

Multiple Solutions for a Class of Schrödinger Equations Involving the Fractional p -Laplacian

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We deal with the multiplicity of weak solutions of the non-local elliptic equation

$$(-\Delta)_p^s u + V(x) |u|^{p-2} u = g(x, u)$$

in \mathbb{R}^N , where $(-\Delta)_p^s$ is the so-called fractional p -Laplacian, V is a suitable continuous potential and the nonlinearity g grows as $|u|^{p-2} u$ at infinity. Our results extend the classical local counterpart, that is when $s = 1$.

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1. Introduction

In the present paper we look for solutions of the non-local quasilinear problem

$$(-\Delta)_p^s u + V(x) |u|^{p-2} u = \lambda |u|^{p-2} u + f(x, u) \quad \text{in } \mathbb{R}^N, \quad (1)$$

where $0 < s < 1 < p < +\infty$, V is a continuous potential, λ is a real parameter, f is a suitable nonlinear term and $(-\Delta)_p^s$ denotes the fractional p -Laplacian which (up to normalization factors) may be defined for any $x \in \mathbb{R}^N$ as

$$(-\Delta)_p^s \varphi(x) = 2 \lim_{\varepsilon \searrow 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|\varphi(x) - \varphi(y)|^{p-2} (\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} dy$$

along any $\varphi \in C_0^\infty(\mathbb{R}^N)$, where $B_\varepsilon(x) = \{y \in \mathbb{R}^N : |x - y| < \varepsilon\}$. We refer the reader to [17, 18, 23] and references therein for further details on the fractional p -Laplacian.

Recently, a deep interest was shown in the literature for nonlinear Schrödinger equations driven by non-local integro-differential operators, alike (1), with different nonlinearities on the right-hand side. We refer to [9, 11, 12, 13, 15, 16, 21, 24, 30] and the references therein, for the particular case $p = 2$ of the fractional Laplacian $(-\Delta)^s$.

In particular in [15] the authors study the uniqueness and non-degeneracy of the ground state of $(-\Delta)^s u + u = u^{\alpha+1}$ in \mathbb{R} for all $s \in]0, 1[$, where $0 < \alpha < 4s/(1-2s)$ for $s \in]0, 1/2[$ and $0 < \alpha < +\infty$ for $s \in [1/2, 1[$. More recently, the result of [15] has been extended in any dimension when s is sufficiently close to 1 in [12] and later for all $s \in]0, 1[$ in [16]. In [13] the existence of positive solutions of (1) is proved, through variational techniques, when $\lambda = 0$, $V(x) \equiv 1$ and f satisfies the Ambrosetti–Rabinowitz condition. In [11] the authors consider the same problem of [13] with $f(x, t) = |u|^{p-1} u$, providing also symmetry properties of solutions. For the case in which the potential V is non-constant, we could refer to [28] where the existence of ground state solutions of the case $p = 2$ of (1) is obtained when the potential verifies $V(x) \rightarrow +\infty$ as $|x| \rightarrow +\infty$ and the Ambrosetti–Rabinowitz condition still holds.

For asymptotically fractional Schrödinger problems it is worth mentioning [9], where potential V is a continuous and positive function for which there exists $r_0 > 0$ such that

$$\lim_{|y| \rightarrow +\infty} \text{meas}(\{x \in B_{r_0}(y) : V(x) \leq M\}) = 0, \quad \text{for any } M > 0. \quad (2)$$

Condition (2) is weaker than the coercivity assumption in [30], however the existence of a ground state solution can be proved by a suitable variant of the mountain pass theorem. In [21] the existence of a non-negative solution for an asymptotically linear Schrödinger equation like (1) is given when V is a constant, by looking at Pohozaev manifold in place of Nehari constraint (used for example in [30]).

For an application in fractional quantum mechanics, we refer to [19, 20] where Laskin proposed a nonlinear fractional Schrödinger equation as a result of expanding Feynman path integral, from the Brownian-like to the Lévy-like quantum mechanical paths. In Laskin’s studies, the Feynman path integral leads to the classical Schrödinger equation and the path integral over Lévy trajectories leads to the fractional Schrödinger equation. In particular, if u is a solution of (1), with a general nonlinearity $g(x, u)$ on the right-hand side, then the function $\psi(x, t) = e^{-ict}u(x)$ is the so-called wave solution of the following time-dependent fractional Schrödinger equation

$$i \frac{\partial \psi}{\partial t} + (-\Delta)_p^s \psi + (V(x) - c)\psi = g(x, u) \quad \text{in } \mathbb{R}^N,$$

with c a suitable constant and i the imaginary unit.

Motivated by this interest in the current literature, we would like to focus here our attention in p -fractional asymptotically linear problems, also in presence of

resonance. For this, we point out that the existence and the number of solutions of (1) are strictly linked to the interaction between λ and f with the spectrum of the non-local operator

$$\mathcal{H}_p : u \longmapsto \mathcal{H}_p(u) := (-\Delta)_p^s u + V(x) |u|^{p-2} u.$$

Concerning the potential function, we assume $V : \mathbb{R}^N \rightarrow \mathbb{R}$ is a Lebesgue measurable function such that:

$$(V_1) \quad \text{ess inf}_{x \in \mathbb{R}^N} V(x) > 0;$$

$$(V_2) \quad \lim_{|x| \rightarrow +\infty} \int_{B_1(x)} \frac{1}{V(y)} dy = 0, \quad \text{where } B_1(x) = \{y \in \mathbb{R}^N : |x - y| < 1\}.$$

We remark that the assumptions $(V_1) - (V_2)$ are weaker than those for potential V described above, such as coercivity and (2), as shown in [29, Proposition 3.1].

