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An Extragradient Algorithm With Double Inertial Extrapolations For Generalized Mixed Equilibrium Problem And Fixed Point Problem Of Nonexpansive Mappings

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Abstract. The principal aim of this work is to find a simultaneous solution to two distinct problems: the generalized mixed equilibrium inequalities (where the main operator is assumed to be monotone and uniformly continuous) and the fixed point problem associated with nonexpansive semigroups in a real Hilbert space. We introduce an iner-tial extragradient method featuring two-step inertial extrapolations. The global strong convergence of the algorithm is rigorously demonstrated under standard constraints. Finally, we provide numerical experiments that confirm the superior performance of our approach.

Key word. Generalized mixed equilibrium problem; fixed point problem; strong con-vergence; nonexpansive mapping; uniformly continuous.

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1 Introduction

Consider H, a real Hilbert space, equipped with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. Let C be a non-empty, closed, and convex subset of H. We also consider two bifunctions $T, \psi : C \times C \to \mathbb{R}$ and a nonlinear mapping $G : C \to H$. The generalized mixed equilibrium problem (GMEP) is formulated as finding an element $x \in C$ such that

$$T(x, y) + \psi(y, x) - \psi(x, x) + \langle G(x), y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The solution set for problem (1.1) will be denoted as $GMEP(T, \psi, G)$. The GMEP, which has been extensively investigated in works like [10, 22, 25], offers a robust mathematical structure. This framework successfully unifies and generalizes several challenges fundamental to continuous optimization and variational analysis, including the simpler

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mixed equilibrium problem and the generalized equilibrium problem. Setting the functions $\psi = 0$ and $G = 0$ simplifies (1.1) to the classic equilibrium problem (EP), which seeks to find $x \in C$ such that

$$T(x, y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

We denote the solution set of (1.2) as $EP(T)$. The concept of the equilibrium problem, initially introduced by Ky Fan [21] and significantly expanded by Blum and Oetlli [2], serves as an incredibly versatile tool for modeling a wide range of optimization scenarios across both theoretical and applied disciplines. Next, we examine the fixed point problem for a mapping M, which is defined as:

$$\text{find} \quad x \in C \quad \text{such that} \quad Mx = x. \tag{1.3}$$

The fixed point set of M is denoted by $F(M)$. Fixed point theory has captured significant attention and remains an active research frontier in modern mathematics due to its diverse and far-reaching applications; seminal works include [3, 4, 5, 6, 7, 8, 9, 11,14, 15, 17, 23, 27, 33, 37, 38]. Recently, there has been considerable effort to devise iter-ative solution procedures capable of approximating common points that simultaneously satisfy both the generalized mixed equilibrium problem and the fixed point problem, as evidenced by studies like [13, 18, 31, 34, 47].

The generalized system of modified variational inclusion problems (GSMVIP), introduced by Kheawborisut and Kangtunyakarn [26] in 2021, consists of finding $v \in H$ where

$$0 \in (N + B_1)v \quad \text{and} \quad 0 \in (N + B_2)v \tag{1.4}$$

and $N : H \to H$ is a mapping and $B_1, B_2 : H \to 2^H$ are set-valued mappings. Let Γ the solution set of problem (1.4).

A novel iterative scheme combining the inertial technique and the subgradient extragradient method was recently developed by Husain and Asad [25]. Inspired by the works of [22] and [26], this algorithm approximates common solutions to modified variational inclusion problems and mixed equilibrium problems within real Hilbert space. Assume that $f : C \to H$ be a nonexpansive mapping and $G : C \to H$ be a L-Lipschitz continuous and monotone mapping. Let the sequence $\{x_n\}$ be generated for any $x_0, x_1, x \in H\psi$by

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ v_n = S^T_{r_n}(w_n - r_n G(w_n)), \\ z_n = S^T_{r_n QQ_n}(w_n - r_n G(v_n)), \\ x_{n+1} = \lambda_n u + \eta_n z_n + \gamma_n f(u_n) \end{cases}$$

where $QQ_n = \{x \in H : \langle w_n - r_n G(w_n) - v_n, x - v_n\rangle \geq r_n T(v_n, x)\}$ and $\{\lambda_n\}$, $\{\eta_n\}$, $\{\gamma_n\} \subset (0, 1)$ with $\lambda_n + \eta_n + \gamma_n = 1, r_n \leq \frac{1}{L}, \theta_n \in [0, 1)$. They demonstrated that the sequence $\{x_n\}$ converges strongly to $q = P_{GMEP(T,\psi,G)\bigcap\Gamma}x$, provided that certain conditions on the parameters are met.

Observe that the convergence analysis for the algorithms proposed in [22] and [25] relies on the Lipschitz continuity of the mapping G. This strong regularity condition limits the practical scope of these algorithms. Our core motivation is to overcome this limitation by designing a simplified methodology that operates without the Lipschitz continuity requirement, thereby making the method more broadly applicable and less dependent on the specific operator properties. Driven by the aforementioned body of research and a thorough review of related literature, we put forward an inertial extragradient scheme employing double extrapolation steps. This scheme is designed to compute the common solution shared by the generalized mixed equilibrium problem (1.1) (for operators that are monotone and uniformly continuous) and the fixed point problem (1.3) of a nonexpansive mapping. A key differentiator is that, unlike previous approaches that often necessitate weak sequential continuity, our convergence proof only demands the underlying operator for (1.1) to be monotone and uniformly continuous. Our algorithm further distinguishes itself from standard methods that rely on time-consuming Armijo-type line search routines by utilizing a straightforward, self step size rule. This novel rule adaptively adjusts the step size, producing a sequence that is not restricted to being monotonically increasing or decreasing. Crucially, our technique provides a closed-form step size update, eliminating the need for both complicated search procedures and prior knowledge of the Lipschitz constant of G. Finally, the chosen step size strategy incorporates elements found in successful existing methods [28, 32, 41, 42].

2 Preliminaries

We commence with an overview of fundamental definitions and established mathematical results. These concepts will form the analytical groundwork for our subsequent investigation of the proposed method.

