb{R}^N)$ is a compact operator.

In order to use the Schafer's Fixed Point Theorem, we consider the equation $v = \mu Su$ for $\mu \in [0, 1]$. We may consider $\mu \ne 0$ and so $\frac{1}{\mu}v = Sv$, that is,

$$
\begin{cases} -\Delta v = \mu\lambda P(x)\tilde{f}(T_\lambda v) \text{ in } \mathbb{R}^N \\ \lim_{|x|\to\infty} v(x) = 0, \end{cases} \tag{23}
$$

which provides that, for each fixed $\lambda > 0$ as above, the set $\{v \in L^2_p(\mathbb{R}^N);\ v = \mu S(v)$ for all $0 \le \mu \le 1\}$ is bounded. Invoking the Schaefer's Fixed Point Theorem, it follows that there is $v \in D^{1,2}(\mathbb{R}^N)$ such that $v = S(v)$, that is,

$$-\Delta v = \lambda P(x)\tilde{f}(Tv) \text{ in } \mathbb{R}^N. \tag{24}$$

Now, we need show that $v \in$ M. For this, consider $z := \bar{v} - v$. We have

$$-\Delta z \ge \lambda P(x)\tilde{f}(\bar{v}) - \lambda P(x)\tilde{f}(Tv) \text{ in } \mathbb{R}^N. \tag{25}$$

Multiplying both sides of the last inequality by $-(\bar{v} - v)^-$ and integrating, we get $v \le \bar{v}$. A similar reasoning provides that $\underline{v} \le v$. Therefore $T_\lambda v = v$. Setting $v_\lambda := v$, we have $\underline{v} \le v_\lambda \le \bar{v}$, that is, $v_\lambda \in$ M.

Theorem 4.3. *Under the above assumptions, let $u_\lambda = h(v_\lambda)$ be a nontrivial solution to (P) as we before. Then for every $p \ge 1$ one has $u_\lambda \to a_1$ in $L^p_{loc}(\mathbb{R}^N)$, as $\lambda \to +\infty$.*

Proof. Note that

$$P(x)\tilde{f}(v_\lambda) \to 0, \text{ in } D'(\mathbb{R}^N), \text{ as } \lambda \to \infty.$$

In fact, since $-\Delta v_\lambda = \lambda P(x)\tilde{f}(v_\lambda)$ in $\mathbb{R}^N$, we have

$$\int_{\mathbb{R}^N} P(x)\tilde{f}(v_\lambda)\varphi dx = \frac{1}{\lambda}\int_{\mathbb{R}^N} \nabla v_\lambda \nabla \varphi dx, \quad \forall \varphi \in D(\mathbb{R}^N)$$

and so, because $\varphi \in D(\mathbb{R}^N)$ and $\|v_\lambda\|_\infty \le h^{-1}(a_1)$, we get

$$\int_{\mathbb{R}^N} P(x) f(v_\lambda)\varphi dx = \frac{1}{\lambda}\int_{\mathbb{R}^N} v\lambda(-\Delta\varphi)dx \to 0, \text{ when } \lambda \to \infty,$$

that is, $P(x)f(v_\lambda) \to 0$ in $D'(\mathbb{R}^N)$.

On the other hand, let $K \subset \mathbb{R}^N$ be a compact set. For each $\eta > 0$ we have

$$|\{x \in K;\ v_\lambda(x) \le h^{-1}(a_1) - \eta\}| \to 0, \text{ as } \lambda \to \infty.$$

Indeed, if $\varphi \in D(\mathbb{R}^N)$ with $\varphi \equiv 1$ in K, then

$$\int_{\mathbb{R}^N} P(x)\tilde{f}(v_\lambda)\varphi dx \ge \int_K P(x)\tilde{f}(v_\lambda)dx \ge \int_{\{v_\lambda \le h^{-1}(a_1)-\eta\}} P(x)\tilde{f}(v_\lambda)dx, \tag{26}$$

where $\{v_\lambda \le h^{-1}(a_1) - \eta\} = \{x \in K;\ v_\lambda(x) \le h^{-1}(a_1) - \eta\}$. In the set $\{v_\lambda \le a_1 - \eta\}$ it occurs that $\tilde{f}(v_\lambda) \ge \inf_{[0,h^{-1}(a_1)-\eta)} \tilde{f} =: C > 0$, which provides

$$\int_{\mathbb{R}^N} P(x)\tilde{f}(v_\lambda)\varphi dx \ge C \int_{\{v_\lambda \le h^{-1}(a_1)-\eta\}} P(x)dx. \tag{27}$$