Firstly we are interested in treating the non-resonant asymptotically p -linear case, thus we assume the nonlinearity $f : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function satisfying:

$$(f_1) \quad \sup_{|t| \leq a} |f(\cdot, t)| \in L^\infty(\mathbb{R}^N) \quad \text{for any } a > 0;$$

$$(f_2) \quad \lim_{|t| \rightarrow +\infty} \frac{f(x, t)}{|t|^{p-2} t} = 0 \quad \text{uniformly for a.e. } x \in \mathbb{R}^N;$$

$$(f_3) \quad \lim_{t \rightarrow 0} \frac{f(x, t)}{|t|^{p-2} t} = \gamma \in \mathbb{R} \setminus \{0\} \quad \text{uniformly for a.e. } x \in \mathbb{R}^N;$$

$$(f_4) \quad \lambda \notin \sigma(\mathcal{H}_p);$$

$$(f_5) \quad \text{there exist } h, k \in \mathbb{N} \text{ with } k \geq h, \text{ such that}$$

$$\min \{\lambda + \gamma, \lambda\} < \alpha_h \leq \beta_k < \max \{\lambda + \gamma, \lambda\},$$

where $(\alpha_n)_n$ and $(\beta_n)_n$ are given in Section 2;

$$(f_6) \quad f(x, \cdot) \text{ is odd for a.e. } x \in \mathbb{R}^N.$$

We observe that thanks to (f_2) the nonlinear right-hand side of problem (1) is asymptotical, that is grows as $|u|^{p-2} u$ at infinity. Furthermore, by (f_3) the trivial function $u = 0$ solves (1), so we state the following multiplicity result by seeking non-trivial solutions.

Theorem 1.1. *Let $s \in]0, 1[$ and $p > 1$. Assume that $(V_1) - (V_2)$ and $(f_1) - (f_6)$ hold. Then, problem (1) has at least $k - h + 1$ distinct pairs of non-trivial weak solutions.*

We point out that condition (f_4) gives a non-resonant setting. We are also able to study problem (1) in a resonant case, namely when $\lambda \in \sigma(\mathcal{H}_p)$, requiring the following stronger assumptions for the nonlinearity f :

$$(f'_1) \quad \sup_{|t| \leq a} |\lambda |t|^{p-2} t + f(\cdot, t)| \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) \quad \text{for any } a > 0;$$

$$(f_7) \quad \lim_{|x| \rightarrow +\infty} (f(x, t)t - pF(x, t)) = +\infty \quad \text{uniformly a.e. in } \mathbb{R}^N, \text{ where}$$

$$F(x, t) = \int_0^t f(x, \tau) d\tau.$$

These two new assumptions will allow us to overcome some technical difficulties in the proof of the compactness property due to Cerami.

Theorem 1.2. *Let $s \in]0, 1[$ and $p > 1$. Assume that $(V_1) - (V_2)$ and (f'_1) , $(f_2) - (f_3)$, $(f_5) - (f_7)$ hold. If $\lambda \in \sigma(\mathcal{H}_p)$, then problem (1) has at least $k - h + 1$ distinct pairs of non-trivial weak solutions.*

The proofs of both Theorem 1.1 and 1.2 are mainly based on the mini-max theorem [1, Theorem 2.9]. Moreover, they can be seen as a generalization of [3, Theorems 1.2 and 1.3] on a fractional framework.

The paper is organized as follows. In Section 2 we introduce the variational tools needed for the proof of our main results. In Section 3 and 4 we prove respectively Theorem 1.1 and 1.2.

2. Variational formulation

In this section we introduce the variational formulation of (1) and the functional space where finding weak solutions.

Throughout this section we denote by $(X, \|\cdot\|_X)$ a Banach space, by $(X', \|\cdot\|_{X'})$ its dual, by I a C^1 functional on X , by $I^b = \{u \in X : I(u) \leq b\}$ the sublevel of I corresponding to $b \in \mathbb{R}$ and by $K_c = \{u \in X : I(u) = c, dI(u) = 0\}$ the set of the critical points of I in X at the critical level $c \in \mathbb{R}$.

Firstly, we recall the so-called Cerami's variant of the Palais-Smale condition: even if it is a condition weaker than the classical one, it is enough in order to state a Deformation Theorem and some critical point theorems (cf. [1]).

Definition 2.1. The functional I satisfies the *Cerami's variant of the Palais-Smale condition at level c* ($c \in \mathbb{R}$), if any sequence $(u_n)_n \subseteq X$ such that

$$\lim_{n \rightarrow +\infty} I(u_n) = c \quad \text{and} \quad \lim_{n \rightarrow +\infty} \|dI(u_n)\|_{X'}(1 + \|u_n\|_X) = 0 \quad (3)$$

converges in X , up to subsequences. In general, if $-\infty \leq a < b \leq +\infty$, I satisfies (CPS) in $]a, b[$ if so is at each level $c \in]a, b[$.

In the proof of our main theorems we use [1, Theorem 2.9] for Banach spaces where the index theory related to the genus acts. Before stating it, we recall some notions of the index theory on Banach spaces X for an even functional with symmetry group $\mathbb{Z}_2 = \{\text{id}, -\text{id}\}$.

We set

$$\Sigma = \Sigma(X) = \{A \subseteq X : A \text{ closed and symmetric w.r.t. the origin}\}$$

and

$$\mathcal{H} = \{h \in C(X, X) : h \text{ odd}\}.$$

Taking $A \in \Sigma$, $A \neq \emptyset$, the genus of A is

$$\gamma(A) = \inf\{k \in \mathbb{N} : \exists \psi \in C(A, \mathbb{R}^k \setminus \{0\}) \text{ s.t. } \psi(-u) = -\psi(u) \text{ for all } u \in A\},$$

if such an infimum exists, otherwise $\gamma(A) = +\infty$. Assume $\gamma(\emptyset) = 0$.