Assumption 2.1 *[2] Let $T : C \times C \to \mathbb{R}$ and $\psi : C \times C \to \mathbb{R}$ be two bifunctions satisfying the following properties:*

(A_1) $T(u,u) = 0, \forall u \in C$;

(A_2) T *is monotone, i.e.,* $T(u, v) + T(v, u) \leq 0, \forall u, v \in C$;

(A_3) For each $u, v, w \in C, \lim_{t\to 0} T(tw + (1-t)u, v) \leq T(u, v)$;

(A_4) the bifunction $T(\cdot,\cdot)$ *is weakly continuous. For each* $u \in C, v \to T(u, v)$ *is convex and lower semicontinuous.*

(B_1) the bifunction $\psi(\cdot,\cdot)$ is weakly continuous and the bifunction $\psi(\cdot, v)$ is convex, $\forall v \in C$;

(B_2) the bifunction ψ is generalized skew-symmetric, i.e.,

$$\psi(u,u) - \psi(u,v) + \psi(v,v) - \psi(v,w) + \psi(w,w) - \psi(w,u) \geq 0, \forall u, v, w \in C.$$

The bifunction $T : C \times C \to \mathbb{R}$ is said to be 2-monotone if

$$T(u,v) + T(v,w) + T(w,u) \leq 0, \forall u, v, w \in C. \tag{2.1}$$

Lemma 2.1 *[20] Assume that $T, \psi : C \times C \to \mathbb{R}$ be two bifunctions satisfying Assumption 2.1. For $r > 0$ and $\forall u \in H$, the operator $S_r^T : H \to C$ is uniquely defined by:*

$$S_r^T(u) = \Big\{ w \in C : T(w,v) + \psi(v,w) - \psi(w,w) + \frac{1}{r}\langle v - w, w - u\rangle \geq 0, \forall v \in C \Big\}. \tag{2.2}$$

Then the following properties hold:

(i) S_r^T is non-empty and single-valued;

(ii) S_r^T is firmly nonexpansive, i.e.,

$$\|S_r^T(u) - S_r^T(v)\|^2 \leq \langle S_r^T(u) - S_r^T(v), u - v\rangle, \forall u, v \in H;$$

(iii) The fixed point set of S_r^T coincides with the solution set of the Equilibrium Problem: $F(S_r^T) = EP(T)$;

(iv) The solution set $EP(T)$ is closed and convex.

Remark 2.1 *Applying the definition of S_r^T immediately leads to the inequality:*

$$T(S_r^T(u), v) + \psi(v, S_r^T(u)) - \psi(S_r^T(u), S_r^T(u)) + \frac{1}{r}\langle v - S_r^T(u), S_r^T(u) - u\rangle \geq 0, \forall v \in C$$

By algebraic manipulation and using the identity $2\langle a, b\rangle = \|a\|^2 + \|b\|^2 - \|a - b\|^2$, we derive the useful non-expansivity relation:

$$\begin{aligned} \|v - S_r^T(u)\|^2 \leq\ & \|v - u\|^2 - \|S_r^T(u) - u\|^2 \\ & + 2r\Big(T(S_r^T(u), v) + \psi(v, S_r^T(u)) - \psi(S_r^T(u), S_r^T(u))\Big), \forall v \in C. \end{aligned} \tag{2.3}$$

Lemma 2.2 [30] *Opial's Condition: Every Hilbert space satisfies this property, meaning that for any sequence $\{x_n\}$ in the space that converges weakly ($x_n \rightharpoonup x_0$), the following strict inequality holds:*

$$\liminf_{n\to\infty} \|x_n - x_0\| < \liminf_{n\to\infty} \|x_n - y\| \tag{2.4}$$

for all points y in the space such that $y \neq x_0$.

Lemma 2.3 [46] *Uniform continuity of a function G on a convex domain is equivalent to the following: for every $\varrho > 0$, there exists a finite constant L such that for all points a and b in the domain, the inequality $\|G(a) - G(b)\| \leq L\|a - b\| + \varrho$ holds.*

Lemma 2.4 [40] *Let $\{c_n\}$ and $\{\omega_n\}$ be two non-negative real sequences satisfying the recurrence relation:*

$$c_{n+1} \leq c_n + \omega_n, \quad \forall n \geq 1.$$

If the sum of the error terms is finite ($\sum_{n=0}^{\infty} \omega_n < \infty$), then the sequence $\{c_n\}$ converges to a limit ($\lim_{n\to\infty} c_n$ exists).

Lemma 2.5 [35] *Let $\{c_n\}$ be a positive real sequence, $\{\omega_n\}$ be a sequence in $(0, 1)$ such that $\sum_{n=1}^{\infty} \omega_n = \infty$ and ϑ_n is a sequence of real numbers. If the sequence satisfies the inequality:*

$$c_{n+1} \leq (1 - \omega_n)c_n + \omega_n \vartheta_n, \quad \forall n \geq 1.$$

and if $\limsup_{k\to\infty} \vartheta_{n_k} \leq 0$ for all subsequences $\{c_{n_k}\}$ of $\{c_n\}$ that satisfy the condition $\liminf_{k\to\infty}(c_{n_k+1} - c_{n_k}) \geq 0$. Then, the sequence converges strongly to zero ($\lim_{n\to\infty} c_n = 0$).

Lemma 2.6 *[24] Let H be a real Hilbert space and K be a nonempty closed convex subset of H. Let $M : C \to C$ be a nonexpansive mapping with a non-empty fixed point set (Fix(M)$\neq \emptyset$). Then the mapping $I - M$ is demiclosed at 0. That is, for any sequence $\{x_n\}$ in C such that:*

(1) The sequence converges weakly: $x_n \rightharpoonup x$, and

(2) The difference converges strongly to zero: $\{(I - M)x_n\} \to 0$,

it follows that x is a fixed point: $(I - M)x = 0$, or equivalently, $x \in$Fix(M).

Finally, we utilize the following standard algebraic relations and inequalities applicable in Hilbert spaces.

