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# Three solutions for a $(p(x,.), q(x,.))$—Kirchhof type elliptic system in general fractional Sobolev space with variable exponents

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In this work, we introduce the general weighted fractional Sobolev spaces with variable exponents $W^{s,p(x,y)}_{K,\omega}(\Omega)$ and we prove their continuous and compact embeddings into weighted Lebesgue spaces with variable exponent $L^{\gamma(.)}_{\omega}(\Omega)$. Moreover, we consider a class of fractional elliptic systems involving general $(p(x,.), q(x,.))$—Laplacian operators. Our main tool is based on the Three Critical Points Theorem introduced by B. Ricceri and on the theory of fractional Sobolev spaces with variable exponents.

*Keywords:* Elliptic systems, General fractional Sobolev spaces, Nonlocal problem, General fractional $p(x,.)$-Laplacian, Kirchhoff type Problems, Three Critical Points Theorem.

*Mathematics Subject Classification* (*2010*)*:* Primary 35J48; Secondary 35J50; 46E35.

## 1. Introduction

In this paper, we aim to introduce the general weighted fractional Sobolev spaces with variable exponents $W^{s,p(x,y)}_{K,\omega}(\Omega)$ and we prove their continuous and compact embeddings into weighted Lebesgue spaces with variable exponent $L^{\gamma(.)}_{\omega}(\Omega)$. Moreover, we investigate the existence and multiplicity of weak solutions for the following fractional elliptic system of $(p(x,.), q(x,.))$—Kirchhoff type

$$(\mathrm{P}^{K}_{\omega})\begin{cases} M\left(I^{\omega}_{K,p(x,y)}(u)\right)\left(\circledcirc^{p(x,.)}_{K}(u)+\omega(x)|u|^{\underline{p}(x)-2}u\right)=\lambda F_{u}(x,u,v)+\mu G_{u}(x,u,v), & \text{in } \Omega,\\ M\left(I^{\omega}_{K,q(x,y)}(v)\right)\left(\circledcirc^{q(x,.)}_{K}(v)+\omega(x)|v|^{\underline{q}(x)-2}v\right)=\lambda F_{v}(x,u,v)+\mu G_{v}(x,u,v), & \text{in } \Omega,\\ u=v=0, & \text{in } \mathbb{R}^{N}\setminus\Omega, \end{cases}$$

$$I^{\omega}_{K,r(x,y)}(u)=\int_{Q}\frac{1}{r(x,y)}|u(x)-u(y)|^{r(x,y)}K(x,y)\,dxdy+\int_{\Omega}\omega(x)\frac{|u(x)|^{\overline{r}(x)}}{\overline{r}(x)}dx,$$

where

- $\Omega$ is an open bounded subset in $\mathbb{R}^N$, $N \geq 2$, with Lipschitz boundary $\partial\Omega$, $s \in (0,1)$, $M : [0, +\infty) \to (0, +\infty)$ is a strictly nondecreasing continuous function and $\omega$ is a continuous wight function belongs to $L^1(\Omega)$ such that $\operatorname*{essinf}_{x\in\Omega} \omega(x) = \omega_0 > 0$ .
- $Q := \left(\mathbb{R}^N \times \Omega\right) \cup \left(\Omega \times \mathbb{R}^N\right)$ . The set $Q$ captures a broader range of the domain compared to $\Omega \times \Omega$. This is particularly important when dealing with non-local operators and problems defined over the entire space $\mathbb{R}^N$ with a specific domain of interest $\Omega$. Integrating over $Q$ allows for a more comprehensive analysis of the interactions between points in the domain and its complement.
- $p, q : \overline{Q} \to (1, +\infty)$ are symmetric continuous functions such that

$$
\begin{aligned}
&1 < p^- \le p(x,y) \le p^+ < \min\{ p_s^*(x), q_s^*(x)\} < +\infty, \\
&1 < q^- \le q(x,y) \le q^+ < \min\{ p_s^*(x), q_s^*(x)\} < +\infty,
\end{aligned} \tag{1.1}
$$

where

$$
\begin{aligned}
p^- &= \inf_{(x,y)\in\overline{Q}} p(x,y), \quad & p^+ &= \sup_{(x,y)\in\overline{Q}} p(x,y), \\
q^- &= \inf_{(x,y)\in\overline{Q}} q(x,y), \quad & q^+ &= \sup_{(x,y)\in\overline{Q}} q(x,y),
\end{aligned}
$$

and $p_s^*$, $q_s^*$ are critical fractional Sobolev exponents given by

$$
p_s^*(x) = \begin{cases} \dfrac{N\overline{p}(x)}{N - s\overline{p}(x)} & if \quad N > s\overline{p}(x), \\ +\infty & if \quad N \le s\overline{p}(x). \end{cases} \qquad q_s^*(x) = \begin{cases} \dfrac{N\overline{q}(x)}{N - s\overline{q}(x)} & if \quad N > s\overline{q}(x), \\ +\infty & if \quad N \le s\overline{q}(x). \end{cases}
$$

where

$$
\overline{p}(x) = p(x,x) \quad \text{for any } x \in \overline{\Omega}, \quad \text{and} \quad \overline{q}(x) = q(x,x) \quad \text{for any } x \in \overline{\Omega}.
$$

- $F, G : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are functions such that $F(., t, \xi)$, $G(.,t, \xi)$ are measurable in $\Omega$ for all $t, \xi \in \mathbb{R}$ and $F(x,.,.)$, $G(x,.,.)$ are continuously differentiable in $\mathbb{R}^2$ for a.e. $x \in \Omega$. $F_u(x, u, v)$, $F_v(x, u, v)$, $G_u(x, u, v)$ and $G_v(x, u, v)$ are the partial derivatives of $F$ and $G$ with respect to $u$ and $v$ respectively. $\lambda$, $\mu$ are two real parameters.
- $\mathcal{L}_K^{p(x,.)}$ is a nonlocal integro-differential operator of elliptic type defined by

$$
\mathcal{L}_K^{p(x,.)}\left(u(x)\right) = p.v. \int_{\mathbb{R}^N} |u(x) - u(y)|^{p(x,y)-2} (u(x) - u(y)) K(x,y)\, dy,
$$

  where *p.v.* is a commonly used abbreviation in the principal value sense, ensuring that the singularity at $x = y$ is handled properly by considering the limit of the integral as $y$ approaches $x$ symmetrically from all directions within a neighborhood, typically represented as a ball centered at $x$. This approach effectively "subtracts out" the singularity, ensuring a balanced treatment around $x$.
- The singular kernel $K : \mathbb{R}^N \times \mathbb{R}^N \to (0, +\infty)$ is a measurable function with the following properties:

$$
K(x,y) = K(y,x) \text{ for any } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N, \tag{1.2}
$$

  there exists $k_0 > 0$ such that

$$K(x,y)\square k_0|x-y|^{-(N+sp(x,y))} \quad \text{for any } (x,y)\in\mathbb{R}^N\times\mathbb{R}^N \text{ and } x\neq y, \tag{1.3}$$

$$\text{and } mK\in L^1\left(\mathbb{R}^N\times\mathbb{R}^N\right), \text{ where } m(x,y)=\min\left\{1,|x-y|^{p(x,y)}\right\} \tag{1.4}$$

A typical example for $K$ is given by the singular kernel $K(x, y) = |x - y|^{-(N+sp(x,y))}$. In this case $\circledcirc_K^{p(x,.)} = \left(-\Delta_{p(x,.)}\right)^s$ is the fractional $p(x,.)$—Laplacian defined as

$$\left(-\Delta_{p(x,.)}\right)^s u(x) = p.v.\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p(x,y)-2}\,(u(x)-u(y))}{|x-y|^{N+sp(x,y)}}dy \quad \text{for all } x\in\mathbb{R}^N. \tag{1.5}$$

The birth of fractional calculus was in a letter dated in 1695 when L’Hôpital wrote to Leibniz and asked him about the nth-derivative of the function $f(x) = x$, $\frac{D^n x}{Dx^n}$. What could be the result if $n=\frac{1}{2}$ ?

Leibniz respond: “An apparent paradox from which, one day, useful consequences will be drawn.” After this first discussion between L’Hôpital and Leibniz, fractional calculus becomes above all for big mathematicians and which can be traced back to L’Hôpital (1695), Wallis (1697), Euler (1738), Laplace (1812), Lacroix (1820), Fourier (1822), Abel (1823), Liouville (1832), Riemann (1847), Leibniz (1853), Grunwald (1867), Letnikov (1868) and many others. We refer the reader to [17, 25, 32] and the references therein for a detailed exposition about the history of the classical fractional calculus.

Thanks to these classical definitions, fractional calculus becomes a venerable branch of mathematics in the last century, and fractional operators give more development to the fields of Potential theory, Probability, Hyper singular integrals, Harmonic analysis, Functional analysis, Pseudo-differential operators and Semigroup theory In particular, the fractional Laplacian operator $(-\Delta)^s$, $s \in (0,1)$ is a pseudo-differential operator which has various definitions in different fields: Fourier transform, distributional definition, Bochner’s definition, Balakrishnan’s definition, singular integral definition, quadratic definition, semigroup definition, definition as harmonic extension, and definition as the inverse of Riez potential. We refer to Kwasnicki [22] who collected all these definitions and established the equivalence between them.

In the last decade, great attention has been given to the study of nonlinear nonlocal case. More precisely, the problems involving the fractional $p$—Laplacian operator $(-\Delta)_p^s$. We refer to Di Nezza et al. [24] for a comprehensive introduction to the study of nonlocal problems. Basic properties, embedding theorems and regularity results are established. The paper opens the door to many mathematicians to deal with general problems in the field of partial differential equations. In the context of non homogeneous materials (such that electrorheological fluids and smart fluids), the use of Lebesgue and Sobolev spaces $L^p$ and $W^{s,p}$ seems to be inadequate, which leads to the study of variable exponent functional spaces. The study of problems which involves the $p(.)$-Laplacian and the corresponding elliptic equations constitutes promising a domain of research. The interest in studying such problems was stimulated by their applications in many physical phenomena such as conservation laws, ultramaterials and water waves, optimization, population dynamics, soft thin films, mathematical finance, phases transitions, stratified materials, anomalous diffusion, crystal dislocation, semipermeable membranes, flames propagation, ultra-relativistic limits of quantum mechanics, electrorheological fluid etc. we refer the reader to [6, 24, 30, 31, 36] for details.

Now, what results can be recovered if the $p(.)$—Laplace operator is replaced by the fractional $p(x,.)$—Laplacian of the form $\left(-\Delta_{p(x,.)}\right)^{s}$. In 2017, Kaufmann et al. [19] have introduced the fractional Sobolev spaces with variable exponent $W^{s,q(x),p(x,y)}(\Omega)$. They are established continuous and compact embedding theorems of these spaces into variable exponent Lebesgue spaces with the restriction $p(x, x) < q(x)$, and as applications, they also prove an existence result for nonlocal problems involving the fractional p(x; y)-Laplacian. In [5], Bahrouni et al. presented some further qualitative properties of both on this function space and the related $p(x, .)$—Laplacian operator $\circledcirc_{p(x,.)}$.

Under the restricted condition $p(x, x) < q(x)$, the space $W^{s,q(x),p(x,y)}(\Omega)$ is in fact not a generalization of the typical fractional Sobolev space $W^{s,p}(\Omega)$ with a constant exponent. However, Ho et al. [18] and Azroul et al. [3] provided some fundamental embeddings for the fractional Sobolev space with variable exponent to cover the case $p(x, x) = q(x)$ and their applications such as a priori bounds and multiplicity of solutions of the fractional $p(x, .)$—Laplacian problems. We refer also to [12] in which the authors proved a trace theorem in fractional Sobolev spaces with variable exponents.