The index theory $(\Sigma, \mathcal{H}, \gamma)$ related to \mathbb{Z}_2 is also called *genus*.

According to [6], the pseudo-index related to the genus, an even functional $I : X \rightarrow \mathbb{R}$ and $S \in \Sigma$ is the triplet $(S, \mathcal{H}^*, \gamma^*)$ such that \mathcal{H}^* is a group of odd homeomorphisms and $\gamma^* : \Sigma \rightarrow \mathbb{N} \cup \{+\infty\}$ is the map defined by

$$\gamma^*(A) = \min_{h \in \mathcal{H}^*} \gamma(h(A) \cap S) \quad \text{for all } A \in \Sigma.$$

Since

$$\gamma(h(A) \cap S) = \gamma(A \cap h^{-1}(S)) \quad \text{for all } h \in \mathcal{H}^*,$$

then

$$\gamma^*(A) = \min_{h \in \mathcal{H}^*} \gamma(A \cap h(S)) \quad \text{for all } A \in \Sigma. \quad (4)$$

Theorem 2.2. Consider $a, b, c_0, c_\infty \in \bar{\mathbb{R}}$, $-\infty \leq a < c_0 < c_\infty < b \leq +\infty$. Let I be an even functional, $(\Sigma, \mathcal{H}, \gamma)$ the genus theory on X , $S \in \Sigma$, $(S, \mathcal{H}^*, \gamma^*)$ the pseudo-index theory related to the genus, I and S , with

$$\mathcal{H}^* = \{h \in \mathcal{H} : h \text{ bounded homeomorphism s.t. } h(u) = u \text{ if } u \notin I^{-1}(]a, b])\}.$$

Assume that:

- (i) the functional I satisfies (CPS) in $]a, b[$;
- (ii) $S \subseteq I^{-1}(]c_0, +\infty[)$;
- (iii) there exist $\tilde{k} \in \mathbb{N}$ and $\tilde{A} \in \Sigma$ such that $\tilde{A} \subseteq I^{c_\infty}$ and $\gamma^*(\tilde{A}) \geq \tilde{k}$.

Then the numbers

$$c_i = \inf_{A \in \Sigma_i^*} \sup_{u \in A} I(u), \quad i \in \{1, \dots, \tilde{k}\}, \quad (5)$$

with $\Sigma_i^* = \{A \in \Sigma : \gamma^*(A) \geq i\}$, are critical values for I and

$$c_0 \leq c_1 \leq \dots \leq c_{\tilde{k}} \leq c_\infty.$$

Furthermore, if $c = c_i = \dots = c_{i+r}$, with $i \geq 1$ and $i+r \leq \tilde{k}$, then $\gamma(K_c) \geq r+1$.

Remark 2.3. In order to apply the theorem above, we need a lower bound for the pseudo-index of a suitable \tilde{A} as in (iii) of Theorem 2.2. Thus, let us consider the genus theory $(\Sigma, \mathcal{H}, \gamma)$ on X and E, F two closed subspaces of X . If

$$\dim E < +\infty \quad \text{and} \quad \text{codim } F < +\infty,$$

then, for every odd bounded homeomorphism h on X and every open bounded symmetric neighbourhood B of 0 in X , it results

$$\gamma(E \cap h(\partial B \cap W)) \geq \dim E - \text{codim } F$$

(cf. [1, Theorem A.2] and [2, Theorem 2.7]).

Now we focus on the framework of our result and consider the weighted fractional Sobolev space

$$W_V := W_V^{s,p}(\mathbb{R}^N) = \left\{ u \in W^{s,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^p dx < \infty \right\} \quad (6)$$

endowed with the norm

$$\|u\|_V = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy + \int_{\mathbb{R}^N} V(x)|u|^p dx \right)^{\frac{1}{p}}, \quad (7)$$

where, defining the Gagliardo seminorm by

$$[u]_{s,p} = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{\frac{1}{p}},$$

the fractional Sobolev space is defined by

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < +\infty\}.$$

We observe that $(W_V, \|\cdot\|_V)$ is well defined thanks to (V_1) and for the same assumption it is a separable and uniformly convex Banach space (see, e.g., [26, Lemma 10]). Furthermore, by (V_1) and above all (V_2) we are able to overcome the usual lack of compactness due on unbounded domain. That is, we have the following crucial lemma.

Lemma 2.4. *Assume that (V_1) and (V_2) hold. Then the embedding of $(W_V, \|\cdot\|_V)$ in $(L^p(\mathbb{R}^N), |\cdot|_p)$ is compact.*

Proof. The proof is a consequence of [7, Theorem 3.1], where we can just replace $H^{1,p}(\mathbb{R}^N)$ with $W^{s,p}(\mathbb{R}^N)$ \square

For further use we recall that $p_s^* = \frac{pN}{N-sp}$ if $sp < N$ and $p_s^* = +\infty$ otherwise.