(i)

$$\|x + v\|^2 \leq \|x\|^2 + 2\langle v, x + v\rangle, \qquad \forall x, v \in H. \tag{2.5}$$

(ii) For each $v_1, \cdots, v_m \in H$ and coefficients $\delta_1, \cdots, \delta_m \in [0, 1]$ with $\sum_{i=1}^{m} \delta_i = 1$, the following identity holds (often used for convex combinations):

$$\|\delta_1 v_1 + \cdots + \delta_m v_m\|^2 = \sum_{i=1}^{m} \delta_i \|v_i\|^2 - \sum_{1\leq i\leq j\leq m} \delta_i \delta_j \|v_i - v_j\|^2. \tag{2.6}$$

3 Main result

Consider two bifunctions, T and ψ, defined on $C \times C$ to $\mathbb{R}$, which are assumed to satisfy Assumption 2.1. Let $h : H \to H$ be a contraction mapping with contraction factor $k \in [0, 1)$. Additionally, let M, A, and D be nonexpansive mappings from H to H. We also define several positive sequences: $\{\delta_n\}$, $\{\beta_n\}$, $\{\lambda_n\}$, $\{\epsilon_n\}$, $\{\tau_n\}$ and $\{\rho_n\}$, all of which satisfy the specified properties:

(a) $\delta_n + \beta_n + \lambda_n = 1$, and $\liminf_{n\to\infty}\beta_n\lambda_n > 0$;

(b) Let $\{\epsilon_n\}$ and $\{\xi_n\}$ be positive real sequences with the property that $\lim\limits_{n\to\infty} \dfrac{\epsilon_n}{\delta_n} = 0$ and $\lim\limits_{n\to\infty} \dfrac{\xi_n}{\delta_n} = 0$;

(c) $\delta_n \in (0, 1)$, $\lim\limits_{n\to\infty} \delta_n = 0$ and $\sum\limits_{n=1}^{\infty} \delta_n = \infty$;

(d) $\sum\limits_{n=1}^{\infty} \tau_n < \infty$, $\lim\limits_{n\to\infty} \rho_n = 0$.

Under the standard assumptions:

(C1) The feasible set C is nonempty closed and convex.

(C2) The operator $G : H \to H$ be monotone and uniformly continuous on H and satisfies the following property: whenever $\{t_n\} \in C, t_n \rightharpoonup t^*$, one has $\|G(t^*)\| \leq \liminf\limits_{n\to\infty} \|G(t_n)\|$.

(C3) The bifunction ψ is generalized skew-symmetric and T is 2-monotone.

(C4) The solution set $\Omega = F(M) \bigcap GMEP(T, \psi, G) \neq \emptyset$.

We introduce the following algorithm, designed to find the common solutions to problem (1.1) and problem (1.3).

Algorithm 3.1.

Step 0. Choose initial parameters and arbitrary starting points:
Given constants $\chi \in (0,1), \gamma \in (0,2), r_1 > 0, \theta > 0, \varpi > 0$. Select $t_0, t_1 \in H$ be arbitrary.
Use t_{n-1}, t_n for the current iteration.

Step 1. Determine adaptive extrapolation coefficients θ_n and ϖ_n such that

$$\theta_n := \begin{cases} \min\{\theta, \frac{\epsilon_n}{\|t_n - t_{n-1}\|}\} & \text{if } t_n \neq t_{n-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{3.1}$$

And

$$\varpi_n := \begin{cases} \max\{\varpi, \frac{\xi_n}{\|t_n - t_{n-1}\|}\} & \text{if } t_n \neq t_{n-1}, \\ \varpi, & \text{otherwise.} \end{cases} \tag{3.2}$$

Step 2. Calculate

$$c_n = t_n + \theta_n\Big(A(t_n) - A(t_{n-1})\Big),$$

$$d_n = t_n + \varpi_n\Big(D(t_n) - D(t_{n-1})\Big),$$

and compute

$$z_n = S^T_{r_n}\Big(c_n - r_n G(c_n)\Big),$$

if $z_n = c_n$ then stop, c_n is a solution of (1.1). Else, do Step 3.

Step 3. Compute

$$u_n = S^T_{r_n Q_n}\Big(c_n - r_n G(z_n)\Big)$$

such that

$$Q_n = \Big\{x \in H : \langle c_n - r_n G(c_n) - z_n, x - z_n\rangle \leq r_n T(z_n, x) + r_n \psi(x, z_n) - r_n \psi(z_n, z_n)\Big\}$$

and $S^T_{r_n Q_n}$ is obtained from (2.2) by setting $C = Q_n$.

Step 4. Compute the next iterate

$$t_{n+1} = \delta_n h(d_n) + \beta_n u_n + \lambda_n M(u_n).$$

Then, update the parameter r_n.

$$r_{n+1} := \begin{cases} \min\Big\{\frac{(\rho_n + \chi)(\|c_n - z_n\|^2 + \|u_n - z_n\|^2)}{2\langle G(c_n) - G(z_n), u_n - z_n\rangle}, r_n + \tau_n\Big\} & \text{if } \langle G(c_n) - G(z_n), u_n - z_n\rangle > 0, \\ r_n + \tau_n & \text{otherwise.} \end{cases} \tag{3.3}$$

Increase n to $n+1$ and return to Step 1.

In the case where ψ represents the indicator function of a closed convex set C and $T = 0$, the operator $S^{T}_{r_n}$ is equivalent to P_C [1]. Thus, Algorithm 3.1 simplifies to the following scheme

Algorithm 3.2.

Step 0. Choose initial parameters and arbitrary starting points:
Given constants $\chi \in (0,1), \gamma \in (0,2), r_1 > 0, \theta > 0, \varpi > 0$. Select $t_0, t_1 \in H$ be arbitrary.
Use t_{n-1}, t_n for the current iteration.

