Minimax Theory and its Applications
Volume 10 (2025), No. 1, 135-163

Multiple Solutions to A Kirchhoff-Boussinesq Type Problem With Competing Potentials

Giovany M. Figueiredo and Segundo Manuel A. Salirrosas

Departamento de Matemtica, Universidade de Brasília - UnB, 70910-900, Brasília - DF, Brazil
giovany@unb.br
semaarsa@gmail.com

Received on November 23, 2023
Accepted on February 5, 2024

In this work we study the existence and multiplicity of nontrivial solutions for the following class of critical eliptic Kirchhoff-Boussinesq type problem given by

$$\varepsilon^4 \Delta^2 u \pm \varepsilon^p \Delta_p u + V(x)u = Q(x) f(u) + K(x)|u|^{2_{**}-2} u \text{ in } \mathbb{R}^N,\ u \in H^2(\mathbb{R}^N),$$

where ε is a positive parameter, $2 < p < 2^* = \dfrac{2N}{N-2}$, $2_{**} = \dfrac{2N}{N-4}$, $N \geq 5$, $V, Q, K : \mathbb{R}^N \to \mathbb{R}$ are continuous functions and $f : \mathbb{R} \to \mathbb{R}$ is a function of C^1 class. We employ the concept of Ljusternik-Schnirelmann category to investigate the multiplicity of solutions to this elliptic equation and techniques, including variational methods, critical point theory and functional analysis.

Keywords: Kirchhoff-Boussinesq type problem, Ljusternik-Schnirelmann category theory.

2010 Mathematics Subject Classification: Primary 35J60; Secondary 35J10, 35J20.

Giovany M. Figueiredo was supported by CNPq, Capes and Fapdf - Brazil. Segundo Manuel A. Salirrosas was supported by CNPq, Conselho Nacional de Desenvolvimento Cientlfico e Tecnologico - Brazil (150632/2022-3).

1. Introduction

In an intriguing article, Chueshov and Lasiecka [9] delve into the captivating realm of a nonlinear plate equation known as the Kirchhoff-Boussinesq (K-B) model:

$$w_{tt} + kw_t + \Delta^2 w = \operatorname{div}\left(|\nabla w|^{p-2} \nabla w\right) + \sigma\Delta(w^2) - f(w) \tag{1.1}$$

The study focuses on this dynamic equation, which is defined on a bounded domain $\Omega \subset \mathbb{R}^2$ with a sufficiently smooth boundary and suitably chosen initial data. The (KB) model represents a crucial mathematical framework for understanding the complex behavior of plates subjected to various forces and nonlinear effects.

As revealed in [10], the model (1.1) emerges naturally as the limit of Mindlin-Timoshenko equations, which describe the dynamic response of plates that account for transverse shear effects. These effects are crucial for accurately characterizing the behavior of thin elastic plates subject to mechanical loads. The connection between the (K-B)

ISSN 2199-1421 (electronic)

model and the Mindlin-Timoshenko equations establishes the model's relevance in the study of elastic structures and its ability to capture essential physical phenomena.

Mindlin-Timoshenko equations, widely studied in the literature [15], [16], form the basis for investigating the dynamics of slender structures, including beams and plates. The K-B model inherits its fundamental properties from these well-established equations, making it a powerful tool for analyzing the intricate interplay between transverse shear, nonlinearity, and dispersion in plate dynamics.

Throughout the investigation of the (K-B) model, Chueshov and Lasiecka [9] uncover novel insights into the underlying mathematics and physical phenomena. Their comprehensive study encompasses a diverse range of analytical techniques and numerical simulations, shedding light on the stability, existence, and qualitative properties of solutions to the (K-B) model.

The significance of this work extends beyond theoretical mathematics, finding practical applications in engineering disciplines. The (K-B) model's ability to describe the dynamic behavior of plates has proven crucial in various engineering domains, including structural mechanics, aerospace engineering, and materials science. Understanding the interplay of various effects within this model has empowered engineers to design more robust and efficient structures capable of withstanding diverse loading conditions.

For a comprehensive understanding of Mindlin-Timoshenko plates and their connection to the (K-B) model, the interested reader is encouraged to explore the seminal works by Lagnese [15], [16] and the references therein. Lagnese's contributions have been pivotal in advancing the understanding of these dynamic systems, providing essential background knowledge for researchers in this field.

In summary, Chueshov and Lasiecka's investigation into the Kirchhoff-Boussinesq model [9] marks a significant milestone in the study of dynamic plate equations. Their findings deepen our understanding of the intricate dynamics of plates and contribute to a broader scientific and engineering understanding of elastic structures.

More recently, some authors are focusing on stationary Kirchhoff-Boussinesq, that is, those in which there is only a single space variable $x \in \Omega$ unlike in evolution problems, where the unknown also depends on the time variable $t \geq 0$. For example, Liu, Sun and Wu [19] are concerned with the following biharmonic equation with p-Laplacian and Neumann boundary condition given by

$$\begin{cases} \Delta^2 u - \lambda \Delta_p u = f(x,u) - \dfrac{\mu}{|\Omega|} \displaystyle\int_\Omega f(y,u(y))dy \ \text{ in } \ \Omega, \\ \dfrac{\partial \Delta u}{\partial \eta} = \dfrac{\partial u}{\partial \eta} = 0 \text{ on } \ \partial\Omega. \end{cases}$$

Using the Fountain Theorem, the authors obtain the existence of infinitely many sign-changing high energy solutions.

Sun and Wu [22] also study a class of biharmonic equations with p—Laplacian and singular sign-changing potential as follows

$$\Delta^2 u - \beta \Delta_p u + V\lambda u = 0 \ \text{ in } \ \mathbb{R}^N,$$

where $N \geq 5$, $\beta < 0$ with $V_\lambda(x) = \lambda a(x) - b(x)$ with $\lambda > 0$. Under some suitable assumptions on a and b, the authors obtain the existence of nontrivial solutions for λ large enough.

Sun, Chu and Wu [11] also study problem

$$\Delta^2 u - \beta\Delta_p u + V_\lambda u = f(x,u) \;\; in \;\; \mathbb{R}^N$$

with the same arguments used in [22]. Yang [24] also obtain a nontrivial weak solution to a critical biharmonic system involving *p*—Laplacian and Hardy potential via variational methods.

In [4] Carlos, Figueiredo and Ruviaro studied the Kirchhoff-Boussinesq equation considering the nonlinearity of the Berestycki-Lions type and showed the existence of a solution in the zero mass case and in the positive mass case.

In [5], Carlos and Figueiredo studied the problem

$$\Delta^2 u \pm \Delta_p u = f(u) \;\; in \;\; \Omega, \; u = 0 \;\; on \;\; \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary and *f* exhibits subcritical and critical exponential growth.

In this paper we deal with the existence and multiplicity of nontrivial solutions for the following the problem

$$(C_\varepsilon)\begin{cases} \varepsilon^4\Delta^2 u \pm \varepsilon^p \Delta_p u + V(x)u = Q(x)f(u) + K(x)|u|^{2_{**}-2}u \text{ in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases}$$

where $\varepsilon > 0$, $2 < p < 2^* = \dfrac{2N}{N-2}$, $2_{**} = \dfrac{2N}{N-4}$, and $N \geq 5$.

In order to state the main results, we need to introduce some notions by setting

$$0 < V_{\min} := \min_{x\in\mathbb{R}^N} V(x),\; V_{\max} := \max_{x\in\mathbb{R}^N} V(x),\; V_\infty := \liminf_{|x|\to\infty} V(x),$$

$$0 < Q_{\min} := \min_{x\in\mathbb{R}^N} Q(x),\; Q_{\max} := \max_{x\in\mathbb{R}^N} Q(x),\; Q_\infty := \limsup_{|x|\to\infty} Q(x),$$

$$0 < K_{\min} := \min_{x\in\mathbb{R}^N} K(x),\; K_{\max} := \max_{x\in\mathbb{R}^N} K(x),\; K_\infty := \limsup_{|x|\to\infty} K(x).$$

We assume that *V*, *Q* and *K* satisfy the following conditions:

(H1) $\mathcal{M} = \{x \in \mathbb{R}^N : Q(x) = Q_{\max}, K(x) = K_{\max}\} \neq \emptyset$.

(H2) $V_{\min} < V_\infty$, $Q_{\max} > Q_\infty$ *and* $K_{\max} > K_\infty$.

The nonlinearity $f : \mathbb{R} \to \mathbb{R}$ fulfills the following hypotheses:

(f_1) We suppose that

$$\lim_{|t|\to 0} \frac{f(t)}{t} = 0.$$

(f_2) There exists $q \in (p, 2^{**})$ such that

$$\lim_{|t|\to\infty} \frac{f(t)}{t^{q-1}} = 0.$$

(f_3) The function $t \mapsto \dfrac{f(t)}{t^{p-1}}$ is increasing in $(0,+\infty)$ and decreasing in $(-\infty, 0)$.

(f_4) There are $p < r < 2^{**}$ and $\sigma > 0$ such that

$$f(t) \geq \sigma\, |t|^{r-2}\, t,$$

for all $t \geq 0$.

Since we deal with the multiplicity of solutions of (C_ε), we recall that if Y is a given closed subset of a topological space X, we denote by $\mathrm{cat}_X(Y)$ the Ljusternik-Schnirelmann category of Y in X, that is the least number of closed and contractible sets in X which cover Y.

Let us denote by

$$M = \{ x \in \widetilde{M} : V(x) = V_{\min} \} \text{ and } M_\delta = \{ x \in \mathbb{R}^N : \mathrm{dist}(x, M) \leq \delta \}, \text{ for } \delta > 0,$$

where $\widetilde{\mathrm{M}}$ appeared in the hypothesis (H_1). Our main result is:

Theorem 1.1. *Assume that V, Q and K are continuous potentials and satisfy* (H_1) − (H_2). *Assume also that conditions* (f_1) − (f_4) *hold. Then, for any* $\delta > 0$ *given, there exists* $\varepsilon_\delta > 0$ *and* $\sigma^* > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_\delta)$ *and for all* $\sigma > \sigma^*$*, problem* (C_ε) *has at least* $\mathrm{cat}_{M_\delta}(M)$ *nontrivial solutions.*

An interesting question, which motivates the present work, is to verify whether we can obtain a multiplicity result involving the category of the set where the points are both the minimum points of the potential V and the maximum points of the potentials Q and K.

Wang and Zeng [23] showed the existence of solutions for the problem

$$(WZ) \qquad \square^2 \Delta u + V(x)u = K(x)|u|^{p-2}u + Q(x)|u|^{q-2}u \text{ in } \mathbb{R}^N$$

with $2 < q < p < 2^*$. They also showed the concentration points are located on the middle ground of the competing potential functions and in some cases are given explicitly in terms of these functions. Cingolani and Lazzo [7] obtained a multiplicity result involving the set of global minima of a function which provides some kind of global median value between the minimum of V and the maximum of K and Q.

Our arguments were strongly influenced by [7] and [23]. Below we list what we believe to be the main contributions of our paper.

- i) We show a multiplicity result involving the category of the set where the points are both the minimum points of the potential V and the maximum points of the potentials Q and K for a class of Kirchhoff-Boussinesq type problem, as Cingolani Lazzo made in [7].
- ii) Since in the Kirchhoff-Boussinesq type problem appear the term $\pm\varepsilon^{p}\Delta_p u$ and since we have the presence of the weight functions V, Q and K, some estimates are more refined. See for example, Lemma 2.6, Proposition 2.7, Lemma 3.2 and Proposition 3.3.
- iii) In the literature, there are no results of and multiplicity of solutions for Kirchhoff-Boussinesq type problem involving the competition of potentials with weights.
- iv) The concentration result that can be found in [23] is an open problem when considering Kirchhoff-Boussinesq type problem. The reason is the lack of regularity results caused by the presence of the term $\pm\varepsilon^p\Delta_p u$.

v) We also complete the study that can be found [11], [19], [22], [24]. But, different from these articles, here we consider two $\Delta^2 u \pm \Delta_p u$ cases, without to use parameters in from of the operator p-Laplacian.

Before concluding this introduction, it is very important to say that in the literature, we find many papers where the authors study the existence and multiplicity of solution by using the category of Lusternik-Schnirelman for problems involving several operators, see, for example, [2], [3], [6], [8], [17], [21] and references therein.

The paper is organized as follows. In section 2 we present the variational framework for our problem and we study the results related to autonomous problem associated to (C_ε). In section 3 we study the compactness result to the functional associated to our problem. The multiplicity of solutions is studied in section 4.

2. Variational Framework

For the proof of ours results, we shall consider an equivalent problem to (C_ε). Changing variables by $x \mapsto \varepsilon x$, we can rewrite the problem (C_ε) in to the following equivalent form

$$(P_\varepsilon) \quad \begin{cases} \Delta^2 u \pm \Delta_p u + V(\varepsilon x)u = Q(\varepsilon x) f(u) + K(\varepsilon x)|u|^{2_{**}-2} u \text{ in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N). \end{cases}$$

If u is a solution of problem (P_ε), then $\hat{u}(x) = u(x/\varepsilon)$ is a solution of problem (C_ε). Thus, to study problem (C_ε), it suffices to study problem (P_ε).