Now, by Lemma 2.4 we can adapt the arguments in [8, Section 5] and [5, Section 2] to our operator \mathcal{H}_p on W_V . More precisely, we can construct a family $(\alpha_n)_n$ of quasi-eigenvalues, for \mathcal{H}_p on W_V , and of functions $(\psi_n)_n$ such that:

- the increasing sequence $(\alpha_n)_n$ diverges positively;
- fixing any $n \geq 1$ and setting

$$X_n = \text{span}\{\psi_1, \dots, \psi_n\} = \left\{ v \in W_V : \exists b_1, \dots, b_n \in \mathbb{R} \text{ s.t. } v = \sum_{i=1}^n b_i \psi_i \right\},$$

there exists a subspace Y_n of W_V such that

$$W_V = X_n \oplus Y_n; \quad (8)$$

- taking $n \geq 1$, for all $w \in Y_n$ we get

$$\alpha_{n+1} \int_{\mathbb{R}^N} |w|^p \, dx \leq \|w\|_V^p; \quad (9)$$

- the sequence $(\psi_n)_n$ generates the whole space W_V .

Furthermore, following [22] we introduce another sequence of positive numbers. For all $n \in \mathbb{N}$, taking ψ_1 as above, we set

$$\mathbb{W}_n = \{Z : Z \text{ is a subspace of } W_V, \psi_1 \in Z \text{ and } \dim Z \geq n\} \quad (10)$$

and

$$\beta_n = \inf_{Z \in \mathbb{W}_n} \sup_{u \in \mathcal{S} \cap Z} \|u\|_V^p, \quad (11)$$

with $\mathcal{S} = \left\{ v \in W_V : \int_{\mathbb{R}^N} |v|^p \, dx = 1 \right\}$. By the previous definitions, $\mathbb{W}_{n+1} \subseteq \mathbb{W}_n$ and we have

$$\beta_1 = \inf_{v \in \mathcal{S}} \|v\|_V^p = \alpha_1,$$

where the last equality follows by the construction given in [8, Section 5]. Hence, also $(\beta_n)_n$ is an increasing sequence of quasi-eigenvalues.

Remark 2.5. For $p = 2$ the sequences $(\alpha_n)_n$ and $(\beta_n)_n$ reduce to the known sequence of eigenvalues $(\lambda_n)_n$ of $(-\Delta)^s$; we refer to [31, Section 3] for the case of bounded domains.

For the proof of Theorems 1.1 and 1.2, we observe that problem (1) has a variational structure. Indeed we observe that by (f_2) and (f_3) it follows that for all $\sigma > 0$ there exist $R_\sigma, \delta_\sigma > 0$ (without loss of generality $R_\sigma \geq 1$) such that

$$|f(x, t)| \leq \sigma |t|^{p-1} \quad \text{if } |t| > R_\sigma, \text{ for a.e. } x \in \mathbb{R}^N \quad (12)$$

and

$$|f(x, t) - \gamma |t|^{p-2} t| \leq \sigma |t|^{p-1} \quad \text{if } |t| < \delta_\sigma, \text{ for a.e. } x \in \mathbb{R}^N. \quad (13)$$

Moreover, by (f_1) , there exists $a_{R_\sigma} > 0$ such that

$$|f(x, t)| \leq a_{R_\sigma} |t|^{p-1} \quad \text{if } \delta_\sigma \leq |t| \leq R_\sigma, \text{ for a.e. } x \in \mathbb{R}^N. \quad (14)$$

By (12)–(14), for any $\sigma > 0$ there exists $A_\sigma > 0$ large enough such that

$$|f(x, t)| \leq A_\sigma |t|^{p-1} \quad \text{for a.e. } x \in \mathbb{R}^N \text{ and for all } t \in \mathbb{R}. \quad (15)$$

Therefore (1) is the Euler-Lagrange equation of the C^1 functional $J : W_V \rightarrow \mathbb{R}$ defined on $u \in W_V$ as follows

$$J(u) = \frac{1}{p} \|u\|_V^p - \frac{\lambda}{p} \int_{\mathbb{R}^N} |u(x)|^p \, dx - \int_{\mathbb{R}^N} F(x, u(x)) \, dx, \quad (16)$$

whose derivative is given by

$$\begin{aligned} dJ(u)[\varphi] &= \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} dx dy \\ &+ \int_{\mathbb{R}^N} V(x) |u(x)|^{p-2} u(x) \varphi(x) dx - \lambda \int_{\mathbb{R}^N} |u(x)|^{p-2} u(x) \varphi(x) dx \\ &- \int_{\mathbb{R}^N} f(x, u(x)) \varphi(x) dx, \end{aligned} \quad (17)$$

for any $\varphi \in W_V$.

Also, for the sake of simplicity we introduce the operator $A : W_V \rightarrow (W_V)^*$, defined for all $u, \varphi \in W_V$ by

$$\langle A(u), \varphi \rangle = \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} dx dy.$$

3. Proof of Theorem 1.1

From now on we assume $(V_1) - (V_2)$, without further mentioning.

Reasoning as in [3, Proposition 4.1], it follows that the operator \mathcal{H}_p is of type (S) on W_V (cf., e.g., [3, Definition 4.1]). In the next proposition we prove that the functional J satisfies the (CPS) condition in the non-resonant case.

Proposition 3.1. *Assume that (f_1) , (f_2) and (f_4) hold. Then the functional J in (16) satisfies (CPS) in \mathbb{R} .*

Proof. Let $c \in \mathbb{R}$ and $(u_n)_n$ be a sequence in W_V such that (3) holds, so in particular (7) and (17) imply

$$\|u_n\|_V^p = \lambda |u_n|_p^p + \int_{\mathbb{R}^N} f(x, u_n) u_n dx + o(1) \quad (18)$$

and

$$\begin{aligned} \langle A(u_n), \varphi \rangle &+ \int_{\mathbb{R}^N} V(x) |u_n|^{p-2} u_n \varphi dx \\ &= \lambda \int_{\mathbb{R}^N} |u_n|^{p-2} u_n \varphi dx + \int_{\mathbb{R}^N} f(x, u_n) \varphi dx + o(1), \end{aligned} \quad (19)$$

for all $\varphi \in W_V$ (here, $o(1)$ denotes any infinitesimal sequence).