The equation presented by Kirchhoff in 1883 (see [20]), of the following form

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2}, \tag{1.6}$$

is an extension of the classical d'Alembert's wave equation by considering the changes in the length of the string during vibrations. In (1.6), $L$ is the length of string, $h$ is the area of the cross section, $E$ is the Young modulus of the material, $\rho$ is the mass density, and $P_0$ is the initial tension. The Kirchhoff 's model takes into account the length changes of the string produced by transverse vibrations. Some interesting results can be found, for example in [8]. On the other hand, Kirchhoff-type boundary value problems model several physical and biological systems where $u$ describes a process which depend on the average of itself, as for example, the population density. We refer the reader to [1, 16, 28] for some related works.

In the nonlocal case, many publications [7, 11, 15, 23, 33, 34] have appeared concerning elliptic systems of Kirchhoff type. Existence and multiplicity results have been investigated. For example, when the exponents $p(.)$ and $q(.)$ are reduced to be constants, the authors have considered in [7] the following boundary problem involving $(p, q)$-Kirchhoff type

$$(P_1)\quad\begin{cases}\left[M\left(\int_\Omega|\nabla u|^p\,dx\right)\right]^{p-1}(-\Delta)_p u=\lambda F_u(x,u,v)+\mu G_u(x,u,v), & \text{in }\Omega,\\ \left[M\left(\int_\Omega|\nabla v|^q\,dx\right)\right]^{q-1}(-)_q v=\lambda F_v(x,u,v)+\mu G_v(x,u,v), & \text{in }\Omega,\\ u=v=0, & \text{on }\partial\Omega,\end{cases}$$

where $\Omega$ is a bounded smooth domain, $\lambda$, $\mu$, $\in [0, +\infty)$, $p > N$, $q > N$, and $(-\Delta)_p$ is the $p$-Laplacian operator $(-\Delta)p(u) = -div(|\nabla u|^{p-2}\,\nabla u)$. Based on Ricceri three critical points theorem [28], they established the existence of three weak solutions.

In [23], using the same approach, Liu and Shi proved the existence of three solutions for a $(p(.), q(.))$-Laplacian system of the form

$$(P_2)\quad \begin{cases} (-\Delta)_{p(x)}u = \lambda F_u(x,u,v) + \mu G_u(x,u,v), & \text{in } \Omega, \\ (-\Delta)_{q(x)}v = \lambda F_v(x,u,v) + \mu G_v(x,u,v), & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases}$$

In our paper, we aim to prove the existence of three positive solutions for problem $(P^K_\omega)$ using the three critical points theorem introduced by Ricceri.

The rest of our paper is organized as follows. In section 2 we briefly review some notations and basic properties of the spaces $L^{p(x)}(\Omega)$, $W^{s,p(x,y)}(\Omega)$ and $W_K^{s,p(x,y)}(\Omega)$. Moreover, we introduce the weighted fractional Sobolev spaces with variable exponent $W^{s,p(x,y)}_{K,\omega}(\Omega)$. In Section 3 we study the existence of three weak solutions for system $(P^K_\omega)$ by using the three critical points theorem.

## 2. Preliminaries and basic assumptions

We will study our problem $(P^K_\omega)$ in the general fractional weighted Sobolev spaces with variable exponent, so we recall some basic properties of these spaces and we need to establish some new embedding results of these spaces into generalized Lebesgue spaces with weight.

### Variable exponent Lebesgue spaces

In this subsection, we recall some properties of variable exponent spaces. For more details, we refer to [13, 14, 21, 26], and the references therein. Let $\Omega$ be a Lipschitz bounded open set in $\mathbb{R}^N$. Set

$$C_+(\bar{\Omega}) = \left\{\gamma \in C(\bar{\Omega}) : \gamma(x) > 1, \forall x \in \bar{\Omega}\right\}.$$

For any $\gamma \in C_+(\bar{\Omega})$, we define the generalized Lebesgue space $L^{\gamma(x)}(\Omega)$ as

$$L^{\gamma(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ is measurable} : \int_\Omega |u(x)|^{\gamma(x)}\, dx < +\infty \right.$$

this space equipped with the *Luxemburg* norm

$$\|u\|_{\gamma(x)} = \inf\left\{ \lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{\gamma(x)} dx \le 1 \right\},$$

is a separable reflexive Banach space.

Let $\hat{\gamma} \in C_+(\Omega)$ be the conjugate exponent of $\gamma$, i.e. $\dfrac{1}{\gamma(x)} + \dfrac{1}{\hat{\gamma}(x)} = 1$. Then we have the following Hölder-type inequality.

**Lemma 2.1.** (*Hölder inequality*). *If* $v \in L^{\hat{\gamma}(x)}(\Omega)$ *and* $v \in L^{\hat{\gamma}(x)}(\Omega)$, *so*

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{\gamma^-} + \frac{1}{\hat{\gamma}^-}\right)\|u\|_{\gamma(x)}\|v\|_{\hat{\gamma}(x)} \le 2\|u\|_{\gamma(x)}\|v\|_{\hat{\gamma}(x)}.$$

The modular of $L^{\gamma(x)}(\Omega)$ is defined by

$$\rho_{\gamma(.)} : L^{\gamma(x)}(\Omega) \to \mathbb{R}$$
$$u \to \rho_{\gamma(.)}(u) = \int_\Omega |u(x)|^{\gamma(x)}\, dx.$$

**Proposition 2.2.** *Let* $u \in L^{\gamma(x)}(\Omega)$ *, then we have*

1. $\|u\|_{\gamma(x)} < 1 \ (resp = 1, > 1) \Leftrightarrow \rho_{\gamma(.)}(u) < 1 (resp = 1, > 1).$
2. $\|u\|_{\gamma(x)} < 1 \Rightarrow \|u\|_{\gamma(x)}^{\gamma^+} \square \rho_{\gamma(.)}(u) \square \|u\|_{\gamma(x)}^{\gamma^-};$
3. $\|u\|_{\gamma(x)} > 1 \Rightarrow \|u\|_{\gamma(x)}^{\gamma^-} \square \rho_{\gamma(.)}(u) \square \|u\|_{\gamma(x)}^{\gamma^+};$

**Proposition 2.3.** If $u$, $u_k \in L^{\gamma(x)}(\Omega)$ and $k \in \mathrm{N}$, then the following assertions are equivalent

1. $\lim_{k\to+\infty} \|u_k - u\|_{\gamma(x)} = 0.$
2. $\lim_{k\to+\infty} \rho_{\gamma(.)}(u_k - u) = 0.$
3. $u_k \to u$ *in measure in* $\Omega$ *and* $\lim_{k\to+\infty} \rho_{\gamma(.)}(u_k) = \rho_{\gamma(.)}(u).$

**Proposition 2.4.** Let $\Omega$ be a bounded open subset of $\mathrm{R}^N$, $\gamma \in C_+(\overline{\Omega})$, then $L^{\gamma(x)}(\Omega)$, $||u||_{\gamma(x)}$ is a reflexive uniformly convex and separable Banach space.

**Proposition 2.5.** Let $\gamma$ , $r \in C_+(\overline{\Omega})$ such that $1 \leq r(x)\gamma(x) \leq +\infty$ for a.e. $x \in \Omega$. Let $u \in L^{\gamma(x)}(\Omega)$, $u \neq 0$. Then

1. $\|u\|_{r(x)\gamma(x)} \square 1 \Rightarrow \|u\|_{r(x)\gamma(x)}^{r^+} \square \||u|^{r(x)}\|_{\gamma(x)} \square \|u\|_{r(x)\gamma(x)}^{r^-},$
2. $\|u\|_{r(x)\gamma(x)} \square 1 \Rightarrow \|u\|_{r(x)\gamma(x)}^{r^-} \square \||u|^{r(x)}\|_{\gamma(x)} \square \|u\|_{r(x)\gamma(x)}^{r^+}.$

In particular, if $r(x) = r$ is constant, then

$$\||u|^r\|_{\gamma(x)} = \|u\|_{r\gamma(x)}^r.$$

Next, let $\omega \in C(\Omega, \mathrm{R})$ with $\omega(x) > 0$. we define the weighted variable exponent Lebesgue space $L_\omega^{\gamma(x)}(\Omega)$ by

$$L_\omega^{\gamma(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ is measurable } : \int_\Omega \omega(x)|u(x)|^{\gamma(x)}\, dx < +\infty \right\},$$

with the norm

$$\|u\|_{\gamma(x),\omega} = \inf \left\{ \lambda > 0 : \int_\Omega \omega(x) \left| \frac{u(x)}{\lambda} \right|^{\gamma(x)} dx \square 1 \right\}.$$

From now on, we suppose that $\omega \in L^1(\Omega)$ and $essinf_{x\in\Omega}\omega(x) = \omega_0 > 0$ . Then obviously $L_\omega^{\gamma(x)}(\Omega)$ is a Banach space. In that context, we refer the reader to [2, 10] for more details.

Moreover, the weighted modular on $L_{\omega}^{\gamma(x)}(\Omega)$ is the mapping $\rho_{\gamma(x),\omega} : L_{\omega}^{\gamma(x)}(\Omega) \to \mathbb{R}$ defined as follows

$$\rho_{\gamma,\omega}(u) = \int_{\Omega} \omega(x) |u(x)|^{\gamma(x)}\, dx.$$

As a simple example of $\omega$, we can take $\omega(x) = (1 + |x|)^{\alpha}$ for all $x \in \Omega$ where $\alpha \in \mathsf{R}^{+}$ or $\omega(x) = (1 + |x|)^{\alpha(x)}$ with $\alpha \in C_{+}(\overline{\Omega})$.

The following proposition is similar to Proposition 2.2, and it follows easily from the definition of $\|u\|_{\gamma(x),\omega}$ and $\rho_{\gamma,\omega}$.

**Proposition 2.6.** *Let u,* $\{u_n\} \subset L_{\omega}^{\gamma(x)}(\Omega)$*, then we have*

1. $\|u\|_{\gamma(x),\omega} < 1 \; (resp. = 1, > 1) \Leftrightarrow \rho_{\gamma(x),\omega}(u) < 1 \; (resp. = 1, > 1),$
2. $\|u\|_{\gamma(x),\omega} < 1 \Rightarrow \|u\|_{\gamma(x),\omega}^{\gamma^{+}} \leq \rho_{\gamma(x),\omega}(u) \leq \|u\|_{\gamma(x),\omega}^{\gamma^{-}},$
3. $\|u\|_{\gamma(x),\omega} > 1 \Rightarrow \|u\|_{\gamma(x),\omega}^{\gamma^{-}} \leq \rho_{\gamma(x),\omega}(u) \leq \|u\|_{\gamma(x),\omega}^{\gamma^{+}},$
4. $\lim\limits_{n\to+\infty} \|u_n\|_{\gamma(x),\omega} = 0 \quad \Leftrightarrow \quad \lim\limits_{n\to+\infty} \rho_{\gamma(x),\omega}(u_n) = 0,$
5. $\lim\limits_{n\to+\infty} \|u_n\|_{\gamma(x),\omega} = \infty \quad \Leftrightarrow \quad \lim\limits_{n\to+\infty} \rho_{\gamma(x),\omega}(u_n) = \infty.$

### Fractional Sobolev spaces with variable exponents

In this subsection, we recall the definition and some results on general fractional Sobolev spaces with variable exponent introduced in [4], and we prove continuous and compact embeddings of these spaces into weighted Lebesgue spaces with variable exponent $L_{\omega}^{\gamma}(\Omega)$. Moreover, we introduce the general weighted fractional Sobolev spaces with variable exponents $W_{K,\omega}^{s,p(x,y)}(\Omega)$.

We define the usual fractional Sobolev space with variable exponent as:

$$\begin{aligned} W &= W^{s,p(x,y)}(\Omega) \\ &= \left\{ u \in L^{\bar{p}(x)}(\Omega) : \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\mu^{p(x,y)} |x-y|^{N+sp(x,y)}} dxdy < \infty, \; for \; some \; \mu > 0 \right\}, \end{aligned}$$

which we endowed with the Luxemberg norm

$$\|u\|_{s,p(x,y)} = \|u\|_{\bar{p}(x)} + [u]_{s,p(x,y)},$$

where $[u]_{s,p(x,y)}$ is a Gagliardo semi-norm with variable exponent which is given by

$$[u]_{s,p(x,y)} = \inf \left\{ \mu > 0 : \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\mu^{p(x,y)} |x-y|^{N+sp(x,y)}} dxdy \leq 1 \right\}.$$

The space $(W, \|\,\|_{s,p(x,y)})$ is separable reflexive banach space.