Since $P(x) > 0$ in $\mathbb{R}^N$ there is a constant $C_1 > 0$ such that,

$$P(x) \ge C_1 > 0, \forall x \in K, \tag{28}$$

which imply

$$\int_{\mathbb{R}^N} P(x)\tilde{f}(v_\lambda)\varphi dx \ge \bar{C}\,|\{v_\lambda \le h^{-1}(a_1) - \eta\}| \tag{29}$$

and, consequently

$$|\{v_\lambda \le h^{-1}(a_1) - \eta\}| \to 0, \text{ as } \lambda \to \infty. \tag{30}$$

Since

$$\begin{aligned} |v_\lambda - h^{-1}(a_1)|_{p,K}^p &= \int_K |v_\lambda - h^{-1}(a_1)|^p \, dx \\ &= \int_{\{v_\lambda \le h^{-1}(a_1)-\eta\}} |v_\lambda - h^{-1}(a_1)|^p \, dx + \int_{\{v_\lambda > h^{-1}(a_1)-\eta\}} |v_\lambda - h^{-1}(a_1)|^p \, dx \\ &\le h^{-1}(a_1)^p |\{v_\lambda \le h^{-1}(a_1) - \eta\}| + \eta^p |K| \end{aligned}$$

when $\lambda \to \infty$, we have

$$\lim_{\lambda\to\infty} |u_\lambda - a_1|_{p,K}^p \le \eta^p |K|, \quad \forall \eta > 0, \tag{31}$$

which implies

$$\lim_{\lambda\to\infty} |v_\lambda - h^{-1}(a_1)|_{p,K}^p = 0. \tag{32}$$

Therefore, since h is a C^2 function we conclude that

$$\lim_{\lambda\to\infty} |u_\lambda - a_1|_{p,K}^p = 0. \tag{33}$$

Proof of Theorem 1.3

In order to to prove Theorem 1.3 it will suffice to consider the case that $k = 2$. Now, we have that if u is a solution of

$$\begin{cases} -\Delta u - \Delta(u^2) = P(x) f_2(u) \text{in } \mathbb{R}^N \\ \lim\limits_{|x|\to\infty} u(x) = 0, \end{cases} \tag{34}$$

then we get by [2] that $u \in C^1(\mathbb{R}^N)$ because $f_2 \in L^\infty(\mathbb{R}^N)$. Performing the change of variables $v = h^{-1}(u)$ again, where h is defined by (8), we obtain

$$\begin{cases} -\Delta v = \lambda P(x) h'(v) f_2(h(v)) \text{ in } \mathbb{R}^N \\ \lim\limits_{|x|\to\infty} v(x) = 0, \end{cases} \tag{35}$$

Before proving the theorem, we need some important auxiliary results. Initially, we establish a version of a sub-supersolution method to the problem (35).

Proposition 5.1. *[15] Consider that problem (35) has a sub-solution $\underline{v}$ and a super-solution $\overline{v}$ satisfying $-\infty < \underline{c} \le \underline{v} \le \overline{v} \le \overline{c} < +\infty$. Then problem (35) has a solution $v \in D^{1,2}(\mathbb{R}^N)$ with $\underline{v} \le v \le \overline{v}$. Moreover, there is a minimal solution $\underline{w}$ and a maximal solution $\overline{w}$ such that, for all solution v with $\underline{v} \le v \le \overline{v}$, we have*

$$\underline{v} \le \underline{w} \le v \le \overline{w} \le \overline{v} \text{ in } \mathbb{R}^N. \tag{36}$$

