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Multivalued Minimax Relations with Applications

Irene Benedetti *
Department of Mathematics and Computer Science, University of Perugia,
Via Vanvitelli 1, 06123 Perugia, Italy
irene.benedetti@dmi.unipg.it

Anna Martellotti
Department of Mathematics and Computer Science, University of Perugia,
Via Vanvitelli 1, 06123 Perugia, Italy
anna.martellotti@unipg.it

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We prove variational inclusions and matching type results, as well as, minimax inequalities for multivalued maps, relaxing the usual hypotheses of compactness and convexity. We end the discussion adding several applications to variational inequalities, maximization of binary relations and non-cooperative equilibrium for n-person games with discontinuous payoffs.

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1. Introduction

The main aim of this paper is to relax the usual hypotheses of compactness and convexity assumed to obtain variational inclusions and matching type results, i.e. solutions to the following problems:

given two Hausdorff topological vector spaces X and Y, a nonempty closed convex subset M of Y, a multimap $G : X \times X \multimap Y$, find $x_o \in X$ such that

$$G(x_o, y) \subset M, \quad \text{for every } y \in X \text{ (variational inclusion);}$$

or

$$G(x_o, y) \cap M \neq \emptyset, \quad \text{for every } y \in X \text{ (variational matching).}$$

This result is based on a slight extension of the remarkable FKKM Theorem, obtained using a generalization of the classical concept of KKM multimap introduced by Tian in [26] (see Theorems 2.6 and 2.8). This new concept is more general than the usual one of KKM multimap. As pointed out by Tian in [26],

*Corresponding author

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for instance the multimap $U : Z \multimap Z$, $U(x) = \{y \in Z \ : \ y \succeq x\}$, where $\succeq$ is a preference relation on a topological space Z, can be exhibited as a counterexample. Moreover, we distribute the hypotheses over two multimaps, improving the hypothesis of KKM multimap contained in Theorem 2.2 of [17]. From the obtained variational inclusions and matching type results we deduce several minimax inequalities in this framework. In particular, Section 5 contains minimax inequalities of various type, including multivalued version of Ky Fan minimax inequalities (Theorems 5.1–5.5) as well as single valued versions involving one map (Theorems 5.12 and 5.13), two maps (Theorem 5.9) and even three maps (Theorem 5.10). The same section contains also minimax inequalities of vector-type, that is for functions ranging in ordered vector spaces; such minimax relations have been considered in the late 80's by several authors ([4], [5], [18], [19]); here we generalize some of those results, for instance Theorem 13 in [19]. To this aim, Section 4 considers a class of generalized convex (concave) multivalued maps called diagonal transfer quasi-convex (quasi-concave). These concepts were introduced in [6] for the single valued case and were used very recently to find equilibria for strategic form games, see e.g., [23]. We extended them to the multivalued setting. In particular, to explain the novelty of the diagonal multivalued transfer quasi-convexity (quasi-concavity), Section 4 is enriched with several examples, comparison results with the classical notion of convexity for multivalued maps and characterizations by means of suitable single valued maps.

Since the Ky Fan minimax inequality finds application in several frameworks such as non-cooperative game theory, fixed point theorems, theoretical economics, variational inequalities and so on, several applications of the same kind follow from our minimax inequalities.

More precisely, we generalize a class of existence theorems on the maximal elements of binary relations, price equilibria, fixed points and variational inequalities by relaxing the compactness and convexity of choice sets and the continuity of the involved maps. In particular, applying our versions of the Ky-Fan minimax inequality to the case of one single valued map (Theorems 5.12 and 5.13) we are able to prove a generalization of the classical Schauder Fixed Point Theorem and of the Hartmann-Stampacchia existence result for variational inequalities, and we show two maximization results of a binary relation, generalizing the classical result of Sonneschein (see [24]). Finally, using the minimax-type inequality for three single valued maps, we obtain an existence result for a non-cooperative equilibrium for a n-person game where the payoffs are not necessarily continuous functions. Remarkable examples of such games are the Bertrand oligopoly (see [8]) and the Hotelling linear city model (see [14]), which have been a source of inspiration for a wide literature on equilibrium existence in games with discontinuous payoffs. For instance, Baye et al. in [6] showed that a compact game has a Nash equilibrium if its aggregator function is diagonally transfer continuous and diagonally transfer quasi-concave and, recently, essential Nash equilibria of discontinuous games are studied in [22]. In this paper we prove an existence result for a non-cooperative equilibrium generalizing the hypotheses assumed in

[6] and in [27]. Moreover, we show that the aggregator function does not satisfy the hypotheses of compactness required in Theorem 2 of [6] (see Example 6.13).

2. KKM-type theorems

The Knaster-Kuratowski-Mazurkiewicz (KKM) Theorem (proved in 1929) is a classical result for nonlinear analysis, which is equivalent to many basic theorems such as Brouwer's fixed point theorem. We recall it here for the sake of completeness.

Theorem 2.1. *Let $P_n = \overline{co}\{a_1, \ldots, a_n\}$ be a closed n-simplex and let $F_1, \ldots, F_n$ be n closed subsets of P_n. If for each set $A \subset \{0, \ldots, n\}$ we have*

$$\overline{co}\{a_i,\ i \in A\} \subset \bigcup_{i\in A} F_i, \quad \textit{then} \quad \bigcap_{i=0}^{n} F_i \neq \emptyset.$$

The most important generalization of this result is the Fan-Knaster-Kuratowski-Mazurkiewicz (FKKM) Theorem which was obtained by Fan in 1961, see [12], extending the KKM-Theorem from $\mathbb{R}^n$ to Hausdorff topological vector spaces. It is based on the notion of KKM multimap, namely

Definition 2.2. Let X be a convex set in a Hausdorff topological vector space and $Y \subset X$ be a non empty set. A multimap $F : Y \multimap X$ is said to be a *KKM multimap* if for every $\{y_1, \ldots, y_n\} \subset Y$

$$\mathrm{co}\{y_1, \ldots, y_n\} \subset \bigcup_{j=1}^{n} F(y_j).$$

Theorem 2.3. *Let X be an arbitrary set in a Hausdorff topological vector space Y, and let $F : X \multimap Y$ be a closed valued KKM multimap such that $F(x)$ is compact for at least one $x \in X$. Then $\bigcap_{x\in X} F(x) \neq \emptyset$.*

Later, in 1984, he relaxed the compactness condition on the multimap F. More precisely, he proved in [13] the following result.

Theorem 2.4. *Let X be an arbitrary set in a topological vector space Y, and let $F : X \multimap Y$ be a relatively closed valued KKM multimap such that there is a nonempty subset X_0 of X such that the intersection $\bigcap_{x\in X_0} F(x)$ is compact and X_0 is contained in a compact convex subset of Y. Then $\bigcap_{x\in X} F(x) \neq \emptyset$.*

Since this celebrated result many intersection theorems have appeared (see e.g. [1, 3, 7, 17, 21, 25, 26]). In particular, Tian in [26, Definition 3] introduces a generalization considering two different sets of points in the domain and in the image of the multivalued map F. To be more precise, he proposes the following definition.

Definition 2.5. Let X be a convex set in a Hausdorff topological vector space and $Y \subset X$ be a non empty set. A multimap $F : Y \multimap X$ is said to be *transfer FS convex* if for every $\{y_1, \dots, y_n\} \subset Y$ there exists $\{x_1, \dots, x_n\} \subset X$ such that for every $J \subset \{1, \dots, n\}$

$$\mathrm{co}\{x_j, j \in J\} \subset \bigcup_{j \in J} F(y_j).$$

It is evident that if $\{y_1, \dots, y_n\} = \{x_1, \dots, x_n\}$ we go back to the original KKM assumption. Following the same line of the proof of Lemma 1 in [26] it is possible to prove that if a multimap $G : Y \multimap X$ is transfer FS convex, then the family $\{G(y), y \in Y\}$ has the finite intersection property. Then, by the classical FKKM Theorem we get the following result.

Theorem 2.6. *Let X be a convex set in a Hausdorff topological vector space, $Y \subset X$ be a non empty set, and $F, G : Y \multimap X$ be two multimaps such that*

(1) *$F(y) \subseteq G(y)$ for every $y \in Y$;*
(2) *G is closed valued;*
(3) *F is transfer FS convex;*
(4) *there is a nonempty finite subset Y_o of Y such that $\bigcap_{y \in Y_o} \overline{F(y)}^X$ is compact (where $\overline{F(y)}^X$ represents the closure in the topology induced on X).*

Then $\bigcap_{y \in Y} G(y) \neq \emptyset$.

The above theorem is a generalization to transfer FS-convex multimaps of Theorem 2 of [25]. This is a meaningful generalization, in fact it is possible to produce examples of multimaps that are transfer FS-convex but not KKM.