For $\varepsilon > 0$, we define the rescaled weighted Sobolev space X_ε by

$$X_\varepsilon = \left\{ u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\varepsilon x)|u|^2 \, dx < \infty \right\}$$

endowed with the inner product

$$\langle u, v \rangle = \int_{\mathbb{R}^N} \left[\Delta u \Delta v + V(\varepsilon x) uv \right] dx$$

and the norm

$$\|u\|_\varepsilon^2 = \int_{\mathbb{R}^N} \left(|\Delta u|^2 + V(\varepsilon x)|u|^2 \right) dx.$$

An important result in this paper is a Gagliardo-Nirenberg interpolation inequality, which the proof can be seen in [13] and [20]:

Theorem 2.1. *Suppose that N, j, m are non-negative integers and that* $1 \le k_1, k_2, k_3 \le \infty$ *and* $\theta \in [0,1]$ *are real numbers such that*

$$\frac{1}{k_1} = \frac{j}{N} + \left(\frac{1}{k_3} - \frac{m}{N} \right) \theta + \frac{1-\theta}{k_2}$$

and

$$\frac{j}{m} \le \theta \le 1.$$

Then, there exist a constant C > 0 *independent of* u *such that*

$$\| D^j u\|_{k_1} \le C\|D^m u\|_{k_3}^{\theta} \|u\|_{k_2}^{1-\theta}, \ \forall u \in L^{k_2}(\mathbb{R}^N) \cap W^{m,k_3}(\mathbb{R}^N).$$

Using Sobolev embedding theorem and the fact that $V(x) \ge V_{min}$, we have

$$X_\varepsilon \hookrightarrow H^2(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N), \text{ for } 2 \le s \le 2_{**}. \tag{2.1}$$

Since $2 < p < 2^*$, by using Theorem 2.1 for $j = 1$, $m = 2$, $\frac{1}{2} < \theta \le 1$, $k_1 = p$, $k_2 = k_3 = 2$, we have the following continuous embedding

$$X_\varepsilon \hookrightarrow W^{1,p}(\mathbb{R}^N). \tag{2.2}$$

From conditions (f_1) and (f_2), given $\varrho > 0$, there exists a positive constant $C_\varrho > 0$ such that

$$|f(t)| \le q|t| + C_q |t|^{q-1}. \tag{2.3}$$

From condition (f_3), we can show that

$$tf(t) - pF(t) \text{ is increasing for } t > 0 \text{ and decreasing for } t < 0, \tag{2.4}$$

where $F(t) = \int_0^t f(s)ds$ and

$$f'(t)t^2 - (p-1)f(t)t > 0 \text{ for all } t \ne 0. \tag{2.5}$$

Moreover,

$$tf(t) - pF(t) \ge 0, \text{ for all } t \in \mathbb{R}. \tag{2.6}$$

From (2.1), (2.2), (2.3) and the hypotheses on the functions V, Q and K we get that the functional associated to problem (P_ε) given by

$$\begin{aligned} I_\varepsilon(u) = &\frac{1}{2}\int_{\mathbb{R}^N}\left(|\Delta u|^2 + V(\varepsilon x)|u|^2\right)dx \pm \frac{1}{p}\int_{\mathbb{R}^N}|\nabla u|^p dx \\ &-\int_{\mathbb{R}^N} Q(\varepsilon x)F(u)dx - \frac{1}{2_{**}}\int_{\mathbb{R}^N} K(\varepsilon x)|u|^{2_{**}} dx \end{aligned}$$

is well defined for $u \in X_\varepsilon$. Moreover, using standard arguments, we can see that $I_\varepsilon \in C^1(X_\varepsilon, \mathbb{R})$ with

$$\begin{aligned} I_\varepsilon'(u)\phi = &\int_{\mathbb{R}^N}\left(\Delta u\Delta\phi + V(\varepsilon x)u\phi\right)dx \pm \int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\nabla\phi dx \\ &- \int_{\mathbb{R}^N} Q(\varepsilon x)f(u)\phi dx - \int_{\mathbb{R}^N} K(\varepsilon x)|u|^{2_{**}-2}u\phi dx \end{aligned}$$

for all $\phi \in X_\varepsilon$. Then, the critical points of I_ε are weak solutions of (P_ε).

In order to use critical points theory we firstly derive results related to the Palais-Smale compacteness condition for the functional I_ε. A sequence $(u_n) \subset X_\varepsilon$ is a Palais-Smale sequence at level c_ε for the functional I_ε if

$$I_\varepsilon(u_n) \to c_\varepsilon$$

and

$$I_\varepsilon'(u_n) \to 0 \quad \text{in} \quad X_\varepsilon',$$

where

$$c_\varepsilon = \inf_{\gamma \in \Gamma_\varepsilon} \max_{t\in[0,1]} I_\varepsilon(\gamma(t)) > 0$$

and

$$\Gamma_\varepsilon = \{\gamma \in C([0,1], X_\varepsilon) : \gamma(0) = 0, I_\varepsilon(\gamma(1)) < 0\}.$$

If every Palais-Smale sequence of I_ε has a strongly convergent subsequence then one says that I_ε satisfies the Palais-Smale condition ((*PS*) for short).

Lemma 2.2. The functional I_ε satisfies the following conditions:

- *i.* *There exist* $\alpha, p > 0$, *such that* $I_\varepsilon(u) \geq \alpha$ if $\|u\|_\varepsilon = \rho$.
- *ii.* *There exists an* $e \in X_\varepsilon$ with $\|e\|_\varepsilon > \rho$ *such that* $I_\varepsilon(e) < 0$.

Proof. By using (2.1) and (2.3), there exists $C_1 > 0$ such that

$$\begin{aligned} I_\varepsilon(u) \geq{}& \frac{1}{2}\|u\|_\varepsilon^2 \pm \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p\,dx \\ &- \frac{q}{2} Q_{\max} \|u\|^2_{L^2(\mathbb{R}^N)} - \frac{C_q}{q} Q_{\max} \|u\|^q_{L^q(\mathbb{R}^N)} - \frac{K_{\max}}{2_{**}}\int_{\mathbb{R}^N} |u|^{2^{**}}\,dx \\ \geq{}& \frac{1}{2}(1 - qC_1)\|u\|_\varepsilon^2 \pm \frac{1}{p}\|\nabla u\|^p_{L^p(\mathbb{R}^N)} - C_1\frac{C_q}{q}\|u\|_\varepsilon^q - \frac{C_1}{2_{**}}\|u\|_\varepsilon^{2^{**}}. \end{aligned}$$

In the case that the second term in the associated functional Iε is positive, we get

$$I_\varepsilon(u) \geq \frac{1}{2}(1 - qC_1)\|u\|_\varepsilon^2 - \frac{1}{q}C_q C_1\|u\|_\varepsilon^q - \frac{C_1}{2_{**}}\|u\|_\varepsilon^{2^{**}}$$

and by taking $\varrho > 0$ sufficiently small, the proof of item (i) follows by choosing $\rho > 0$ small enough.

In the case that the second term in the associated functional I_ε is negative, we can use (2.2) to get $C_2 > 0$ such that

$$I(u) \geq \frac{1}{2}(1 - qC_1)\|u\|_\varepsilon^2 - C_2\|u\|_\varepsilon^p - \frac{1}{q}C_q C_1\|u\|_\varepsilon^q - \frac{C_1}{2_{**}}\|u\|_\varepsilon^{2^{**}}.$$

Since $2 < p < q < 2^{**}$, then item (i) holds.

In order to prove (ii), fix $\varphi \in C_0^{\infty}(\mathbb{R}^N,\mathbb{R})$ with $\varphi \neq 0$. Now, from (f) we have

$$\frac{I_\varepsilon(t\varphi)}{t^p} \leq \frac{1}{t^p}\left(\frac{t^2}{2}\|\varphi\|_\varepsilon^2 \pm \frac{t^p}{p}\int_{\mathbb{R}^N}|\nabla\varphi|^p\,dx - \frac{t^r}{r}Q_{\min}\int_{\mathbb{R}^N}|\varphi|^r\,dx - \frac{t^{2_{**}}}{2_{**}}K_{\min}\int_{\mathbb{R}^N}|\varphi|^{2_{**}}\,dx\right)$$

$$= \frac{1}{2t^{p-2}}\|\varphi\|_\varepsilon^2 \pm \frac{1}{p}\int_{\mathbb{R}^N}|\nabla\varphi|^p\,dx - \frac{t^{r-p}}{r}Q_{\min}\int_{\mathbb{R}^N}|\varphi|^r\,dx - \frac{t^{2_{**}-p}}{2_{**}}K_{\min}\int_{\mathbb{R}^N}|\varphi|^{2_{**}}\,dx$$

for all $t > 0$. Since $2 < p < r < 2_{**}$, there exists $t^* > 0$ sufficiently large such that $e = t^*\varphi$ satisfies $I_\varepsilon(e) < 0$ and $\|e\|_\varepsilon > \rho$.

From Lemma 2.2, I_ε has the mountain pass geometry. Hence, there exists a Palais-Smale sequence $(u_n) \subset X_\varepsilon$ at level c_ε.

Lemma 2.3. Let $(u_n) \subset X_\varepsilon$ be a $(PS)c_\varepsilon$ *sequence for* I_ε. *Then* (u_n) is *bounded in* X_ε.

Proof. Let (u_n) be a $(PS)c_\varepsilon$ sequence of I_ε, that is

$$I_\varepsilon(u_n) \to c_\varepsilon \text{ and } I_\varepsilon'(u_n) \to 0 \text{ in } X_\varepsilon' \text{ as } n \to \infty.$$

Therefore, by using (2.6) we get

$$\begin{aligned} I_\varepsilon(u_n) - \frac{1}{p}I_\varepsilon'(u_n)u_n &= \left(\frac{1}{2}-\frac{1}{p}\right)\|u_n\|_\varepsilon^2 + \frac{1}{p}\int_{\mathbb{R}^N}Q(\varepsilon x)\left[f(u_n)u_n - pF(u_n)\right]dx \\ &+ \left(\frac{1}{p}-\frac{1}{2_{**}}\right)\int_{\mathbb{R}^N}K(\varepsilon x)|u_n|^{2_{**}}\,dx \\ &\geq \left(\frac{1}{2}-\frac{1}{p}\right)\|u_n\|_\varepsilon^2. \end{aligned}$$

Then,

$$\left(\frac{1}{2}-\frac{1}{p}\right)\|u_n\|_\varepsilon^2 \leq I_\varepsilon(u_n) - \frac{1}{p}I_\varepsilon'(u_n)u_n = c_\varepsilon + o(\|u_n\|_\varepsilon),$$

where we conclude that (u_n) is bounded in X_ε.

To characterize the least energy, we define the Nehari manifold associated to the functional I_ε given by

$$\mathrm{U}_\varepsilon = \left\{u \in X_\varepsilon \setminus \{0\} : I_\varepsilon'(u)u = 0\right\}.$$

In order to obtain a least energy solution, we need a characterization of the least energy. Define

$$c_\varepsilon^* = \inf_{u \in \mathrm{U}_\varepsilon} I_\varepsilon(u) \quad \text{and} \quad c_\varepsilon^{**} = \inf_{u \in X_\varepsilon \setminus \{0\}} \max_{t \geq 0} I_\varepsilon(tu).$$

Lemma 2.4. For any $u \in X_\varepsilon \setminus \{0\}$, we have

i. There exists a unique $t_u = t(u) > 0$ such that $t_u u \in \mathrm{N}_\varepsilon$. Moreover

$$I_\varepsilon(t_u u) = \max_{t \geq 0} I_\varepsilon(tu).$$

ii. $c_\varepsilon = c_\varepsilon^{*} = c_\varepsilon^{**} > 0.$

Proof. For each $u \in X_\varepsilon \setminus \{0\}$ and $t > 0$, let us introduce the function $\gamma_u(t) = I_\varepsilon(tu)$. Then, $tu \in \mathrm{N}_\varepsilon$ if, and only if $\gamma'_u(t) = 0$. Taking $\varrho > 0$ sufficiently small in (2.3) and using (2.1), there exists $C_3 > 0$ such that

$$\gamma_u(t) \geq \frac{t^2}{2}\left(1-\mathrm{q}C_3\right)\|u\|_\varepsilon^2 \pm \frac{t^p}{p}\int_{\mathbb{R}^N}|\nabla u|^p\,dx - C_3C_\mathrm{q}\frac{t^q}{q}\|u\|_\varepsilon^q - \frac{t^{2_{**}}}{2_{**}}\int_{\mathbb{R}^N}K(\varepsilon x)|u|^{2_{**}}\,dx.$$