In order to prove this statement, it is enough to show that $(\|u_n\|_V)_n$ is bounded (see e.g. [25, Lemma 3.3] and [3, Proposition 4.2]). If by contradiction we assume that

$$\|u_n\|_V \rightarrow +\infty \quad \text{as } n \rightarrow +\infty, \quad (20)$$

then, setting $w_n = \frac{u_n}{\|u_n\|_V}$, $(w_n)_n$ is bounded in W_V hence there exists $w \in W_V$ such that, up to subsequences,

$$w_n \rightharpoonup w \quad \text{weakly in } W_V. \quad (21)$$

Therefore Lemma 2.4 implies that

$$w_n \rightarrow w \quad \text{strongly in } L^p(\mathbb{R}^N). \quad (22)$$

Dividing (18) by $\|u_n\|_V^p$ and (19) by $\|u_n\|_V^{p-1}$ respectively, we get

$$1 = \|w_n\|_V^p = \lambda |w_n|_p^p + \int_{\mathbb{R}^N} \frac{f(x, u_n)u_n}{\|u_n\|_V^p} dx + o(1) \quad (23)$$

and

$$\begin{aligned} \langle A(w_n), \varphi \rangle + \int_{\mathbb{R}^N} V(x)|w_n|^{p-2}w_n\varphi dx \\ = \lambda \int_{\mathbb{R}^N} |w_n|^{p-2}w_n\varphi dx + \int_{\mathbb{R}^N} \frac{f(x, u_n)}{\|u_n\|_V^{p-1}}\varphi dx + o(1) \end{aligned} \quad (24)$$

for all $\varphi \in W_V$. Let us remark that by (15) for a fixed $\sigma > 0$ it follows that

$$\left| \int_{\mathbb{R}^N} \frac{f(x, u_n)u_n}{\|u_n\|_V^p} dx \right| \leq A_\sigma |w_n|_p^p. \quad (25)$$

Now we claim that $w \not\equiv 0$; otherwise, if $w \equiv 0$, by (22) and (23) it follows

$$1 = \int_{\Omega} \frac{f(x, u_n)u_n}{\|u_n\|_V^p} dx + o(1). \quad (26)$$

On the other hand, (22) and (25) imply

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} \frac{f(x, u_n)u_n}{\|u_n\|_V^p} dx = 0,$$

in contradiction to (26).

Let us test equation (24) for $\varphi = w_n - w$; then we have that

$$\begin{aligned} \langle A(w_n), w_n - w \rangle + \int_{\mathbb{R}^N} V(x)|w_n|^{p-2}w_n(w_n - w) dx \\ = \lambda \int_{\mathbb{R}^N} |w_n|^{p-2}w_n(w_n - w) dx + \int_{\mathbb{R}^N} \frac{f(x, u_n)}{\|u_n\|_V^{p-1}}(w_n - w) dx + o(1). \end{aligned} \quad (27)$$

Since the Hölder inequality and (22) imply

$$\left| \int_{\mathbb{R}^N} |w_n|^{p-2}w_n(w_n - w) dx \right| \leq |w_n|_p^{p-1} |w_n - w|_p = o(1),$$

and (15), (22) give

$$\left| \int_{\mathbb{R}^N} \frac{f(x, u_n)}{\|u_n\|_V^{p-1}}(w_n - w) dx \right| \leq A_\sigma |w_n|_p^{p-1} |w_n - w|_p = o(1),$$

by (27) we infer that

$$\langle A(w_n), w_n - w \rangle + \int_{\mathbb{R}^N} V(x) |w_n|^{p-2} w_n (w_n - w) \, dx = o(1);$$

therefore, by (21), because \mathcal{H}_p is of type (S), it follows that

$$w_n \rightarrow w \quad \text{strongly in } W_V. \quad (28)$$

Now, we claim that

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} \frac{f(x, u_n)}{\|u_n\|_V^{p-1}} \varphi \, dx = 0 \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N). \quad (29)$$

In fact, for any $\sigma > 0$ by (f_2) there exists $R_\sigma > 0$ such that (12) holds; moreover, by (f_1) there exists $k_\sigma > 0$ such that

$$|f(x, t)| \leq k_\sigma \quad \text{if } |t| \leq R_\sigma, \text{ for a.e. } x \in \mathbb{R}^N. \quad (30)$$

Therefore, taking $\varphi \in C_0^\infty(\mathbb{R}^N)$, by (12), (30) and the Hölder inequality we have that

$$\begin{aligned} \left| \int_{\mathbb{R}^N} \frac{f(x, u_n)}{\|u_n\|_V^{p-1}} \varphi \, dx \right| &\leq \sigma \int_{\{|u_n| > R_\sigma\}} |w_n|^{p-1} |\varphi| \, dx + k_\sigma \int_{\{|u_n| \leq R_\sigma\}} \frac{|\varphi|}{\|u_n\|_V^{p-1}} \, dx \\ &\leq \sigma |w_n|_p^{p-1} |\varphi|_p + \frac{k_\sigma |\varphi|_1}{\|u_n\|_V^{p-1}}. \end{aligned}$$

So, by (20) and (22), for all $\sigma > 0$ there exists $n_\sigma \in \mathbb{N}$ such that

$$\left| \int_{\mathbb{R}^N} \frac{f(x, u_n)}{\|u_n\|_V^{p-1}} \varphi \, dx \right| \leq \sigma (k |\varphi|_p + |\varphi|_1) \quad \text{for all } n \geq n_\sigma,$$

for a suitable $k > 0$. Then, (29) holds and, by passing to the limit in (24), from (28) we get

$$\langle A(w), \varphi \rangle + \int_{\mathbb{R}^N} V(x) |w|^{p-2} w \varphi \, dx = \lambda \int_{\mathbb{R}^N} |w|^{p-2} w \varphi \, dx$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. By a density argument, λ is an eigenvalue of \mathcal{H}_p in W_V , against assumption (f_4) . So, the proof is complete. \square

We divide the proof of Theorem 1.1 into two technical lemmas.