Example 2.7. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a map defined as

$$f(x, y) = \begin{cases} x - y + 1 & x \leq y \\ y - x + 1 & y < x < y + 1 \\ 1 & x \geq y + 1 \end{cases}$$

The multimap $\Phi : \mathbb{R} \multimap \mathbb{R}$ defined as $\Phi(x) = \{y \colon f(x, y) \geq 1\}$ is transfer FS-convex, but not KKM. Indeed let $x_1, x_2 \in \mathbb{R}$, and $\lambda \in]0, 1[$ with $x_1 < x_\lambda < x_2 < x_\lambda + 1$, hence we have

$$f(x_1, x_\lambda) = x_1 - x_\lambda + 1 < 1, \; f(x_2, x_\lambda) = x_\lambda - x_2 + 1 < 1$$

implying $x_\lambda \notin \Phi(x_1)$ and $x_\lambda \notin \Phi(x_2)$. On the other hand, for every $\{x_1, \dots, x_n\} \in \mathbb{R}_+$, let $Y = \{x_1 - 1, \dots, x_n - 1\}$ for every $J \subset \{1, \dots, n\}$ and for every $y \in \mathrm{co}\{y_j, j \in J\}$ it follows $y \leq x_p - 1$, $p = 1, \dots, n$, implying $f(x_p, y) = 1$, $p = 1, \dots, n$. Hence

$$y_\lambda \in \Phi(x_p) \subset \bigcup_{i \in J} \Phi(x_i).$$

If assumption (3) of Theorem 2.6 is replaced by the classical KKM condition, there is no need to assume in (4) that the set Y_o is finite. More precisely, we obtain the following extension to two multimaps of the classical FKKM Theorem (Theorem 2.4).

Theorem 2.8. *Let X be a convex set in a Hausdorff topological vector space, $Y \subset X$ be a non empty set, and $F, G : Y \multimap X$ be two multimaps such that*

(1) $F(y) \subseteq G(y)$ *for every* $y \in Y$;
(2) G *is closed valued;*
(3) F *is a KKM multimap;*
(4) *there is a compact subset* Y_o *of* Y *such that* $\bigcap_{y \in Y_o} \overline{F(y)}^X$ *is compact.*

Then $\bigcap_{y \in Y} G(y) \neq \emptyset$.

Remark 2.9. In the case of one multimap, i.e. if in Theorems 2.6 and 2.8 $F \equiv G : Y \multimap X$, the intersection $\bigcap_{y \in Y} G(y)$ is nonempty and compact. Indeed it is a closed set contained in the compact set $\bigcap_{y \in Y_o} G(y) = \bigcap_{y \in Y_o} \overline{F(y)}^X$.

Remark 2.10. Notice that in Theorem 2.8 it is possible to weaken hypothesis (2) assuming the following condition

(2') $\bigcap_{y \in Y} G(y) = \bigcap_{y \in Y} \overline{G(y)}^X$,

thus obtaining a generalization of Lemma 1 in [26] to non compact multivalued maps.

3. Variational inclusion and matching type results

In this section, as a consequence of Theorem 2.6, we prove several variational inclusions and matching type results. We are able to consider quite general hypotheses of convexity and compactness, in particular we generalize Theorem 3.1 in [11].

In the whole section X and Y denote two Hausdorff topological vector spaces, K a non empty convex subset of X, and M a non empty closed subset of Y.

Theorem 3.1. *Let $F, G : K \times K \multimap Y$ be two multimaps such that*

(1) $F(x, y) \subseteq G(x, y)$ *for every* $(x, y) \in K \times K$;
(2) *for every* $(x_o, y_o) \in K \times K$ *such that* $F(x_o, y_o) \cap M^c \neq \emptyset$ *there exist a neighborhood* $\mathcal{I}$ *of* x_o *such that*

$$F(z, y_o) \cap M^c \neq \emptyset, \text{ for every } z \in \mathcal{I};$$

(3) *for every* $\{y_1, \ldots, y_n\} \subset K$ *there exists* $\{x_1, \ldots, x_n\} \subset K$ *such that for every* $J \subset \{1, \ldots, n\}$ *it follows*

$$co\{x_j, j \in J\} \subset \bigcup_{j \in J} \{x \in K \ : \ G(x, y_j) \subset M\};$$

(4) *there exists a relatively compact set* $A \subset K$ *and a finite set* $Y_o \subset K$ *such that for every* $x \notin A$ *there exists* $y \in Y_o$ *and a neighborhood* $\mathcal{I}$ *of* x *such that* $G(z, y) \cap M^c \neq \emptyset$ *for every* $z \in \mathcal{I}$.

Then there exists $x_o \in K$ *such that* $F(x_o, y) \subset M$ *for every* $y \in K$.

Proof. For any $y \in K$ let

$$\Gamma(y) = \{x \in K : G(x, y) \subset M\} \quad \text{and} \quad \Theta(y) = \{x \in K : F(x, y) \subset M\}.$$

By condition (1) $\Gamma(y) \subset \Theta(y)$ for every $y \in K$.
By assumption (2) it follows easily that the multimap $\Theta : K \multimap K$ is closed valued. Observing that for every $\{y_1, \ldots, y_n\} \subset K$

$$\bigcup_{j=1}^{n} \{x \in K \ : \ G(x, y_j) \subset M\} = \bigcup_{j=1}^{n} \Gamma(y_j),$$

by condition (3) it follows trivially that Γ is transfer FS-convex.

By condition (4) it follows that the set $\bigcap_{y \in Y_o} \overline{\Gamma(y)}$ is a compact set. In fact we can show that it is contained in A. For, if $x \in \bigcap_{y \in Y_o} \overline{\Gamma(y)}$ would exist, with $x \notin A$, then there would be $y_o \in Y_o$ and a neighborhood U of x such that $G(z, y_o) \cap M^c \neq \emptyset$ for every $z \in U$, so $x \notin \overline{\Gamma(y_o)}$ obtaining a contradiction.

Now, applying Theorem 2.6, we obtain that $\bigcap_{y \in K} \Theta(y) \neq \emptyset$, i.e. there exists $x_o \in K$ such that $F(x_o, y) \subset M$ for every $y \in K$. □

In a similar way it is possible to prove the following matching result.

Theorem 3.2. *Let* $F, G : K \times K \multimap Y$ *be two multimaps such that*

(1) $F(x, y) \subseteq G(x, y)$ *for every* $(x, y) \in K \times K$;

(2) *for every* $(x_o, y_o) \in K \times K$ *such that* $G(x_o, y_o) \subset M^c$ *there exist a neighborhood* $\mathcal{I}$ *of* x_o *such that* $G(z, y_o) \subset M^c$, *for every* $z \in \mathcal{I}$;

(3) *for every* $\{y_1, \ldots, y_n\} \subset K$ *there exists* $\{x_1, \ldots, x_n\} \subset K$ *such that for every* $J \subset \{1, \ldots, n\}$ *it follows*

$$co\{x_j, j \in J\} \subset \bigcup_{j \in J} \{x \in K \ : \ F(x, y_j) \cap M \neq \emptyset\};$$

(4) *there exists a relatively compact set* $A \subset K$ *and a finite set* $Y_o \subset K$ *such that for every* $x \notin A$ *there exists* $y \in Y_o$ *and a neighborhood* $\mathcal{I}$ *of* x *such that* $F(z,y) \subset M^c$ *for every* $z \in \mathcal{I}$.

Then there exists $x_o \in K$ *such that* $G(x_o, y) \cap M \neq \emptyset$, *for every* $y \in K$.

Proof. The result can be obtained as in the proof of Theorem 3.1 with $\Gamma : K \multimap K$ and $\Theta : K \multimap K$ given by $\Gamma(y) = \{x \in K \colon G(x,y) \cap M \neq \emptyset\}$ and $\Theta(y) = \{x \in K \colon F(x,y) \cap M \neq \emptyset\}$. □

Clearly condition (2) of Theorems 3.1 and 3.2 are satisfied if the multimaps F and G are respectively lower semicontinuous and upper semicontinuous with respect to the first variable.

Remark 3.3. If in condition (3) of Theorems 3.1 and 3.2 above we consider $y_i = x_i$, for every $i = 1, \dots, n$, then in condition (4) there is no need to assume the existence of the finite set Y_o. More precisely, one requires

(4′) there exists a relatively compact set $E \subset K$ such that for every $x \notin E$ there exists $y \in E$ and a neighborhood U of x such that $G(z,y) \cap M^c \neq \emptyset$ ($F(z,y) \subset M^c$) for every $z \in U$.

Notice that setting $E = Y_o \cup A$ condition (4) implies condition (4′). In this case, the conclusion is obtained applying Theorem 2.8 instead of Theorem 2.6.

Remark 3.4. By Remark 2.10, condition (2) of Theorems 3.1 and 3.2 can be slightly weakened assuming that:

for every $(x_o, y_o) \in K \times K$ such that $F(x_o, y_o) \cap M^c \neq \emptyset$ there exist $y_1 \in K$ and a neighborhood $\mathcal{I}$ of x_o such that

$$F(z, y_1) \cap M^c \neq \emptyset, \text{ for every } z \in \mathcal{I};$$

and: for every $(x_o, y_o) \in K \times K$ such that $G(x_o, y_o) \subset M^c$ there exist $y_1 \in K$ and a neighborhood $\mathcal{I}$ of x_o such that

$$G(z, y_1) \subset M^c, \text{ for every } z \in \mathcal{I};$$

in Theorems 3.1 and 3.2, respectively.

In the case of a unique multimap, i. e. $F \equiv G$, we state the following corollary of Theorem 3.1. Moreover, by Remark 2.9, the "solution set" is compact. In Section 5 we will apply this corollary to obtain a minimax inequality for vector valued functions.