Lemma 5.2. *The maximum (minimum) of two solutions of (35) is a sub(super)solution of (35).*

Proof. Consider v_1 and v_2 two solutions of problem (35). We have that $v_1, v_2 \geq 0$ in $\mathbb{R}^N$. Invoking the Riesz Potential Theory it follows that if v is a solution of (35) one has

$$v(x) = C\int_{\mathbb{R}^N} \frac{P(y)h'(v(y))f_2(h(v(y)))}{|x-y|^{N-2}}dy,$$

and so, $v \in C(\mathbb{R}^N)$, see [1, Lemma 3.1]. Since

$$-\Delta v = P(x)h'(v)f_2(h(v)),$$

$P(x) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $h'(v)f_2(h(v)) \in L^\infty(\mathbb{R}^N)$, we have $P(x)h'(v)f_2(h(v)) \in L^p(\mathbb{R}^N)$, for all $p \geq 1$. Thus, we have that $v \in W^{2,p}_{loc}(\mathbb{R}^N)$.

Now, similarly to the ideas of [17], we will show that v^* defined by

$$v^*(x) = \max\{v_1(x), v_2(x)\} \tag{37}$$

is a sub-solution of (34).

From Kato's inequality [27] we have

$$-\int_{\mathbb{R}^N} |w|\Delta\varphi dx \leq -\int_{\mathbb{R}^N} sign(w)\Delta(w)\varphi dx, \text{ for all } w \in W^{2,1}_{loc}(\mathbb{R}^N), \varphi \in D^+(\mathbb{R}^N), \tag{38}$$

where $D^+(\mathbb{R}^N) = \{\varphi \in C^\infty(\mathbb{R}^N); \varphi \geq 0\}$.

For $u_1, u_2 \in W^{2,1}_{loc}(\mathbb{R}^N)$ we obtain that

$$\begin{aligned}
-\int_{\mathbb{R}^N} v^*\Delta\varphi dx &= -\frac{1}{2}\int_{\mathbb{R}^N}(v_1+v_2+|v_1-v_2|)\Delta\varphi dx\\
&\leq -\frac{1}{2}\int_{\mathbb{R}^N}(\Delta v_1+\Delta v_2+sign(v_1-v_2)(\Delta v_1-\Delta v_2))\varphi dx\\
&\leq \int_{\mathbb{R}^N} P(x)(\chi_{[v_1>v_2]}h'(v_1)f_2(h(v_1))\\
&\quad +\chi_{[v_1<v_2]}h'(v_2)f_2(h(v_2))+\frac{1}{2}\chi_{[v_1=v_2]}(h'(v_1)f_2(u_1)+h'(v_2)f_2(u_2)))\varphi dx\\
&= \int_{\mathbb{R}^N} P(x)h'(v^*)f_2(h(v^*))\varphi dx, \text{ for } \varphi \in D^+(\mathbb{R}^N),
\end{aligned}$$

which implies that

$$\int_{\mathbb{R}^N}\nabla v^*\nabla\varphi dx \leq \int_{\mathbb{R}^N} P(x)h'(v^*)f_2(h(v^*))\varphi dx, \tag{39}$$

for all $\varphi \in D^+(\mathbb{R}^N)$. Therefore, it is clear that v^* is a sub-solution of (35). Similarly we show that $v_*(x) = \min\{v_1(x), v_2(x)\}$ is a supersolution of (35), using that $v_*(x) = v_1 + v_2 - |v_1 - v_2|$.

Remark 1. From what has been exposed so far, if w is a solution of (35), then $w \in W^{2,p}_{loc}(\mathbb{R}^N)$ for all $p \geq 1$. Thus, by the Sobolev embedding theorem we have $w \in C^1(\mathbb{R}^N)$.