On the other hand, by using (f_4) we have

$$\gamma_u(t) \leq \frac{t^2}{2}\|u\|_\varepsilon^2 \pm \frac{t^p}{p}\int_{\mathbb{R}^N}|\nabla u|^p\,dx - \sigma\frac{t^r}{r}\int_{\mathbb{R}^N}Q(\varepsilon x)|u|^r\,dx - \frac{t^{2_{**}}}{2_{**}}\int_{\mathbb{R}^N}K(\varepsilon x)|u|^{2_{**}}\,dx.$$

Thus, because $2 < p < q$, $r < 2_{**}$, we have

$$\liminf_{t\to 0^+}\frac{\gamma_u(t)}{t^2} > 0 \quad \text{and} \quad \limsup_{t\to\infty}\frac{\gamma_u(t)}{t^p} < 0.$$

Then, there exists at least one $t_u > 0$ such that $\gamma'_u(t) = 0$, i.e. $t_u u \in \mathrm{N}_\varepsilon$. Moreover, since $p > 2$, we get

$$\begin{aligned}\gamma'_u(t) &= t\|u\|_\varepsilon^2 \pm t^{p-1}\int_{\mathbb{R}^N}|\nabla u|^p\,dx - \int_{\mathbb{R}^N}Q(\varepsilon x)f(tu)u\,dx - t^{2_{**}-1}\int_{\mathbb{R}^N}K(\varepsilon x)|u|^{2_{**}}\,dx\\ &= t^{p-1}\Big(\frac{1}{t^{p-2}}\|u\|_\varepsilon^2 \pm \int_{\mathbb{R}^N}|\nabla u|^p\,dx - \int_{\mathbb{R}^N}Q(\varepsilon x)\frac{f(tu)}{(tu)^{p-1}}u^p\,dx\\ &\quad - t^{2_{**}-p}\int_{\mathbb{R}^N}K(\varepsilon x)|u|^{2_{**}}\,dx\Big).\end{aligned}$$

From (f_3) we conclude that $\dfrac{\gamma_u'(t)}{t^{p-1}}$ is decreasing. Then, it vanishes exactly once, and consequently there is not other $t > 0$ such that $tu \in \mathrm{N}_\varepsilon$. Note, in particular, that t_u is a global maximum point of γ_u and $\gamma_u(t_u) > 0$, i.e. $I_\varepsilon(t_u u) > 0$. Since $t_u = 1$ if $u \in \mathrm{N}_\varepsilon$, we deduce that $I_\varepsilon(u) > 0$ for every u e N_e. Now we can argue as in [25] to complete the proof.

Lemma 2.5. For *all* $u \in \mathrm{N}_\varepsilon$, *there exists* $\mu > 0$, which is *independent of* ε *and u, such that*

$$\|u\|_\varepsilon \geq \mu.$$

Proof. Using (2.3), for any $u \in \mathrm{N}_\varepsilon$ we have

$$\begin{aligned}\|u\|_\varepsilon^2 \pm \int_{\mathbb{R}^N}|\nabla u|^p\,dx &= \int_{\mathbb{R}^N}Q(\varepsilon x)f(u)u\,dx + \int_{\mathbb{R}^N}K(\varepsilon x)|u|^{2_{**}}\,dx\\ &\leq \mathrm{q}Q_{\max}\|u\|^2_{L^2(\mathbb{R}^N)} + C_\mathrm{q}Q_{\max}\|u\|^q_{L^q(\mathbb{R}^N)} + K_{\max}\|u\|^{2_{**}}_{L^{2_{**}}(\mathbb{R}^N)}.\end{aligned}$$

From (2.1), (2.2) and taking $\varrho > 0$ sufficiently small we can deduce that

$$(1-\mathrm{q}C_4)\|u\|_\varepsilon^2 \leq C_5\|u\|_\varepsilon^p + C_\mathrm{q}C_6\|u\|_\varepsilon^q + C_7\|u\|_\varepsilon^{2_{**}}$$

for some constants $C_4, C_5, C_6, C_7 > 0$.

Now, if $||u|| \geq 1$ the result follows. If $||u|| < 1$, then

$$(1-qC_4)\|u\|_\varepsilon^2 \leq (C_5 + C_q C_6 + C_7)\|u\|_\varepsilon^p ,$$

which implies that $||u||_\varepsilon \geq \mu$ for some $\mu > 0$.

2.1. On the Autonomous Problem

In order to prove the main result of this section, we will need some basic results of the autonomous problem. Precisely, for any ξ, θ, $\tau > 0$ we consider the following problem

$$(P_{\xi\theta\tau}) \quad \begin{cases} \Delta^2 u \pm \Delta_p u + \xi u = \theta f(u) + \tau |u|^{2_{**}-2} u \text{ in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N). \end{cases}$$

It is well know that the solutions of problem $(P_{\xi\theta\tau})$ are critical points of the functional

$$I_{\xi\theta\tau}(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\Delta u|^2 + \xi |u|^2 \right) dx \pm \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p \, dx - \theta \int_{\mathbb{R}^N} F(u) dx - \frac{\tau}{2_{**}} \int_{\mathbb{R}^N} |u|^{2_{**}} \, dx.$$

Clearly, $I_{\xi\theta\tau} \in C^1(X_\xi, \mathrm{R})$ and its differential is given by

$$\begin{aligned} I'_{\xi\theta\tau}(u)\varphi &= \int_{\mathbb{R}^N} (\Delta u \Delta\varphi + \xi u \varphi) dx \pm \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \varphi dx \\ &- \theta \int_{\mathbb{R}^N} f(u)\varphi dx - \tau \int_{\mathbb{R}^N} |u|^{2_{**}-2} u \varphi dx \end{aligned}$$

for any $\varphi \in X_\xi = H^2(\mathrm{R}^N)$. We denote the norm in X_ξ by

$$\|u\|_\xi^2 = \int_{\mathbb{R}^N} \left(|\Delta u|^2 + \xi |u|^2 \right) dx.$$

Arguing as in Lemma 2.2, we can show that $I_{\xi\theta\tau}$ has the mountain pass geometry. Therefore, we can set the minimax level $c_{\xi\theta\tau}$ in the following way

$$c_{\xi\theta\tau} = \inf_{\gamma \in \Gamma_{\xi\theta\tau}} \max_{t \in [0,1]} I_{\xi\theta\tau}(\gamma(t)) > 0,$$

where $\Gamma_{\xi\theta\tau} = \left\{ \gamma \in C([0,1], X_\xi) : \gamma(0) = 0, I_{\xi\theta\tau}(\gamma(1)) < 0 \right\}$.

The Nehari manifold associated to $I_{\xi\theta\tau}$ is defined by

$$\square_{\xi\theta\tau} = \left\{ u \in X_\xi \setminus \{0\} : I'_{\xi\theta\tau}(u)u = 0 \right\}.$$

Thus, the least energy associated to $(P_{\xi\theta\tau})$ may be defined as

$$c^*_{\xi\theta\tau} = \inf_{u \in \square_{\xi\theta\tau}} I_{\xi\theta\tau}(u).$$

The number $c^*_{\xi\theta\tau}$ and the manifold $\mathrm{M}_{\xi\theta\tau}$ have similar properties as those of c_ε and I_ε stated in Lemma 2.4. Hence, for each $u \in X_\xi \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathrm{M}_{\xi\theta\tau}$. Moreover, we also have

$$c_{\xi\theta\tau} = c^*_{\xi\theta\tau} = c^{**}_{\xi\theta\tau} := \inf_{u \in X_\xi \setminus \{0\}} \max_{t \geq 0} I_{\xi\theta\tau}(tu).$$

We denote by S the best constant of the immersion, $D^{2,2}(\mathbb{R}^N)$ into $L^{2^{**}}(\mathbb{R}^N)$, that is,

$$S = \inf_{\substack{u\in D^{2,2}(\mathbb{R}^N)\\ u\neq 0}} \frac{\int_{\mathbb{R}^N} |\Delta u|^2\,dx}{\left(\int_{\mathbb{R}^N} |u|^{2^{**}}\,dx\right)^{2/2^{**}}}.$$

The infimum $S > 0$ is achieved by the functions

$$u_{\delta,y}(x) = \frac{C_N \delta^{\frac{N-4}{4}}}{[\delta + |x-y|^2]^{\frac{N-4}{2}}},$$

for any $\delta > 0$ and $y \in \mathbb{R}^N$ (see [12]). Now, we define the following constant

$$\hat{S} = \inf_{\substack{u\in H^2(\mathbb{R}^N)\\ u\neq 0}} \frac{\int_{\mathbb{R}^N} |\Delta u|^2\,dx + \int_{\mathbb{R}^N} |\nabla u|^p\,dx}{\left(\int_{\mathbb{R}^N} |u|^{2^{**}}\,dx\right)^{2/2^{**}}}$$

Note that

$$\frac{\int_{\mathbb{R}^N} |\Delta u|^2\,dx}{\left(\int_{\mathbb{R}^N} |u|^{2^{**}}\,dx\right)^{2/2^{**}}} \leq \frac{\int_{\mathbb{R}^N} |\Delta u|^2\,dx + \int_{\mathbb{R}^N} |\nabla u|^p\,dx}{\left(\int_{\mathbb{R}^N} |u|^{2^{**}}\,dx\right)^{2/2^{**}}}$$

implies

$$0 < S < \hat{S}.$$

From this we can deduce the following result:

Lemma 2.6. Let $(u_n) \subset M_{\xi\theta\tau}$ *be a sequence such that* $I_{\xi\theta\tau}(u_n) \to c_{\xi\theta\tau}$ *with*

$$c_{\xi\theta\tau} < \min\left\{\left(\frac{1}{p} - \frac{1}{2_{**}}\right)\hat{S}^{N/4}\tau^{(4-N)/4}, \left(\frac{1}{2} - \frac{1}{p}\right)\min\left\{\frac{1}{2C^2}, \frac{S\tau^{(4-N)/N}}{2}\right\}^{p/(p-2)}, \frac{p-2}{2p}\right\},$$

for some $C > 0$. *Then we have either*

i. $\|u_n\|_\xi \to 0$, or

ii. *there exists a sequence* $(y_n) \subset \mathbb{R}^N$ *and constants* $R, \eta > 0$ *such that*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^2\,dx \geq \eta.$$

Proof. Suppose that (*ii*) does not hold. Since (u_n) is bounded in X_ξ , then, by [18, Lemma I.1], we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^s\,dx = 0$$

for all $s \in (2, 2_{**})$. Given $\varrho > 0$, we can use (2.3) to get

$$0 \le \left|\int_{\mathbb{R}^N} f(u_n)u_n dx\right| \le \varrho\int_{\mathbb{R}^N} |u|^2 dx + C_\varrho\int_{\mathbb{R}^N} |u_n|^q dx$$

for some constant $C_\varrho > 0$. Since (u_n) is bounded in $L^2(\mathbb{R}^N)$, $u_n \to 0$ in $L^q(\mathbb{R}^N)$ and ϱ is arbitrary, we conclude that $\int_{\mathbb{R}^N} f(u_n)u_n dx = o_n(1)$. Thus, from $I'_{\xi\theta\tau}(u_n)u_n = 0$, we obtain

$$\int_{\mathbb{R}^N}\left(|\Delta u_n|^2 + \xi|u_n|^2\right)dx \pm \int_{\mathbb{R}^N}|\nabla u|^p dx = \tau\int_{\mathbb{R}^N}|\Delta u_n|^{2_{**}} dx + o_n(1). \tag{2.7}$$

In the case that the second term on the left hand side in (2.7) is positive, we have

$$\|u_n\|_\xi^2 + \int_{\mathbb{R}^N}|\nabla u|^p dx = \tau\int_{\mathbb{R}^N}|u_n|^{2_{**}} dx + o_n(1).$$

Taking a subsequence, we obtain $l \ge 0$ such that

$$\|u_n\|_\xi^2 + \int_{\mathbb{R}^N}|\nabla u|^p dx \to l \quad \text{and} \quad \tau\int_{\mathbb{R}^N}|u_n|^{2_{**}} dx \to l. \tag{2.8}$$

From (2.3), we also have $\int_{\mathbb{R}^N} F(u_n)dx \to 0.$ Thus

$$\begin{aligned} c_{\xi\theta\tau} + o_n(1) &= I_{\xi\theta\tau}(u_n) \\ &\ge \frac{1}{p}\left(\|u_n\|_\xi^2 + \int_{\mathbb{R}^N}|\nabla u_n|^p dx\right) - \frac{\tau}{2_{**}}\int_{\mathbb{R}^N}|u_n|^{2_{**}} dx + o_n(1). \end{aligned}$$

Letting $n \to \infty$ and using (2.8), we get

$$c_{\xi\theta\tau} \ge \left(\frac{1}{p} - \frac{1}{2_{**}}\right) l. \tag{2.9}$$

Recalling the definition of $\hat{S}$ we have

$$\begin{aligned} \|u_n\|_\xi^2 + \int_{\mathbb{R}^N}|\nabla u_n|^p dx &\ge \int_{\mathbb{R}^N}|\Delta u_n|^2 dx + \int_{\mathbb{R}^N}|\nabla u_n|^p dx \\ &\ge \frac{\hat{S}}{\tau^{(N-4)/N}}\left(\tau\int_{\mathbb{R}^N}|u_n|^{2_{**}} dx\right)^{2/2_{**}}. \end{aligned}$$

Taking the limit we conclude that $l \ge \frac{\hat{S}}{\tau^{(N-4)/N}} l^{2/2_{**}}$. If $l > 0$ we obtain from (2.9) that

$$c_{\xi\theta\tau} \ge \left(\frac{1}{p} - \frac{1}{2_{**}}\right)\hat{S}^{N/4}\tau^{(4-N)/4},$$

which is a contradiction. Hence $l = 0$ and therefore $u_n \to 0$ in X_ξ.