Corollary 3.5. *Let* $G : K \times K \multimap Y$ *a multimap such that*

(1) *the set* $\Theta(y) = \{x \in K \colon G(x,y) \subset M\}$ *is closed for every* $y \in K$;

(2) *for every* $\{y_1, \dots, y_n\} \subset K$ *there exists* $\{x_1, \dots, x_n\} \subset K$ *such that for every* $J \subset \{1, \dots, n\}$ *it follows*

$$co\{x_j, j \in J\} \subset \bigcup_{j \in J} \{x \in K \colon G(x, y_j) \subset M\};$$

(3) *there exists a relatively compact set* $A \subset K$ *and a finite set* $Y_o \subset K$ *such that for every* $x \notin A$ *there exists* $y \in Y_o$ *and there exists a neighborhood* $\mathcal{I}(x)$ *such that* $G(z, y) \subset M^c$, *for every* $z \in \mathcal{I}(x)$.

Then there exists $x_o \in K$ *such that* $G(x_o, y) \subset M$ *for every* $y \in K$.

Furthermore, the set of such x_o *is compact.*

4. Convexity for multivalued maps

In this section we propose an extension to multimaps of the diagonal transfer quasi-convexity (quasi-concavity) introduced in [6] for single valued maps. This is a generalization of the usual quasi-convexity (quasi-concavity). To explain it we recall some classical definitions.

Throughout this section X is a convex subset of a topological vector space and $F : X \multimap \mathbb{R}$ a multimap with nonempty values.

The classical definitions of multivalued convexity and multivalued concavity, due to Nikodem (see [20]) are the following:

Definition 4.1. F is said to be *convex à la Nikodem* (N-conv) (*concave à la Nikodem* (N-conc)), if for every $x_1, x_2 \in X$ and every $\lambda \in]0, 1[$

$$\lambda F(x_1) + (1 - \lambda) F(x_2) \subseteq F(\lambda x_1 + (1 - \lambda) x_2)$$
$$(\lambda F(x_1) + (1 - \lambda) F(x_2) \supseteq F(\lambda x_1 + (1 - \lambda) x_2)).$$

Observe that this concept generalizes that of affine function in the single valued case. Besides the above definitions we can set the following (see [10]).

Definition 4.2. F is said to be *Borwein convex* (B-convex) (*Borwein concave* (B-concave)) if for every pair $x_1, x_2 \in X$, every $\lambda \in]0, 1[$ and every choice $t_i \in F(x_i)$ there exists some $t \in F(\lambda x_1 + (1 - \lambda)x_2)$ with

$$t \le \lambda t_1 + (1 - \lambda)t_2 \quad (t \ge \lambda t_1 + (1 - \lambda)t_2).$$

In other words

$$F(\lambda x_1 + (1 - \lambda)x_2) \cap (-\infty, \lambda t_1 + (1 - \lambda)t_2] \neq \emptyset$$
$$(F(\lambda x_1 + (1 - \lambda)x_2) \cap [\lambda t_1 + (1 - \lambda)t_2, +\infty) \neq \emptyset).$$

It is clear that a multimap that is convex à la Nikodem is both Borwein convex and Borwein concave. Indeed, it is enough to choose $t = \lambda t_1 + (1 - \lambda)t_2$ in Definition 4.2.

A generalization of B-convexity (B-concavity) is the concept of multivalued quasi-convexity (multivalued quasi-concavity) (see [10]).

Definition 4.3. F is said to be *multivalued quasi-convex* (*multivalued quasi-concave*) if for every pair $x_1, x_2 \in X$, every $\lambda \in]0, 1[$ and every choice $t_i \in F(x_i)$, with $t_1 \le t_2$ there exists some $t \in F(\lambda x_1 + (1 - \lambda)x_2)$ with $t \le t_2$ $(t \ge t_1)$.

It is immediate to observe that, equivalently, F is multivalued quasi-convex (multivalued quasi-concave) if for every finite set $\{x_1, \dots, x_n\} \subset X$, every $x \in \text{co}\{x_1, \dots, x_n\}$ and every choice $t_i \in F(x_i)$, $i = 1, \dots, n$ there exists some $t \in F(x)$ with

$$t \leq \max_{1 \leq i \leq n} t_i \quad \left(t \geq \min_{1 \leq i \leq n} t_i \right).$$

Also note that the multivalued quasi-convexity (multivalued quasi-concavity) is equivalent to the convexity of the set

$$L^-_\alpha = \{x \in X \mid F(x) \cap (-\infty, \alpha] \neq \emptyset\} \quad \big(L^+_\alpha = \{x \in X \mid F(x) \cap [\alpha, +\infty) \neq \emptyset\}\big)$$

for every $\alpha \in \mathbb{R}$.

We are now able to give the following multivalued extensions of the definitions of diagonal quasi-convexity (quasi-concavity) and diagonal transfer quasi-convexity (quasi-concavity).

Definition 4.4. A multimap $F : X \times X \multimap \mathbb{R}$ is said to be *diagonal multivalued quasi-convex* (*diagonal multivalued quasi-concave*) in the first variable if for every finite set $\{x_1, \dots, x_n\} \subset X$, for every $x \in \text{co}\{x_1, \dots, x_n\}$, and for every $t_i \in F(x_i, x)$, $i = 1, \dots, n$ there exists some $t \in F(x, x)$ such that

$$t \leq \max_{1 \leq i \leq n} t_i \quad \left(t \geq \min_{1 \leq i \leq n} t_i \right).$$

The above definition is trivially a generalization of the multivalued quasi-convexity and quasi-concavity in the first variable. Moreover, a further generalization of Definition 4.4 is the diagonal transfer quasi-convexity and concavity.

Definition 4.5. A multimap $F : X \times X \multimap \mathbb{R}$ is said to be *diagonal multivalued transfer quasi-convex* (*diagonal multivalued transfer quasi-concave*) in the first variable if for every finite set $\{x_1, \dots, x_n\} \subset X$ there exists $\{y_1, \dots, y_n\} \subset X$ such that for every $Y \subset \{y_1, \dots, y_n\}$, say $Y = \{y_{j1}, \dots, y_{js}\}$, for every $y \in \text{co}Y$, for every $t_{j\ell} \in F(x_{j\ell}, y)$, there exists $t \in F(y, y)$ such that

$$t \leq \max_{1 \leq \ell \leq s} t_{j\ell} \quad \left(t \geq \min_{1 \leq \ell \leq s} t_{j\ell} \right).$$

In order to compare the previous definitions, we shall assume at first that the multimap $F : X \times X \multimap \mathbb{R}$ has *bounded values*; then it is possible to define the functions

$$f(x, y) = \inf F(x, y), \quad g(x, y) = \sup F(x, y).$$

We can then easily characterize diagonal multivalued quasi-convexity (quasi-concavity) and diagonal multivalued transfer quasi-convexity (quasi-concavity) by means of the above two single valued maps.

Proposition 4.6. *Let $F : X \times X \multimap \mathbb{R}$ have bounded values; then the following implications hold*

(1) *If F is diagonal multivalued quasi-convex then f is diagonal quasi-convex;*
(2) *if F is diagonal multivalued transfer quasi-convex then f is diagonal transfer quasi-convex.*

It follows easily that if f is a selection of F, for instance if F has closed values, then the conditions 1 and 2 in Proposition 4.6 immediately becomes an "iff". Hence, using the above result it is possible to provide counterexamples of diagonal multivalued quasi-convex multimaps that fail to be multivalued quasi-convex, as well as diagonal multivalued transfer quasi-convex multimaps that are not diagonal multivalued quasi-convex.

Example 4.7. The map $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ defined as

$$f(x,y) = \begin{cases} -\left(x - \dfrac{y}{2}\right)^2 & 0 \leq x \leq y \\ -\dfrac{y^2}{4} & x > y \end{cases}$$

is diagonal quasi-convex but not quasi-convex. Indeed for every $x_1, x_2 \in \mathbb{R}$ and any $\lambda \in]0,1[$, denoting with $x_\lambda = \lambda x_1 + (1-\lambda)x_2$ we have that

$$f(x_\lambda, x_\lambda) = -\frac{x_\lambda^2}{4} = \min_{\mathbb{R}_+} f(x, x_\lambda) \leq f(x_1, x_\lambda) \vee f(x_2, x_\lambda).$$

On the other hand let $\overline{y} \in \mathbb{R}$, $x_1 \in \left]0, \dfrac{\overline{y}}{2}\right[$ and $x_2 = \overline{y} - x_1$, then

$$f\left(\frac{x_1 + x_2}{2}, \overline{y}\right) = f\left(\frac{\overline{y}}{2}, \overline{y}\right) = \max_{\mathbb{R}_+} f(x, \overline{y}) > f(x_1, \overline{y}) \vee f(x_2, \overline{y}).$$

Example 4.8. The map $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined in Example 2.7 is diagonal transfer quasi-convex, but not diagonal quasi-convex. Indeed let $x_1, x_2 \in \mathbb{R}$, and $\lambda \in]0,1[$ with $x_1 < x_\lambda < x_2 < x_\lambda + 1$, hence we have

$$f(x_\lambda, x_\lambda) = 1 = \max_{x \leq y+1} f(x,y) > f(x_1, x_\lambda) \vee f(x_2, x_\lambda).$$

On the other hand, for every $\{x_1, \ldots, x_n\} \in \mathbb{R}_+$, let $Y = \{x_1 - 1, \ldots, x_n - 1\}$ for every $J \subset \{1, \ldots, n\}$ and for every $y \in \text{co}\{y_j, j \in J\}$ it follows $y \leq x_p - 1$, implying $f(x_p, y) = 1$. Hence

$$\max_{j \in J} f(x_j, y) \geq 1 = f(y,y).$$

Analogous characterizations for diagonal quasi-concavity can be proven.