Lemma 5.3. *Let $v \in C^1(\mathbb{R}^N)$ be a nonnegative weak solution of (35). If $f(0) > 0$, then v is positive in $\mathbb{R}^N$.*

Proof. Consider $x_0 \in \mathbb{R}^N$ such that $v(x_0) = 0$. Let $D = B_r(y_0) \subset \mathbb{R}^N$ be an open ball such that $x_0 \in \partial D$ and g a strictly decreasing continuous function, with $g \le f_2$, defined on $[0, \infty)$ such that $g(0) = h'(0)f_2(h(0)) = h'(0)f(h(0)) > 0$ and $\gamma := g\left(\frac{a_1}{2}\right) = \inf\left\{g(s); 0 \le s \le \frac{a_1}{2}\right\} > 0$. Define the function b on D as

$$b(x) = \varepsilon\left(e^{-\left|\frac{x-y_0}{R}\right|^2} - e^{-1}\right),$$

where ε is sufficiently small such that

$$\sup_{x\in D} |\Delta b(x)| \le \gamma .$$

Then b is a subsolution of

$$-\Delta w = P(x)g(w) \text{ in } D,\ w = 0 \text{ on } \partial D.$$

It follows for all $\varphi \ge 0$ in $H_0^1(D)$ that

$$\int_D (\nabla b - \nabla v)\nabla\varphi dx \le \int_D P(x)(g(b) - h'(v) f_2(h(v)))\varphi dx.$$

Choosing $\varphi = (b - v)^+$, and using the fact that the Laplacian operator is monotone and g is strictly decreasing, we obtain

$$0 \le \int_{D^+} |\nabla(b - v)|^2\, dx \le \int_{D^+} P(x)(g(b) - h'(v) f_2(h(v)))(b - v)dx \le 0,$$

where $D^+ = \{x \in D;\ b(x) > v(x)\}$. Therefore, D^+ is empty or equivalently $v \ge b$ in D. Since $v(x_0) = b(x_0) = 0$, and $b > 0$ in D we have that the normal derivative with respect to the boundary of D satisfies $\frac{\partial v}{\partial \nu}(x_0) \le \frac{\partial b}{\partial \nu}(x_0) < 0$, implying that $|\nabla v(x_0)| \neq 0$, which contradicts the fact that $v(x_0) = 0$ is a minimum value of v in $\mathbb{R}^N$.

Remark 2. Using the ideas of [2, Lemma 3.1], note that: if $v \in D^{1,2}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ is a positive solution of (35), there exists $\varepsilon > 0$ such that

$$v(x) \ge \varepsilon > 0, \forall x \in \overline{B}_{a_2}(0).$$

Now, if we set $w(x) = v(x) - \varepsilon|x|^{2-N}$ and

$$\tilde{w} = \begin{cases} 0, & |x| \le a_2 \\ -w^-(x), & |x| > a_2 \end{cases}$$

it follows that $\tilde{w} \in D^{1,2}(\mathbb{R}^N)$, $supp(\tilde{w}) \subset B^c_{a_2}(0)$ and $\tilde{w} \ge 0$. Therefore

$$\int_{\mathbb{R}^N} \nabla w \nabla \tilde{w} dx = \int_{\mathbb{R}^N} \nabla v \nabla \tilde{w} dx = \int_{\mathbb{R}^N} P(x)h'(v) f_2(h(v))\tilde{w}\, dx = \int_{B^c_{a_2}(0)} P(x)h'(v) f_2(h(v))\tilde{w}\, dx = 0$$

and so

$$\int_{B^c_{a_2}(0)} |\nabla w^-|^2\, dx = 0.$$

Hence, $w^- = 0$ in $R^N \setminus B_a(0)$, which provides that

$$v(x) \geq \varepsilon |x|^{2-N},\ |x| \geq a_2. \tag{40}$$

The previous remark is very useful in the proof of Theorem 1.3.