In the case that the second term on the left hand side in (2.7) is negative, we have

$$\int_{\mathbb{R}^N}\left(|\Delta u_n|^2 + \xi|u_n|^2\right)dx = \int_{\mathbb{R}^N}|\nabla u|^p dx + \tau\int_{\mathbb{R}^N}|u_n|^{2_{**}} dx + o_n(1).$$

Taking a subsequence, we obtain $\hat{l} \geq 0$ such that

$$\|u_n\|_\xi^2 \to \hat{l} \text{ and } \int_{\mathbb{R}^N} |\nabla u_n|^p \, dx + \tau \int_{\mathbb{R}^N} |u_n|^{2^{**}} dx \to \hat{l} \tag{2.10}$$

with $\hat{l} = l_1 + l_2$, $\int_{\mathbb{R}^N} |\nabla u_n|^p \, dx \to l_1$ and $\tau \int_{\mathbb{R}^N} |u_n|^{2_{**}} dx \to l_2$.

Since $\int_{\mathbb{R}^N} F(u_n) dx \to 0$ we have

$$c_{\xi\theta\tau} + o_n(1) = I_{\xi\theta\tau}(u_n) \geq \frac{1}{2}\|u_n\|_\xi^2 - \frac{1}{p}\left(\int_{\mathbb{R}^N} |\nabla u_n|^p \, dx + \tau \int_{\mathbb{R}^N} |u_n|^{2_{**}} dx\right) + o_n(1).$$

Letting $n \to \infty$ and using (2.10) we get

$$c_{\xi\theta\tau} \geq \left(\frac{1}{2} - \frac{1}{p}\right)\hat{l}. \tag{2.11}$$

If $\hat{l} \geq 1$, then

$$c_{\xi\theta\tau} \geq \frac{p-2}{2p},$$

which is a contradiction by the hypotheses. Then, we have $0 \leq \hat{l} < 1$.

On the other hand, we can use the definition of S to get

$$\|u_n\|_\xi^2 \geq \int_{\mathbb{R}^N} |\Delta u|^2 \, dx \geq \frac{S}{\tau^{(N-4)/N}}\left(\tau \int_{\mathbb{R}^N} |u_n|^{2_{**}} dx\right)^{2/2_{**}}. \tag{2.12}$$

Now from (2.2), there exits $C > 0$ such that

$$\left(\int_{\mathbb{R}^N} |\nabla u_n|^p \, dx\right)^{1/p} \leq C\|u_n\|_\xi.$$

Then,

$$\|u_n\|_\xi^2 \geq \frac{1}{C^2}\left(\int_{\mathbb{R}^N} |\nabla u_n|^p \, dx\right)^{2/p}. \tag{2.13}$$

Using (2.12) and (2.13) we obtain

$$\|u_n\|_\xi^2 \geq \frac{1}{2}\min\left\{\frac{1}{C^2}, S\tau^{(4-N)/N}\right\}\left\{\left(\int_{\mathbb{R}^N} |\nabla u_n|^p \, dx\right)^{2/p} + \left(\tau \int_{\mathbb{R}^N} |u_n|^{2_{**}} dx\right)^{2/2_{**}}\right\}.$$

Taking the limit and using (2.10), we have

$$\hat{l} \geq \frac{1}{2}\min\left\{\frac{1}{C^2}, S\tau^{(4-N)/N}\right\}\left(l_1^{2/p} + l_2^{2/2_{**}}\right).$$

Suppose that $0 < \hat{l} < 1$. In this case $0 < l_1, l_2 < 1$. Then, since $2 < p < 2^{**}$, we have

$$\hat{l} \geq \frac{1}{2}\min\left\{\frac{1}{C^2}, S\tau^{(4-N)/N}\right\}\hat{l}^{2/p}.$$

Since $\hat{l} \geq 0$, we obtain

$$\hat{l}^{(p-2)/p} \geq \min\left\{\frac{1}{2C^2}, \frac{S\tau^{(4-N)/N}}{2}\right\}$$

and from (2.11) we have that

$$c_{\xi\theta\tau} \geq \left(\frac{1}{2} - \frac{1}{p}\right)\min\left\{\frac{1}{2C^2}, \frac{S\tau^{(4-N)/N}}{2}\right\}^{p/(p-2)},$$

which is a contradiction. Hence $\hat{l} = 0$ and therefore (*i*) holds.

Proposition 2.7. There *exists* σ > 0 *such that for all* σ > σ*

$$c_{\xi\theta\tau} < \min\left\{\left(\frac{1}{p} - \frac{1}{2_{**}}\right)\hat{S}^{N/4}\tau^{(4-N)/4}, \left(\frac{1}{2} - \frac{1}{p}\right)\min\left\{\frac{1}{2C^2}, \frac{S\tau^{(4-N)/N}}{2}\right\}^{p/(p-2)}, \frac{p-2}{2p}\right\}.$$

Proof. Since $I_{\xi\theta\tau}$ has the mountain pass geometry, there exists $e \in X_\xi$ such that $I_{\xi\theta\tau}(e) < 0$. We define $\gamma(t) = te$ for $t \in [0, 1]$. Then $\gamma \in \Gamma_{\xi\theta\tau}$ and by using (f_4) we get

$$\begin{aligned} 0 < c_{\xi\theta\tau} &\leq \max_{t\in[0,1]} I_{\xi\theta\tau}(\gamma(t)) \\ &\leq \max_{t\in[0,1]}\left\{\frac{t^2}{2}\|e\|_\xi^2 \pm \frac{t^p}{p}\int_{\mathbb{R}^N}|\nabla e|^p\,dx - \theta\sigma\frac{t^r}{r}\int_{\mathbb{R}^N}|e|^r\,dx\right\}. \end{aligned}$$

In the case that the second term in the associated functional $I_{\xi\theta\tau}$ is positive, we have

$$\begin{aligned} 0 < c_{\xi\theta\tau} &\leq \max_{t\in[0,1]}\left\{\frac{t^2}{2}\|e\|_\xi^2 + \frac{t^p}{p}\int_{\mathbb{R}^N}|\nabla e|^p\,dx - \theta\sigma\frac{t^r}{r}\int_{\mathbb{R}^N}|e|^r\,dx\right\} \\ &\leq \max_{t\in[0,1]}\left\{\frac{t^2}{2}\left(\|e\|_\xi^2 + \|\nabla e\|^p_{L^p(\mathbb{R}^N)}\right) - \theta\sigma\frac{t^r}{r}\int_{\mathbb{R}^N}|e|^r\,dx\right\} \\ &\leq \left(\frac{1}{2} - \frac{1}{r}\right)\frac{1}{(\theta\sigma)^{\frac{2}{r-2}}}\frac{\left(\|e\|_\xi^2 + \|\nabla e\|^p_{L^p(\mathbb{R}^N)}\right)^{\frac{r}{r-2}}}{\|e\|^{\frac{2r}{r-2}}_{L^r(\mathbb{R}^N)}}. \end{aligned}$$

Thus, $c_{\xi\theta\tau} < \left(\frac{1}{p} - \frac{1}{2_{**}}\right)\hat{S}^{N/4}\tau^{(4-N)/4}$, for all σ > σ*, where

$$\sigma^* = \frac{1}{\theta}\left[\frac{\frac{1}{2} - \frac{1}{r}}{\left(\frac{1}{p} - \frac{1}{2_{**}}\right)\hat{S}^{N/4}\tau^{(4-N)/4}}\right]^{\frac{r-2}{2}}\frac{\left(\|e\|_\xi^2 + \|\nabla e\|^p_{L^p(\mathbb{R}^N)}\right)^{r/2}}{\|e\|^r_{L^r(\mathbb{R}^N)}}.$$

In the case that the second term in the associated functional $I_{\xi\theta\tau}$ is negative, we have

$$0 < c_{\xi\theta\tau} \le \max_{t \ge 0} \left\{ \frac{t^2}{2} \|e\|_\xi^2 - \theta\sigma \frac{t^r}{r} \int_{\mathbb{R}^N} |e|^r dx \right\}$$

$$= \left(\frac{1}{2} - \frac{1}{r} \right) \frac{1}{(\theta\sigma)^{\frac{2}{r-2}}} \frac{\|e\|_\xi^{\frac{2r}{r-2}}}{\|e\|_{L^r(\mathbb{R}^N)}^{\frac{2r}{r-2}}}.$$

Thus, $c_{\xi\theta\tau} < \min\left\{ \left(\frac{1}{2} - \frac{1}{p} \right) \min\left\{ \frac{1}{2C^2}, \frac{S\tau^{(4-N)/N}}{2} \right\}^{p/(p-2)}, \frac{p-2}{2p} \right\}$, for all $\sigma > \sigma^*$, where

$$\sigma^* = \frac{1}{\theta} \left[\frac{\frac{1}{2} - \frac{1}{r}}{\min\left\{ \left(\frac{1}{2} - \frac{1}{p} \right) \min\left\{ \frac{1}{2C^2}, \frac{S\tau^{(4-N)/N}}{2} \right\}^{p/(p-2)}, \frac{p-2}{2p} \right\}} \right]^{\frac{r-2}{2}} \frac{\|e\|_\xi^r}{\|e\|_{L^r(\mathbb{R}^N)}^r}.$$

The proof is complete.

Proposition 2.8. Let $(u_n) \subset \mathcal{M}_{\xi\theta\tau}$ *be a sequence satisfying* $I_{\xi\theta\tau}(u_n) \to I_{\xi\theta\tau}$. *Then, there exists a sequence* $(y_n) \subset \mathbb{R}^N$ *such that, up to a subsequence,* $v_n(x) = u_n(x + y_n)$ *converges strongly in* X_ξ. *In particular, there exists a minimizer to* $c_{\xi\theta\tau}$.

Proof. Applying Ekeland's Variational Principle [25, Theorem 8.5], we may suppose that (u_n) is a $(PS)c_{\xi\theta\tau}$ for $I_{\xi\theta\tau}$. In similar arguments to Lemma 2.3, we can show that (u_n) is bounded in X_ξ. Then, up to a subsequence, we may suppose that $u_n \rightharpoonup u$ weakly in X_ξ. By using a density argument, we can conclude that u is a critical point of $I_{\xi\theta\tau}$. Now, will divide our study in two cases.

Case 1. $u \not\equiv 0$.

In this case $I'_{\xi\theta\tau}(u)u = 0$ and therefore $u \in \mathcal{M}_{\xi\theta\tau}$. We are going to prove that

$$\|u\|_\xi = \lim_{x\to\infty} \|u_n\|_\xi. \tag{2.14}$$

Suppose, by contradiction, that (2.14) does not hold. Then by using (2.4) and Fatou's lemma we get

$$\begin{aligned} c_{\xi\theta\tau} &\le I_{\xi\theta\tau}(u) - \frac{1}{p} I'_{\xi\theta\tau}(u)u \\ &= \left(\frac{1}{2} - \frac{1}{p} \right) \|u\|_\xi^2 + \frac{\theta}{p} \int_{\mathbb{R}^N} \big(f(u)u - pF(u) \big) dx + \tau \left(\frac{1}{p} - \frac{1}{2_{**}} \right) \int_{\mathbb{R}^N} |u|^{2_{**}} dx \\ &< \liminf_{n\to\infty} \Bigg[\left(\frac{1}{2} - \frac{1}{p} \right) \|u_n\|_\xi^2 + \frac{\theta}{p} \int_{\mathbb{R}^N} \big(f(u_n)u_n - pF(u_n) \big) dx + \\ &\qquad \tau \left(\frac{1}{p} - \frac{1}{2_{**}} \right) \int_{\mathbb{R}^N} |u_n|^{2_{**}} dx \Bigg] \\ &= \liminf_{n\to\infty} \left[I_{\xi\theta\tau}(u_n) - \frac{1}{p} I'_{\xi\theta\tau}(u_n)u_n \right] = c_{\xi\theta\tau}, \end{aligned}$$

which is a contradiction. Hence, $u_n \to u$ in X_ξ. Consequently $I_{\xi\theta\tau}(u) = c_{\xi\theta\tau}$ and $y_n = 0$ for all $n \in \mathbb{N}$.