Proposition 4.9. *Let $F : X \times X \multimap \mathbb{R}$ have bounded values, then the following implications hold*

(1) *If F is diagonal multivalued quasi-concave then g is diagonal quasi-concave;*

(2) *if F is multivalued transfer diagonal quasi-concave then g is diagonal transfer quasi-concave.*

Also in this case if g is a selection of F, for instance if F has closed values, then the conditions (1) and (2) in Proposition 4.9 immediately become an "iff".

5. Minimax relations

In this section, considering $M = [\mu, +\infty)$, or $M = (-\infty, \mu]$, we apply the variational inclusion and matching type results proved in the Section 3 to obtain several generalizations of the classical Ky-Fan minimax inequality. In particular, we generalize Theorems 2.1 and 2.3 in [16].

Next at the end of the section, we consider an ordered topological vector space, (Y, C), with order cone C, and we apply the theorems of Section 3 with $M = -C$ obtaining a minimax inequality for vector valued functions.

In the whole section K is a convex set of a Hausdorff topological vector space.

Theorem 5.1. *Let $\Phi, \Psi : K \times K \multimap \mathbb{R}$ be two multimaps with nonempty values such that*

(1) $\Phi(x, y) \subseteq \Psi(x, y)$ *for every* $(x, y) \in K \times K$;
(2) *there exists $\mu \in \mathbb{R}$ such that $\Psi(x, x) \subset [\mu, +\infty)$ for every $x \in K$;*
(3) *for every $(x_o, y_o) \in K \times K$ such that $\Phi(x_o, y_o) \cap (-\infty, \mu[\neq \emptyset$, there exists a neighborhood $\mathcal{I}$ of x_o such that $\Phi(z, y_o) \cap (-\infty, \mu[\neq \emptyset)$ for every $z \in \mathcal{I}$;*
(4) *Ψ is diagonal multivalued transfer quasi-convex in the second variable;*
(5) *there exists a relatively compact set $A \subset K$ and a finite set $Y_o \subset K$ such that for every $x \notin A$ there exists $y \in Y_o$ and a neighborhood $\mathcal{I}$ of x such that $\Psi(z, y) \cap (-\infty, \mu[\neq \emptyset$ for every $z \in \mathcal{I}$.*

Then there exists $x_o \in K$ such that $\Phi(x_o, y) \subset [\mu, +\infty)$ for every $y \in K$.

Proof. We will prove that all the conditions of Theorem 3.1 with $M = [\mu, +\infty)$ are satisfied.
Conditions (2) and (4) together imply condition (3) of Theorem 3.1. Indeed, by contradiction assume that there exists a finite set of points $\{y_1, \ldots, y_n\} \in K$ such that for every $\{x_1, \ldots, x_n\} \subset K$ there exists $J \subset \{1, \ldots, n\}$ such that

$$\mathrm{co}\{x_j, j \in J\} \not\subset \bigcup_{j \in J} \{x \in K \;:\; \Psi(x, y_j) \subset [\mu, +\infty)\}.$$

Hence there exists $\overline{x} \in \mathrm{co}\{x_j, j \in J\}$ such that $\Psi(\overline{x}, y_j) \not\subset [\mu, +\infty)$ for each $y_j, j \in J$. Thus, for every $j \in J$ there exists $t_j \in \Psi(\overline{x}, y_j)$ such that $t_j < \mu$.

Let $\{x_1, \ldots, x_n\} \subset K$ satisfy the condition in the Definition 4.5, hence by the diagonal multivalued transfer quasi-convexity of $\Psi(x, \cdot)$, it follows that there exists $t \in \Psi(\overline{x}, \overline{x})$ such that

$$t \leq \max_{j \in J} t_j < \mu.$$

Therefore $\Psi(\overline{x},\overline{x}) \cap (-\infty,\mu[\neq \emptyset$ in contradiction with the definition of μ. Condition (3) and (5) are exactly condition (2) and (4) of Theorem 3.1. Then applying Theorem 3.1 we obtain the claimed result. □

Remark 5.2. Notice that condition (5) in Theorem 5.1 is weaker than condition (iv) in Theorem 2.3 of [16], precisely:

(iv) there exists a compact set K_o and an element $y_o \in \overline{K} \cap K_o$ such that

$$\Psi(x,y_o) \cap (-\infty,\mu[\neq \emptyset, \text{ for every } x \in \overline{K} \setminus K_o.$$

Indeed assume that the latter condition holds and consider $A = \overline{K} \cap K_o$. Then by the closedness of A, there exists an open set $\mathcal{I} \subset A^c$ such that

$$\Psi(z,y_o) \cap (-\infty,\mu[\neq \emptyset, \text{ for every } z \in \mathcal{I}.$$

Similarly the following result holds.

Theorem 5.3. *Let $\Phi,\Psi : K \times K \multimap \mathbb{R}$ be two multimaps with nonempty values such that*

(1) $\Phi(x,y) \subseteq \Psi(x,y)$ *for every* $(x,y) \in K \times K$;
(2) *there exists* $\mu \in \mathbb{R}$ *such that* $\Psi(x,x) \subset (-\infty,\mu]$ *for every* $x \in K$;
(3) *for every* $(x_o,y_o) \in K \times K$ *such that* $\Phi(x_o,y_o) \cap]\mu,+\infty) \neq \emptyset$, *there exists a neighborhood* $\mathcal{I}$ *of* x_o *such that* $\Phi(z,y_o) \cap]\mu,+\infty) \neq \emptyset$ *for every* $z \in \mathcal{I}$;
(4) Ψ *is diagonal multivalued transfer quasi-concave in the second variable;*
(5) *there exists a relatively compact set* $A \subset K$ *and a finite set* $Y_o \subset K$ *such that for every* $x \notin A$ *there exists* $y \in Y_o$ *and a neighborhood* $\mathcal{I}$ *of* x *such that* $\Psi(z,y) \cap]\mu,+\infty) \neq \emptyset$ *for every* $z \in \mathcal{I}$.

Then there exists $x_o \in K$ *such that* $\Phi(x_o,y) \subset (-\infty,\mu]$ *for every* $y \in K$.

Analogously, applying Theorem 3.2, it is possible to prove the following dual results.

Theorem 5.4. *Let $\Phi,\Psi : K \times K \multimap \mathbb{R}$ be two multimaps with nonempty values such that*

(1) $\Phi(x,y) \subseteq \Psi(x,y)$ *for every* $(x,y) \in K \times K$;
(2) *there exists* $\mu \in \mathbb{R}$ *such that* $\Phi(x,x) \subset [\mu,+\infty)$ *for every* $x \in K$;
(3) *for every* $(x_o,y_o) \in K \times K$ *such that* $\Psi(x_o,y_o) \subset (-\infty,\mu[$ *there exists a neighborhood* $\mathcal{I}$ *of* x_o *such that* $\Psi(z,y_o) \subset (-\infty,\mu[$ *for every* $z \in \mathcal{I}$;
(4) Φ *is diagonal multivalued transfer quasi-convex in the second variable;*
(5) *there exist a relatively compact set* $A \subset K$ *and a finite set* $Y_o \subset K$ *such that for every* $x \notin A$ *there exist* $y \in Y_o$ *and a neighborhood* $\mathcal{I}$ *of* x *such that* $\Phi(z,y) \subset (-\infty,\mu[$ *for every* $z \in \mathcal{I}$.

Then there exists $x_o \in K$ *such that* $\Psi(x_o,y) \cap [\mu,+\infty) \neq \emptyset$ *for every* $y \in K$.

Theorem 5.5. *Let $\Phi, \Psi : K \times K \multimap \mathbb{R}$ be two multimaps with nonempty values such that*

(1) $\Phi(x,y) \subseteq \Psi(x,y)$ *for every* $(x,y) \in K \times K$;
(2) *there exists* $\mu \in \mathbb{R}$ *such that* $\Phi(x,x) \subset (-\infty, \mu]$ *for every* $x \in K$;
(3) *for every* $(x_o, y_o) \in K \times K$ *such that* $\Psi(x_o, y_o) \subset]\mu, +\infty)$ *there exists a neighborhood* $\mathcal{I}$ *of* x_o *such that* $\Psi(z, y_o) \subset]\mu, +\infty)$ *for every* $z \in \mathcal{I}$;
(4) Φ *is diagonal multivalued transfer quasi-concave in the second variable;*
(5) *there exist a relatively compact set* $A \subset K$ *and a finite set* $Y_o \subset K$ *such that for every* $x \notin A$ *there exist* $y \in Y_o$ *and a neighborhood* $\mathcal{I}$ *of* x *such that* $\Phi(z,y) \subset]\mu, +\infty)$ *for every* $z \in \mathcal{I}$.

Then there exists $x_o \in K$ *such that*

$$\Psi(x_o, y) \cap (-\infty, \mu] \neq \emptyset \quad \textit{for every } y \in K.$$

Observe that it is not possible to compare the previous theorem with Theorem 2.2 in [16]. Indeed, in the cited paper Krystaly and Varga assume the weaker following condition

(KV) $\Phi(x,x) \cap (-\infty, 0] \neq \emptyset$ for every $x \in X$

instead of condition (2) above with $\mu = 0$. On the other hand, they assume that the multimap Φ is concave à la Nikodem in the second variable, an assumption much stronger than condition (4). With the following simple example in the case of one multimap, i.e. $\Phi \equiv \Psi$, and with $\mu = 0$, we show that it is not possible to replace condition (2) with assumption (KV).