Proof of Theorem 1.3

Observe that $\beta : R^N \to R$ defined by

$$\beta(x) = \begin{cases} 0, & |x| \leq b_1, \\ \dfrac{\varepsilon}{(a_2 - b_1)a_2^{N-2}}(|x| - b_1), & b_1 \leq |x| \leq a_2 \\ \varepsilon |x|^{2-N}, & |x| > a_2. \end{cases}$$

is a subsolution of (35). Moreover, v is a supersolution of (35). Therefore (35) has a minimal solution σ with $\beta(x) \leq \sigma(x) \leq v(x)$ for all $x \in R^N$. Thus, for all solution v of (35) with $\beta(x) \leq w(x) \leq v(x)$ it occurs that $w \geq \sigma$. See Lemma 5.1 and Remark 2.

We claim that this implies that σ is radially symmetric, that is, $\sigma(x_1) = \sigma(x_2)$ for all $x_1, x_2 \in R^N$ with $|x_1| = |x_2|$. Suppose that this not occurs, then there exist $x_1, x_2 \in R^N$ with $|x_1| = |x_2|$ such that $\sigma(x_1) > \sigma(x_2)$. Let T be an $N \times N$ matrix in $SO(N\ ;\ R)$, the special orthogonal group, such that $x_2 = Tx_1$. Note that the transpose matrix T^t of T is also its inverse matrix. Consider $v_1(x) = \sigma(Tx)$. Since for all $x \in R^N$

$$\nabla v_1(x) = T\nabla\sigma(Tx),$$

and the map $x \mapsto Tx$ is an isometry, it follows that

$$|\nabla v_1(x)| = |T\nabla\sigma(Tx)| = |\nabla\sigma(Tx)|.$$

We next show that v_1 is a weak solution of (35). That is, we need to prove that for all $\varphi \in D^{1,2}(R^N)$ it occurs

$$\int_{R^N} \nabla v_1(x)\nabla\varphi(x)dx = \int_{R^N} P(x)h'(v_1)f_2(h(v_1))\varphi(x)dx. \tag{41}$$

Define $\psi(x) = \varphi(T^t x) \in D^{1,2}(R^N)$. Note that

$$\begin{aligned}
\int_{R^N} \nabla v_1(x)\nabla\varphi(x)dx &= \int_{R^N} T\nabla\sigma(Tx)\nabla\varphi(x)dx \\
&= \int_{R^N} T\nabla\sigma(Tx)\cdot(T\nabla\psi(TX))dx \\
&= \int_{R^N} \nabla\sigma(y)\nabla\psi(y)\det T dy \\
&= \int_{R^N} P(TT^t y)h'(\sigma(TT^t y))f_2(h(\sigma(TT^t y)))\psi(TT^t y)dy \\
&= \int_{R^N} P(Tx)h'(\sigma(Tx))f_2(h(\sigma(Tx)))\psi(Tx)\det T^t dx \\
&= \int_{R^N} P(x)h'(v_1(x))f_2(h(v_1(x)))\varphi(x)dx.
\end{aligned}$$

Hence, (41) is holds. And so, (35) has two solutions v and v_1. It follows that (35) has another solution v_2 such that $\beta \le v_2 \le \min\{v, v_1\}$. Since σ is the minimal solution with respect to the pair of sub-supersolutions (v, β), we get $\sigma(x_1) \le v_2(x_1) \le v_1(x_1) = \sigma(x_2) < \sigma(x_1)$. This contradiction shows that σ must be radially symmetric.

Next, define a C^1−function $\Theta : [0, \infty) \to \mathbb{R}^+$ by $\Theta(|x|) = \sigma(x)$ for all $x \in \mathbb{R}^N$. Using the chain rule for classical diferentiation, we have for all $r \in (0, \infty)$ that

$$\frac{\partial\sigma}{\partial x_i} = \frac{d\Theta}{dr}\frac{\partial r}{\partial x_i} = h'\frac{x_i}{r}, \quad i = 1, \dots, N,$$

and

$$|\nabla\sigma| = |\Theta'|\left(\sum_{i=1}^{N}\frac{x_i^2}{r^2}\right)^{1/2} = |\Theta'|.$$