Case 2. $u = 0$.

In this case we cannot have that $u_n \to u$ strongly in X_ξ because $c_{\xi\theta\tau} > 0$. Hence, by using Lemma 2.6, there exists a sequence $(y_n) \subset \mathbb{R}^N$ such that

$$v_n \rightharpoonup v \text{ weakly in } X_\xi,$$

where $v_n(x) = u_n(x + y_n)$. Therefore, (v_n) is also a $(PS)_{c_{\xi\theta\tau}}$ sequence for $I_{\xi\theta\tau}$ and $v \neq 0$. Repeating the same arguments used in Case 1, it follows that $v_n \to v$ strongly in X_ξ and the proof of proposition is finished.

The following lemma describes a comparison between the least value of different parameters ξ, θ, τ, which will play an important role in proving the existence result in the next subsection

Lemma 2.9. Let $\xi_j, \theta_j, \tau_j > 0$ $(j = 1, 2)$, with $\xi_1 < \xi_2$, $\theta_1 > \theta_2$ *and* $\tau_1 > \tau_2$. *Then* $c_{\xi_1\theta_1\tau_1} \leq c_{\xi_2\theta_2\tau_2}$. *In particular, if one of inequalities is strict, then* $c_{\xi_1\theta_1\tau_1} < c_{\xi_2\theta_2\tau_2}$.

Proof. From Proposition 2.8, there exists $u \in \mathcal{M}_{\xi_2\theta_2\tau_2}$ be such that

$$c_{\xi_2\theta_2\tau_2} = I_{\xi_2\theta_2\tau_2}(u) = \max_{t\geq 0} I_{\xi_2\theta_2\tau_2}(tu).$$

According to Lemma 2.4-(i), there exists $t_0 > 0$ such that $u_0 = t_0 u \in \mathcal{M}_{\xi_1\theta_1\tau_1}$ satisfying

$$I_{\xi_1\theta_1\tau_1}(u_0) = \max_{t\geq 0} I_{\xi_1\theta_1\tau_1}(tu_0).$$

Clearly, from the above facts we get

$$\begin{aligned} c_{\xi_2\theta_2\tau_2} &= \max_{t\geq 0} I_{\xi_2\theta_2\tau_2}(tu) \geq I_{\xi_2\theta_2\tau_2}(u_0) \\ &= I_{\xi_1\theta_1\tau_1}(u_0) + \frac{\xi_2 - \xi_1}{2}\int_{\mathbb{R}^N} |u_0|^2\, dx + (\theta_1 - \theta_2)\int_{\mathbb{R}^N} F(u_0)\,dx + \frac{\tau_1 - \tau_2}{2_{**}}\int_{\mathbb{R}^N} |u_0|^{2_{**}}\, dx \\ &\geq c_{\xi_1\theta_1\tau_1} + \frac{\xi_2 - \xi_1}{2}\int_{\mathbb{R}^N} |u_0|^2\, dx + (\theta_1 - \theta_2)\int_{\mathbb{R}^N} F(u_0)\,dx + \frac{\tau_1 - \tau_2}{2_{**}}\int_{\mathbb{R}^N} |u_0|^{2_{**}}\, dx, \end{aligned}$$

this implies that $c_{\xi_2\theta_2\tau_2} \geq c_{\xi_1\theta_1\tau_1}$.

3. Palais-Smale Compactness Condition

In this section, for convenience of notation, we define

$$\chi_{K_{\max}} := \hat{S}^{N/4} K_{\max}^{(4-N)/4} \qquad \text{and} \qquad \Pi_{K_{\max}} := \min\left\{ \frac{1}{2C^2}, \frac{S K_{\max}^{(4-N)/N}}{2} \right\}^{p/(p-2)}.$$

Lemma 3.1. *Let* $(u_n) \subset X_\varepsilon$ *be a* $(PS)_{c_\varepsilon}$ *sequence for* I_ε *with*

$$c_\varepsilon < \min\left\{ \left(\frac{1}{p} - \frac{1}{2_{**}}\right)\chi_{K_{\max}}, \left(\frac{1}{2} - \frac{1}{p}\right)\Pi_{K_{\max}}, \frac{p-2}{2p} \right\}$$

and $u_n \rightharpoonup 0$ *weakly in* X_ε. *Then we have either*

i. $\|u_n\|_\varepsilon \to 0$, *or*

ii. *there exists a sequence* $(y_n) \subset \mathbb{R}^N$ *and constants* $R, \eta > 0$ *such that*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^2 dx \geq \eta.$$

Proof. It suffices to note that $\|u_n\|_\varepsilon^2 \geq \int_{\mathbb{R}^N} |\Delta u_n|^2 dx$ and argue as in the proof of Lemma 2.6.

Lemma 3.2. *Let* $(u_n) \subset X_\varepsilon$ *be a* $(PS)_d$ *sequence for* I_ε *with*

$$d < \min\left\{\left(\frac{1}{p} - \frac{1}{2_{**}}\right)\chi_{K_{\max}}, \left(\frac{1}{2} - \frac{1}{p}\right)\Pi_{K_{\max}}, \frac{p-2}{2p}\right\}$$

and $u_n \rightharpoonup 0$ weakly in X_ε. If $u_n \not\to 0$ strongly in X_ε, then $d \geq c_{V_\infty Q_{max} K_{max}}$, where $c_{V_\infty Q_{max} K_{max}}$ is the infimum of $I_{V_\infty Q_{max} K_{max}}$ over $\mathrm{M}_{V_\infty Q_{max} K_{max}}$.

Proof. Consider the auxiliary functional

$$I_{\varepsilon,\infty}(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\Delta u|^2 + V_\infty |u|^2\right) dx \pm \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx$$
$$- \int_{\mathbb{R}^N} Q(\varepsilon x) F(u) dx - \frac{1}{2_{**}} \int_{\mathbb{R}^N} K(\varepsilon x) |u|^{2_{**}} dx.$$

The corresponding Nehari manifold and the minimax level are $\mathrm{N}_{\varepsilon,\infty}$ and $c_{\varepsilon,\infty}$, respectively. Since $Q(x) \leq Q_{max}$ and $K(x) \leq K_{max}$, we have that $I_{\varepsilon,\infty}(u) \geq I_{V_\infty Q_{max} K_{max}}(u)$. From the characterization of the value $c_{V_\infty Q_{max} K_{max}}$, we get $c_{\varepsilon,\infty} \geq c_{V_\infty Q_{max} K_{max}}$.

From Lemma 2.4, for each u_n there is a unique $t_n > 0$ such that $t_n u_n \in \mathrm{N}_{\varepsilon,\infty}$. We start by proving that

$$t_0 = \limsup_{n\to\infty} t_n \leq 1.$$

Indeed, assume by contradiction that the above conclusion does not hold. Then, there exists $\lambda > 0$ and a subsequence of (t_n), still denoted by itself, such that

$$t_n \geq 1 + \lambda \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.1}$$

According to Lemma 2.3, (u_n) is bounded in X_ε, then $I'_\varepsilon(u_n)u_n = o_n(1)$, or equivalently

$$\int_{\mathbb{R}^N} \left(|\Delta u_n|^2 + V(\varepsilon x)|u_n|^2\right) dx \pm \int_{\mathbb{R}^N} |\nabla u_n|^p dx$$
$$= \int_{\mathbb{R}^N} Q(\varepsilon x) f(u_n) u_n dx + \int_{\mathbb{R}^N} K(\varepsilon x) |u_n|^{2_{**}} dx + o_n(1). \tag{3.2}$$

From the fact that $t_n u_n \in \mathrm{N}_{\varepsilon,\infty}$, we can infer that

$$t_n^{2-p} \int_{\mathbb{R}^N} \left(|\Delta u_n|^2 + V_\infty |u_n|^2\right) dx \pm \int_{\mathbb{R}^N} |\nabla u_n|^p dx$$
$$= \int_{\mathbb{R}^N} Q(\varepsilon x) \frac{f(t_n u_n)}{(t_n u_n)^{p-1}} u_n^p dx + t_n^{2_{**}-p} \int_{\mathbb{R}^N} K(\varepsilon x) |u_n|^{2_{**}} dx.$$

Putting together (3.2), (3.3) and by using the fact that $Q(x) > Q_{min}$ and $K(x) > K_{min}$, we obtain

$$\left(t_n^{2-p}-1\right)\int_{\mathbb{R}^N}|\Delta u_n|^2\,dx + \int_{\mathbb{R}^N}\left(t_n^{2-p}V_\infty - V(\varepsilon x)\right)|u_n|^2\,dx + o_n(1)$$
$$\geq Q_{\min}\int_{\mathbb{R}^N}\left[\frac{f(t_n u_n)}{(t_n u_n)^{p-1}} - \frac{f(u_n)}{(u_n)^{p-1}}\right]u_n^p\,dx + K_{\min}\left(t_n^{2_{**}-p}-1\right)\int_{\mathbb{R}^N}|u_n|^{2_{**}}\,dx. \tag{3.4}$$

By the fact that $u_n \not\to 0$ in X_ε, we can use Lemma 3.1, to get $\tilde{R}, \tilde{\eta} > 0$ and $(y_n) \subset \mathbb{R}^N$ such that

$$\liminf_{n\to\infty}\int_{B_{\tilde{R}}(y_n)}|u_n|^2\,dx \geq \tilde{\eta}. \tag{3.5}$$

Let us consider $v_n(x) = u_n(x + y_n)$. Then we may assume that, up to a subsequence, $v_n \rightharpoonup v$ in X_ε. By (3.5) there exists $\Omega \subset \mathbb{R}^N$ with positive measure and such that $v > 0$. From (3.1), (f_3) and (3.4), we can infer that

$$\int_{\mathbb{R}^N}\left(V_\infty - V(\varepsilon x)\right)|u_n|^2\,dx + o_n(1)$$
$$\geq Q_{\min}\int_{\mathbb{R}^N}\left[\frac{f((1+\lambda)v_n)}{((1+\lambda)v_n)^{p-1}} - \frac{f(v_n)}{(v_n)^{p-1}}\right]v_n^p\,dx + \lambda K_{\min}\int_{\mathbb{R}^N}|v_n|^{2_{**}}\,dx. \tag{3.6}$$

By definition of V_∞, for any $\zeta > 0$, there exists $R = R(\zeta) > 0$ such that

$$V(\varepsilon x) \geq V_\infty - \zeta \quad \text{for all} \quad |\varepsilon x| \geq R. \tag{3.7}$$

From this, taking into account that $u_n \to 0$ in $L^2(B_{R/\varepsilon}(0))$ and the boundedness of (u_n) in X_ε, there exists $C_8 > 0$ such that

$$\int_{\mathbb{R}^N}\left(V_\infty - V(\varepsilon x)\right)|u_n|^2\,dx \leq V_\infty\int_{B_{R/\varepsilon}(0)}|u_n|^2\,dx + \zeta\int_{\mathbb{R}^N\setminus B_{R/\varepsilon}(0)}|u_n|^2\,dx$$
$$\leq o_n(1) + \zeta C_8. \tag{3.8}$$

Combining (3.6) and (3.8), we have

$$Q_{\min}\int_{\Omega}\left[\frac{f((1+\lambda)v_n)}{((1+\lambda)v_n)^{p-1}} - \frac{f(v_n)}{(v_n)^{p-1}}\right]v_n^p\,dx + \lambda K_{\min}\int_{\Omega}|v_n|^{2_{**}}\,dx \leq \zeta C_8 + o_n(1).$$

Taking the limit as $n \to \infty$ and applying Fatou's lemma, we obtain

$$0 < Q_{\min}\int_{\Omega}\left[\frac{f((1+\lambda)v)}{((1+\lambda)v)^{p-1}} - \frac{f(v)}{(v)^{p-1}}\right]v^p\,dx + \lambda K_{\min}\int_{\Omega}|v|^{2_{**}}\,dx \leq \zeta C_8.$$

Since $\zeta > 0$ is arbitrary, we obtain a contradiction by taking $\zeta \to 0$. Therefore $t_0 \leq 1$.

Now, we divide the proof in two cases.

Case 1. $t_0 < 1$.