We define the general fractional Sobolev space with variable exponent as in [4] as follows

$$W_K^{s,p(x,y)}(\Omega) = \left\{ u \in L^{\bar{p}(x)}(\Omega) : \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\mu^{p(x,y)}} K(x,y) dxdy < \infty, \; for \; some \; \mu > 0 \right\},$$

The norm in $W_K^{s,p(x,y)}(\Omega)$ can be defined as follows:

$$\|u\|_{K,p(x,y)} = \|u\|_{\overline{p}(x)} + [u]_{K,p(x,y)},$$

where

$$[u]_{K,p(x,y)} = \inf\left\{\mu > 0 : \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\mu^{p(x,y)}} K(x,y)dxdy \square 1\right\},$$

**Corollary 2.7.**

- $\left(W_K^{s,p(x,y)}(\Omega); \|u\|_{K,p(x,y)}\right)$ is a separable and reflexive uniformly convex space.
- if $\Omega \subset \mathbb{R}^n$ is a *domain of class* $C^{0,1}$*, then* $\left(W_K^{s,p(x,y)}(\Omega); \|u\|_{K,p(x,y)}\right)$ is a *Banach space.*

**Theorem 2.8.** [4] *Let* $\Omega$ *be a Lipschitz bounded open domain in* $\mathbb{R}^n$ *and let* $s \in (0, 1)$*. Let* $p(x, .)$ *satisfies* (1.1) *with* $sp^+ < N$*. Let* $r : \Omega \to (1, +\infty)$ be a *continuous variable exponent such that*

$$1 < r^- < r(x) < p_s^*(x) \text{ for all } x \in \overline{\Omega}$$

*Let* $K : \mathbb{R}^N \times \mathbb{R}^N \to (0, +\infty)$ *be a measurable function satisfying* (1.2)–(1.4). *Then, there exists a constant* $C_0 = C_0(N, s, p, r, \Omega) > 0$ *such that for any* $u \in W_K^{s,p(x,y)}(\Omega)$,

$$\|u\|_{r(x)} \leq C_0 \|u\|_{s,p(x,y)} \leq C_0 \max\left\{1, \tilde{k}_0\right\} \|u\|_{K,p(x,y)}.$$

*That is, the space* $W_K^{s,p(x,y)}(\Omega)$ *is continuously embedded in* $L^{r(x)}(\Omega)$. *Moreover, this embedding is compact.*

**Corollary 2.9.** Let $\Omega$ be a *Lipschitz bounded open domain in* $\mathbb{R}^N$ and $s \in (0,1)$. Let $p \in C_+(\overline{Q})$ *satisfies* (1.1) *with* $sp^+ < N$*. Let* $h \in C_+(\overline{\Omega})$ *with* $1 < h^- < h(x) < p_s^*(x)$ *for all* $x \in \overline{\Omega}$*. Let* $a \in L^{\alpha(x)}(\Omega)$ *where* $\alpha \in C_+(\Omega)$ *satisfying:*

$$\overline{p}(x) \leq n(x) = \frac{\alpha(x)h(x)}{\alpha(x)-1} < p_s^*(x).$$

*Suppose that* $K : \mathbb{R}^N \times \mathbb{R}^N \to (0, +\infty)$ *is a measurable function satisfying* (1.2)–(1.4). *Then, there exists a constant* $C_1 = C_1(N, s, p, h, r, \Omega) > 0$ *such that for any* $u \in W_K^{s,p(x,y)}(Q)$

$$\|u\|_{h(x),a} \leq C_1 \|u\|_{s,p(x,y)} \leq C_1 \max\left\{1, \tilde{k}_0\right\} \|u\|_{K,p(x,y)}.$$

*where* $\tilde{k}_0$ *is a positive constant.*

*That is, the space* $W_K^{s,p(x,y)}(\Omega)$ *is continuously embedded in* $L_a^{h(x)}(\Omega)$*. Moreover, this embedding is compact.*

*Proof.* We shall prove the continuous embedding, we just need to prove that $W_K^{s,p(x,y)}(\Omega) \subset L_a^{h(x)}(\Omega)$. Indeed, since $\overline{p}(x) \square n(x) < p_s^*(x)$ *for all* $x \in \Omega$ , then by the above theorem we have that the embedding $W_K^{s,p(x,y)}(\Omega) \hookrightarrow L^{n(x)}(\Omega)$ is continuous. Thus for all $u \in W_K^{s,p(x,y)}(\Omega)$, we have $u \in L^{n(x)}(\Omega)$. Then

$$\int_\Omega \left(|u|^{h(x)}\right)^{\frac{n(x)}{h(x)}} dx = \int_\Omega |u|^{n(x)}\, dx < +\infty.$$

As a result, $|u|^{h(x)} \in L^{\frac{n(x)}{h(x)}}(\Omega)$. Since $a \in L^{\alpha(x)}(\Omega)$ and $\frac{1}{\alpha(x)} + \frac{h(x)}{n(x)} = 1$, then by Lemma 2.1, we have

$$_{h,a}(u) 2 \|a\|_{(x)} \left\| |u|^{h(x)} \right\|_{\frac{n(x)}{h(x)}} < +\infty. \tag{2.1}$$

Therefore, $u \in L_a^{h(x)}(\Omega)$, that is, the embedding $W_K^{s,p(x,y)}(\Omega), \hookrightarrow L_a^{h(x)}(\Omega)$ is continuous. Next, let $\{u_n\} \subset W_K^{s,p(x,y)}(\Omega)$ such that $u_n \rightharpoonup 0$ in $W_K^{s,p(x,y)}(\Omega)$. Since $W_K^{s,p(x,y)}(\Omega) \hookrightarrow\hookrightarrow L^{n(x)}(\Omega)$, we have that

$$u_n \to 0 \text{ in } L^{n(x)}(\Omega).$$

Then, it follows that $\left\| |u_n|^{h(x)} \right\|_{\frac{n(x)}{h(x)}} \to 0$, as $n \to +\infty$. By equation (2.1), one has $\rho_{h,a}(u_n) \to 0$. From Proposition 2.6, we deduce that

$$\|u_n\|_{h(x),a} \underset{n\to+\infty}{\to} 0.$$

Consequently, the embedding $W_K^{s,p(x,y)}(\Omega) \hookrightarrow L_a^{h(x)}(\Omega)$ is compact.

## 3. Mains Result

### General fractional weighted variable exponent Sobalev spaces

We introduce the general fractional weighted variable exponent Sobolev space as follows:

$$W_{K,\omega}^{s,p(x,y)}(\Omega) = \{ u \in L_\omega^{\bar{p}(x)}(x)(\Omega) : \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\mu^{p(x,y)}} K(x,y)dxdy < \infty, \textit{ for some } \mu > 0 \},$$

which endowed with the norm

$$\|u\|_{K,\omega,p(x,y)} = \|u\|_{W_{K,\omega}^{s,p(x,y)}(\Omega)}$$

$$= \inf \left\{ \gamma > 0 : \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\gamma^{p(x,y)}} K(x,y)dxdy + \int_\Omega w(x) \left| \frac{u(x)}{\gamma} \right|^{\underline{p}(x)} dx \le 1 \right\}.$$

Moreover, the space $(W_{K,\omega}^{s,p(x,y)}(\Omega), \|.\|_{K,\omega,p(x,y)})$ is a separable reflexive Banach space. For any $u \in W_K^{s,p(x,y)}(\Omega)$, we set

$$\rho_{K,p(.,.)}^{\omega}(u) = \int_Q |u(x)-u(y)|^{p(x,y)}\, K(x,y)dxdy + \int_\Omega w(x)|u(x)|^{p(x)}\, dx,$$

which is a modular on $\overline{W}_K^{s,p(x,y)}(\Omega)$, and it satisfies the following inequalities

**Proposition 3.1.** Let $u$, $\{u_n\} \subset W^{s,p(x,y)}_{K,\omega}(\Omega)$, *then we have*

1. $\|u\|_{K,\omega,p(x,y)} < 1\ (resp. = 1, > 1) \Leftrightarrow \rho^{\omega}_{K,p(.,.)}(u) < 1\ (resp. = 1, > 1)$,
2. $\|u\|_{K,\omega,p(x,y)} < 1 \Rightarrow \|u\|^{p^+}_{K,\omega,p(x,y)} \leq \rho^{\omega}_{K,p(.,.)}(u) \leq \|u\|^{p^-}_{K,\omega,p(x,y)}$,
3. $\|u\|_{K,\omega,p(x,y)} > 1 \Rightarrow \|u\|^{p^-}_{K,\omega,p(x,y)} \leq \rho^{\omega}_{K,p(.,.)}(u) \leq \|u\|^{p^+}_{K,\omega,p(x,y)}$,
4. $\lim\limits_{n\to+\infty} \|u_n - u\|_{K,\ ,p(x,y)} = 0 \quad \Leftrightarrow \quad \lim\limits_{n\to+\infty}\ {}_{K,p(.,.)}(u_n - u) = 0.$

*Proof.* The proof is similar to [4, Lemma 4].

**Lemma 3.2.** Let $\Omega$ *be a Lipschitz bounded open domain in* $\mathbb{R}^N$ *and* $s \in (0,1)$. Let $p \in \underline{C}_+(\overline{Q})$ *satisfies* (1.1) *with* $sp^+ < N$. *Let* $h \in C_+(\Omega)$ *with* $1 < h^- < h(x) < p^*_s(x)$ *for all* $x \in \Omega$. *Let* $a \in L^{\alpha(x)}(\Omega)$ *with* $\alpha$ *satisfying:*

$$\bar{p}(x) \leq n(x) = \frac{\alpha(x)h(x)}{\alpha(x)-1} < p^*_s(x).$$

*Suppose that* $K : \mathbb{R}^N \times \mathbb{R}^N \to (0, +\infty)$ *is a measurable function satisfying* (1.2)–(1.4). *Then, there exists a constant* $C_2 = C_2(N, s, p, h, r, \omega, \Omega) > 0$ *such that for any* $u \in W^{s,p(x,y)}_{K,\omega}(\Omega)$

$$\|u\|_{L_a^{h(x)}(\Omega)} \leq C_1 \max\left\{1, \tilde{k}_0\right\} \|u\|_{K,p(x,y)} \leq C_2 \max\left\{1, \tilde{k}_0\right\} \|u\|_{K,\omega,p(x,y)}.$$

*That is, the embedding from* $W^{s,p(x,y)}_{K,\omega}(\Omega)$ *into* $L^{h(x)}_a(\Omega)$ *is continuous and compact.*

*Proof.* Since $essinf_{x\in\Omega}\omega(x) = \omega_0 > 0$, then $L^{\bar{p}(x)}_{\omega}(\Omega) \hookrightarrow L^{\bar{p}(x)}(\Omega)$. Thus $W^{s,p(x,y)}_{K,\omega}(\Omega) \hookrightarrow W^{s,p(x,y)}_{K}(\Omega)$. By the use of Corollary 2.9 we have

$$W^{s,p(x,y)}_{K,\omega}(\Omega) \hookrightarrow W^{s,p(x,y)}_{K}(\Omega) \hookrightarrow\hookrightarrow L^{h(x)}_a(\Omega)$$

Since the latter embedding is compact, then the embedding $W^{s,p(x,y)}_{K,\omega}(\Omega)$, $\hookrightarrow L^{h(x)}_a(\Omega)$ is also compact. ,

We define $E = W^{s,p(x,y)}_{K,\omega}(\Omega) \times W^{s,q(x,y)}_{K,\omega}(\Omega)$ as the solution space corresponding to $\left(\mathrm{P}^K_\omega\right)$, equipped with the norm $\|(u,v)\|_E = \|u\|_{K,\omega,p(x,y)} + \|v\|_{K,\omega,q(x,y)}$. Clearly $(E, \|(u,v)\|_E)$ is a reflexive, separable Banach space.