For any $w \in C_0^\infty(0, \infty)$, put

$$\alpha(r) = \frac{w(r)}{r^{N-1}}, \quad r \in (0,\infty), \quad \alpha(0) = 0,$$

and

$$\bar{w}(x) = w(|x|), \quad \bar{\alpha}(x) = \alpha(|x|), \quad x \in \mathbb{R}^N.$$

As a weak solution of (34), σ satisfies

$$\int_{\mathbb{R}^N} \nabla\sigma\nabla\bar{\alpha}\,dx = \int_{\mathbb{R}^N} P(x)h'(\sigma)f_2(h(\sigma))\bar{\alpha}\,dx.$$

On the other hand, $\dfrac{\partial\bar{\alpha}(x)}{\partial x_i} = \alpha'\dfrac{x_i}{|x|}$, which provides that

$$\int_0^\infty \Theta' w' r^{N-1}dr = \int_0^\infty P(r)h'(\Theta)f_2(h(\Theta))wr^{N-1}dr.$$

Substituting $\alpha = \dfrac{w}{r^{N-1}}$ and $\alpha' = \dfrac{w'}{r^{N-1}} - \dfrac{N-1}{r^N}w$ into the previous equation, we get

$$\int_0^\infty \Theta'\left(\frac{w'}{r^{N-1}} - \frac{N-1}{r^N}w\right)r^{N-1}dr = \int_0^\infty P(r)h'(\Theta)f_2(h(\Theta))\frac{w}{r^{N-1}}r^{N-1}dr,$$

or

$$\int_0^\infty \Theta' w' dr - \int_0^\infty \frac{N-1}{r}h' v dr = \int_0^\infty P(r)f_2(\Theta)w dr,$$

for all $w \in C_0^\infty(0, \infty)$. This implies that $\Theta \in C^1$ weak solution of the equation

$$-\Theta'' = \frac{N-1}{r}\Theta' + P(r)h'(\Theta)f_2(h(\Theta)), \tag{42}$$

and by the continuity of the right-hand side, the distributional derivative ∂ above becomes a classical derivative and hence h is a classical solution of (42).

Since σ is radially symmetric, $\Theta'(0) = 0$. Hence, Θ is a solution of (42) subject to the condition $\Theta'(0) = 0 = \lim_{r\to\infty} \Theta(r)$. Let $r_0 \in [0, \infty)$ be such that $v_{max} = \Theta(r_0) = \max\{\Theta(r); r [0, \infty)\}$. Multiplying both sides of (42) by Θ' and integrating it, we have

$$-\left(\int_{r_0}^{r}\Theta'\Theta''dt + (N-1)\int_{r_0}^{r}\frac{|\Theta'|^2}{t}dt\right) = \int_{r_0}^{r} P(t)h(\Theta)f_2(h(\Theta))\Theta' dt,$$

for all $0 < r < \infty$. Now, since $v_{max} = \Theta(r_0)$ is greater than $h^{-1}(a_1)$, we can choose $r \in (0, \infty)$ such that $\Theta(r) = h^{-1}(a_1)$. The previous equality becomes

$$\int_{v_{max}}^{h^{-1}(a_1)} h'(s)f(h(s))ds \le -\int_0^{\Theta'(r)} S\,ds - (N-1)\int_0^r \frac{|\Theta'|^2}{t}dt < 0.$$

This equation shows that $\int_{h^{-1}(a_1)}^{v_{max}} h'(s)f(h(s))ds > 0$. Since $f \le 0$ in $(h^{-1}(a_1), h^{-1}(b_1)]$, $u_{max} \in (h^{-1}(b_1), h^{-1}(a_2)]$ and f is nonnegative in $[u_{max}, h^{-1}(a_2)]$, we get

$$\int_{h^{-1}(a_1)}^{h^{-1}(a_2)} h'(s)f(h(s))ds \ge \int_{s_0}^{u_{max}} h'(s)f(h(s))ds > 0,$$

Therefore, applying the change of variable $t = h(s)$, we obtain

$$\int_{a_1}^{a_2} f(t)ds > 0.$$

Acknowledgments

We would like to express our gratitude to an anonymous reviewer for his/her suggestions that improved this paper.

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