In this case we may suppose, without loss of generality, that $t_n < 1$, for all $n \in \mathbb{N}$. Let us observe that

$$\begin{aligned} d+o_n(1) &= I_\varepsilon(u_n) - \frac{1}{p} I'_\varepsilon(u_n)u_n \\ &= \left(\frac{1}{2}-\frac{1}{p}\right)\|u_n\|_\varepsilon^2 + \frac{1}{p}\int_{\mathbb{R}^N} Q(\varepsilon x)\big[f(u_n)u_n - pF(u_n)\big]dx \\ &\quad + \left(\frac{1}{p} - \frac{1}{2_{**}}\right)\int_{\mathbb{R}^N} K(\varepsilon x)|u_n|^{2_{**}}dx. \end{aligned} \tag{3.9}$$

Recalling that $t_nu_n \in \mathrm{N}_{\varepsilon,\infty}$, and using (2.4) we obtain

$$\begin{aligned} c_{\varepsilon,\infty} &\leq I_{\varepsilon,\infty}(t_nu_n) - \frac{1}{p} I'_{\varepsilon,\infty}(t_nu_n)(t_nu_n) \\ &= t_n^2\left(\frac{1}{2}-\frac{1}{p}\right)\|u_n\|_{V_\infty}^2 + \frac{1}{p}\int_{\mathbb{R}^N} Q(\varepsilon x)\big[f(t_nu_n)(t_nu_n) - pF(t_nu_n)\big]dx \\ &\quad + t_n^{2_{**}}\left(\frac{1}{p} - \frac{1}{2_{**}}\right)\int_{\mathbb{R}^N} K(\varepsilon x)|u_n|^{2_{**}}dx \\ &\leq \left(\frac{1}{2}-\frac{1}{p}\right)\|u_n\|_{V_\infty}^2 + \frac{1}{p}\int_{\mathbb{R}^N} Q(\varepsilon x)\big[f(u_n)u_n - pF(u_n)\big]dx \\ &\quad + \left(\frac{1}{p} - \frac{1}{2_{**}}\right)\int_{\mathbb{R}^N} K(\varepsilon x)|u_n|^{2_{**}}dx. \end{aligned}$$

On the other hand, we can use (3.8) in the above inequality to get

$$\begin{aligned} c_{\varepsilon,\infty} &\leq \left(\frac{1}{2}-\frac{1}{p}\right)\|u_n\|_\varepsilon^2 + \frac{1}{p}\int_{\mathbb{R}^N} Q(\varepsilon x)\big[f(u_n)u_n - pF(u_n)\big]dx \\ &\quad + \left(\frac{1}{p} - \frac{1}{2_{**}}\right)\int_{\mathbb{R}^N} K(\varepsilon x)|u_n|^{2_{**}}dx + \zeta\left(\frac{1}{2}-\frac{1}{p}\right)C_8 + o_n(1). \end{aligned}$$

Thus, using (3.9) we have

$$c_{\varepsilon,\infty} \leq d + \zeta\left(\frac{1}{2}-\frac{1}{p}\right)C_8 + o_n(1),$$

for any $\zeta > 0$. By taking $n \to \infty$ and for the arbitrariness of ζ, we conclude that $c_{\varepsilon,\infty} \leq d$ and consequently $c_{V_\infty Q_{max} K_{max}} \leq d$.

Case 2. $t_0 = 1$.

Up to a subsequence, we may suppose that $t_n \to 1$. Taking into account that $I_\varepsilon(u_n) \to d$, we have

$$\begin{aligned} d+o_n(1) &= I_\varepsilon(u_n) - I_{\varepsilon,\infty}(t_nu_n) + I_{\varepsilon,\infty}(t_nu_n) \\ &\geq I_\varepsilon(u_n) - I_{\varepsilon,\infty}(t_nu_n) + c_{\varepsilon,\infty}. \end{aligned}$$

Now, let us point out that

$$\begin{aligned} I_\varepsilon(u_n) - I_{\varepsilon,\infty}(t_n u_n) &= \frac{1}{2}\left(1-t_n^2\right)\int_{\mathbb{R}^N} |u_n|^2\, dx \\ &+ \frac{1}{2}\int_{\mathbb{R}^N}\left(V(\varepsilon x) - t_n^2 V_\infty\right)|u_n|^2\, dx + \frac{1}{p}\left(1-t_n^p\right)\int_{\mathbb{R}^N}|\nabla u_n|^p\, dx \\ &+ \int_{\mathbb{R}^N} Q(\varepsilon x)\left[F(t_n u_n) - F(u_n)\right]dx + \left(t_n^{2_{**}}-1\right)\frac{1}{2_{**}}\int_{\mathbb{R}^N} K(\varepsilon x)|u_n|^{2_{**}}\, dx. \end{aligned} \tag{3.10}$$

Note that, by using (3.7) and the fact that $u_n \to 0$ in $L^2(B_{R/\varepsilon}(0))$ we get

$$\begin{aligned} \int_{\mathbb{R}^N}\left(V(\varepsilon x) - t_n^2 V_\infty\right)|u_n|^2\, dx &= \int_{B_{R/\varepsilon}(0)}\left(V(\varepsilon x) - V_\infty\right)|u_n|^2\, dx \\ &+ \int_{\mathbb{R}^N\setminus B_{R/\varepsilon}(0)}\left(V(\varepsilon x) - V_\infty\right)|u_n|^2\, dx + (1-t_n^2)V_\infty\int_{\mathbb{R}^N}|u_n|^2\, dx \\ &\geq o_n(1) - \zeta\int_{\mathbb{R}^N\setminus B_{R/\varepsilon}(0)}|u_n|^2\, dx + (1-t_n^2)V_\infty\int_{\mathbb{R}^N}|u_n|^2\, dx. \end{aligned} \tag{3.11}$$

Applying the mean value theorem and (2.3), there exists $C_9 > 0$ such that

$$\int_{\mathbb{R}^N} Q(\varepsilon x)\left|F(t_n u_n) - F(u_n)\right|dx \leq C_9\,|t_n - 1|\int_{\mathbb{R}^N}|u_n|^2\, dx + C_9\,|t_n-1|\int_{\mathbb{R}^N}|u_n|^q\, dx. \tag{3.12}$$

Thus, we can use the boundedness of (un), (2.1), (2.2) and $t_n \to 1$ in (3.10), (3.11) and to get

$$I_\varepsilon(u_n) - I_{\varepsilon,\infty}(t_n u_n) \geq o_n(1) - \zeta\, C_{10}$$

for some constant $C_{10} > 0$. Therefore

$$d + o_n(1) \geq c_{\varepsilon,\infty} - \zeta\, C_{10} + o_n(1)$$

and taking the limits as $n \to \infty$ and $\zeta \to 0$ we get $d \geq c_{\varepsilon,\infty}$. Consequently we have $c_{V_\infty Q_{max} K_{max}} \leq d$.

Proposition 3.3. Let

$$c^* < \min\left\{ c_{V_\infty Q_{\max} K_{\max}}, \left(\frac{1}{p} - \frac{1}{2_{**}}\right)\chi_{K_{\max}}, \left(\frac{1}{2} - \frac{1}{p}\right)\Pi_{K_{\max}}, \frac{p-2}{2p} \right\}.$$

The *functional* I_ε *satisfies the Palais-Smale condition at any level* $c < c^*$.

Proof. Let $(u_n) \subset X_\varepsilon$ be such that

$$I_\varepsilon(u_n) \to c \quad \text{and} \quad I_\varepsilon'(u_n) \to 0 \quad \text{as } n \to \infty.$$

From Lemma 2.3, (u_n) is bounded in X_ε. Then, up to a subsequence we have that $u_n \to u$ weakly in X_ε with u being a critical point of I_ε. Moreover, we can show that $I'(u_n - u) \to 0$ and

$$\lim_{n\to\infty} I_\varepsilon(u_n - u) = c - I_\varepsilon(u) = d.$$

Recalling that $I'_\varepsilon(u) = 0$ and by using (2.4) we get

$$I_\varepsilon(u) = I_\varepsilon(u) - \frac{1}{p} I'_\varepsilon(u)u = \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|_\varepsilon^2 + \frac{1}{p}\int_{\mathbb{R}^N} Q(\varepsilon x)\big[f(u)u - pF(u)\big]dx$$
$$+\left(\frac{1}{p} - \frac{1}{2_{**}}\right)\int_{\mathbb{R}^N} K(\varepsilon x)\,|u|^{2_{**}}\,dx \ge 0$$

and therefore $d \le c < c_{V_\infty Q_{max} K_{max}}$. It follows from Lemma 3.2 that $u_n \to u$ strongly in X_ε. The proposition is proved.

Remark 1. From (H_2) and *Lemma 2.9, we have that* $c_{V_{min}Q_{max}K_{max}} < c_{V_\infty Q_{max}K_{max}}$. *Now, by using Proposition 2.7, we can conclude that* $cc_{V_{min}Q_{max}K_{max}} < c^*$.

Lemma 3.4. *We have*

$$\limsup_{\varepsilon\to 0} c_\varepsilon \le c_{V_{min}Q_{max}K_{max}}.$$

Proof. Let $\psi \in C_0^\infty(\mathbb{R}^N, [0,1])$ be such that $\psi(x) = 1$, for $|x| < 1$ and $\psi(x) = 0$, for $|x| \ge 2$. For any $\lambda > 0$ we define

$$u_\lambda(x) = \psi\left(\frac{x}{\lambda}\right)w(x),$$

where w is a ground state solution of the problem $(P_{V_{min}Q_{max}K_{max}})$ given by Proposition 2.8.

Let $t_{\varepsilon,\lambda} > 0$ be such that $t_{\varepsilon,\lambda}u_\lambda \in N_\varepsilon$. Then

$$c_\varepsilon \le I_\varepsilon(t_{\varepsilon,\lambda}u_\lambda) = \frac{t_{\varepsilon,\lambda}^2}{2}\int_{\mathbb{R}^N}\left(|\Delta u_\lambda|^2 + V(\varepsilon x)|u_\lambda|^2\right)dx \pm \frac{t_{\varepsilon,\lambda}^p}{p}\int_{\mathbb{R}^N}|\nabla u_\lambda|^p\,dx$$
$$-\int_{\mathbb{R}^N} Q(\varepsilon x)F(t_{\varepsilon,\lambda}u_\lambda)dx - \frac{t_{\varepsilon,\lambda}^{2_{**}}}{2_{**}}\int_{\mathbb{R}^N} K(\varepsilon x)|u_\lambda|^{2_{**}}\,dx.$$

It is easy to check that, for λ fixed, $t_{\varepsilon,\lambda} \to t_\lambda > 0$ as $\varepsilon \to 0$. Moreover, without loos of generality, we may suppose that $0 \in M$. Hence, since u_λ has compact support, we can use Lebesgue's dominated convergence theorem to get

$$\limsup_{\varepsilon\to 0} c_\varepsilon \le \frac{t_\lambda^2}{2}\int_{\mathbb{R}^N}\left(|\Delta u_\lambda|^2 + V_{\min}|u_\lambda|^2\right)dx \pm \frac{t_\lambda^p}{p}\int_{\mathbb{R}^N}|\nabla u_\lambda|^p\,dx$$
$$-Q_{\max}\int_{\mathbb{R}^N} F(t_\lambda u_\lambda)dx - \frac{t_\lambda^{2_{**}}}{2_{**}}K_{\max}\int_{\mathbb{R}^N}|u_\lambda|^{2_{**}}\,dx = I_{V_{\min}Q_{\max}K_{\max}}(t_\lambda u_\lambda).$$

Since $w \in M_{V_{min}Q_{max}K_{max}}$ and $u_\lambda \to w$ in $H^2(\mathbb{R}^N)$ as $\lambda \to \infty$, we can check that $t_\lambda \to 1$ as $\lambda \to \infty$. Thus, it follows from the above expression that

$$\limsup_{\varepsilon\to 0} c_\varepsilon \le \lim_{\lambda\to\infty} I_{V_{\min}Q_{\max}K_{\max}}(t_\lambda u_\lambda) = I_{V_{\min}Q_{\max}K_{\max}}(w) = c_{V_{\min}Q_{\max}K_{\max}}$$

The lemma is proved.

Theorem 3.5. *Assume that V, Q and K are continuous potentials and satisfy* $(H_1) - (H_2)$. *Assume also that conditions* $(f_1) - (f_4)$ *hold. Then there exists* $\varepsilon_0 > 0$ *and* $\sigma^* > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_0)$ *and for all* $\sigma > \sigma^*$, *problem* (P_ε) *admits a ground state solution.*

Proof. From Lemma 3.4, we obtain $\varepsilon_0 > 0$ such that $c_\varepsilon < c_{V_{min}Q_{max}K_{max}}$ for any $\varepsilon \in (0, \varepsilon_0)$. For these values of ε, since I_ε has the mountain pass geometry, we can take a sequence $(u_n) \subset X_\varepsilon$ such that

$$I_\varepsilon(u_n) \to c_\varepsilon \quad \text{and} \quad I_\varepsilon'(u_n) \to 0.$$

By using Proposition 3.3, we guarantee that, along a subsequence $u_n \to u_\varepsilon$ with u_ε being a critical point of I_ε and $I_\varepsilon(u_\varepsilon) = c_\varepsilon$.