Lemma 3.2. *Assume that $(f_1) - (f_3)$ hold. Let α_h be as in (f_5) and Y_{h-1} as in (8). Then, there exist $\rho > 0$ and $c_0 > 0$ such that, setting $S_\rho = \{u \in W_V : \|u\|_V = \rho\}$, the functional J in (16) verifies*

$$J(u) \geq c_0 \quad \text{for all } u \in S_\rho \cap Y_{h-1}. \quad (31)$$

Proof. By (f_2) and (f_3) it follows that

$$\lim_{|t| \rightarrow +\infty} \frac{F(x, t)}{|t|^p} = 0 \quad \text{uniformly with respect to a.e. } x \in \mathbb{R}^N \quad (32)$$

and

$$\lim_{t \rightarrow 0} \frac{F(x, t)}{|t|^p} = \frac{\gamma}{p} \quad \text{uniformly with respect to a.e. } x \in \mathbb{R}^N.$$

Then, for all $\sigma > 0$ there exist $R_\sigma, \delta_\sigma > 0$ such that

$$|F(x, t)| \leq \frac{\sigma}{p} |t|^p \quad \text{if } |t| > R_\sigma, \text{ for a.e. } x \in \mathbb{R}^N \quad (33)$$

and

$$\left| F(x, t) - \frac{\gamma}{p} |t|^p \right| \leq \frac{\sigma}{p} |t|^p \quad \text{if } |t| < \delta_\sigma, \text{ for a.e. } x \in \mathbb{R}^N. \quad (34)$$

Taking any $r \in [0, p_s^* - p[$ (with $p_s^* - p = +\infty$ if $p \geq N$), by (f_1) there exists $b_\sigma > 0$ such that

$$|F(x, t)| \leq b_\sigma |t|^{r+p} \quad \text{if } \delta_\sigma \leq |t| \leq R_\sigma, \text{ for a.e. } x \in \mathbb{R}^N. \quad (35)$$

If $\gamma < 0$, (f_5) becomes

$$\gamma + \lambda < \alpha_h \leq \beta_k < \lambda. \quad (36)$$

For any $\sigma > 0$, by (33)–(35) for a.e. $x \in \mathbb{R}^N$ and for all $t \in \mathbb{R}$ we have that

$$-\frac{(\sigma - \gamma)}{p} |t|^p - B_\sigma |t|^{r+p} \leq F(x, t) \leq \frac{(\sigma + \gamma)}{p} |t|^p + B_\sigma |t|^{r+p}, \quad (37)$$

for a large enough $B_\sigma > 0$, then

$$\int_{\mathbb{R}^N} F(x, u) \, dx \leq \frac{(\sigma + l_0)}{p} \|u\|_p^p + B_\sigma \|u\|_{r+p}^{r+p} \quad \text{for all } u \in W_V. \quad (38)$$

Since $W_V \hookrightarrow W^{s,p}(\mathbb{R}^N)$, by the Sobolev embedding theorem on \mathbb{R}^N and (38) we get

$$J(u) \geq \frac{1}{p} \|u\|_V^p - \frac{\lambda + \gamma + \sigma}{p} \|u\|_p^p - B'_\sigma \|u\|_V^{r+p} \quad \text{for all } u \in W_V, \quad (39)$$

for a suitable $B'_\sigma > 0$. Taking α_h as in (36), by (8) we have that $W_V = X_{h-1} \oplus Y_{h-1}$, where $X_{h-1} = \text{span}\{\psi_1, \dots, \psi_{h-1}\}$ and Y_{h-1} is its complement. Thus, by (9) and (39)

$$J(u) \geq \frac{1}{p} \left(1 - \frac{\lambda + \gamma + \sigma}{\alpha_h} \right) \|u\|_V^p - B'_\sigma \|u\|_V^{r+p} \quad \text{for all } u \in Y_{h-1}.$$

Therefore, by (36), if σ is small enough then there exists $B''_\sigma > 0$ such that

$$J(u) \geq B''_\sigma \|u\|_V^p - B'_\sigma \|u\|_V^{r+p} \quad \text{for all } u \in Y_{h-1}.$$

So, setting $S_\rho = \{u \in W_V : \|u\|_V = \rho\}$, if ρ is small enough there exists $c_0 > 0$ such that (31) holds. \square

Lemma 3.3. *Assume that $(f_1) - (f_2)$ hold. Let β_k as in (f_5) , \mathbb{W}_k as in (10) and c_0 as in Lemma 3.2. Then, there exist a k -dimensional space $E \in \mathbb{W}_k$ and $c_\infty > c_0$ such that the functional J in (16) verifies*

$$J(u) \leq c_\infty \quad \text{for all } u \in E. \tag{40}$$

Proof. Let us take β_k as in (36) and $\sigma > 0$ such that $\beta_k + \sigma < l_\infty$. By (11), for such a fixed $\sigma > 0$ there exists a subspace $Z_k^\sigma \in \mathbb{W}_k$ (see (10)), with $\dim Z_k^\sigma \geq k$, such that

$$\beta_k \leq \sup_{u \in S \cap Z_k^\sigma} \|u\|_V^p < \beta_k + \frac{\sigma}{2}. \tag{41}$$