Example 5.6. The multimap $\Phi : [0,1] \times [0,1] \multimap \mathbb{R}$ defined as

$$\Phi(x,y) = \begin{cases} 1 & x \neq y \\ [-1,1] & x = y \end{cases}$$

is upper semicontinuous in the first variable, diagonal multivalued quasi-concave in the second variable, but fails to be concave à la Nikodem. Moreover it satisfies (KV) but not condition (2) and does not exists an element $x_o \in [0,1]$ such that

$$\Phi(x_o, y) \cap (-\infty, 0] \neq \emptyset \quad \text{for every } y \in X.$$

Remark 5.7. In all the above minimax type inequalities if condition (4) is replaced by

(4′) Φ (or Ψ) is diagonal multivalued quasi-convex (or quasi-concave) in the second variable

then in condition (5) there is no need to assume the existence of the finite set Y_o. More precisely one requires, for instance in Theorem 5.1, that there holds

(5′) there exists a relatively compact set E such that for every $x \notin E$ there is $y \in E$ and a neighborhood $\mathcal{I}$ of x such that $\Psi(z,y) \cap (-\infty, \mu[\neq \emptyset$ for each $z \in \mathcal{I}$.

Note that this assumption is automatically guaranteed by condition (5). Indeed one sets $E = A \cup Y_o$.

Remark 5.8. Introducing a generalization to the multivalued case of the diagonal transfer continuity for single valued maps (see [6]), it is possible to slightly weaken the hypotheses of lower/upper semicontinuity type in Theorems 5.1, 5.3, 5.4, and 5.5. More precisely, in Theorems 5.1 and 5.4 it is possible to substitute condition (3) with the following

(3') for every $(x_o, y_o) \in K \times K$ such that $\Phi(x_o, y_o) \cap (-\infty, \mu[\neq \emptyset$, there exists $y_1 \in Y$ and a neighborhood $\mathcal{I}$ of x_o such that $\Phi(z, y_1) \cap (-\infty, \mu[\neq \emptyset)$ for every $z \in \mathcal{I}$.

(3') for every $(x_o, y_o) \in K \times K$ such that $\Psi(x_o, y_o) \subset (-\infty, \mu[$, there exists $y_1 \in Y$ and a neighborhood $\mathcal{I}$ of x_o such that $\Psi(z, y_1) \subset (-\infty, \mu[$ for every $z \in \mathcal{I}$.

respectively. Considering the interval $]\mu, +\infty)$ it is possible to give analogous conditions for Theorem 5.3 and 5.5.

As a consequence of Theorem 5.3 we state the following minimax type results for the single valued case.

Theorem 5.9. *Let $\phi, \psi : K \times K \to \mathbb{R}$ be two maps such that*

(1) *$\phi(x, y) \leq \psi(x, y)$ for every $(x, y) \in K \times K$;*

(2) *there exists $\mu \in \mathbb{R}$ such that $\psi(x, x) \leq \mu$ for every $x \in K$;*

(3) *for every $(x_o, y_o) \in X \times Y$ such that $\phi(x_o, y_o) > \mu$ there exists a neighborhood $\mathcal{I}$ of x_o such that $\phi(z, y_o) > \mu$ for every $z \in \mathcal{I}$;*

(4) *ψ is diagonal transfer quasi-concave in the second variable;*

(5) *there exist a relatively compact set $A \subset K$ and a finite set $Y_o \subset K$ such that for every $x \notin A$ there exist $y \in Y_o$ and a neighborhood $\mathcal{I}$ of x such that $\psi(z, y) > \mu$ for every $z \in \mathcal{I}$.*

Then there exists $x_o \in K$ such that $\phi(x_o, y) \leq \mu$ for every $y \in K$.

Proof. Let $\Phi : K \times K \multimap \mathbb{R}$ and $\Psi : K \times K \multimap \mathbb{R}$ two multimaps defined respectively as

$$\Phi(x, y) = (-\infty, \phi(x, y)], \quad \Psi(x, y) = (-\infty, \psi(x, y)].$$

By condition (1) it follows that $\Phi(x, y) \subset \Psi(x, y)$ for every $(x, y) \in K \times K$.
Moreover if $(x_o, y_o) \in K \times K$ is such that $\Phi(x_o, y_o) \cap]\mu, +\infty) \neq \emptyset$, it follows that $\phi(x_o, y_o) > \mu$, hence by condition (3) there exists a neighborhood $\mathcal{I}$ of x_o such that $\phi(z, y_o) > \mu$ for every $z \in \mathcal{I}$, implying $\Phi(z, y_o) \cap]\mu, +\infty) \neq \emptyset$ for every $z \in \mathcal{I}$, i.e. condition (3) of Theorem 5.3.
Furthermore by Proposition 4.9 and the fact that Ψ has closed values it follows that Ψ is diagonal multivalued transfer quasi-concave with respect to the second variable.

Trivially, condition (5) implies condition (5) of Theorem 5.3. Hence by Theorem 5.3 we have that there exists $x_o \in K$ such that

$$(-\infty, \phi(x_o, y)] \subset (-\infty, \mu] \text{ for every } y \in K,$$

i.e. $\phi(x_o, y) \leq \mu$ for every $y \in K$. □

We can prove also the following version of a minimax inequality with three maps.

Theorem 5.10. *Let $\phi, \psi, h : K \times K \to \mathbb{R}$ be three maps such that*

(1) *$\phi(x, y) \leq \psi(x, y) \leq h(x, y)$ for every $(x, y) \in K \times K$;*

(2) *there exists $\mu \in \mathbb{R}$ such that $h(x, x) \leq \mu$ for every $x \in K$;*

(3) *for every $(x_o, y_o) \in X \times Y$ such that $\phi(x_o, y_o) > \mu$ there exists a neighborhood $\mathcal{I}$ of x_o such that $\phi(z, y_o) > \mu$ for every $z \in \mathcal{I}$;*

(4) *h is diagonal transfer quasi-concave in the second variable;*

(5) *there exist a relatively compact set $A \subset K$ and a finite set $Y_o \subset K$ such that for every $x \notin A$ there exist $y \in Y_o$ and a neighborhood $\mathcal{I}$ of x such that $\psi(z, y) > \mu$ for every $z \in \mathcal{I}$.*

Then there exists $x_o \in K$ such that $\phi(x_o, y) \leq \mu$ for every $y \in K$.

Proof. Let $\Phi : K \times K \multimap \mathbb{R}$, $\Psi : K \times K \multimap \mathbb{R}$ and $H : K \times K \multimap \mathbb{R}$ be three multimaps defined respectively as

$$\Phi(x, y) = (-\infty, \phi(x, y)],\ \Psi(x, y) = (-\infty, \psi(x, y)],\ H(x, y) = (-\infty, h(x, y)]$$

By condition (1) it follows that $\Phi(x, y) \subset \Psi(x, y) \subset H(x, y)$ for every $(x, y) \in K \times K$. Now, define the corresponding multimaps $\Gamma, \Theta, \Lambda : K \multimap K$ as follows:

$$\Gamma(y) = \{x \in K \ : \ \Phi(x, y) \subset (-\infty, \mu]\}\,;$$
$$\Theta(y) = \{x \in K \ : \ \Psi(x, y) \subset (-\infty, \mu]\}\,;$$
$$\Lambda(y) = \{x \in K \ : \ H(x, y) \subset (-\infty, \mu]\}\,.$$

Notice that $\Lambda(y) \subset \Theta(y) \subset \Gamma(y)$, for every $y \in K$. By condition (3), it follows easily that the set $\Gamma(y)$ is closed for every $y \in K$; by condition (2) and (4) that for every $n \in \mathbb{N}$

$$\bigcap_{i=1}^{n} \Lambda(y_i) \neq \emptyset$$

and by condition (5) that

$$D := \bigcap_{y \in Y_o} \overline{\Theta(y)}$$

is a compact set. Hence

$$\emptyset \neq \bigcap_{y \in Y_o} \Lambda(y) \cap \bigcap_{i=1}^{n} \Lambda(y) \subset D \cap \bigcap_{i=1}^{n} \Gamma(y).$$

Hence we have proven that the family of sets $\{D \cap \Gamma(y), y \in K\}$ has the finite intersection property. Then, by the compactness of $D \cap \Gamma(y)$ we have that

$$\emptyset \neq \bigcap_{y \in K} D \cap \Gamma(y) \subset \bigcap_{y \in K} \Gamma(y),$$

obtaining the claimed result. □

Remark 5.11. Also for the single valued case in Theorems 5.9 and 5.10 it is possible to weaken continuity to transfer continuity.

The single valued versions of Theorems 5.1, 5.4 and 5.5 follow trivially. Observe that in the case of only one multimap, i.e. $\Phi \equiv \Psi$, the Theorems 5.1 and 5.4 in the case of one single valued map reduce to the same theorem. The same is true for Theorems 5.3 and 5.5. Moreover, by Remark 2.9, the "solution set" is compact. More precisely, we can state the following minimax inequalities for one single valued map.