## Existence result

Before starting our results, we introduce some assumptions that we will use to prove that problem $\left(\mathrm{P}^K_\omega\right)$ has at least three weak solutions by using Theorem 3.6. We assume the following conditions hold:

$(M_0)$ $M : \mathbb{R}^+ \to \mathbb{R}$ is nondecreasing continuous function and there exists a constant $m_0 > 0$ such that

$$M(t) \geq m_0, \qquad \forall t \geq 0$$

$(H_0)$ $F(x, 0, 0) = 0$, for a.e. $x \in \Omega$,

$(H_1)$ there exists $d \geq 0$ such that $F(x, s, t) \geq 0$ for a.e. $x \in \Omega$ and all $s, t \in [0, d] \times [0, d]$,

$(H_2)$ There are functions $\alpha(.), \beta(.) \in C_+(\bar{\Omega})$, $1 < \alpha^- < \alpha^+ < p^-$ and $1 < \beta^- < \beta^+ < q^-$ such that

$$F(x, s, t) < C(1 + \alpha(x)\,|s|^{\alpha(x)} + b(x)\,|t|^{\beta(x)}), \qquad \text{for a.e. } x \in \Omega,$$

where

$$a(x) \in L^{r_1(.)}(\Omega), \quad p(x) \leq \frac{r_1(x)\alpha(x)}{r_1(x)-1} < p_s^*(x),$$

and

$$b(x) \in L^{r_2(.)}(\Omega), \quad q(x) \leq \frac{r_2(x)\beta(x)}{r_2(x)-1} < q_s^*(x).$$

$(H_3)$ There exist $p_1, q_1 \in C_+(\Omega)$, $p^+ < p_1^- < p_1^+ < p_s^*(x)$ and $q^+ < q_1^- < q_1^+ < q_s^*(x)$ such that

$$\limsup_{(s,t)\to(0,0)} \frac{\sup_{x\in\Omega} F(x,s,t)}{a(x)|s|^{p_1(x)} + b(x)|t|^{q_1(x)}} < +\infty.$$

**Definition 3.3.** For $sp^+; sq^+ < N$, we denote by A the class of functions $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ such that $F_u = \frac{\partial F}{\partial u}$ and $F_v = \frac{\partial F}{\partial v}$ are two Caratheodory functions and

$$\sup_{(x,s,t)\in\Omega\times\mathbb{R}\times\mathbb{R}} \frac{|F_s(x,s,t)|}{1+|s|^{\eta_1(x)-1}+|t|^{\eta_2-1}} < +\infty,$$

$$\sup_{(x,s,t)\in\Omega\times\mathbb{R}\times\mathbb{R}} \frac{|F_t(x,s,t)|}{1+|s|^{\eta_1(x)-1}+|t|^{\eta_2-1}} < +\infty,$$

for all $\eta_1(.) \in (p^+, p_s^*(x))$ and $\eta_2(.) \in (q^+, q_s^*(x))$.

**Definition 3.4.** We say that $(u,v) \in E$ is a weak solution of $\left(P_\omega^K\right)$ if

$$\begin{aligned}
&M(I_\omega^{K,p(x,y)}(u)) \times \left[\int_Q |u(x)-u(y)|_{p(x,y)-2}\,(u(x)-u(y))(\varphi(x)-\varphi(y))K(x,y)\,dxdy \right. \\
&\qquad \left. + \int_\Omega \omega(x)|u|^{p(x)-2}\,u(x)\varphi(x)\,dx\right] \\
&+ M(I_\omega^{K,q(x,y)}(v)) \times \left[\int_Q |v(x)-v(y)|_{q(x,y)-2}\,(v(x)-v(y))(\psi(x)-\psi(y))K(x,y)\,dxdy \right. \\
&\qquad \left. + \int_\Omega \omega(x)|v|^{q(x)-2}\,u(x)\psi(x)\,dx\right] \\
&= \lambda\int_\Omega F_u(x,u,v)\varphi(x)\,dx + \mu\int_\Omega G_u(x,u,v)\varphi(x)\,dx + \lambda\int_\Omega F_v(x,u,v)\psi(x)\,dx + \mu\int_\Omega G_v(x,u,v)\psi(x)\,dx,
\end{aligned}$$

for all $(\varphi, \psi) \in E$.

Our result is formulated as follows.

**Theorem 3.5.** Let $\Omega$ be a *bounded open set of* $\mathbb{R}^N$ and let $s \in (0,1)$. Let $p,q : \bar{Q} \to (1,+\infty)$ be two *continuous variable exponents with* $sp^+ < N$ *and* $sq^+ < N$ *for all* $(x,y) \in \bar{Q}$ *satisfying* (1.1) *and* $F \in \mathrm{A}$. *Assume that the assumptions* $(M_0)$ *and* $(H_0) - (H_3)$ *hold. Then, there exists an open* $\Lambda \subset (0, +\infty)$ *and a positive real number* $\delta > 0$ *such that, for each* $\lambda \in \Lambda$ *and every function* $\Psi \in \mathrm{A}$, *there exists* $\mu^* > 0$ *such that for any* $\mu \in [0, \mu^*]$, *system* $\left(\mathrm{P}_{\omega}^{K}\right)$ *has at least three weak solutions whose norms are less than* $\delta$.

In the literature, different versions of the Three Critical Points Theorem are established, we refer the reader to [27, 29] for example. In order to prove our existence results, we will use the following result proved in [28] that, on the basis of [9, Theorem 1], is equivalently stated as follows.

**Theorem 3.6.** *Let X be a reflexive real Banach space. Let* $\Phi : X \to \mathbb{R}$ *be a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on* $X^*$ *and* $\Phi$ *is bounded on each bounded subset of X;* $J : X \to \mathbb{R}$ *be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that:*

i. $\lim\limits_{\|u\|_X \to +\infty} \Phi(u) + \lambda J(u) = +\infty$ *for all* $\lambda > 0$;

ii. *there exist* $\gamma \in \mathbb{R}$ and $u_0, u_1 \in X$ *such that*

$$\Phi(u_0) < \gamma < \Phi(u_1);$$

iii. $$\inf_{u \in \Phi^{-1}((-\infty,\gamma])} J(u) > \frac{\left(\Phi(u_1) - \gamma\right) J(u_0) + \left(\gamma - \Phi(u_0)\right) J(u_1)}{\Phi(u_1) - \Phi(u_0)}.$$

*Then there exist an open interval* $\Lambda \subset (0, \infty)$ and a *positive real number* $\delta$ *such that, for each* $\lambda \in \Lambda$ *and every continuously Gâteaux differentiable functional* $\Psi : X \to \mathbb{R}$ *with compact derivative, there exists* $\mu^* > 0$ *such that for each* $\mu \in [0, \mu^*]$, *the equation*

$$\Phi'(u) + \lambda J'(u) + \mu \Psi'(u) = 0$$

*has at least three solutions in X whose norms are less than* $\delta$.

It is clear that problem $\left(\mathrm{P}_{\omega}^{K}\right)$ has a variational structure. The energy functional corresponding to problem $\left(\mathrm{P}_{\omega}^{K}\right)$ is defined as $H : E \to \mathbb{R}$

$$H(u,v) = \Phi(u,v) + \lambda J(u,v) + \mu \Psi(u,v),$$

where

$$J(u,v) = -\int_{\Omega} F(x,u,v)dx \quad ; \quad \Psi(u,v) = -\int_{\Omega} G(x,u,v)dx$$

and $\quad \Phi(u,v) = \hat{M}(I^{\omega}_{K,p(x,y)}(u)) + \hat{M}(I^{\omega}_{K,q(x,y)}(v))$

with $\quad \hat{M}(t) = \int_0^t M(s)ds, \qquad \forall t \geq 0$

In the following auxiliary lemmas are useful to prove our existence theorem.

**Lemma 3.7.** *the following proprieties hold true:*

i. *The functional* $\Phi \in C^1\left(E, \mathbb{R}^n\right)$ *and its Gateaux derivative* $\Phi' : E \to E^*$ *is given by:*
$\left\langle \Phi'(u,v),(\varphi,\psi) \right\rangle = \left\langle \Phi_p(u),\varphi \right\rangle + \left\langle \Phi_q(u),\psi \right\rangle,$
*where*

$$\begin{aligned}&\left\langle \Phi_p(u),\varphi \right\rangle\\ &= M(I^{\omega}_{K,p(x,y)}(u)) \times \Big[\iint_Q |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))K(x,y)dxdy\\ &\quad + \int_\Omega \omega(x)|u|^{\bar{p}(x)-2}u(x)\varphi(x)dx\Big]\end{aligned}$$

*and*

$$\begin{aligned}&\left\langle \Phi_q(u),\psi \right\rangle\\ &= M(I^{\omega}_{K,q(x,y)}(u)) \times \Big[\iint_Q |v(x)-v(y)|^{q(x,y)-2}(v(x)-v(y))(\psi(x)-\psi(y))K(x,y)dxdy\\ &\quad + \int_\Omega \omega(x)|v|^{\bar{q}(x)-2}u(x)\psi(x)dx\Big].\end{aligned}$$

ii. *The functional* $\Phi$ *is sequential ly weakly lower semi-continuous.*

iii. *The functional* $\Phi$ *bounded on each bounded subset of E.*

*Proof.* i. By a standard argument, we have that $\Phi$ is differentiable and its Gateaux derivative is given by

$$\begin{aligned}&\Phi'(u,v),(\varphi,\psi)\\ &= M(I^{\omega}_{K,p(x,y)}(u)) \times \Big[\iint_Q |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))K(x,y)dxdy\\ &\qquad + \int_\Omega \omega(x)|u|^{\bar{p}(x)-2}u(x)\varphi(x)dx\Big]\\ &+ M(I^{\omega}_{K,q(x,y)}(v)) \times \Big[\iint_Q |v(x)-v(y)|^{q(x,y)-2}(v(x)-v(y))(\psi(x)-\psi(y))K(x,y)dxdy\\ &\qquad + \int_\Omega \omega(x)|v|^{\bar{q}(x)-2}v(x)\psi(x)dx\Big],\end{aligned}$$

for all $(u, v)$, $(\phi, \psi) \in E$. Now, we prove that $\Phi'$ is continuous. The continuity of $I^{\omega}_{K,p(x,y)}$ and of $I^{\omega}_{K,q(x,y)}$ is similar to the continuity of the $p(x, .)$ and $q(x, .)$-Laplacian operators, and since $M$ is continuous, it remains to show that $\left(I^{\omega}_{K,p(x,y)}\right)'$ and $\left(I^{\omega}_{K,q(x,y)}\right)'$ are continuous.