Step 1. Determine adaptive extrapolation coefficients θ_n and ϖ_n such that

$$\theta_n := \begin{cases} \min\{\theta, \frac{\epsilon_n}{\|t_n - t_{n-1}\|}\} & \text{if } t_n \neq t_{n-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{3.4}$$

And

$$\varpi_n := \begin{cases} \max\{\varpi, \frac{\xi_n}{\|t_n - t_{n-1}\|}\} & \text{if } t_n \neq t_{n-1}, \\ \varpi, & \text{otherwise.} \end{cases} \tag{3.5}$$

Step 2. Calculate

$$c_n = t_n + \theta_n \Big(A(t_n) - A(t_{n-1}) \Big),$$
$$d_n = t_n + \varpi_n \Big(D(t_n) - D(t_{n-1}) \Big),$$

and compute

$$z_n = P_C \Big(c_n - r_n G(c_n) \Big),$$

if $z_n = c_n$ then stop, c_n is a solution of (1.1). Else, do Step 3.

Step 3. Compute

$$u_n = P_{C_n} \Big(c_n - r_n G(z_n) \Big)$$

such that

$$C_n = \Big\{ x \in H : \langle c_n - r_n G(c_n) - z_n, x - z_n \rangle \leq 0 \Big\}.$$

Step 4. Compute the next iterate

$$t_{n+1} = \delta_n h(d_n) + \beta_n u_n + \lambda_n M(u_n).$$

Then, update the parameter r_n.

$$r_{n+1} := \begin{cases} \min \Big\{ \frac{(\rho_n + \chi)(\|c_n - z_n\|^2 + \|u_n - z_n\|^2)}{2\langle G(c_n) - G(z_n), u_n - z_n \rangle}, r_n + \tau_n \Big\} & \text{if } \langle G(c_n) - G(z_n), u_n - z_n \rangle > 0, \\ r_n + \tau_n & \text{otherwise.} \end{cases} \tag{3.6}$$

Increase n to $n+1$ and return to Step 1.

Remark 3.2 *From condition (b) and (3.4) we have*

$$\theta_n\|t_n - t_{n-1}\| \leq \epsilon_n \quad and \quad \varpi_n\|t_n - t_{n-1}\| \leq \xi_n.$$

then

$$\lim_{n\to\infty} \frac{\theta_n}{\delta_n}\|t_n - t_{n-1}\| = 0 \tag{3.7}$$

and

$$\lim_{n\to\infty} \frac{\varpi_n}{\delta_n}\|t_n - t_{n-1}\| = 0. \tag{3.8}$$

Thus, there exist $N_1 > 0$ and $N_2 > 0$ such that

$$\frac{\theta_n}{\delta_n}\|t_n - t_{n-1}\| \leq N_1, \forall n \in \mathbf{N} \tag{3.9}$$

and

$$\frac{\varpi_n}{\delta_n}\|t_n - t_{n-1}\| \leq N_2, \forall n \in \mathbf{N}. \tag{3.10}$$

For the proof of global convergence of our approach, we first state the necessary lemmas.

Lemma 3.1 *[10] Given that $\{r_n\}$ is defined by (3.6). Then, we have $\lim_{n\to\infty} r_n = r$, where $r \in \Big[\min\Big(\frac{\chi}{M}, r_1\Big), r_1 + \sum_{n=1}^{\infty} \tau_n\Big]$.*

Remark 3.3 *From Lemma 3.1 and condition (e) we have*

$$\lim_{n\to\infty}\left(1 - \frac{(\rho_n + \chi)r_n}{r_{n+1}}\right) = 1 - \chi > 0, \tag{3.11}$$

for sufficiently large n (i.e., for all $n \geq n_0$ for some n_0) we have $1 - \frac{(\rho_n+\chi)r_n}{r_{n+1}} > \frac{1-\chi}{2} > 0$.

Lemma 3.2 *Assume $\{t_n\}$ is the sequence produced by the Algorithm 3.1 and $t^* \in \Omega$. The following inequality holds*

$$\|u_n - t^*\|^2 \leq \|c_n - t^*\|^2 - \Big(1 - \frac{(\rho_n + \chi)r_n}{r_{n+1}}\Big)\|u_n - z_n\|^2 - \Big(1 - \frac{(\rho_n + \chi)r_n}{r_{n+1}}\Big)\|z_n - c_n\|^2. \tag{3.12}$$

Proof. It follows from (2.3) that

$$\begin{aligned}
\|u_n - t^*\|^2 &= \|S^T_{r_nQ_n}\Big(c_n - r_nG(z_n)\Big) - t^*\|^2 \\
&\leq \|c_n - r_nG(z_n) - t^*\|^2 - \|u_n - c_n + r_nG(z_n)\|^2 \\
&\quad +2r_nT(u_n, t^*) + 2r_n[\psi(t^*, u_n) - \psi(u_n, u_n)] \\
&= \|c_n - t^*\|^2 - 2r_n\langle u_n - t^*, G(z_n)\rangle - \|u_n - c_n\|^2 \\
&\quad +2r_nT(u_n, t^*) + 2r_n[\psi(t^*, u_n) - \psi(u_n, u_n)].
\end{aligned}$$

Since $t^* \in GMEP(T,\psi,G)$, we obtain

$$T(t^*, z_n) + \langle G(t^*), z_n - t^*\rangle + \psi(z_n, t^*) - \psi(t^*, t^*) \geq 0. \tag{3.13}$$

and applying the property of monotonicity of G, we get

$$\begin{aligned}\langle G(z_n), u_n - t^*\rangle &= \langle G(z_n) - G(t^*), z_n - t^*\rangle + \langle G(t^*), z_n - t^*\rangle + \langle G(z_n), u_n - z_n\rangle \\ &\geq \langle G(t^*), z_n - t^*\rangle + \langle G(z_n), u_n - z_n\rangle. \end{aligned} \tag{3.14}$$

From $u_n \in Q_n$, we have

$$\langle c_n - r_n G(c_n) - z_n, u_n - z_n\rangle \leq r_n T(z_n, u_n) + r_n\psi(u_n, z_n) - r_n\psi(z_n, z_n).$$