Theorem 5.12. *Let $\phi : K \times K \to \mathbb{R}$ a map such that*

(1) *there exists $\mu \in \mathbb{R}$ such that $\phi(x, x) \geq \mu$ for every $x \in K$;*

(2) *for every $(x_o, y_o) \in X \times Y$ such that $\phi(x_o, y_o) < \mu$ there exists a neighborhood $\mathcal{I}$ of x_o such that $\phi(z, y_o) < \mu$ for every $z \in \mathcal{I}$;*

(3) *is diagonal transfer quasi-convex in the second variable;*

(4) *there exist a relatively compact set $A \subset K$ and a finite set Y_o such that for every $x \notin A$ there exist $y \in Y_o$ and a neighborhood $\mathcal{I}$ of x such that $\phi(z, y) < \mu$ for every $z \in \mathcal{I}$.*

Then there exists $x_o \in K$ such that $\phi(x_o, y) \geq \mu$ for every $y \in K$.
Furthermore, the set of such x_o is compact.

Theorem 5.13. *Let $\phi : K \times K \to \mathbb{R}$ a map such that*

(1) *there exists $\mu \in \mathbb{R}$ such that $\phi(x, x) \leq \mu$ for every $x \in K$;*

(2) *for every $(x_o, y_o) \in X \times Y$ such that $\phi(x_o, y_o) > \mu$ there exists a neighborhood $\mathcal{I}$ of x_o such that $\phi(z, y_o) > \mu$ for every $z \in \mathcal{I}$;*

(3) *is diagonal transfer quasi-concave in the second variable;*

(4) *there exist a relatively compact set $A \subset K$ and a finite set Y_o such that for every $x \notin A$ there exist $y \in Y_o$ and a neighborhood $\mathcal{I}$ of x such that $\phi(z, y) > \mu$ for every $z \in \mathcal{I}$.*

Then there exists $x_o \in K$ such that $\phi(x_o, y) \leq \mu$ for every $y \in K$.
Furthermore the set of such x_o is compact.

As mentioned at the beginning we shall conclude this section presenting minimax inequalities for vector valued functions in the stream of ideas of [4], [5], [18], [19]. Let (Y, C) be an ordered topological vector space with order cone C. First of all we adapt the definition of diagonal quasi-concavity.

Definition 5.14. A map $f : K \times K \to Y$ is said to be *diagonal quasi-concave* in the second variable if for every $\{y_1, \dots, y_n\} \subset K$, for every $y \in \text{co}\{y_1, \dots, y_n\}$ there exists $i \in \{1, \dots, n\}$ such that $f(y, y_i) \in f(y, y) - C$.

For the semicontinuity condition the results in [18], [19] assumed that the interior of C is non-empty. Then one states the following definition.

Definition 5.15. A map $f : K \to Y$ is said to be *lower semicontinuous (l.s.c.) at x_o* if for every $\varepsilon \in C^o$ there exists a neighborhood U of x_o such that

$$f(x) - f(x_o) + \varepsilon \in C^o$$

for every $x \in U$. The map f is said to be *lower semicontinuous* if it is lower semicontinuous at x_o for every $x_o \in K$.

Alternatively, one defines weak semicontinuity as follows.

Definition 5.16. A map $f : K \to Y$ is said to be *weakly lower semicontinuous (weakly l.s.c.)* if for every $z \in Y$ the level set $L(z) = \{x \colon f(x) \in z - C\}$ is closed.

Theorem 5.17. *Let $f : K \times K \to Y$ be a map such that*

(1) *f is weakly lower semicontinuous with respect to the first variable;*
(2) *f is diagonal quasi-concave with respect to the second variable;*
(3) *$f(x, x) \in -C$ for every $x \in K$;*
(4) *there exists a compact set $A \subset K$ such that for every $x \notin A$ there exists $y \in A$ such that $f(x, y) \in -C$.*

Then there exists $x_o \in K$ such that $f(x_o, y) \in -C$ for every $y \in K$.
Furthermore the set of such x_o is compact.

Proof. Let $\Theta : K \multimap K$ be the multimap defined as

$$\Theta(y) = \{x \in K \ : \ f(x, y) \in -C\}.$$

We shall prove that Corollary 3.5 applies with $M = -C$.

The set $\Theta(y)$ is closed by Definition 5.16. Also condition (2) of Corollary 3.5 is satisfied.

By contradiction assume that there exists a finite set of points $\{y_1, \dots, y_n\} \subset K$ such that for every $\{x_1, \dots, x_n\} \subset K$ there exists $J \subset \{1, \dots, n\}$ such that

$$\text{co}\{x_j, j \in J\} \not\subset \bigcup_{j \in J} \{x \in K \colon f(x, y_j) \in -C\}.$$

Hence there exists $\overline{x} \in \text{co}\{x_j, j \in J\}$ such that $f(\overline{x}, y_j) \notin -C$. Thus, by condition (2) this implies that $f(\overline{x}, \overline{x}) \notin -C$ obtaining a contradiction with condition (3). Finally, condition (4) is the analogous of condition (3) of Corollary 3.5.
Hence all conditions of Corollary 3.5 with $M = -C$ are satisfied and we obtain that there exists $x_o \in K$ such that $f(x_o, y) \in -C$, for every $y \in K$. □

Clearly Theorem 5.17 can be proven under the weaker condition

(2*) for each finite set $\{y_1, \ldots, y_n\}$ and each $y \in \mathrm{co}\{y_i, , i\}$ there is $i \in \{1, \ldots, n\}$ for which $f(y, y_i) \in -C$.

Under this weaker formulation one generalizes for example Theorem 13 in [18]:

Theorem 5.18. *Let (Y, C) be a Banach lattice with $C^o \neq \emptyset$ and let $f : K \times K \to Y$ satisfy*

(i) *f is lower semicontinuous with respect to the first variable;*
(ii) *for each $x \in K$ the set $B(x) = \{y \in K : f(x, y) \notin -C\}$ is convex;*
(iii) $\sup_{x \in K} f(x, x) \in -C$

Then there exists $x_o \in K$ such that $f(x_o, y) \in -C$ for every $y \in K$.

Proof. Indeed, assumption (i) implies that f is weakly l.s.c. (see Prop 2.3 in [18]), while assumption (iii) is equivalent to (3) of Theorem 5.17.

Finally, under assumption (ii), also (2*) is fulfilled: suppose that $\{y_1, \ldots, y_n\} \subset K$ and $\bar{y} \in \mathrm{co}\{y_i, i\}$ can be found, such that $f(y, y_i) \notin -C$ for each $i = 1, \ldots, n$.

Thus $y_i \in B(\bar{y})$ for each i, and by (ii) $\bar{y} \in B(\bar{y})$ which contradicts (iii). □

6. Applications

In this section we provide several applications of the minimax inequalities proved in the previous section. In particular, we generalize a class of existence theorems on the maximal elements of binary relations, price equilibria, fixed points and variational inequalities by relaxing the compactness and convexity of choice sets and the continuity of the involved maps.

6.1. Fixed point theorems and variational inequalities

First, we prove a basic result from which a generalization of Shauder fixed point Theorem and of the classical Hartmann-Stampacchia existence result for variational inequalities follows.

Theorem 6.1. *Let K be a convex subset of a real normed space E, $f : K \to E$ a map such that*

(1) *for every $(\overline{x}, \overline{y})$ such that $\|\overline{x} - f(\overline{x})\| > \|\overline{y} - f(\overline{x})\|$ there exists a neighborhood $\mathcal{I}$ of $\overline{x}$ such that $\|z - f(z)\| > \|\overline{y} - f(z)\|$ for every $z \in \mathcal{I}$;*
(2) *there exist a relatively compact set $A \subset K$ and a finite set Y_o such that for every $x \notin A$ there exist $y \in Y_o$ and a neighborhood $\mathcal{I}$ of x such that $\|z - f(z)\| > \|y - f(z)\|$ for every $z \in \mathcal{I}$.*

Hence there exists a point $x_o \in K$ such that

$$\|x_o - f(x_o)\| \leq \|y - f(x_o)\| \quad \textit{for every}\, y \in K.$$

Furthermore the set of such x_o is relatively compact.

Proof. All the assumptions of Theorem 5.12 for the map $\phi : X \times X \to \mathbb{R}$ defined as

$$\phi(x, y) = \|y - f(x)\| - \|x - f(x)\|$$

and $\mu = 0$ can be easily verified. Hence, applying the cited theorem we obtain the assertion. □

Remark 6.2. If the function $f : X \to E$ is continuous then condition (1) of the previous theorem is verified and condition (2) becomes

(2) there exist a relatively compact set $A \subset K$ and a finite set Y_o such that for every $x \notin A$ there exists $y \in Y_o$ such that $\|x - f(x)\| > \|y - f(x)\|$.

Two consequences of this result follow easily, namely a generalization of the Schauder Fixed Point Theorem and of the classical Hartmann-Stampacchia existence result for variational inequalities.