We define

$$L_1(u) = \int_Q \frac{1}{p(x,y)}|u(x)-u(y)|^{p(x,y)}K(x,y)\,dxdy$$

$$\text{and} \quad L_2(u) = \int_\Omega \omega(x)\frac{|u(x)|^{\bar{p}(x)}}{\bar{p}(x)}dx$$

Then,

$$I^{\omega}_{K,p(x,y)}(u) = L_1(u) + L_2(u) \text{ and } \left(I^{\omega}_{K,p(x,y)}\right)'(u) = L_1'(u) + L_2'(u). \qquad (3.1)$$

Let $u_n \subset W_{K,\omega}^{s,p(x,y)}(\Omega)$ be a sequence such that $u_n$ converges to $u$ in $W_{K,\omega}^{s,p(x,y)}(\Omega)$ then we have

$$\begin{aligned}\left\langle L_1'(u_n) - L_1'(u), \varphi \right\rangle &= \int_Q \Big[ |u_n(x) - u_n(y)|^{p(x,y)-2}(u_n(x) - u_n(y)) - |u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y)) \Big] \\ &\quad \times (\varphi(x) - \varphi(y)) K(x,y)\, dxdy.\end{aligned}$$

Set

$$H_n(x,y) = |u_n(x) - u_n(y)|^{p(x,y)-2}(u_n(x) - u_n(y)) K(x,y)^{\frac{1}{\hat{p}(x,y)}} \in L^{\hat{p}(x,y)}(Q),$$

$$H(x,y) = |u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y)) K(x,y)^{\frac{1}{\hat{p}(x,y)}} \in L^{\hat{p}(x,y)}(Q),$$

$$\bar{\varphi}(x,y) = (\varphi(x) - \varphi(y)) K(x,y)^{\frac{1}{p(x,y)}} \in L^{p(x,y)}(Q),$$

where $\hat{p} : Q \to (1, +\infty)$ is the conjugate exponent of $p$, that is, $\dfrac{1}{\hat{p}(x,y)} + \dfrac{1}{p(x,y)} = 1.$

By Holder inequality, we get

$$\left\langle L_1'(u_n) - L_1'(u), \varphi \right\rangle \leq \|H_n(x,y) - H(x,y)\|_{\hat{p}(x,y)} \, \|\bar{\varphi}(x,y)\|_{p(x,y)}.$$

Thus

$$\|L_1'(u_n) - L_1'(u)\|_{\left(W_K^{s,p(x,y)}(\Omega)\right)^*} \leq \|H_n(x,y) - H(x,y)\|_{\hat{p}(x,y)}.$$

Now, let

$$t_n(x,y) = (u_n(x) - u_n(y)) K(x,y)^{\frac{1}{p(x,y)}} \in L^{p(x,y)}(Q),$$

and
$$t(x,y) = (u(x) - u(y)) K(x,y)^{\frac{1}{p(x,y)}} \in L^{p(x,y)}(Q).$$

Since $u_n \to u$ in $W_K^{s,p(x,y)}(\Omega)$. Then $t_n \to t$ in $L^{p(x,y)}(Q)$.
Hence, for a subsequence of $(t_n)_{n\geq 0}$, we get
$t_n(x,y) \to t(x,y)$ a.e. in $Q$ and $\exists g \in L^{p(x,y)}(Q)$ such that

$$|t_n(x,y)| \leq g(x,y).$$

So, we have

$$H_n(x,y) \to H(x,y) \ a.e. \ in Q,$$

and
$$H_n(x,y)| \leq |t_n(x,y)|^{p(x,y)-1} \leq |g(x,y)|^{p(x,y)-1}.$$

Then, by the dominated convergence theorem, we deduce that

$$H_n(x,y) \to H(x,y) \ in \ L^{\hat{p}(x,y)}(Q),$$

Consequently

$$L_1^{'}(u_n) \to L_1^{'}(u) \, in \left(W_K^{s,p(x,y)}(\Omega)\right)^*.$$

By the same argument, we show that

$$L_2^{'}(u_n) \to L_2^{'}(u) \, in \left(L_\omega^{p(x)}(\Omega)\right)^*.$$

Then, we deduce that $\left(I^{\omega}_{K,p(x,y)}\right)'$ is continuous.

Using the same argument we conclude also that $\left(I^{\omega}_{K,q(x,y)}\right)'$ is continuous.

ii. Since $t \to |t|^{p(x,y)}$ is convex, we have that $I^{\omega}_{K,p(x,y)}$ is convex on $W^{s,p(x,y)}_{K,\omega}(\Omega)$ and $I^{\omega}_{K,q(x,y)}$ is also convex on $W^{s,q(x,y)}_{K,\omega}(\Omega)$. Let $(u_n, v_n) \subset E$ be a sequence such that $u_n \rightharpoonup u$ in $W^{s,p(x,y)}_{K,\omega}(\Omega)$ and $v_n \rightharpoonup v$ in $W^{s,q(x,y)}_{K,\omega}(\Omega)$. Then, by the convexity of $I^{\omega}_{K,p(x,y)}$, we have

$$I^{\omega}_{K,p(x,y)}(u_n) - I^{\omega}_{K,p(x,y)}(u) \geq \left\langle I^{\omega}_{K,p(x,y)}{}'(u), u_n - u \right\rangle.$$

Hence, by passing to limit, we get

$$I^{\omega}_{K,p(x,y)}(u) \leq \liminf_{n\to+\infty} I^{\omega}_{K,p(x,y)}(u_n).$$

Thus, $I^{\omega}_{K,p(x,y)}$ is sequentially weakly lower semi-continous. As well, $I^{\omega}_{K,q(x,y)}$ is also sequentially weakly lower semi-continous. Additionally, by the continuity and monotonicity of the function $t \mapsto \hat{M}(t)$, we get

$$\begin{aligned}\liminf_{n\to+\infty} \Phi(u_n, v_n) &\geq \liminf_{n\to+\infty} \hat{M}(I^{\omega}_{K,p(x,y)}(u_n)) + \liminf_{n\to+\infty} \hat{M}(I^{\omega}_{K,q(x,y)}(v_n)) \\ &\geq \hat{M}(\liminf_{n\to+\infty}(I^{\omega}_{K,p(x,y)}(u_n))) + \hat{M}(\liminf_{n\to+\infty}(I^{\omega}_{K,q(x,y)}(v_n))) \\ &\geq \hat{M}(I^{\omega}_{K,p(x,y)}(u)) + \hat{M}(I^{\omega}_{K,q(x,y)}(v)) = \Phi(u,v),\end{aligned}$$

that is, $\Phi$ is sequentially weakly lower semi-continous.

(iii) Obviously, $\Phi$ is bounded on each bounded subset.

**Lemma 3.8.** The *functional* $\Phi' : E \to E^*$ is a *homeomorphism.*

*Proof.* we show that the operator $\Phi' : E \to E^*$ is invertible on $E$. By Minty-Browder Theorem (see. [35]), it suffices to prove that $\Phi'$ is strictly monotone, hemicontinuous and coercive in the sense of monotone operators. So, let $(u,v) \in E$, with $u \neq v$ and $\lambda, \mu \in [0,1]$ with $\lambda + \mu = 1$, since $I^{\omega}_{K,p(x,y)}$ is strictly convexe and $\hat{M}$ is strictly nondecreasing, then

$$\begin{aligned}\hat{M}(I^{\omega}_{K,p(x,y)}(\lambda u + \mu v)) &< \hat{M}(\lambda I^{\omega}_{K,p(x,y)}(u) + \mu I^{\omega}_{K,p(x,y)}(v)) \\ &\leq \lambda \hat{M}(I^{\omega}_{K,p(x,y)}(u)) + \mu \hat{M}(I^{\omega}_{K,p(x,y)}(v)).\end{aligned}$$

This shows that is $\hat{M}$ is strictly convex.

Now, let $(u_1,v_2)$, $(u_2, v_2) \in E$ with $(u_1,v_1) \neq (u_2,v_2)$ and $\lambda, \mu$ € $[0,1]$ with $\lambda + \mu = 1$, by the strict convexity of $\hat{M}$, we have

$$\begin{aligned}\Phi(\lambda(u_1,v_1)+\mu(u_2,v_2)) &= \Phi(\lambda u_1+\mu u_2, \lambda v_1+\mu v_2)\\ &= \hat{M}(I^{\omega}_{K,p(x,y)}(\lambda u_1+\mu u_2)) + \hat{M}(I^{\omega}_{K,q(x,y)}(\lambda v_1+\mu v_2))\\ &< \lambda\hat{M}(I^{\omega}_{K,p(x,y)}(u_1)) + \mu\hat{M}(I^{\omega}_{K,p(x,y)}(u_2))\\ &\quad + \lambda\hat{M}(I^{\omega}_{K,q(x,y)}(v_1)) + \mu\hat{M}(I^{\omega}_{K,q(x,y)}(v_2))\\ &= \lambda\left[\hat{M}(I^{\omega}_{K,p(x,y)}(u_1)) + \hat{M}(I^{\omega}_{K,q(x,y)}(v_1))\right]\\ &\quad + \mu\left[\hat{M}(I^{\omega}_{K,p(x,y)}(u_2)) + \hat{M}(I^{\omega}_{K,p(x,y)}(v_2))\right]\\ &= \lambda\Phi(u_1,v_1)+\mu\Phi(u_2,v_2).\end{aligned}$$

Therefore, $\Phi$ is strictly convex, so by proposition 25.10 in [35], $\Phi'$ is strictly monotone. Let $(u,v) \in E$ such that $\|u\|_{K,\omega,p(x,y)}, \|v\|_{K,\omega,q(x,y)} > 1$ by Proposition 3.1, we have

$$\begin{aligned}\frac{\langle \Phi'(u,v),(u,v)\rangle}{\|(u,v)\|_E} &= \frac{M(I^{\omega}_{K,p(x,y)}(u))\,\rho^{\omega}_{K,p(x,y)}(u) + M(I^{\omega}_{K,q(x,y)}(v))\,\rho^{\omega}_{K,q(x,y)}(v)}{\|(u,v)\|_E}\\ &\geq \frac{m_0\left(\rho^{\omega}_{K,p(x,y)}(u)) + \rho^{\omega}_{K,q(x,y)}(v))\right)}{\|(u,v)\|_E}\\ &\geq \frac{m_0\left(\|u\|^{p^-}_{K,\omega,p(x,y)} + \|v\|^{q^-}_{K,\omega,q(x,y)}\right)}{\|(u,v)\|_E}\\ &\geq \frac{m_0\left(\|u\|_{K,\omega,p(x,y)} + \|v\|_{K,\omega,q(x,y)}\right)^{\min(p^-,q^-)}}{\|(u,v)\|_E}\\ &\geq \frac{m_0\,\|(u,v)\|_E^{\min(p^-,q^-)}}{\|(u,v)\|_E}\\ &\geq m_0\,\|(u,v)\|_E^{\min(p^-,q^-)-1}.\end{aligned}$$

Hence,

$$\lim_{\|(u,v)\|_E \to +\infty} \frac{\Phi'(u,v),(u,v)}{\|(u,v)\|_E} = +\infty,$$

and $\Phi'$ is coercive. Since $\Phi \in C^1(E, \mathbb{R})$, then $\Phi'$ is hemicontinuous. Hence, in the light of Minty-Browder Theorem, $\Phi'$ is surjective. By the monotonicity, $\Phi'$ is aloso an injection. Thus $\Phi'$ is invertible and $\Phi'^{-1} : E^* \to E$ is bounded. It remains to show that $\Phi'^{-1}$ is continuous. Let us first prove that $\Phi'^{-1}$ satisfies the following property:

(S): If $(u_n, v_n) \rightharpoonup (u, v)$ in $E$ and $\Phi'(u_n, v_n) \to \Phi'(u, v)$, then $(u_n, v_n) \to (u, v)$. Indeed, let $(u_n, v_n) \sim (u, v)$ in $E$ and $\Phi'(u_n, v_n) \to \Phi'(u, v)$ in $E^*$. Then, using the compact embedding in Lemma 3.2, we get $u_n(x) \to u(x)$ a.e. $x \in \Omega$ and $v_n(x) \to v(x)$ a.e. $x \in \Omega$. By Fatou's lemma we obtain

$$\begin{aligned}&\liminf_{n\to+\infty}\left[\int_Q |u_n(x)-u_n(y)|^{p(x,y)} K(x,y)dxdy + \int_\Omega \omega(x)|u_n(x)|^{\bar{p}(x)}dx\right]\\ &\geq \int_Q |u(x)-u(y)|^{p(x,y)} K(x,y)dxdy + \int_\Omega \omega(x)|u(x)|^{\bar{p}(x)}dx, \qquad (3.2)\end{aligned}$$

and

$$\begin{aligned}\liminf_{n\to+\infty}&\left[\int_Q |v_n(x)-v_n(y)|^{q(x,y)}K(x,y)dxdy+\int_\Omega \omega(x)|v_n(x)|^{\underline{q}(x)}dx\right]\\ &\geq \int_Q |v(x)-v(y)|^{q(x,y)}K(x,y)dx+\int_\Omega \omega(x)|v(x)|^{q(x)}dx. \qquad (3.3)\end{aligned}$$