We can assume that Z_k^σ is a finite dimensional subspace and claim that

$$\text{there exists } c_\infty > c_0 \quad \text{such that } J(u) \leq c_\infty \quad \text{for all } u \in Z_k^\sigma. \tag{42}$$

In fact, taking any $u \in W_V \setminus \{0\}$, by (32) we get

$$\lim_{R \rightarrow +\infty} \frac{F(x, Ru(x))}{R^p} = 0 \quad \text{for a.e. } x \in \mathbb{R}^N;$$

then, since by (33)–(35) the Lebesgue’s dominated convergence theorem applies, it follows that

$$\lim_{R \rightarrow +\infty} \int_{\mathbb{R}^N} \frac{F(x, Ru(x))}{R^p} dx = 0.$$

Thus, for $\sigma > 0$ fixed, there exists $\bar{R}_\sigma > 0$ such that

$$\left| \int_{\mathbb{R}^N} \frac{F(x, Ru(x))}{R^p} dx \right| < \frac{\sigma}{2} \quad \text{if } R > \bar{R}_\sigma$$

and thus

$$\left| \int_{\mathbb{R}^N} F(x, u) dx \right| = |u|_p^p \left| \int_{\mathbb{R}^N} \frac{F(x, |u|_p \frac{u}{|u|_p})}{|u|_p^p} dx \right| < \frac{\sigma}{2} |u|_p^p \quad \text{if } |u|_p > \bar{R}_\sigma. \tag{43}$$

So, (16) and (41) imply that

$$J(u) \leq \frac{1}{p} \left(\beta_k + \frac{\sigma}{2} - \lambda \right) |u|_p^p + \left| \int_{\mathbb{R}^N} F(x, u) dx \right| \quad \text{for all } u \in Z_k^\sigma.$$

By using (37) and (43) respectively, for $u \in Z_k^\sigma$ we get that

$$J(u) \leq \frac{1}{p} \left(\beta_k + \frac{3}{2}\sigma - \lambda - \gamma \right) |u|_p^p + B_\sigma |u|_{s+r}^{s+r} \leq c \quad \text{if } |u|_p \leq \bar{R}_\sigma,$$

for a suitable $c > 0$, and

$$J(u) \leq \frac{1}{p} (\beta_k + \sigma - \lambda) |u|_p^p \quad \text{if } |u|_p > \bar{R}_\sigma.$$

Thus, the claim (42) is proved. □

Proof of Theorem 1.1. By (f_6) and Proposition 3.1 the functional J is even and satisfies (CPS) in \mathbb{R} . Let us consider $\alpha_h, Y_{h-1}, \rho, c_0$ as in Lemma 3.2 and $\beta_k, \mathbb{W}_k, Z_k^\sigma, c_\infty$ as in Lemma 3.3. Now, considering the pseudo-index theory $(S_\rho \cap Y_{h-1}, \mathcal{H}^*, \gamma^*)$ related to the genus and $S_\rho \cap Y_{h-1}$, by Remark 2.3 (applied to $X = Z_k^\sigma, \partial B = S_\rho$ and $Y = Y_{h-1}$) we get

$$\gamma(Z_k^\sigma \cap h(S_\rho \cap Y_{h-1})) \geq \dim Z_k^\sigma - \operatorname{codim} Y_{h-1} \quad \text{for all } h \in \mathcal{H}^*;$$

hence,

$$\gamma^*(Z_k^\sigma) \geq k - h + 1.$$

So, Theorem 2.2 applies with $\tilde{A} := Z_k^\sigma$ and $S := S_\rho \cap Y_{h-1}$ and J has at least $k - h + 1$ distinct pairs of critical points.

If $\lambda > 0$, (f_5) becomes

$$\lambda < \alpha_h \leq \beta_k < \gamma + \lambda.$$

In this case, fixing $\sigma > 0$ such that

$$\lambda + \sigma < \alpha_h \leq \beta_k + \sigma < \gamma + \lambda,$$

arguments similar to those ones used in Lemmas 3.2 and 3.3 imply that J is bounded from below on Y_{h-1} and

$$J(u) \leq -\frac{1}{p}(\lambda + \gamma - \beta_k - \sigma)|u|_p^p + B_\sigma|u|_{r+p}^{r+p} \quad \text{for all } u \in Z_k^\sigma \cap S_\rho,$$

for a suitable positive constant B_σ . Thus, Theorem 2.2 applies to the functional $-J$, with $\tilde{A} := Y_{h-1}$ and $S := S_\rho \cap Z_k^\sigma$. \square

4. Proof of Theorem 1.2

The only difference in the proof with respect to the non-resonant case relies on the proof of the (CPS) condition. In fact, when we admit $\lambda \in \sigma(\mathcal{H}_p)$, the other arguments used in Section 3 work (we point out that (f'_1) implies (f_1)). The proof is a combination of [22, Proposition 3.1] (for the case of the classical p -Laplacian on bounded domains) and of [2, Proposition 5.1] (again for the classical p -Laplacian, but on \mathbb{R}^N).

Proposition 4.1. *Assume that $(f'_1), (f_2)$ and (f_7) hold. If $\lambda \in \sigma(\mathcal{H}_p)$, then the functional J in (16) satisfies (CPS) in \mathbb{R} .*

Proof. Let $c \in \mathbb{R}$ and $(u_n)_n$ be a sequence in W_V such that (3) holds. Setting

$$g(x, t) = \lambda|t|^{p-2}t + f(x, t)$$

we have that

$$\frac{1}{p}\|u_n\|_V^p - \int_{\mathbb{R}^N} G(x, u_n) \, dx = c + o(1) \tag{44}$$

and

$$\|u_n\|_V^p - \int_{\mathbb{R}^N} g(x, u_n)u_n \, dx = o(1), \tag{45}$$

where $G(x, t) = \int_0^t g(x, \tau) \, d\tau$.