Corollary 6.3. *Let X be a convex set of a real normed space E, $f : X \to X$ be a map satisfying hypothesis (1) and (2) of Theorem 6.1. Then there exists a fixed point $x_o \in X, x_o = f(x_o)$. Furthermore the set of fixed points is relatively compact.*

Proof. By Theorem 6.1, for $y = f(x_o)$ we obtain $\|x_o - f(x_o)\| \leq 0$, and thus $x_o = f(x_o)$. □

Remark 6.4. In an analogous way it is possible to obtain a generalization of Kakutani Fixed Point Theorem for a multimap $G : X \multimap X$ applying Theorem 5.1 to the map $\Psi \equiv \Phi : X \times X \to \mathbb{R}$ defined as

$$\Phi(x, y) = d(y, G(x)) - d(x, G(x)),$$

where d is the usual distance $d(y, G(x)) = \inf_{z \in G(x)} \|y - z\|$.

Corollary 6.5. *Let X_R be the closed ball of radius R of a Hilbert space H, $f : X_R \to H$ be a map satisfying hypotheses* (1) *and* (2) *of Theorem* 6.1. *Then either*

(1) *there exists a fixed point $x_o \in X_R$, $x_o = f(x_o)$,*

or

(2) *there exists a solution $x_o \in X_R$, of the following variational inequality.*

$$\langle f(x_o), x - x_o \rangle \leq 0 \ \textit{ for every } x \in X_R.$$

Furthermore the set of such x_o is relatively compact.

Proof. From Theorem 6.1, we obtain the existence of a point $x_o \in X_R$ such that

$$\|x_o - f(x_o)\| = \inf_{x \in X_R} \|x - f(x_o)\|. \tag{1}$$

That is, x_o is the metric projection of $f(x_o)$ on X_R. Hence, if $f(x_o) \in X_R$ we obtain $x_o = f(x_o)$.

If $f(x_o) \in H \setminus X_R$, by the characterization of the metric projection (1) (see, e.g., Theorem 2.3 [15]) we have that $\langle x_o - f(x_o), x - x_o\rangle \geq 0$ for every $x \in X_R$ and $\|x_o\| = R$. Hence

$$\begin{aligned}\langle f(x_o), x - x_o\rangle &\leq \langle x_o, x - x_o\rangle \leq \langle x_o, x\rangle - \|x_o\|^2 \\ &\leq \|x_o\|(\|x\| - \|x_o\|) = R(\|x\| - R) \leq 0\end{aligned}$$

for every $x \in X_R$, obtaining the claimed result. The compactness follows directly from Theorem 6.1. □

6.2. Maximization of binary relations

A *binary relation* U on a set K is a multimap, $U : K \multimap K$, from K into itself with possibly empty values. We can write $y \in U(x)$ to mean that y stands in the relation U to x. A *maximal element* of the binary relation U is a point x such that no point y satisfies $y \in U(x)$, i.e. $U(x) = \emptyset$. Thus, denoting with $K_o = \{x : U(x) \neq \emptyset\}$, the set of maximal elements of U is equal to $K \setminus K_o$ (see, e.g. [9]).

Theorem 6.6. *Let K be a convex subset of a metric space, $U : K \multimap K$ be a relation, $\alpha > 0$ be the Kuratowski measure of non-compactness of $U(K)$. Assume that*

(a) *$x \notin U(x)$, for every $x \in K$;*
(b) *there exists $t \in (0, 1)$ such that diam$(U(x)) < t\alpha$, for every $x \in K$;*
(c) *$U^{-1}(y)$ is open for every $y \in K$;*
(d) *there exist a relatively compact set $A \subset K$ and a finite set Y_o such that for every $x \notin A$ there exist $y \in Y_o$ and a neighborhood $\mathcal{I}$ of x such that $\bigcap_{z\in\mathcal{I}\cap K_o} U(z) \neq \emptyset$ for every $z \in \mathcal{I}$.*

Then $K \setminus K_o$ is nonempty and relatively compact.

Proof. Let $f : K \times K \to \mathbb{R}$ be the map defined as $f(x, y) = \begin{cases} 1 & y \in U(x), \\ 0 & \text{otherwise.}\end{cases}$

We will show that all the hypotheses of Theorem 5.13 with $\mu = 0$ are satisfied.

By assumption (a) $f(x, x) = 0$, for every $x \in K$.

Let $(x_o, y_o) \in K$ such that $f(x_o, y_o) > 0$, hence $y_o \in U(x_o)$ and so $x_o \in K_o$. By (c) there exists a neighborhood $\mathcal{I}(x_o)$ such that $y_o \in U(z)$ for every $z \in \mathcal{I}(x_o)$, i.e. $f(z, y_o) = 1 > 0$ for every $z \in \mathcal{I}(x_o)$.

Now, according to Definition 4.5, we have to prove that for every finite set $\{y_1, \ldots, y_n\} \subset X$ there exists $\{x_1, \ldots, x_n\} \subset K$ such that for every subset $J \subset \{x_1, \ldots, x_n\}$, say $J = \{x_{j1}, \ldots, x_{js}\}$, for every $\xi \in \text{co}J$,

$$0 = f(\xi, \xi) \geq \min_{1\leq\ell\leq s} f(\xi, y_{j\ell}).$$

Assume by contradiction that there exists a finite set $\{y_1,\dots,y_n\} \subset X$ such that for every corresponding set $\{x_1,\dots,x_n\}$ there exists $J \subset \{1,\dots,n\}$ and there exists $x \in \mathrm{co}\{x_j,\, j \in J\}$ such that $f(x,y_j) = 1$, i.e. such that $y_j \in U(x)$, for every $j \in J$. Then, it follows that

$$U(K) \subset \bigcup_{i=1}^{n} B\left(y_i, t\alpha\right) \tag{2}$$

thus, obtaining a contradiction with the definition of $\alpha(U(K))$.

Indeed, let $x \in U(K)$, then there exists $z \in K$ such that $x \in U(z)$, considering $\{x_1,\dots,x_n\} = \{z\}$, we have that $y_i \in U(z)$ for every $i = 1,\dots,n$ then by condition (b) we have that $U(z) \subset B(y_i, t\alpha)$, proving (2).

Finally, directly from (d) we obtain condition (4) of Theorem 5.13 and hence the thesis. □

Notice that by (b) K in the above theorem is necessarily non compact. We will see now another result of the same kind without assuming this condition.

Theorem 6.7. *Let K be a convex subset of a metric space, $U : K \multimap K$ be a relation. Assume that*

(a) $x \notin U(x)$, *for every* $x \in K$;
(b) $x \in U(y)$ *if and only if* $y \in U(x)$, *for every* $x, y \in K$;
(c) *for every* $\{y_1,\dots,y_n\} \subset K$, $K \setminus \bigcup_{i=1}^{n} U(y_i) \neq \emptyset$;
(d) $U^{-1}(y)$ *is open for every* $y \in K$;
(e) *there exist a relatively compact set $A \subset K$ and a finite set Y_o such that for every $x \notin A$ there exist $y \in Y_o$ and a neighborhood $\mathcal{I}$ of x such that* $\bigcap_{z \in \mathcal{I} \cap K_o} U(z) \neq \emptyset$ *for every* $z \in \mathcal{I}$.

Then $K \setminus K_o$ is nonempty and relatively compact.

Proof. We define a map $f : K \times K \to \mathbb{R}$ as in Theorem 6.6 and we prove that all the hypotheses of Theorem 5.13 with $\mu = 0$ are satisfied.

We obtain conditions (1), (2) and (4) of Theorem 5.13 exactly as in the proof of Theorem 6.6, we have to prove only condition (3).

For any $\{y_1,\dots,y_n\} \subset K$, let $\overline{x} \in K \setminus \bigcup_{i=1}^{n} U(y_i)$ and $\{x_1,\dots,x_n\} = \{\overline{x}\}$, then $\overline{x} \notin U(y_i)$ for every $i = 1,\dots,n$. Hence, by condition (b), $y_i \notin U(\overline{x})$, for every $i = 1,\dots,n$. This ensures that for every $J \subseteq \{1,\dots,n\}$ $\min_{j\in J} f(\overline{x}, y_j) = 0$. Then

$$0 = f(\overline{x}, \overline{x}) = \min_{j \in J} f(\overline{x}, y_j)$$

and we obtain the claimed result. □

The following example shows that the above result does not hold without assumption (c).

Example 6.8. Let $K = [0,1]$, $U(x) = \{y \ : \ |x-y| > 0\}$. Then $K \setminus K_o = \emptyset$, although U satisfies conditions (a)-(b)-(d)-(e) of Theorem 6.7. In fact condition (c) does not hold.

Remark 6.9. Replacing in Theorem 5.13 assumption (2) by means of the so called μ-transfer lower semicontinuity (see [17, 25]), i.e.:

> for every $(x_o, y_o) \in X \times Y$ such that $\phi(x_o, y_o) > \mu$ there exists $y_1 \in Y$ and a neighborhood U of x_o such that $\phi(z, y_1) > \mu$ for every $z \in U$,

it is possible to weaken conditions (c) and (d) in Theorems 6.6 6.7 respectively as

> for every $x \in K_o$ and any $y \in U(x)$ there exists a neighborhood of x, $\mathcal{I}(x)$, such that $y \in \bigcap_{z \in \mathcal{I} \cap K_o} U(z)$.

Let now $\succeq$ be an ordering preference relation on a topological space Y determined by an utility function $u : Y \to \mathbb{R}$, i.e. $x \succeq y$ if and only if $u(x) \geq u(y)$. Consider the binary relation $U : Y \multimap Y$ that associates to y the weakly upper contour set $U(y) = \{x \ : \ u(x) \geq u(y)\}$. In this context a maximal element with respect to $\succeq$ is a common point for the family of values of U. As observed in the Introduction the multimap U is transfer FS convex. Hence the next result follows easily from Theorem 2.6.