Using (3.13), (3.14) and (3.15), we obtain

$$\begin{aligned}\|u_n - t^*\|^2 &\leq \|c_n - t^*\|^2 - 2r_n\Big(\langle G(t^*), z_n - t^*\rangle + \langle G(z_n), u_n - z_n\rangle\Big) - \|u_n - c_n\|^2 \\ &\quad +2r_n T(u_n, t^*) + 2r_n[\psi(t^*, u_n) - \psi(u_n, u_n)] \\ &\quad +2r_n\Big(T(t^*, z_n) + \langle G(t^*), z_n - t^*\rangle + \psi(z_n, t^*) - \psi(t^*, t^*)\Big) \\ &\quad -2\langle c_n - r_n G(c_n) - z_n, u_n - z_n\rangle + 2r_n\Big(T(z_n, u_n) + \psi(u_n, z_n) - \psi(z_n, z_n)\Big) \\ &= \|c_n - t^*\|^2 - 2r_n\langle G(z_n), u_n - z_n\rangle - \|u_n - z_n\|^2 - \|z_n - c_n\|^2 - 2\langle u_n - z_n, z_n - c_n\rangle \\ &\quad -2\langle c_n - r_n G(c_n) - z_n, u_n - z_n\rangle + 2r_n\Big(T(u_n, t^*) + T(t^*, z_n) + T(z_n, u_n)\Big) \\ &\quad -2r_n\Big(\psi(t^*, t^*) - \psi(t^*, u_n) + \psi(u_n, u_n) - \psi(u_n, z_n) + \psi(z_n, z_n) - \psi(z_n, t^*)\Big) \\ &= \|c_n - t^*\|^2 - \|u_n - z_n\|^2 - \|z_n - c_n\|^2 + 2r_n\langle G(c_n) - G(z_n), u_n - z_n\rangle \\ &\quad +2r_n\Big(T(u_n, t^*) + T(t^*, z_n) + T(z_n, u_n)\Big) \\ &\quad -2r_n\Big(\psi(t^*, t^*) - \psi(t^*, u_n) + \psi(u_n, u_n) - \psi(u_n, z_n) + \psi(z_n, z_n) - \psi(z_n, t^*)\Big)\end{aligned}$$

By invoking the 2-monotonicity of T and the generalized skew symmetry of ψ in the above inequality, we have

$$\|u_n - t^*\|^2 \leq \|c_n - t^*\|^2 - \|u_n - z_n\|^2 - \|z_n - c_n\|^2 + 2r_n\langle G(c_n) - G(z_n), u_n - z_n\rangle. \tag{3.15}$$

From (3.6), we get

$$\|u_n - t^*\|^2 \leq \|c_n - t^*\|^2 - \Big(1 - \frac{(\rho_n + \chi)r_n}{r_{n+1}}\Big)\|u_n - z_n\|^2 - \Big(1 - \frac{(\rho_n + \chi)r_n}{r_{n+1}}\Big)\|z_n - c_n\|^2.$$

we get the assertion of this lemma. □

Lemma 3.3 *Consider the sequence* $\{t_n\}$ *resulting from Algorithm 3.1 and* $t^* \in \Omega$. *Then, we have* $\forall n \geq n_0$ $\{t_n\}$ *is bounded.*

Proof. Combining condition (a) and Lemma 3.2 yields

$$\begin{aligned}
\|t_{n+1}-t^*\| &= \|\delta_n h(d_n)+\beta_n u_n+\lambda_n M(u_n)-t^*\| \\
&\leq \delta_n\|h(d_n)-t^*\|+\beta_n\|u_n-t^*\|+\lambda_n\|M(u_n)-t^*\| \\
&\leq \delta_n\|h(d_n)-h(t^*)\|+\delta_n\|h(t^*)-t^*\|+\beta_n\|u_n-t^*\|+\lambda_n\|M(u_n)-M(t^*)\| \\
&\leq \delta_n k\|d_n-t^*\|+\delta_n\|h(t^*)-t^*\|+(1-\delta_n)\|u_n-t^*\| \\
&\leq \delta_n k\|d_n-t^*\|+\delta_n\|h(t^*)-t^*\|+(1-\delta_n)\|c_n-t^*\|. \qquad (3.16)
\end{aligned}$$

Observe from (3.9), we have

$$\begin{aligned}
\|c_n-t^*\| &= \|t_n+\theta_n(A(t_n)-A(t_{n-1}))-t^*\| \\
&\leq \|t_n-t^*\|+\delta_n\Big(\frac{\theta_n}{\delta_n}\|t_n-t_{n-1}\|\Big) \\
&\leq \|t_n-t^*\|+\delta_n N_1. \qquad (3.17)
\end{aligned}$$

Also, from (3.10), we have

$$\begin{aligned}
\|d_n-t^*\| &= \|t_n+\varpi_n(D(t_n)-D(t_{n-1}))-t^*\| \\
&\leq \|t_n-t^*\|+\delta_n\Big(\frac{\varpi_n}{\delta_n}\|t_n-t_{n-1}\|\Big) \\
&\leq \|t_n-t^*\|+\delta_n N_2. \qquad (3.18)
\end{aligned}$$

Applying (3.17) and (3.18) in (3.16), we have

$$\begin{aligned}
\|t_{n+1}-t^*\| &\leq \Big(1-(1-k)\delta_n\Big)\|t_n-t^*\|+\delta_n\|h(t^*)-t^*\|+\delta_n\Big((1-\delta_n)N_1+k\delta_n N_2\Big) \\
&\leq \Big(1-(1-k)\delta_n\Big)\|t_n-t^*\|+\delta_n\|h(t^*)-t^*\|+\delta_n(N_1+N_2) \\
&= \Big(1-(1-k)\delta_n\Big)\|t_n-t^*\|+\delta_n(1-k)\frac{\|h(t^*)-t^*\|+(N_1+N_2)}{1-k} \\
&\leq \Big(1-(1-k)\delta_n\Big)\max\Big(\|t_n-t^*\|,\frac{\|h(t^*)-t^*\|+(N_1+N_2)}{1-k}\Big) \\
&= \max\Big(\|t_n-t^*\|,\frac{\|h(t^*)-t^*\|+(N_1+N_2)}{1-k}\Big). \qquad (3.19)
\end{aligned}$$

We proceed by induction on n to conclude

$$\|t_n-t^*\|\leq\max\Big(\|t_{n_0}-t^*\|,\frac{\|h(t^*)-t^*\|+(N_1+N_2)}{1-k}\Big),\forall n\geq n_0.$$

Hence $\{t_n\}$ is is bounded, we can immediately conclude that $\{c_n\},\{z_n\},\{u_n\},\{h(d_n)\}$ and $\{M(u_n)\}$ are bounded. □

4 Convergence Analysis

We proceed with the strong convergence demonstration of the new method. A key feature of this proof is its independence from the two-cases analysis frequently employed in the literature to establish such convergence.