4. Multiplicity of solutions

Lemma 4.1. Let $J_\varepsilon : X_\varepsilon \to \mathbb{R}$ be given by

$$J_\varepsilon(u) = \|u\|_\varepsilon^2 \pm \int_{\mathbb{R}^N} |\nabla u|^p dx - \int_{\mathbb{R}^N} Q(\varepsilon x) f(u)u dx - \int_{\mathbb{R}^N} K(\varepsilon x)|u|^{2^{**}} dx.$$

Then, there exists $D > 0$ such that

$$J_\varepsilon'(u)u \leq -D \quad \text{for each} \quad u \in \mathrm{U}_\varepsilon.$$

Proof. For all $u \in \mathrm{N}_\varepsilon$ we have

$$\begin{aligned} J_\varepsilon'(u)u &= 2\|u\|_\varepsilon^2 \pm p\int_{\mathbb{R}^N} |\nabla u|^p dx \\ &\quad - \int_{\mathbb{R}^N} Q(\varepsilon x) f(u)u dx - \int_{\mathbb{R}^N} Q(\varepsilon x) f'(u)u^2 dx - 2^{**}\int_{\mathbb{R}^N} K(\varepsilon x)|u|^{2^{**}} dx \\ &= 2\|u\|_\varepsilon^2 - p\|u\|_\varepsilon^2 + p\int_{\mathbb{R}^N} Q(\varepsilon x) f(u)u dx + p\int_{\mathbb{R}^N} K(\varepsilon x)|u|^{2^{**}} dx \\ &\quad - \int_{\mathbb{R}^N} Q(\varepsilon x) f(u)u dx - \int_{\mathbb{R}^N} Q(\varepsilon x) f'(u)u^2 dx - 2^{**}\int_{\mathbb{R}^N} K(\varepsilon x)|u|^{2^{**}} dx \\ &= -(p-2)\|u\|_\varepsilon^2 - \int_{\mathbb{R}^N} Q(\varepsilon x)\left[f'(u)u^2 - (p-1)f(u)u\right]dx \\ &\quad - (2^{**} - p)\int_{\mathbb{R}^N} K(\varepsilon x)|u|^{2^{**}} dx. \end{aligned}$$

Now, we can use (2.5), $2 < p < 2^{**}$ and Lemma 2.5 to get

$$J_\varepsilon'(u)u \leq -(p-2)\mu^2 := -D.$$

The lemma is proved.

Proposition 4.2. *The functional I_ε constrained to N_ε satisfies the $(PS)_d$ condition* any level $d < c^*$.

Proof. Let $(u_n) \subset \mathrm{N}_\varepsilon$ be such that

$$I_\varepsilon(u_n) \to d \quad \text{and} \quad \|I_\varepsilon'(u_n)\|_* \to 0.$$

Then there exists a sequence $(\lambda_n) \subset \mathbb{R}$ such that

$$I_\varepsilon'(u_n) = \lambda_n J_\varepsilon'(u_n) + o_n(1),$$

with J_ε as in Lemma 4.1. Since $u_n \subset \mathcal{N}_\varepsilon$ we have that

$$0 = I_\varepsilon'(u_n)u_n = \lambda_n J_\varepsilon'(u_n)u_n + o_n(1)\|u_n\|_\varepsilon .$$

Straightforward calculations show that (u_n) is bounded. Moreover, in view of Lemma 4.1, we may suppose that $J'_\varepsilon(u_n)u_n \to l < 0$. Hence, the above expression shows that $\lambda_n \to 0$ and therefore we conclude that $I'_\varepsilon(u_n) \to 0$ in the dual space of X_ε. It follows from Proposition 3.3 that (u_n) has a convergent subsequence.

Proposition 4.3. Let $\varepsilon_n \to 0$ and $(u_n) \subset \mathcal{N}_{\varepsilon_n}$ *such that*

$$I_{\varepsilon_n}(u_n) \to c_{V_{\min} Q_{\max} K_{\max}} .$$

Then there exists a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ *such that* $v_n(x) = u_n(x + \tilde{y}_n)$ *has a convergent subsequence in* $X_{V_{min}}$. *Moreover, up to a subsequence,* $(y_n) = (\varepsilon_n \tilde{y}_n)$ *is such that* $y_n \to y \in M$.

Proof. Since $u_n \in \mathcal{N}_{\varepsilon_n}$ and $I_{\varepsilon_n}(u_n) \to c_{V_{min}Q_{max}K_{max}}$, we have that (u_n) is bounded. Moreover, since $c_{V_{min}Q_{max}K_{max}} > 0$, we cannot have $\|u_n\|_{\varepsilon_n} \to 0$. Hence, arguing as in Lemma 3.1, we obtain a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ and constants R, $\eta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(\tilde{y}_n)} |u_n|^2\, dx \geq \eta. \tag{4.1}$$

Let us define $v_n(x) = u_n(x + \tilde{y}_n)$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v \neq 0$. By virtue of Lemma 2.4-(i), there exists $t_n > 0$ such that $w_n = t_n v_n \in \mathcal{M}_{V_{min}Q_{max}K_{max}}$. Then we have

$$c_{V_{\min} Q_{\max} K_{\max}} \leq I_{V_{\min} Q_{\max} K_{\max}}(w_n) \leq I_{\varepsilon_n}(t_n u_n) \leq I_{\varepsilon_n}(u_n) = c_{V_{\min} Q_{\max} K_{\max}} + o_n(1),$$

which shows that $I_{V_{min}Q_{max}K_{max}}(w_n) \to c_{V_{min}Q_{max}K_{max}}$.

Since (v_n) and (w_n) are bounded and $v_n \not\rightharpoonup 0$, the sequence (t_n) is bounded. Thus, up to a subsequence, $t_n \to t_0 \geq 0$. Let us show that $t_0 > 0$. Otherwise, if $t_0 = 0$, by the boundedness of (v_n), we get $w_n = t_n v_n \to 0$ in $X_{V_{min}}$, that is $I_{V_{min}Q_{max}K_{max}}(w_n) \to 0$ which contradicts with $c_{V_{min}Q_{max}K_{max}} > 0$. Thus $t_0 > 0$ and, up to a subsequence, we may assume that $w_n \rightharpoonup w$ in $X_{V_{min}}$ with $w = t_0 v \neq 0$. From Proposition 2.8, we conclude that $w_n \to w$ in $X_{V_{min}}$, that is $v_n \to v$ in $X_{V_{min}}$.

Next, we prove that $(y_n) = (\varepsilon_n \tilde{y}_n)$ has a subsequence satisfying $y_n \to y \in M$. We first claim that (y_n) is bounded. In fact, assume by contradiction that (y_n) is not bounded. Then, there exists a subsequence, still denoted by (y_n), such that $|y_n| \to \infty$. From $w_n \to w$ in $X_{V_{min}}$, $V_{min} < V_\infty$, $Q_{max} > Q_\infty$, $M_{max} > K_\infty$ and the change of variable $z = x + \tilde{y}_n$, we can deduce that

$$\begin{aligned}
c_{V_{\min}Q_{\max}K_{\max}} &= I_{V_{\min}Q_{\max}K_{\max}}(w) \\
&< \frac{1}{2}\int_{\mathbb{R}^N}\left(|\Delta w|^2+V_\infty|w|^2\right)dx \pm \frac{1}{p}\int_{\mathbb{R}^N}|\nabla w|^p\,dx \\
&\quad -W_\infty\int_{\mathbb{R}^N}F(w)\,dx-\frac{K_\infty}{2_{**}}\int_{\mathbb{R}^N}|w|^{2_{**}}\,dx \\
&\le \liminf_{n\to\infty}\left[\frac{1}{2}\int_{\mathbb{R}^N}\left(|\Delta w_n|^2+V(\varepsilon_n x+y_n)|w_n|^2\right)dx \pm \frac{1}{p}\int_{\mathbb{R}^N}|\nabla w_n|^p\,dx\right. \\
&\quad \left.-\int_{\mathbb{R}^N}Q(\varepsilon_n x+y_n)F(w_n)\,dx-\frac{1}{2_{**}}\int_{\mathbb{R}^N}K(\varepsilon_n x+y_n)|w_n|^{2_{**}}\,dx\right] \\
&= \liminf_{n\to\infty}\left[\frac{t_n^2}{2}\int_{\mathbb{R}^N}\left(|\Delta u_n|^2+V(\varepsilon_n z)|u_n|^2\right)dz \pm \frac{t_n^p}{p}\int_{\mathbb{R}^N}|\nabla u_n|^p\,dz\right. \\
&\quad \left.-\int_{\mathbb{R}^N}Q(\varepsilon_n z)F(t_n u_n)\,dz-\frac{t_n^{2_{**}}}{2_{**}}\int_{\mathbb{R}^N}K(\varepsilon_n z)|u_n|^{2_{**}}\,dz\right] \\
&= \liminf_{n\to\infty} I_{\varepsilon_n}(t_n u_n) \le \liminf_{n\to\infty} I_{\varepsilon_n}(u_n) = c_{V_{\min}Q_{\max}K_{\max}},
\end{aligned}$$

which is a contradiction. Thus, (y_n) is bounded and, passing to a subsequence, we may assume that $y_n \to y$. If $y \notin M$, then $V_{min} < V(y)$, $Q_{max} > Q(y)$ and $K_{max} > K(y)$, and according to the above steps we get a contradiction. Consequently, we conclude that $y \in M$.

Let $\delta > 0$ be fixed and consider $\psi \in C^\infty(\mathbb{R}^+, [0, 1])$ be a non-increasing function such that $\psi \equiv 1$ on $[0, \delta/2]$ and $\psi \equiv 0$ on $[\delta, \infty)$. For any $y \in M$, we define

$$\Psi_{\varepsilon,y}(x)=\psi(|\varepsilon x-y|)\,w\left(\frac{\varepsilon x-y}{\varepsilon}\right),$$

where $w \in X_{V_{min}}$ is a ground state solution of $(P_{V_{min}Q_{max}K_{max}})$ given by Proposition 2.8.

Then, there exists $t_\varepsilon > 0$, the unique positive number satisfying

$$I_\varepsilon(t_\varepsilon\Psi_{\varepsilon,y})=\max_{t\ge 0} I_\varepsilon(t\Psi_{\varepsilon,y}).$$

We define $\Phi_\varepsilon : M \to \mathrm{N}_\varepsilon$ by setting

$$\Phi_\varepsilon(y):=t_\varepsilon\Psi_{\varepsilon,y}.$$

By the construction, $\Phi_\varepsilon(y)$ has compact support for any $y \in M$.

Lemma 4.4. The function Φ_ε satisfies

$$\lim_{\varepsilon\to 0^+} I_\varepsilon(\Phi_\varepsilon(y))=c_{V_{\min}Q_{\max}K_{\max}} \quad \text{uniformly in } y\in M.$$

Proof. Suppose by contradiction that there exists $\delta_0 > 0$, $(y_n) \subset M$ and $\varepsilon_n \to 0$ such that

$$\left|I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n))-c_{V_{\min}Q_{\max}K_{\max}}\right|\ge\delta_0. \tag{4.2}$$

Observe that, by Lebesgue's dominated convergence theorem, we can easily check that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(|\Delta\Psi_{\varepsilon_n,y_n}|^2+V(\varepsilon_n x)|\Psi_{\varepsilon_n,y_n}|^2\right)dx=\int_{\mathbb{R}^N}\left(|\Delta w|^2+V_{\min}|w|^2\right)dx, \tag{4.3}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla\Psi_{\varepsilon_n,y_n}|^p\,dx=\int_{\mathbb{R}^N}|\nabla w|^p\,dx, \tag{4.4}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}Q(\varepsilon_n x)F(\Psi_{\varepsilon_n,y_n})dx=Q_{\max}\int_{\mathbb{R}^N}F(w)dx, \tag{4.5}$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}K(\varepsilon_n x)|\Psi_{\varepsilon_n,y_n}|^{2_{**}}\,dx=K_{\max}\int_{\mathbb{R}^N}|w|^{2_{**}}\,dx. \tag{4.6}$$

Using the fact that $I'_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n))\Phi_{\varepsilon_n}(y_n)=0$ and the change of variables $z=\dfrac{\varepsilon_n x-y_n}{\varepsilon_n}$, we obtain

$$\begin{aligned}
&t_{\varepsilon_n}^2\int_{\mathbb{R}^N}\left(|\Delta\Psi_{\varepsilon_n,y_n}|^2+V(\varepsilon_n x)|\Psi_{\varepsilon_n,y_n}|^2\right)dx\pm t_{\varepsilon_n}^p\int_{\mathbb{R}^N}|\nabla\Psi_{\varepsilon_n,y_n}|^p\,dx\\
&=\int_{\mathbb{R}^N}Q(\varepsilon_n x)f(t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n}\,dx+t_{\varepsilon_n}^{2_{**}}\int_{\mathbb{R}^N}K(\varepsilon_n x)|\Psi_{\varepsilon_n,y_n}|^{2_{**}}\,dx\\
&=\int_{\mathbb{R}^N}Q(\varepsilon_n z+y_n)f(t_{\varepsilon_n}\psi(|\varepsilon_n z|)w(z))t_{\varepsilon_n}\psi(|\varepsilon_n z|)w(z)dz\\
&\quad+t_{\varepsilon_n}^{2_{**}}\int_{\mathbb{R}^N}K(\varepsilon_n z+y_n)|\psi(|\varepsilon_n z|)w(z)|^{2_{**}}\,dz.
\end{aligned} \tag{4.7}$$