On the other hand,we have

$$\begin{aligned}\lim_{n\to+\infty}\left[\left\langle \Phi_p(u_n),u_n-u\right\rangle+\left\langle \Phi_q(v_n),v_n-v\right\rangle\right]&=\lim_{n\to+\infty}\left\langle \Phi'(u_n,v_n),(u_n,v_n)-(u,v)\right\rangle\\ &=\lim_{n\to+\infty}\Phi'\left\langle (u_n,v_n)-\Phi'(u,v),(u_n,v_n)-(u,v)\right\rangle=0.\end{aligned}$$

Using young’s inequality we get

$$\begin{aligned}o_n&=\left\langle \Phi_p(u_n),u_n-u\right\rangle+\left\langle \Phi_q(v_n),v_n-v\right\rangle\\ &=M(I^{\omega}_{K,p(x,y)}(u_n))\left[\int_Q |u_n(x)-u_n(y)|^{p(x,y)}K(x,y)dxdy+\int_\Omega \omega(x)|u_n(x)|^{\bar{p}(x)}dx\right.\\ &\quad-\int_Q |u_n(x)-u_n(y)|^{p(x,y)-2}(u_n(x)-u_n(y))(u(x)-u(y))K(x,y)dxdy\\ &\quad\left.-\int_\Omega \omega(x)|u_n(x)|^{\bar{p}(x)-2}u_n(x)u(x)dx\right]\\ &\quad+M(I^{\omega}_{K,q(x,y)}(v_n))\left[\int_Q |v_n(x)-v_n(y)|^{q(x,y)}K(x,y)dxdy+\int_\Omega \omega(x)|v_n(x)|^{\bar{q}(x)}dx\right.\\ &-\int_Q |v_n(x)-v_n(y)|^{q(x,y)-2}(v_n(x)-v_n(y))(v(x)-v(y))K(x,y)dxdy\\ &\quad\left.-\int_\Omega \omega(x)|u_n(x)|^{\bar{q}(x)-2}v_n(x)v(x)dx\right]\\ &\geq M(I^{\omega}_{K,p(x,y)}(u_n))\left[\int_Q |u_n(x)-u_n(y)|^{p(x,y)}K(x,y)dxdy+\int_\Omega \omega(x)|u_n(x)|^{\bar{p}(x)}dx\right.\\ &\quad\left.-\int_Q |u_n(x)-u_n(y)|^{p(x,y)-1}\left|u(x)-u(y)\right|K(x,y)dxdy-\int_\Omega \omega(x)|u_n(x)|^{\underline{p}(x)-1}|u(x)|dx\right]\\ &\quad+M(I^{\omega}_{K,q(x,y)}(v_n))\left[\int_Q |v_n(x)-v_n(y)|^{q(x,y)}K(x,y)dxdy+\int_\Omega \omega(x)|v_n(x)|^{\bar{q}(x)}dx\right.\\ &\quad\left.-\int_Q |v_n(x)-v_n(y)|^{q(x,y)-1}\left|v(x)-v(y)\right|K(x,y)dxdy-\int_\Omega \omega(x)|v_n(x)|^{\underline{q}(x)-1}|v(x)|dx\right]\\ &\geq M(I^{\omega}_{K,p(x,y)}(u_n))\left[\frac{p^- - p^+ +1}{p^-}\int_Q |u_n(x)-u_n(y)|^{p(x,y)}K(x,y)dxdy\right.\\ &\quad+\frac{p^- - p^+ +1}{p^-}\int_\Omega \omega(x)|u_n(x)|^{\bar{p}(x)}dx-\frac{1}{p^-}\int_Q |u(x)-u(y)|^{p(x,y)}K(x,y)dxdy\\ &\quad\left.-\frac{1}{p^-}\int_\Omega \omega(x)|u(x)|^{p(x)}dx\right]\\ &\quad+M(I^{\omega}_{K,q(x,y)}(v_n))\left[\frac{q^- - q^+ +1}{q^-}\int_Q |v(x)-v(y)|^{q(x,y)}K(x,y)dxdy\right.\\ &\quad+\frac{q^- - q^+ +1}{q^-}\int_\Omega \omega(x)|v_n(x)|^{\bar{q}(x)}dx-\frac{1}{q^-}\int_Q |v(x)-v(y)|^{q(x,y)}K(x,y)dxdy\\ &\quad\left.-\frac{1}{q^-}\int_\Omega \omega(x)|v(x)|^{\underline{q}(x)}dx\right].\end{aligned}$$

Going to the limit inf in the above inequality, we obtain

$$\begin{aligned}0 \geq m_0 &\Bigg[\liminf_{n\to+\infty}\int_Q |u_n(x)-u_n(y)|^{p(x,y)}\,K(x,y)dxdy+\int_\Omega \omega(x)|u_n(x)|^{p(x)}\,dx\\ &-\int_Q |u(x)-u(y)|^{p(x,y)}\,K(x,y)dxdy+\int_\Omega \omega(x)|u(x)|^{p(x)}\,dx\Bigg]\\ +m_0&\Bigg[\liminf_{n\to+\infty}\int_Q |v_n(x)-v_n(y)|^{q(x,y)}\,K(x,y)dxdy+\int_\Omega \omega(x)|v_n(x)|^{q(x)}\,dx\\ &-\int_Q |v(x)-v(y)|^{q(x,y)}\,K(x,y)dxdy+\int_\Omega \omega(x)|v(x)|^{q(x)}\,dx\Bigg].\end{aligned}$$

This and (3.2) and (3.3) yeild

$$\begin{aligned}\liminf_{n\to+\infty}&\left[\int_Q |u_n(x)-u_n(y)|^{p(x,y)}\,K(x,y)dxdy+\int_\Omega \omega(x)|u_n(x)|^{p(x)}\,dx\right]\\ &=\int_Q |u(x)-u(y)|^{p(x,y)}\,K(x,y)dxdy+\int_\Omega \omega(x)|u(x)|^{p(x)}\,dx,\end{aligned}$$

and

$$\begin{aligned}\liminf_{n\to+\infty}&\left[\int_Q |v_n(x)-v_n(y)|^{q(x,y)}\,K(x,y)dxdy+\int_\Omega \quad (x)|v_n(x)|^{q(x)}\,dx\right]\\ &=\int_Q |v(x)-v(y)|^{q(x,y)}\,K(x,y)dxdy+\int_\Omega \quad (x)|v(x)|^{q(x)}\,dx.\end{aligned}$$

For a suitable subsequence, we have

$$\begin{aligned}\lim_{n\to+\infty}&\left[\int_Q |u_n(x)-u_n(y)|^{p(x,y)}\,K(x,y)dxdy+\int_\Omega \omega(x)|u_n(x)|^{p(x)}\,dx\right]\\ &=\int_Q |u(x)-u(y)|^{p(x,y)}\,K(x,y)dxdy+\int_\Omega \omega(x)|u(x)|^{p(x)}\,dx,\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to+\infty}&\left[\int_Q |v_n(x)-v_n(y)|^{q(x,y)}\,K(x,y)dxdy+\int_\Omega \omega(x)|v_n(x)|^{q(x)}\,dx\right]\\ &=\int_Q |v(x)-v(y)|^{q(x,y)}\,K(x,y)dxdy+\int_\Omega \omega(x)|v(x)|^{q(x)}\,dx,\end{aligned}$$

that is, $\lim_{n\to+\infty}\rho^{\omega}_{K,p(x,y)}(u_n)=\rho^{\omega}_{K,p(x,y)}(u)$, and $\lim_{n\to+\infty}\rho^{\omega}_{K,q(x,y)}(v_n)=\rho^{\omega}_{K,q(x,y)}(v)$. Proposition 3.1 implies that $(u_n,v_n)\to(u,v)$ in E.

Now we show that $\Phi'^{-1}$ is continuous. Let $\{(f_n, g_n)\}\in E^*$ be a sequence strongly converge to $(f, g)$ in $E^*$ and let $(u_n, v_n), (u, v)\in E$ such that

$$u_n=\Phi_p^{-1}(f_n),\qquad v_n=\Phi_q^{-1}(g_n),\qquad u=\Phi_p^{-1}(f)\text{ and }v=\Phi_q^{-1}(g).$$

Then, we have

$$f_n=\Phi_p(u_n)\to f=\Phi_p(u),$$

and

$$g_n = \Phi_q(v_n) \to g = \Phi_q(v).$$

By the continuity of $\Phi'$ we get

$$\Phi'(u_n, v_n) \to \Phi'(u, v). \tag{3.4}$$

On the other hand, since $\Phi'$ is coercive, then $(u_n, v_n)$ is bounded in E, so, we can assume that it converges weakly to a certain $(u_0, v_0) \in E$. Since $(f_n, g_n)$ converges strongly to $(f, g)$, we have

$$\lim_{n\to+\infty} \left\langle \Phi'(u_n, v_n) - \Phi'(u_0, v_0), (u_n, v_n) - (u_0, v_0) \right\rangle$$
$$= \lim_{n\to+\infty} \left\langle (f_n, g_n), (u_n, v_n) - (u_0, v_0) \right\rangle = 0.$$

In view of property (S) and the continuity of $\Phi'$ we deduce $(u_n, v_n) \to (u_0, v_0)$ in $E$ and

$$\Phi'(u_n, v_n) \to \Phi'(u_0, v_0). \tag{3.5}$$

From (3.4), (3.5) and using the fact that $\Phi'$ is an injection, we conclude that $(u,v) = (u_0, v_0)$.

**Lemma 3.9.** Let $F, G \in \mathcal{A}$. *Then the function* $J, \Psi \in C^1(E, \mathbb{R})$ with the *derivatives given by*

$$\left\langle J'(u,v), (\varphi, \psi) \right\rangle = -\int_\Omega F_u(x,u,v)\varphi dx - \int_\Omega F_v(x,u,v)\psi\, dx,$$

*for all* $(\varphi, \psi) \in E$. *Moreover* $J' : E \to E^*$ *is compact.*

*Proof.* We prove the result for the operator $J$. For $\Psi$, the proof is similar. First, from $F \in \mathcal{A}$ and the embedding result, we have that $J$ is well defined on $E$ . Using a standard argument we have that

$$\left\langle J'(u,v), (\varphi, \psi) \right\rangle = -\int_\Omega F_u(x,u,v)\varphi dx - \int_\Omega F_v(x,u,v)\psi\, dx.$$

Now, we claim that $J'$ is continuous. Let $\{(u_n, v_n)\} \subset E$ be a sequence converging strongly to $(u,v)$ in $E$. By virtue of the compact embedding of the $W^{s,p(x,y)}_{K,\omega}(\Omega)$ and $W^{s,q(x,y)}_{K,\omega}(\Omega)$ into $L^{\gamma(x)}(\Omega)$, where $\max(p^+,q^+) \le \gamma(x) \le \min(p^*_s(x), q^*_s(x))$. Hence, since $F \in \mathcal{A}$, we can define the operator $(u, v) \mapsto F_u(., u(.), v(.)) + F_v(., u(.), v(.))$ from $L^{\gamma(x)}(\Omega) \times L^{\gamma(x)}(\Omega)$ into $L^{\hat{\gamma}(x)}(\Omega)$ with $\frac{1}{\gamma(x)} + \frac{1}{\hat{\gamma}(x)} = 1$. Then, fixing $(\tilde{u}, \tilde{v}) \in E$ with $\|(\tilde{u}, \tilde{v})\|_E \le 1$, by the Holder inequality, we get

$$\begin{aligned}
&\left|\left\langle J'(u_n, v_n) - J'(u,v), (u\sim, v\sim) \right\rangle\right| \\
&\le \left| \int_\Omega \big(F_u(x, u_n(x), v_n(x)) - F_u(x, u(x), v(x))\big) u\sim(x) dx \right| \\
&+ \left| \int_\Omega \big(F_v(x, u_n(x), v_n(x)) - F_v(x, u(x), v(x))\big) v\sim(x) dx \right| \\
&\le C_3 \left\|{}_u(x, u_n(x), v_n(x)) - F_u(x, u(x), v(x))\right\|_{L^{\hat{\gamma}(x)}(\Omega)} \left\|u\sim(x)\right\|_{L^{\gamma(x)}(\Omega)} \\
&+ C_4 \left\|F_v(x, u_n(x), v_n(x)) - F_v(x, u(x), v(x))\right\|_{L^{\hat{\gamma}(x)}(\Omega)} \left\|v\sim(x)\right\|_{L^{\gamma(x)}(\Omega)} n \\
&\le C_5 \left\|F_u(x, u_n(x), v_n(x)) - F_u(x, u(x), v(x))\right\|_{L^{\hat{\gamma}(x)}(\Omega)} \left\|u\sim(x)\right\|_{K,s,p(x,y)} n \\
&+ C_6 \left\|F_v(x, u_n(x), v_n(x)) - F_v(x, u(x), v(x))\right\|_{L^{\hat{\gamma}(x)}(\Omega)} \left\|v\sim(x)\right\|_{K,s,q(x,y)},
\end{aligned}$$

for some constants $C_3$, $C_4$, $C_5$, $C_6 > 0$. Thus, for $\|(\tilde{u}, \tilde{v})\|_E \leq 1$, we obtain

$$\begin{aligned}\|J'(u_n, v_n) - J'(u,v)\|_{E^*} \leq C_7 \|F_u(x, u_n(x), v_n(x)) + F_u(x, u(x), v(x))\|_{L^{\hat{\gamma}(x)}(\Omega)} \\ + C_8 \|F_v(x, u_n(x), v_n(x)) + F_v(x, u(x), v(x))\|_{L^{\hat{\gamma}(x)}(\Omega)}. \quad (3.6)\end{aligned}$$