Theorem 4.1 *Algorithm 3.1 produces a sequence* $\{t_n\}$ *which converges strongly to some point* $\tilde{t} \in \Omega$*, where* $\tilde{t} = P_\Omega[h(\tilde{t})]$.

Proof. Let $\tilde{t} \in \Omega$. According to the definition of c_n, we get

$$\begin{aligned} \|c_n - \tilde{t}\|^2 &\leq \|t_n - \tilde{t}\|^2 + \theta_n^2\|t_n - t_{n-1}\|^2 + 2\theta_n\|t_n - \tilde{t}\|\|t_n - t_{n-1}\| \\ &= \|t_n - \tilde{t}\|^2 + \delta_n\theta_n\frac{\theta_n}{\delta_n}\|t_n - t_{n-1}\|^2 + 2\delta_n\|t_n - \tilde{t}\|\frac{\theta_n}{\delta_n}\|t_n - t_{n-1}\| \\ &= \|t_n - \tilde{t}\|^2 + \delta_n qq_n \end{aligned} \tag{4.1}$$

where

$$qq_n = \theta_n\|t_n - t_{n-1}\|\frac{\theta_n}{\delta_n}\|t_n - t_{n-1}\| + 2\|t_n - \tilde{t}\|\frac{\theta_n}{\delta_n}\|t_n - t_{n-1}\|. \tag{4.2}$$

Also, from the definition of d_n, we obtain

$$\begin{aligned} \|d_n - \tilde{t}\|^2 &\leq \|t_n - \tilde{t}\|^2 + \varpi_n^2\|t_n - t_{n-1}\|^2 + 2\varpi_n\|t_n - \tilde{t}\|\|t_n - t_{n-1}\| \\ &= \|t_n - \tilde{t}\|^2 + \delta_n\varpi_n\frac{\varpi_n}{\delta_n}\|t_n - t_{n-1}\|^2 + 2\delta_n\|t_n - \tilde{t}\|\frac{\varpi_n}{\delta_n}\|t_n - t_{n-1}\| \\ &= \|t_n - \tilde{t}\|^2 + \delta_n pp_n \end{aligned} \tag{4.3}$$

where

$$pp_n = \varpi_n\|t_n - t_{n-1}\|\frac{\varpi_n}{\delta_n}\|t_n - t_{n-1}\| + 2\|t_n - \tilde{t}\|\frac{\varpi_n}{\delta_n}\|t_n - t_{n-1}\|. \tag{4.4}$$

We can easily establish from (3.7) and (3.8) that

$$\lim_{n\to\infty} qq_n = 0 \quad \text{and} \quad \lim_{n\to\infty} pp_n = 0. \tag{4.5}$$

Then from (2.6), Lemma 3.2, (4.1) and (4.3), we have

$$\begin{aligned}
\|t_{n+1}-\tilde{t}\|^2 &= \|\delta_n h(d_n)+\beta_n u_n+\lambda_n M(u_n)-\tilde{t}\|^2 \\
&\leq \delta_n\|h(d_n)-\tilde{t}\|^2+\beta_n\|u_n-\tilde{t}\|^2+\lambda_n\|M(u_n)-\tilde{t}\|^2 \\
&\quad -\beta_n\lambda_n\|M(u_n)-u_n\|^2 \\
&\leq \delta_n\Big(\|h(d_n)-h(\tilde{t})\|+\|h(\tilde{t})-\tilde{t}\|\Big)^2+\beta_n\|u_n-\tilde{t}\|^2 \\
&\quad +\lambda_n\|M(u_n)-M(\tilde{t})\|^2-\beta_n\lambda_n\|M(u_n)-u_n\|^2 \\
&\leq \delta_n\Big(k\|d_n-\tilde{t}\|+\|h(\tilde{t})-\tilde{t}\|\Big)^2+(1-\delta_n)\|u_n-\tilde{t}\|^2 \\
&\quad -\beta_n\lambda_n\|M(u_n)-u_n\|^2 \\
&\leq \delta_n\|d_n-\tilde{t}\|^2+\delta_n\Big(2\|d_n-\tilde{t}\|\|h(\tilde{t})-\tilde{t}\|+\|h(\tilde{t})-\tilde{t}\|^2\Big)+(1-\delta_n)\|u_n-\tilde{t}\|^2 \\
&\quad -\beta_n\lambda_n\|M(u_n)-u_n\|^2 \\
&\leq \delta_n\|d_n-\tilde{t}\|^2+\delta_n\Big(2\|d_n-\tilde{t}\|\|h(\tilde{t})-\tilde{t}\|+\|h(\tilde{t})-\tilde{t}\|^2\Big)+(1-\delta_n)\|c_n-\tilde{t}\|^2 \\
&\quad -(1-\delta_n)\Big(1-\frac{(\rho_n+\chi)r_n}{r_{n+1}}\Big)\|u_n-z_n\|^2-(1-\delta_n)\Big(1-\frac{(\rho_n+\chi)r_n}{r_{n+1}}\Big)\|z_n-c_n\|^2 \\
&\quad -\beta_n\lambda_n\|M(u_n)-u_n\|^2 \\
&\leq \delta_n\Big(\|t_n-\tilde{t}\|^2+\delta_n pp_n\Big)+\delta_n\Big(2\|d_n-\tilde{t}\|\|h(\tilde{t})-\tilde{t}\|+\|h(\tilde{t})-\tilde{t}\|^2\Big) \\
&\quad (1-\delta_n)\Big(\|t_n-\tilde{t}\|^2+\delta_n qq_n\Big)-(1-\delta_n)\Big(1-\frac{(\rho_n+\chi)r_n}{r_{n+1}}\Big)\|u_n-z_n\|^2 \\
&\quad -(1-\delta_n)\Big(1-\frac{(\rho_n+\chi)r_n}{r_{n+1}}\Big)\|z_n-c_n\|^2-\beta_n\lambda_n\|M(u_n)-u_n\|^2 \\
&\leq \|t_n-\tilde{t}\|^2+\delta_n\Big(2\|d_n-\tilde{t}\|\|h(\tilde{t})-\tilde{t}\|+\|h(\tilde{t})-\tilde{t}\|^2+qq_n+pp_n\Big) \\
&\quad -(1-\delta_n)\Big(1-\frac{(\rho_n+\chi)r_n}{r_{n+1}}\Big)\|u_n-z_n\|^2-(1-\delta_n)\Big(1-\frac{(\rho_n+\chi)r_n}{r_{n+1}}\Big)\|z_n-c_n\|^2 \\
&\quad -\beta_n\lambda_n\|M(u_n)-u_n\|^2. \qquad (4.6)
\end{aligned}$$