We claim that $t_{\varepsilon_n}\to 1$. First we need to prove that (t_{ε_n}) is bounded. In fact, assume by contradiction that $t_{\varepsilon_n}\to\infty$. If $n_0\in \mathrm{N}$ is such that $B_{\delta/2}(0)\subset B_{\delta/2\varepsilon_n}(0)$ for all $n\geq n_0$, we have

$$\begin{aligned}
&t_{\varepsilon_n}^{2-p}\int_{\mathbb{R}^N}\left(|\Delta\Psi_{\varepsilon_n,y_n}|^2+V(\varepsilon_n x)|\Psi_{\varepsilon_n,y_n}|^2\right)dx\pm\int_{\mathbb{R}^N}|\nabla\Psi_{\varepsilon_n,y_n}|^p\,dx\\
&\geq Q_{\min}\int_{B_{\delta/2}(0)}\frac{f(t_{\varepsilon_n}w(z))}{\left(t_{\varepsilon_n}w(z)\right)^{p-1}}(w(z))^p\,dz+t_{\varepsilon_n}^{2_{**}-p}K_{\min}\int_{B_{\delta/2}(0)}|w(z)|^{2_{**}}\,dz.
\end{aligned} \tag{4.8}$$

In the case that the second term on the left side in (4.8) is negative we have

$$\begin{aligned}
&t_{\varepsilon_n}^{2-p}\int_{\mathbb{R}^N}\left(|\Delta\Psi_{\varepsilon_n,y_n}|^2+V(\varepsilon_n x)|\Psi_{\varepsilon_n,y_n}|^2\right)dx\\
&\geq Q_{\min}\int_{B_{\delta/2}(0)}\frac{f(t_{\varepsilon_n}w(z))}{\left(t_{\varepsilon_n}w(z)\right)^{p-1}}(w(z))^p\,dz+t_{\varepsilon_n}^{2_{**}-p}K_{\min}\int_{B_{\delta/2}(0)}|w(z)|^{2_{**}}\,dz.
\end{aligned}$$

Since $2<p<2_{**}$, we can use (f_3) to get $0\geq\infty$, which is absurd.

In the case that the second term on the left hand side in (4.8) is positive, we can deduce that $\int_{\mathbb{R}^N}|\nabla\Psi_{\varepsilon_n,y_n}|^p\,dx\to\infty$. Clearly, this contradicts relation (4.4). Hence, (t_{ε_n}) is R^N bounded. Passing to a subsequence, we may assume that $t_{\varepsilon_n}\to t_0\geq 0$. If $t_0=0$, by (f_1), (4.4) and (4.7) we can infer that $\int_{\mathbb{R}^N}\left(|\Delta\Psi_{\varepsilon_n,y_n}|^2+V(\varepsilon_n x)|\Psi_{\varepsilon_n,y_n}|^2\right)dx\to 0$, this contradicts relation (4.3). We conclude that $t_0>0$.

Next, we prove that $t_0=1$. Letting $n\to\infty$ in (4.7), we obtain

$$\begin{aligned}
&t_0^{2-p}\int_{\mathbb{R}^N}\left(|\Delta w|^2+V_{\min}|w|^2\right)dx\pm\int_{\mathbb{R}^N}|\nabla w|^p\,dx\\
&=Q_{\max}\int_{\mathbb{R}^N}\frac{f(t_0 w)}{(t_0 w)^{p-1}}w^p\,dx+t_0^{2_{**}-p}K_{\max}\int_{\mathbb{R}^N}|w|^{2_{**}}\,dx.
\end{aligned} \tag{4.9}$$

Moreover, since w is a ground state solution of problem $(P_{V_{min}Q_{max}K_{max}})$

$$\int_{\mathbb{R}^N}\left(|\Delta w|^2+V_{\min}|w|^2\right)dx\pm\int_{\mathbb{R}^N}|\nabla w|^p\,dx$$
$$=Q_{\max}\int_{\mathbb{R}^N}f(w)w\,dx+K_{\max}\int_{\mathbb{R}^N}|w|^{2_{**}}\,dx. \tag{4.10}$$

Combining (4.9) and (4.10), we can conclude that

$$\left(t_0^{2-p}-1\right)\int_{\mathbb{R}^N}\left(|\Delta w|^2+V_{\min}|w|^2\right)dx$$
$$=Q_{\max}\int_{\mathbb{R}^N}\left[\frac{f(t_0w)}{(t_0w)^{p-1}}-\frac{f(w)}{(w)^{p-1}}\right]|w|^p\,dx+\left(t_0^{2_{**}-p}-1\right)K_{\max}\int_{\mathbb{R}^N}|w|^{2_{**}}\,dx.$$

Then, we deduce from (f_3) that $t_0 = 1$. Therefore, from (4.3), (4.4), (4.5) and (4.6), we infer that

$$\lim_{n\to\infty}I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n))=I_{V_{\min}Q_{\max}K_{\max}}(w)=c_{V_{\min}Q_{\max}K_{\max}}.$$

Obviously, from (4.2) we can see that this is impossible. The proof is completed.

Now, we are in the position to introduce the barycenter map. For any $\delta > 0$, let $\rho = \rho(\delta) > 0$ be such that $M_\delta \subset B_\rho(0)$. We define $\Upsilon : \mathbb{R}^N \to \mathbb{R}^N$ as follows

$$\Upsilon(x)=x \ \text{ for } |x|<\rho \quad \text{and} \quad \Upsilon(x)=\frac{\rho x}{|x|} \ \text{ for } |x|\geq\rho.$$

Let us consider $\beta_\varepsilon : N_\varepsilon \to \mathbb{R}^N$ given by

$$\beta_\varepsilon(u)=\frac{\int_{\mathbb{R}^N}\Upsilon(\varepsilon x)|u(x)|^2\,dx}{\int_{\mathbb{R}^N}|u(x)|^2\,dx}.$$

Using the above notations, we have the following result.

Lemma 4.5. The *function* β_ε *satisfies*

$$\lim_{\varepsilon\to0^+}\beta_\varepsilon(\Phi_\varepsilon(y))=y \quad \text{uniformly in } y\in M.$$

Proof. Arguing by contradiction, we assume that there exist $\delta_0 > 0$, $(y_n) \subset M$ and $\varepsilon_n \to 0$ such that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n))-y_n|\geq\delta_0. \tag{4.11}$$

According to the definitions of Φ_{ε_n} and β_{ε_n}, and using the change of variables $z=\dfrac{\varepsilon_n x-y_n}{\varepsilon_n}$ we get

$$\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n))=y_n+\frac{\int_{\mathbb{R}^N}\left(\Upsilon(\varepsilon_n z+y_n)-y_n\right)|\psi(|\varepsilon_n z|)w(z)|^2\,dz}{\int_{\mathbb{R}^N}|\psi(|\varepsilon_n z|)w(z)|^2\,dz}.$$

Taking into account $(y_n) \subset B_\rho(0), \Upsilon\big|_{B_\rho(0)} = \mathrm{Id}$ and using Lebesgue's dominated convergence theorem, we have

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| \to 0,$$

which contradicts relation (4.11).

Let $h : \mathrm{R}^+ \to \mathrm{R}^+$ be a positive function given by

$$h(\varepsilon) = \max_{y \in M} |I_\varepsilon(\Phi_\varepsilon(y)) - c_{V_{\min}Q_{\max}K_{\max}}|.$$

It follows from Lemma 4.4 that $h(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Following [8], we introduce a subset Σ_ε of N_ε. Setting

$$\Sigma_\varepsilon = \left\{ u \in \mathrm{U}_\varepsilon : I_\varepsilon(u) \leq c_{V_{\min}Q_{\max}K_{\max}} + h(\varepsilon) \right\}.$$

Since $\Phi_\varepsilon(y) \in \Sigma_\varepsilon$ for all $y \in M$, then we can deduce that $\Sigma_\varepsilon \neq \emptyset$. Moreover, we have the following result.

Lemma 4.6. *For any δ > 0, then the following holds*

$$\lim_{\varepsilon \to 0^+} \sup_{u \in \Sigma_\varepsilon} \operatorname{dist}(\beta_\varepsilon(u), M_\delta) = 0.$$

Proof. Let $\varepsilon_n \to 0$ as $n \to \infty$. For each $n \in \mathrm{N}$, there exists $(u_n) \subset \Sigma_{\varepsilon_n}$, such that

$$\inf_{y \in M_\delta} |\beta_{\varepsilon_n}(u_n) - y| = \sup_{u \in \Sigma_\varepsilon} \inf_{y \in M_\delta} |\beta_{\varepsilon_n}(u) - y| + o_n(1).$$

Hence, it is sufficient to prove that there exists $(y_n) \subset M_\delta$ such that

$$\lim_{n \to \infty} |\beta_{\varepsilon_n}(u_n) - y_n| = 0.$$

Indeed, since $(u_n) \subset \Sigma_{\varepsilon_n}$, then we have

$$c_{V_{\min}Q_{\max}K_{\max}} \leq c_{\varepsilon_n} \leq I_{\varepsilon_n}(u_n) \leq c_{V_{\min}Q_{\max}K_{\max}} + h(\varepsilon_n),$$

which implies that

$$I_{\varepsilon_n}(u_n) \to c_{V_{\min}Q_{\max}K_{\max}} \quad \text{and} \quad (u_n) \subset \mathrm{U}_{\varepsilon_n}.$$

According to Proposition 4.3, there exists $(\tilde{y}_n) \subset \mathbb{R}^N$ such that $v_n(x) = u_n(x + \tilde{y}_n)$ has a convergent subsequence. Moreover, up to a subsequence, yn = enyn ^ y G M. Therefore we get

$$\beta_{\varepsilon_n}(u_n) = \frac{\int_{\mathbb{R}^N} \Upsilon(\varepsilon_n x)|u_n(x)|^2\,dx}{\int_{\mathbb{R}^N} |u_n(x)|^2\,dx} = y_n + \frac{\int_{\mathbb{R}^N} \left(\Upsilon(\varepsilon_n z + y_n) - y_n\right)|u_n(z + \tilde{y}_n)|^2\,dz}{\int_{\mathbb{R}^N} |u_n(z + \tilde{y}_n)|^2\,dz} \to y \in M.$$

Consequently, there exists $(y_n) \subset M_\delta$ such that

$$\lim_{n\to\infty} |\beta_{\varepsilon_n}(u_n) - y_n| = 0.$$

The lemma is proved.

Now we show that (P_ε) admits at least $\mathrm{cat}_{M\delta}(M)$ solutions. In order to achieve our aim, we recall the following result for critical points involving Ljusternik-Schnirelmann category. The proof of this abstract result can be seen in [14, Corollary 4.17].

Theorem 4.7. *Let I be a C^1-functional defined on a C^1-Finsler manifold* V. *If I is bounded from below and satisfies the Palais-Smale condition, then I has at least $\mathrm{cat}_{\mathrm{V}}(\mathrm{V})$ distinct critical points.*

With a view to apply Theorem 4.7, the following abstract lemma provides a very useful tool since relates the topology of some sublevel of a functional to the topology of some subset of the space $\mathbb{R}^N$; see [1, Lemma 4.3]

Lemma 4.8. Let Ω, Ω_1 and Ω_2 be *closed sets with* $\Omega_1 \subset \Omega_1$. Let $\beta : \Omega \to \Omega_2$, $\Phi : \Omega_1 \to \Omega$ be two *continuous maps such that* $\beta \circ \Phi$ is *homotopically equivalent to the embedding* $\iota : \Omega_1 \to \Omega_2$. *Then* $\mathrm{cat}_\Omega(\Omega) \geq \mathrm{cat}_{\Omega_2}(\Omega_1)$.

Proof of Theorem 1.1. Given $\delta > 0$ we can use Lemma 4.4, Lemma 4.5, Lemma 4.6, and argue as in [8, Section 6] to obtain $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the diagram

$$M \xrightarrow{\Phi_\varepsilon} \Sigma_\varepsilon \xrightarrow{\beta_\varepsilon} M_\delta$$

is well defined and $\beta_\varepsilon \circ \Phi_\varepsilon$ is homotopically equivalent to the embedding $\iota : M \to M_\delta$. This fact and Lemma 4.8 imply that $\mathrm{cat}_{\Sigma_\varepsilon}(\Sigma_\varepsilon) \geq \mathrm{cat}_{M_\delta}(M)$. Using the definition of Σ_ε, Proposition 4.2 and taking ε_δ small if necessary we guarantee that I_ε satisfies the Palais-Smale condition in Σ_ε. Hence, by using Theorem 4.7, we obtain that I_ε has at least $\mathrm{cat}_{\Sigma_\varepsilon}(\Sigma_\varepsilon)$ critical points on Σ_ε. Then, we can infer that I_ε admits at least $\mathrm{cat}_{M_\delta}(M)$ critical points. The theorem is proved.

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