On the other hand, since $W^{s,p(x,y)}_{K,\omega}(\Omega)$ and $W^{s,q(x,y)}_{K,\omega}(\Omega)$ are compactly embedded in $L^{\gamma(x)}(\Omega)$, we have that $\{(u_n, v_n)\}$ converge strongly in $L^{\gamma(x)}(\Omega)$. It follows that there exist a subsequence of $\{(u_n, v_n)\}$, still denoted by $\{(u_n, v_n)\}$, and $(h, g) \in L^{\gamma(x)}(\Omega) \times L^{\gamma(x)}(\Omega)$ such that

$$u_n(x) \underset{n\to+\infty}{\to} u(x) \text{ a.e. } x \in \Omega, \qquad v_n(x) \underset{n\to+\infty}{\to} v(x) \text{ a.e. } x \in \Omega$$

and

$$|u_n(x)| \leq h(x) \text{ a.e. } x \in \Omega, \quad |v_n(x)| \leq g(x) \text{ a.e. } x \in \Omega.$$

This fact combining with $F \in \mathcal{A}$ implies that

$$F_u(x, u_n(x), v_n(x)) - F_u(x, u(x), v(x)) \to 0 \text{ as } n \to +\infty,$$

$$F_v(x, u_n(x), v_n(x)) - F_v(x, u(x), v(x)) \to 0 \text{ as } n \to +\infty,$$

and

$$|F_u(x, u_n(x), v_n(x)| \leq C_9(1 + |h|^{\gamma(x)-1} + |g|^{\gamma(x)-1}) \in L^{\hat{\gamma}(x)}(\Omega),$$

$$|F_v(x, u_n(x), v_n(x)| \leq C_{10}(1 + |h|^{\gamma(x)-1} + |g|^{\gamma(x)-1}) \in L^{\hat{\gamma}(x)}(\Omega),$$

for almost everywhere $x \in \Omega$. Hence, by applying the dominate convergence theorem, we get

$$\|F_u(x, u_n(x), v_n(x)) - F_u(x, u(x), v(x))\|_{E^*} \to 0 \text{ as } n \to +\infty,$$

$$\|F_v(x, u_n(x), v_n(x)) - F_v(x, u(x), v(x))\|_{E^*} \to 0 \text{ as } n \to +\infty.$$

This proves that $J'$ is continuous. Now, in order to verify the compactness of $J'$, we take $\{(u_n, v_n)\}$ a bounded sequence in $E$. Then there exists a subsequence of $\{(u_n, v_n)\}$, still denoted by $\{(u_n, v_n)\}$, converging weakly in $E$. So, it converges strongly in $L^{\gamma(x)}(\Omega) \times L^{\gamma(x)}(\Omega)$. Using the same argument as above we show that $\{J'(u_n, v_n)\}$ converges strongly and then the operator $J'$ is compact.

**Proof of Theorem 3.5**

To prove our existence result, we take $X = E$ and it is enough to check that $\Phi$, $J$ satisfy the hypotheses of Theorem 3.6. Indeed, by Lemma 3.7 we have that $\Phi \in C^1(E, \mathbb{R})$ is sequentially weakly lower semi-continuous, bounded on each bounded subset and its Gâteaux derivative $\Phi' : E \to E^*$ admits a continuous inverse. Moreover, Lemma 3.9 implies that $J \in C^1(E, \mathbb{R})$ and its derivative $J' : E \to E^*$ is compact.

Next, we will check that condition (i) of Theorem 3.6 is satisfied. Indeed, by Proposition 3.1, for all $\|u\|_{K,\omega,p(x,y)}, \|v\|_{K,\omega,q(x,y)} > 1$, we have

$$\begin{aligned}\Phi(u,v) &= \hat{M}(I^{\omega}_{K,p(x,y)}(u)) + \hat{M}(I^{\omega}_{K,q(x,y)}(v)) \\ &\geq \frac{m_0}{p^+}\rho^{\omega}_{K,p(x,y)}(u) + \frac{m_0}{q^+}\rho^{\omega}_{K,p(x,y)}(v) \\ &\geq m_0 \min\left\{\frac{1}{p^+}, \frac{1}{q^+}\right\}\left(\|u\|^{p^-}_{K,\omega,p(x,y)} + \|v\|^{q^-}_{K,\omega,q(x,y)}\right). \end{aligned} \quad (3.7)$$

On the other hand, for all $\lambda > 0$, From the condition $(H_2)$ we have

$$\begin{aligned}\lambda J(u,v) = -\lambda\int_\Omega F(x,u,v)\,dx &\geq -\lambda\int_\Omega C\Big(1+a(x)|u(x)|^{\alpha(x)}+b(x)|v(x)|^{\beta(x)}\Big)dx\\ &\geq -\lambda C\Big(|\Omega|+\|u\|^{\alpha^+}_{\alpha(x),a(x)}+\|v\|^{\beta^+}_{\beta(x),b}\Big). \qquad (3.8)\end{aligned}$$

Since $W^{s,p(x,y)}_{K,\omega}(\Omega)\hookrightarrow L^{\alpha(x)}_{a(x)}(\Omega)$ and $W^{s,q(x,y)}_{K,\omega}(\Omega)\hookrightarrow L^{\beta(x)}_{b(x)}(\Omega)$, there exist two positive constants $C_{11}, C_{12} > 0$ such that

$$\|u\|_{\alpha(x),a}\leq C_{11}\max\left\{1,\tilde{k}_0\right\}\|u\|_{K,\omega,p(x,y)},$$

and

$$\|v\|_{\beta(x),b}\leq C_{12}\max\left\{1,\tilde{k}_0\right\}\|v\|_{K,\omega,q(x,y)}.$$

Thus,

$$\lambda J(u,v)\geq -\lambda C\Big(|\Omega|+C_{11}\|u\|^{\alpha^+}_{K,\omega,p(x,y)}+C_{12}\|v\|^{\beta^+}_{K,\omega,q(x,y)}\Big). \qquad (3.9)$$

By, (3.7) and (3.9) , it follows that

$$\begin{aligned}\Phi(u,v)+\lambda J(u,v) &\geq m_0\min\left\{\frac{1}{p^+},\frac{1}{q^+}\right\}\Big(\|u\|^{p^-}_{K,\omega,p(x,y)}+\|v\|^{q^-}_{K,\omega,q(x,y)}\Big)\\ &\quad -\lambda C\Big(|\Omega|+C_{11}\|u\|^{\alpha^+}_{K,\omega,p(x,y)}+C_{12}\|v\|^{\beta^+}_{K,\omega,q(x,y)}\Big),\end{aligned}$$

since $\alpha^+ < p^-$ and $\beta^+ < q^-$, we conclude that

$$\lim_{\|(u,v)\|_E\to+\infty}\Phi(u,v)+\lambda J(u,v)=+\infty.$$

Consequently, the assertion (*i*) of Theorem 3.6 is satisfied.

Next, we will prove that the assertions (ii) and (iii) are also satisfied.

Since $\Omega$ is a nonempty bounded open set there is a point $x_0\in\Omega$, let $R > 0$ we put

$$m(x)=\begin{cases}0, & x\in\bar{\Omega}\setminus B(x_0,R),\\ \dfrac{2d}{R}\big(R-|x-x_0|\big) & x\in B(x_0,R)\setminus B(x_0,\frac{R}{2}),\\ d, & x\in B(x_0,\frac{R}{2}).\end{cases} \qquad (3.10)$$

Lets now choose

$$(u_0(x),v_0(x))=(0,0) \qquad\text{and}\qquad (u_1(x),v_1(x))=(m(x),m(x)).$$

It is easy to see that

$$\Phi(u_0(x),v_0(x))=J(u_0(x),v_0(x))=0.$$

From $(H1)$, we have $-J(m(x),m(x))=\int_\Omega F(x,m,m)\,dx>0$ . From $(H_2)$ and $(H_3)$ there exists $C>0$ such that

$$F(x,t,\xi)<C(|t|^{p_1^-}+|\xi|^{q_1^-}),\ \ \forall(t,\xi)\in\mathbb{R}^2\ a.e\ x\in\Omega.$$

Fix $0<r<1$. When

$$m_0\min\{\frac{1}{p^+},\frac{1}{q^+}\}\left(\|u\|^{p^+}_{K,w,p(x,y)}+\|v\|^{q^+}_{K,w,q(x,y)}\right)<r<1,$$

by the continuous embedding of $W^{s,p(x,y)}_{K,\omega}(\Omega)$ in $L^{p_1^-}_{a(.)}(\Omega)$ and of $W^{s,q(x,y)}_{K,\omega}(\Omega)$ in $L^{q_1^-}_{b(.)}(\Omega)$ we have

$$\begin{aligned}-J(u,v)=\int_\Omega F(x,u,v)\,dx &\leq C\int_\Omega\left(a(x)|u|^{p_1^-}+b(x)|v|^{q_1^-}\right)dx\\ &\leq C_{13}\|u\|^{p_1^-}_{K,\omega,p(x,y)}+\|v\|^{q_1^-}_{K,\omega,q(x,y)}\\ &\leq C_{14}\left(r^{\frac{p_1^-}{p^+}}+r^{\frac{q_1^-}{q^+}}\right),\end{aligned}$$

for some positive constants $C_{13}$, $C_{14}$. Since $p_1^->p^+$ and $q_1^->q^+$ we get

$$\lim_{r\to0}\frac{\sup_{m_0\min\{\frac{1}{p^+},\frac{1}{q^+}\}\left(\|u\|^{p^+}_{K,\omega,p(x,y)}+\|v\|^{q^+}_{K,\omega,q(x,y)}\right)<r}-J(u,v)}{r}=0. \tag{3.11}$$

if $m_0\min\left\{\frac{1}{p^+},\frac{1}{q^+}\right\}\left(\|u\|^{p^+}_{K,\omega,p(x,y)}+\|v\|^{q^+}_{K,\omega,q(x,y)}\right)<r$

$$<m_0\min\left\{\frac{1}{p^+},\frac{1}{q^+}\right\}\min\left\{\|m\|^{p^+}_{K,\omega,p(x,y)}+\|m\|^{q^+}_{K,\omega,q(x,y)},\|m\|^{p^-}_{K,\omega,p(x,y)}+\|m\|^{q^-}_{K,\omega,q(x,y)},1\right\},$$

then, if $\|(m,m)\|_E<1$

$$\begin{aligned}\Phi(u_1,v_1)=\Phi(m,m) &=\hat{M}(I^\omega_{K,p(x,y)}(m))+\hat{M}(I^\omega_{K,q(x,y)}(m))\\ &\geq m_0\min\left\{\frac{1}{p^+},\frac{1}{q^+}\right\}\left(\rho^\omega_{K,p(x,y)}(m)+\rho^\omega_{K,q(x,y)}m\right)\\ &\geq m_0\min\left\{\frac{1}{p^+},\frac{1}{q^+}\right\}\left(\|m\|^{p^+}_{K,\omega,p(x,y)}+\|m\|^{q^+}_{K,\omega,q(x,y)}\right)\\ &>r>0,\end{aligned}$$