Assume that $\{\|t_{n_k}-\tilde{t}\|^2\}$ is a subsequence of $\{\|t_n-\tilde{t}\|^2\}$ satisfying

$$\liminf_{k\to\infty}\Big(\|t_{n_k+1}-\tilde{t}\|^2-\|t_{n_k}-\tilde{t}\|^2\Big)\geq 0. \qquad (4.7)$$

From (4.6), we obtain

$$\begin{aligned}
&(1-\delta_{n_k})\Big(1-\frac{(\rho_{n_k}+\chi)r_{n_k}}{r_{n_k+1}}\Big)\|u_{n_k}-z_{n_k}\|^2+(1-\delta_{n_k})\Big(1-\frac{(\rho_{n_k}+\chi)r_{n_k}}{r_{n_k+1}}\Big)\|z_{n_k}-c_{n_k}\|^2 \\
&+\beta_{n_k}\lambda_{n_k}\|M(u_{n_k})-u_{n_k}\|^2 \\
\leq\ &\|t_{n_k}-\tilde{t}\|^2-\|t_{n_k+1}-\tilde{t}\|^2+\delta_{n_k}\Big(2\|d_{n_k}-\tilde{t}\|\|h(\tilde{t})-\tilde{t}\|+\|h(\tilde{t})-\tilde{t}\|^2+qq_{n_k}+pp_{n_k}\Big).
\end{aligned}$$

By utilizing above inequality and (4.7), and since $\lim_{n\to\infty} \delta_n = 0$, we find that

$$\begin{aligned}
& \limsup_{k\to\infty} \Big((1-\delta_{n_k})\Big(1 - \frac{(\rho_{n_k}+\chi)r_{n_k}}{r_{n_k+1}}\Big)\|u_{n_k} - z_{n_k}\|^2 + (1-\delta_{n_k})\Big(1 - \frac{(\rho_{n_k}+\chi)r_{n_k}}{r_{n_k+1}}\Big)\|z_{n_k} - c_{n_k}\|^2 \\
& +\beta_{n_k}\lambda_{n_k}\|M(u_{n_k}) - u_{n_k}\|^2\Big) \\
\leq\ & \limsup_{k\to\infty} \Big(\|t_{n_k} - \tilde{t}\|^2 - \|t_{n_k+1} - \tilde{t}\|^2 + \delta_{n_k}\Big(2\|d_{n_k} - \tilde{t}\|\|h(\tilde{t}) - \tilde{t}\| + \|h(\tilde{t}) - \tilde{t}\|^2 + qq_{n_k} + pp_{n_k}\Big)\Big) \\
=\ & -\liminf_{k\to\infty} \Big(\|t_{n_k+1} - \tilde{t}\|^2 - \|t_{n_k} - \tilde{t}\|^2 \Big) \\
\leq\ & 0.
\end{aligned}$$

Combining the result from (3.11) with condition (a) yields

$$\lim_{k\to\infty} \|u_{n_k} - z_{n_k}\| = 0, \lim_{k\to\infty} \|c_{n_k} - z_{n_k}\| = 0, \lim_{k\to\infty} \|M(u_{n_k}) - u_{n_k}\| = 0. \tag{4.8}$$

It follows from (4.8) that

$$\lim_{k\to\infty} \|c_{n_k} - u_{n_k}\| = 0. \tag{4.9}$$

On the other hand, from (3.9), we have

$$\begin{aligned}
\|c_{n_k} - t_{n_k}\| &= \|\theta_{n_k}(A(t_{n_k}) - A(t_{n_k-1}))\| \\
&\leq \theta_{n_k}\|t_{n_k} - t_{n_k-1}\| \\
&\leq \delta_{n_k} N_1.
\end{aligned}$$

Thus

$$\lim_{k\to\infty} \|c_{n_k} - t_{n_k}\| = 0. \tag{4.10}$$

Since

$$\|u_{n_k} - t_{n_k}\| \leq \|u_{n_k} - c_{n_k}\| + \|c_{n_k} - t_{n_k}\|$$

It follows from (4.9) and (4.10) that

$$\lim_{k\to\infty} \|u_{n_k} - t_{n_k}\| = 0. \tag{4.11}$$

Turning our attention back to the definition of t_{n_k+1}, we can deduce the following

$$\begin{aligned}
\|t_{n_k+1} - t_{n_k}\| &\leq \delta_{n_k}\|h(d_{n_k}) - t_{n_k}\| + \beta_{n_k}\|u_{n_k} - t_{n_k}\| + \lambda_{n_k}\|M(u_{n_k}) - t_{n_k}\| \\
&\leq \delta_{n_k}\|h(d_{n_k}) - t_{n_k}\| + (\beta_{n_k} + \lambda_{n_k})\|u_{n_k} - t_{n_k}\| + \lambda_{n_k}\|M(u_{n_k}) - u_{n_k}\|.
\end{aligned}$$

It follows from (4.8) and (4.11) that

$$\lim_{k\to\infty} \|t_{n_k+1} - t_{n_k}\| = 0. \tag{4.12}$$

Let $\{t_{n_i}\}$ be subsequences of $\{t_n\}$, we proceed to demonstrate that $\omega_w(t_n) \subset \Omega$, where

$$\omega_w(t_n) = \Big\{ t \in H : t_{n_i} \rightharpoonup t \Big\}.$$

The boundedness of $\{t_n\}$ guarantees that $\omega_w(t_n)$ is nonempty. Selecting any $\tilde{t} \in \omega_w(t_n)$, we can find a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ satisfying the weak convergence $t_{n_k} \rightharpoonup \tilde{t}$ as $k \to \infty$.