By (f_7) there exists $\tau_0 > 0$ such that

$$g(x, t)t - pG(x, t) \geq 0 \quad \text{if } |t| \geq \tau_0, \text{ for a.e. } x \in \mathbb{R}^N; \quad (46)$$

moreover, by condition (f'_1) there exists $C_0 = C_0(\tau_0) > 0$ such that

$$\int_{\{|u_n| \leq \tau_0\}} (g(x, u_n)u_n - pG(x, u_n)) \, dx \geq -C_0 \quad \text{for all } n \in \mathbb{N}. \quad (47)$$

Let us fix $\sigma > 0$; then by (f_2) there exists $R_\sigma > 0$ such that

$$|g(x, t)| \leq (|\lambda| + \sigma)|t|^{p-1} \quad \text{if } |t| > R_\sigma, \text{ for a.e. } x \in \mathbb{R}^N. \quad (48)$$

Taking $q \in]p, p_s^*[$, $S > 0$ exists such that

$$|u|_q \leq S^{\frac{1}{p}} \|u\|_V \quad \text{for all } u \in W_V \quad (49)$$

and set

$$K := (pc + C_0)(2(|\lambda| + \sigma)S)^{\frac{q}{q-p}}, \quad (50)$$

with c as in (44) and C_0 as in (47). Now, by (f_7) we get the existence of $\bar{\tau} = \bar{\tau}(K) > \max\{\tau_0, R_\sigma\}$ such that

$$g(x, t)t - pG(x, t) \geq K \quad \text{if } |t| \geq \bar{\tau}, \text{ for a.e. } x \in \mathbb{R}^N, \quad (51)$$

thus for all $n \in \mathbb{N}$ we set

$$A_n = \{x \in \mathbb{R}^N : |u_n(x)| \geq \bar{\tau}\}$$

and

$$B_n = \{x \in \mathbb{R}^N : |u_n(x)| \leq \bar{\tau}\}.$$

As $u_n \in W_V$ implies $\text{meas}(A_n) < +\infty$, by (44)–(47) and (51) it follows that

$$\begin{aligned} pc + o(1) &= \int_{\mathbb{R}^N} (g(x, u_n)u_n - pG(x, u_n)) \, dx \\ &= \int_{A_n} (g(x, u_n)u_n - pG(x, u_n)) \, dx + \int_{\{|u_n| \leq \tau_0\}} (g(x, u_n)u_n - pG(x, u_n)) \, dx \\ &\quad + \int_{\{\tau_0 \leq |u_n| \leq \bar{\tau}\}} (g(x, u_n)u_n - pG(x, u_n)) \, dx \\ &\geq K \text{meas}(A_n) - C_0; \end{aligned}$$

therefore

$$\text{meas}(A_n) \leq \frac{pc + C_0}{K} + o(1) \quad \text{for all } n \in \mathbb{N}. \quad (52)$$

Taking $r > p$, by (44) and (45) we have that

$$\left(\frac{1}{p} - \frac{1}{r}\right) \|u_n\|_V^p - \int_{\mathbb{R}^N} \left(G(x, u_n) - \frac{1}{r}g(x, u_n)u_n\right) \, dx = c + o(1). \quad (53)$$

By (H'_0) there exists $C = C(\bar{r}, r) > 0$ such that

$$\left| \int_{B_n} \left(G(x, u_n) - \frac{1}{r} g(x, u_n) u_n \right) dx \right| \leq C \quad \text{for all } n \in \mathbb{N}. \quad (54)$$

So, by (53) and (54) we infer that

$$c + o(1) \geq \left(\frac{1}{p} - \frac{1}{r} \right) \|u_n\|_V^p - \int_{A_n} \left(G(x, u_n) - \frac{1}{r} g(x, u_n) u_n \right) dx - C$$

and by (46), (48) it follows that

$$\begin{aligned} c + o(1) &\geq \left(\frac{1}{p} - \frac{1}{r} \right) \|u_n\|_V^p - \int_{A_n} \left(\frac{1}{p} g(x, u_n) u_n - \frac{1}{r} g(x, u_n) u_n \right) dx - C \\ &\geq \left(\frac{1}{p} - \frac{1}{r} \right) \left(\|u_n\|_V^p - \int_{A_n} (|l_\infty| + \sigma) |u_n|^p dx \right) - C. \end{aligned}$$

By the Hölder inequality, (49), (50) and (52) we infer that

$$\begin{aligned} c + o(1) &\geq \left(\frac{1}{p} - \frac{1}{r} \right) \left(\|u_n\|_V^p - (|l_\infty| + \sigma) |u_n|_q^p \text{meas}(A_n)^{\frac{q-p}{q}} \right) - C \\ &\geq \left(\frac{1}{p} - \frac{1}{r} \right) \|u_n\|_V^p \left(1 - (|l_\infty| + \sigma) S \left(\frac{1}{(2(|l_\infty| + \sigma) S)^{\frac{q}{q-p}}} + o(1) \right)^{\frac{q-p}{q}} \right) - C. \end{aligned}$$

Therefore the sequence $(\|u_n\|_V)_n$ is bounded and by [25, Lemma 3.3] the proof is complete, because \mathcal{H}_p is of type (S) . \square

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