From (3.11), we know that

$$\begin{aligned}\sup_{m_0\min\{\frac{1}{p^+},\frac{1}{q^+}\}\left(\|u\|^{p^+}_{K,\omega,p(x,y)}+\|v\|^{q^+}_{K,\omega,q(x,y)}\right)<r}-J(u,v) &\leq\frac{r}{2}\cdot\frac{-J(u_1,v_1)}{\max\left\{\frac{1}{p_1^-},\frac{1}{q_1^-}\right\}\left(\|m\|^{p^-}_{K,\omega,p(x,y)}+\|m\|^{q^-}_{K,\omega,q(x,y)}\right)}\\ &\leq\frac{r}{2}\cdot\frac{-J(u_1,v_1)}{\Phi(u_1,v_1)}\\ &<r.\frac{-J(u_1,v_1)}{\Phi(u_1,v_1)},\end{aligned}$$

and, if $\|(m, m)\|_E > 1$

$$\begin{aligned}\Phi(m,m) &= \hat{M}(I^{\omega}_{K,p(x,y)}(m)) + \hat{M}(I^{\omega}_{K,q(x,y)}(m))\\ &\geq m_0 \min\left\{\frac{1}{p^+},\frac{1}{q^+}\right\}\left(\rho^{\omega}_{K,p(x,y)}(m)+\rho^{\omega}_{K,q(x,y)}(m)\right)\\ &\geq m_0 \min\left\{\frac{1}{p^+},\frac{1}{q^+}\right\}\left(\|m\|^{p^-}_{K,\omega,p(x,y)}+\|m\|^{q^-}_{K,\omega,q(x,y)}\right)\\ &> r > 0.\end{aligned}$$

From (3.11), we know that

$$\begin{aligned}\sup_{m_0\min\{\frac{1}{p^+},\frac{1}{q^+}\}\left(\|u\|^{p^+}_{K,\omega,p(x,y)}+\|v\|^{q^+}_{K,\omega,q(x,y)}\right)<r} -J(u,v) &\leq \frac{r}{2}.\frac{-J(u_1,v_1)}{\max\left\{\frac{1}{p^-},\frac{1}{q^-}\right\}\left(\|m\|^{p^+}_{K,\omega,p(x,y)}+\|m\|^{q^+}_{K,\omega,q(x,y)}\right)}\\ &\leq \frac{r}{2}.\frac{-J(u_1,v_1)}{\Phi(u_1,v_1)}\\ &< r.\frac{-J(u_1,v_1)}{\Phi(u_1,v_1)},\end{aligned}$$

Consequently, $\Phi(u_0,v_0) < r < \Phi(u_1,v_1)$. Hence, the condition (ii) in Theorem 3.6 is verified. On the other hand, we have

$$-\frac{\left(\Phi(u_1,v_1)-r\right)J(u_0,v_0)+\left(r-\Phi(u_0,v_0)\right)J(u_1,v_1)}{\Phi(u_1,v_1)-\Phi(u_0,v_0)} = -r\,\frac{J(u_1,v_1)}{\Phi(u_1,v_1)}.$$

For any $(u,v)\in\Phi^{-1}(-\infty,r)$, we get $\Phi(u,v)\leq r$, i.e.

$$\hat{M}(I^{\omega}_{K,p(x,y)}(u(x))) + \hat{M}(I^{\omega}_{K,q(x,y)}(v(x)))) \leq r,$$

then, we get

$$m_0 \min\left\{\frac{1}{p^+},\frac{1}{q^+}\right\}\left(\rho^{\omega}_{K,p(x,y)}(u)+\rho^{\omega}_{K,q(x,y)}(v)\right)\leq r.$$

So,

$$\rho^{\omega}_{K,p(x,y)}(u)+\rho^{\omega}_{K,q(x,y)}(v)\leq \frac{r}{m_0\min\left\{\frac{1}{p^+},\frac{1}{q^+}\right\}} < 1,$$

then

$$\rho^{\omega}_{K,p(x,y)}(u) < 1 \quad \text{and} \quad \rho^{\omega}_{K,q(x,y)}(v) < 1,$$

it follows that, $\|u\|_{K,\omega,p(x,y)}<1$, and $\|v\|_{K,\omega,q(x,y)}<1$. Furthermore, it is clear that

$$m_0\min\left\{\frac{1}{p^+},\frac{1}{q^+}\right\}\left(\|u\|^{p^+}_{K,\omega,p(x,y)}+\|v\|^{q^+}_{K,\omega,q(x,y)}\right)<r.$$

So, we get that

$$\Phi^{-1}(-\infty,r)\subset K(r),\quad K(r)=\left\{(u,v)\in E,\ m_0\min\left\{\frac{1}{p^+},\frac{1}{q^+}\right\}\left(\|u\|^{p^+}_{K,w,p(x,y)}+\|v\|^{q^+}_{K,w,q(x,y)}\right)<r\right\}.$$

Then

$$\begin{aligned}-\inf_{u\in\Phi^{-1}(-\infty,r)}J(u,v)&=\sup_{u\in\Phi^{-1}(-\infty,r)}-J(u,v)\\&\le\sup_{(u,v)\in K(r)}-J(u,v)<-r\frac{J(u_1,v_1)}{\Phi(u_1,v_1)},\end{aligned}$$

that is

$$\inf_{u\in\Phi^{-1}(-\infty,r)}J(u,v)>\frac{\left(\Phi(u_1,v_1)-r\right)J(u_0,v_0)+\left(r-\Phi(u_0,v_0)\right)J(u_1,v_1)}{\Phi(u_1,v_1)-\Phi(u_0,v_0)},$$

which means that condition (iii) in Theorem 3.6 is verified. Then the proof of Theorem 3.5 is achieved.

**Example**

Let $\Omega=(0,1)\times(0,1)$, $p,q:Q\to(1,+\infty)$ be two continuous functions satisfying (1.1) with $sp^+,sq^+<N$. We consider

- $M(t)=a_1+a_2t,\ a_1,a_2>0,$
- $K(x,y)=|x-y|^{-(N+sp(x,y))},$
- $F(x,s,t)=|\sin(s)|^{\gamma_1(x)}|\sin(t)|^{\gamma_2(x)}$ where $\gamma_1,\gamma_2\in C_+(\overline{\Omega})$ such that $\gamma_1^->\max\left\{2,p^+\right\}$, $\gamma_2^->\max\left\{2,q^+\right\}$,
- $\omega(x)=\omega(x_1,x_2)=\dfrac{2+x_1}{\pi}.$

Then, we have

$$\begin{aligned}F_s(x,s,t)&=\gamma_1(x)\cos(s)\sin(s)|\sin(s)|^{\gamma_1(x)-2}|\sin(t)|^{\gamma_2(x)},\\F_t(x,s,t)&=\gamma_2(x)|\sin(s)|^{\gamma_1(x)}\cos(t)\sin(t)|\sin(t)|^{\gamma_2(x)-2},\end{aligned}$$

and

$$\hat{M}=a_1t+\frac{a_2}{2}t^2$$

In this case, the system $\left(\mathrm{P}_{\omega}^{K}\right)$ becomes

$$\left(\mathrm{P}_{a_1,a_2}^{s}\right)\quad \begin{cases} \left(a_1 + a_2\left(I^{\omega}_{s,p(x,y)}(u)\right)\right)\left((-\Delta_{p(x,.)})^{s}(u) + \dfrac{2+x_1}{\pi}|u|_{-}^{p(x)-2}u\right) \\ = \lambda\gamma_1(x)\cos(u)\sin(u)|\sin(u)|^{\gamma_1(x)-2}|\sin(v)|^{\gamma_2(x)} + \mu G_u(x,u,v), & \text{in } \Omega, \\ \left(a_1 + a_2\left(I^{\omega}_{s,q(x,y)}(v)\right)\right)\left((-\Delta_{q(x,.)})^{s}(v) + \dfrac{2+x_1}{\pi}|v|_{-}^{q(x)-2}v\right) & \text{in } \Omega, \\ = \lambda\gamma_2(x)\cos(v)\sin(v)|\sin(u)|^{\gamma_1(x)}|\sin(v)|^{\gamma_2(x)-2} + \mu G_v(x,u,v), \\ u = v = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

Next, we will show that the hypotheses of Theorem 3.5 are achieved. For each $s,t \in \mathsf{R}$, we claim that $F \in$ A. Indeed, we have

$$F_s(x,s,t) \le \gamma_1^{+} \le \gamma_1^{+}\left(1 + |s|^{\eta_1(x)-1} + |t|^{\eta_2(x)-1}\right),$$

and $\quad F_t(x,s,t) \le \gamma_2^{+} \le \gamma_2^{+}\left(1 + |s|^{\eta_1(x)-1} + |t|^{\eta_2(x)-1}\right),$

for all $\eta_1(.) \in (p^{+}, p_s^{*}(x))$, $\eta_2(.) \in (q^{+}, q_s^{*}(x))$ and any $(x,s,t) \in \Omega \times \mathbb{R} \times \mathbb{R}$, then

$$\sup_{(x,s,t)\in\Omega\times\mathbb{R}\times\mathbb{R}} \frac{|F_s(x,s,t)|}{1 + |s|^{\eta_1(x)-1} + |t|^{\eta_2-1}} \le \gamma_1^{+} < +\infty,$$

$$\sup_{(x,s,t)\in\Omega\times\mathbb{R}\times\mathbb{R}} \frac{|F_t(x,s,t)|}{1 + |s|^{\eta_1(x)-1} + |t|^{\eta_2-1}} \le \gamma_2^{+} < +\infty.$$

On the other hand, we have

$$F(x,0,0) = 0,\ F(x,s,t) \ge 0,$$

and

$$F(x,s,t) \le 1 \le 1 + a(x)|s|^{\alpha(x)} + b(x)|t|^{\beta(x)},$$

for any $\alpha, \beta \in C_{+}(\overline{\Omega})$ such that $\alpha^{+} < p^{-}, \beta^{+} < q^{-}, a(x) \in L^{r_1(.)}(\Omega)$ and $b(x) \in L^{r_2(.)}(\Omega)$, where $p(x) \le \dfrac{r_1(x)\alpha(x)}{r_1(x)-1} < p_s^{*}(x)$, and $q(x) \le \dfrac{r_2(x)\beta(x)}{r_2(x)-1} < q_s^{*}(x)$,
Moreover, if we choose $p_1, q_1 \in C_{+}(\overline{\Omega})$, such that $p^{+} < p_1^{-} < p_1^{+} < \gamma_1^{-}$ and $q^{+} < q_1^{-} < q_1^{+} < \gamma_2^{-}$, then

$$\lim_{(s,t)\to(0,0)} \frac{|\sin s|^{\gamma_1^{-}}|\sin t|^{\gamma_2^{-}}}{a(x)|s|^{p_1(x)} + b(x)|t|^{q_1(x)}} = 0 < +\infty.$$

Which means that $(H_0)$, $(H_1)$, $(H_2)$ and $(H_3)$ are verified. Also, for $m_0 = a_1$ the condition $(M_0)$ is satisfied.

Then, there exists an open $\Lambda \subset (0,+\infty)$ and a positive real number $\delta > 0$ such that, for each $\lambda \in \Lambda$ and every function $G \in \mathrm{A}$, there exists $\mu^* > 0$ such that for any $\mu \in [0, \mu^*]$, the system $\left(\mathrm{P}^s_{a_1,a_2}\right)$ has at least three weak solutions whose norms are less than $\delta$.

## Declarations

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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