Theorem 6.10. *Let $u : Y \to \mathbb{R}$ be a continuous map such that there exists $y_o \in Y$ such that $\{x \colon u(x) \geq u(y_o)\}$ is a compact set. Then there exists a maximal element of the relation U, i.e. $\overline{y} \in \bigcap_{y \in Y} U(y)$.*

6.3. Non-cooperative equilibrium in n-person games

In this subsection we consider the decision rules of n players determined by loss functions. Following [27] we denote with E the set of multi-strategies, with E^i the strategy set of the player i and with $\widehat{E}^i$ the strategy set of all the players but i, i.e. $\widehat{E}^i = \Pi_{j \neq i}^n E^i$, $\widehat{x}^i = (x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n) \in \widehat{E}^i$. Hence the set of multi strategies $x := (x^i, \widehat{x}^i)$ may be written as $E = E^i \times \widehat{E}^i$. The loss function is denoted by $f^i : E^i \to \mathbb{R}$ and the associated decision rules are defined by

$$C^i : \widehat{E}^i \multimap E^i, C(\widehat{x}^i) = \{x^i \in E^i : f^i(x^i, \widehat{x}^i) = \inf_{y^i \in E^i} f^i(y^i, \widehat{x}^i)\}.$$

Definition 6.11. A *non-cooperative equilibrium (or Nash-equilibrium)* is a fixed point of the multimap

$$C : E \multimap E, \quad C(x) = \Pi_{i=1}^n C^i(\widehat{x}^i).$$

Defining the function $\varphi : E \times E \to \mathbb{R}$ as

$$\varphi(x,y) = \sum_{i=1}^{n} (f^i(x^i, \widehat{x}^i) - f^i(y^i, \widehat{x}^i)),$$

it follows that $\overline{x} \in E$ is a non-cooperative equilibrium if and only if

$$\varphi(\overline{x}, y) \leq 0, \text{ for every } y \in E,$$

see [2] p. 181.

Theorem 6.12. *In the above setting assume that*

(1) *for every $(\overline{x}, \overline{y}) \in E \times E$ such that $\varphi(x_o, y_o) > 0$ there exists a neighborhood $\mathcal{I}$ of $\overline{x}$ such that $\varphi(z, \overline{y}) > 0$ for every $z \in \mathcal{I}$;*

(2) *for every $i = 1, \dots, n$ there exists a relatively compact set $A^i \subset E^i$ and a finite set Y^i such that for every $x^i \in E^i \setminus A^i$ there exists $u^i \in Y^i$ and a neighborhood $\mathcal{I}^i$ of x^i such that*

$$f^i(z^i, \widehat{x}^i) - f^i(u^i, \widehat{x}^i) > 0, \textit{ for every } z^i \in \mathcal{I}^i, \textit{ and } \widehat{x}^i \in \widehat{E}^i$$

(3) *there exists a map $h : E \times E \to \mathbb{R}$ that is diagonally transfer quasi-concave with respect to the second variable such that*
(3i) *$h(x, x) = 0$ for every $x \in E$*
(3ii) *$f^i(x^i, \widehat{x}^i) - f^i(y^i, \widehat{x}^i) \leq h(x, y)$, $i = 1, \dots, n$, for every $(x, y) \in E \times E$.*

Then there exists a non-cooperative equilibrium.

Proof. First of all we define the map $g : E \times E \to \mathbb{R}$ as

$$g(x, y) = \max_{i=1,\dots,n} \left[f^i(x^i, \widehat{x}^i) - f^i(y^i, \widehat{x}^i) \right].$$

Observing that

$$\varphi(x, y) \leq n g(x, y) \leq n h(x, y), \text{ for every } (x, y) \in E \times E,$$

we will apply Theorem 5.10. We only need to prove that the map g satisfies hypothesis (5) of the cited theorem. Let $A = \Pi_{i=1}^n A^i$ and $Y = \Pi_{i=1}^n Y^i$. Trivially A is compact and Y is finite. Let $x \notin A$, then there exists $\overline{i} \in \{1, \dots, n\}$ such that $x^{\overline{i}} \notin A^{\overline{i}}$. Then by hypothesis (2) there exists a corresponding element $u^{\overline{i}} \in Y^i$ and a neighborhood $\mathcal{I}^{\overline{i}}$ of $x^{\overline{i}}$ such that

$$f^i(z^{\overline{i}}, \widehat{x}^{\overline{i}}) - f^i(u^{\overline{i}}, \widehat{x}^{\overline{i}}) > 0, \text{ for every } z^{\overline{i}} \in \mathcal{I}^{\overline{i}}, \text{ and } \widehat{x}^{\overline{i}} \in \widehat{E}^{\overline{i}}.$$

Then for $u = (u^{\overline{i}}, \widehat{y})$ with $\widehat{y} \in \Pi_{i \neq \overline{i}} Y^i$ and the neighborhood $\mathcal{I}$ of x chosen such that $x^{\overline{i}} \in \mathcal{I}^{\overline{i}}$ we have that

$$g(z, u) \geq f^i(z^{\overline{i}}, \widehat{x}^{\overline{i}}) - f^i(u^{\overline{i}}, \widehat{x}^{\overline{i}}) > 0, \text{ for every } z \in \mathcal{I}.$$

Then by Theorem 5.10 we obtain the existence of $\overline{x} \in E$ such that

$$\varphi(\overline{x}, y) \leq 0 \text{ for every } y \in E,$$

i.e. a non-cooperative equilibrium. □

Now we show an example of a 2-person game to which it is possible to apply Theorem 6.12.

Example 6.13. Consider a 2-person 0-sum game on the set $E = E^1 \times E^2$ with $E^1 = E^2 =]0, +\infty)$ with the payoff $f^i : E \to \mathbb{R}, i = 1, 2$ given by

$$f^1(x_1, x_2) = x_1 x_2, \quad f^2(x_1, x_2) = -x_1 x_2 .$$

Then the function $\varphi : E \times E \to \mathbb{R}$ is given by

$$\varphi((x_1, x_2), (y_1, y_2)) = 2x_1 x_2 - y_1 x_2 - x_1 y_2.$$

Hence it is immediate to observe that φ is continuous with respect to the first variable and so, in particular, it is positive 0-transfer continuous with respect to the first variable.

Now, let for instance $i = 1$ be fixed and let $C_1 =]0, 1]$ and $y_1 < 1$ be fixed. Then for $x \notin C_1$ necessarily there is a neighborhood of x, say $\mathcal{I}(x) =]x - \varepsilon, x + \varepsilon[$, with $x - \varepsilon > 1$ and hence for every $z \in \mathcal{I}(x)$ it follows that

$$f^1(z_1, z_2) - f^1(y_1, z_2) = z_1 z_2 - y_1 z_2 = z_2(z_1 - y_1) > 0$$

whatever y_2 we choose. Therefore condition (2) of Theorem 6.12 is satisfied.

Denoting with $\varphi^i : E \times E \to \mathbb{R}$ the function

$$\varphi^i((x_1, x_2), (y_1, y_2)) = f^i(x^i, \widehat{x}^i) - f^i(y^i, \widehat{x}^i),$$

we choose, as function $h : E \times E \to \mathbb{R}$ in Theorem 6.12, $h((x_1, x_2), (y_1, y_2)) = \max\{\varphi^1((x_1, x_2), (y_1, y_2)), \varphi^2((x_1, x_2), (y_1, y_2))\}$. To prove that it is diagonally transfer quasi-concave with respect to y, let $y_1, \ldots, y_n$ be fixed in E and set $\xi_k = \min\{y_k^i,\ i = 1, \ldots, n\}$, $k = 1, 2$. Thus immediately

$$h((\xi_1, \xi_2), (y_1^i, y_2^i)) = \xi_1 \xi_2 - \min\{y_1^i \xi_2, \xi_1 y_1^i\} \leq 0.$$

Then h is diagonally transfer quasi-concave with $\mathbf{x}_1 = \mathbf{x}_2 = \cdots = \mathbf{x}_n = (\xi_1, \xi_2)$.

Remark 6.14. Note that the map φ of the previous example is concave and therefore diagonally transfer quasi-concave. However, it does not satisfy the condition (4) of Theorem 5.13. In fact, for every choice of $Y_o = (\mathbf{y}^1, \ldots, \mathbf{y}^n)$ in E and for every relatively compact subset $C \subset E$ there should be $x \in E \setminus C$ with $\varphi(\mathbf{x}, \mathbf{y}^i) \leq 0$ for every $i = 1, \ldots, n$. Indeed let $\beta^i = y_2^i / y_1^i$, $i = 1, \ldots, n$ be the n-slopes corresponding to Y_0. Since $0 \notin E^i$ then $\beta^i \in \mathbb{R}$ for every $i = 1, \ldots, n$. Hence, due to the boundedness of C there will necessarily be a point $\mathbf{x} = (x_1, x_2)$ out of C such that the slope $\frac{x_2}{x_1}$ exceeds the maximum of the $\beta_i' s$. In other words

$$x_2 y_1^i \geq x_1 y_2^i,$$

implying

$$\varphi((x_1, x_2), (y_1^i, y_2^i)) \leq 0.$$

Therefore the existence of noncooperative equilibrium can not be deduced neither from Theorem 2 of [6], nor from Theorem 5.13.

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