Given that $\lim_{k\to\infty} \|c_{n_k} - t_{n_k}\| = 0$ implies $c_{n_k} \rightharpoonup \tilde{t}$ as $k \to \infty$. It remains to be shown that $\tilde{t} \in GMEP(T, \psi, G)$. From $z_n = S^T_{r_n}\big(c_n - r_n G(c_n)\big)$, we have

$$T(z_n, x) + \psi(x, z_n) - \psi(z_n, z_n) + \langle G(c_n), y - z_n\rangle + \frac{1}{r_n}\langle x - z_n, z_n - c_n\rangle \geq 0, \quad \forall x \in C.$$

Applying the property of monotonicity of T, we get

$$\psi(x, z_n) - \psi(z_n, z_n) + \langle G(c_n), x - z_n\rangle + \frac{1}{r_n}\langle x - z_n, z_n - c_n\rangle \geq T(y, z_n), \quad \forall x \in C$$

and

$$\psi(x, z_{n_k}) - \psi(z_{n_k}, z_{n_k}) + \langle G(c_{n_k}), x - z_{n_k}\rangle + \langle x - z_{n_k}, \frac{z_{n_k} - c_{n_k}}{r_{n_k}}\rangle \geq T(x, z_{n_k}), \quad \forall x \in C. \tag{4.13}$$

For any $0 < \delta \leq 1$ and $x \in H$, let $b_\delta = \delta x + (1-\delta)\tilde{t}$, we have $b_\delta \in H$. Then from (4.13), we obtain

$$\begin{aligned}
\langle G(b_\delta), b_\delta - z_{n_k}\rangle &\geq \psi(z_{n_k}, z_{n_k}) - \psi(b_\delta, z_{n_k}) + \langle G(b_\delta), b_\delta - z_{n_k}\rangle \\
&\quad - \langle G(c_{n_k}), b_\delta - z_{n_k}\rangle - \langle b_\delta - z_{n_k}, \frac{z_{n_k} - c_{n_k}}{r_{n_k}}\rangle + T(b_\delta, z_{n_k}) \\
&= \psi(z_{n_k}, z_{n_k}) - \psi(b_\delta, z_{n_k}) + \langle G(b_\delta) - G(z_{n_k}), b_\delta - z_{n_k}\rangle \\
&\quad + \langle G(z_{n_k}) - G(c_{n_k}), b_\delta - z_{n_k}\rangle - \langle b_\delta - z_{n_k}, \frac{z_{n_k} - c_{n_k}}{r_{n_k}}\rangle + T(b_\delta, z_{n_k}).
\end{aligned} \tag{4.14}$$

Due to uniform continuity of G and the fact that $\lim_{n\to\infty} \|z_{n_k} - c_{n_k}\| = 0$ (see(4.8)), we obtain $\lim_{k\to\infty} \|G(z_{n_k}) - G(c_{n_k})\| = 0$. From the monotonicity of G, the weakly lower semicontinuity of ψ and $z_{n_k} \rightharpoonup \tilde{t}$, we conclude from (4.14) that

$$\langle G(b_\delta), b_\delta - \tilde{t}\rangle \geq \psi(\tilde{t}, \tilde{t}) - \psi(b_\delta, \tilde{t}) + T(b_\delta, \tilde{t}). \tag{4.15}$$

Hence, from Assumption 2.1 and (4.15), we have

$$\begin{aligned}
0 = T(b_\delta, b_\delta) + \psi(b_\delta, \tilde{t}) - \psi(b_\delta, \tilde{t}) &\leq \delta T(b_\delta, x) + (1-\delta)T(b_\delta, \tilde{t}) + \delta\psi(x, \tilde{t}) + (1-\delta)\psi(\tilde{t}, \tilde{t}) - \psi(b_\delta, \tilde{t}) \\
&= \delta\Big(T(b_\delta, x) + \psi(x, \tilde{t}) - \psi(b_\delta, \tilde{t})\Big) \\
&\quad + (1-\delta)\Big(T(b_\delta, \tilde{t}) + \psi(\tilde{t}, \tilde{t}) - \psi(b_\delta, \tilde{t})\Big) \\
&\leq \delta\Big(T(b_\delta, x) + \psi(x, \tilde{t}) - \psi(b_\delta, \tilde{t})\Big) + (1-\delta)\delta\langle G(b_\delta), x - \tilde{t}\rangle,
\end{aligned} \tag{4.16}$$

then $T(b_\delta, x) + \psi(x, \tilde{t}) - \psi(b_\delta, \tilde{t}) + (1-\delta)\langle G(b_\delta), x - \tilde{t}\rangle \geq 0$. As $\delta \to 0_+$, it follows that

$$T(\tilde{t}, x) + \psi(x, \tilde{t}) - \psi(\tilde{t}, \tilde{t}) + \langle G(\tilde{t}), x - \tilde{t}\rangle \geq 0, \quad \forall x \in C,$$

thus $\tilde{t} \in GMEP(T, \psi, G)$. Our next objective is to prove $\tilde{t} \in F(M)$. Due the fact that $\lim_{k\to\infty} \|u_{n_k} - t_{n_k}\| = 0$ (see(4.11)), we have $u_{n_k} \rightharpoonup \tilde{t}$ as $k \to \infty$. Next, we can assert that

$$\|u_{n_k} - M(\tilde{t})\| \leq \|u_{n_k} - M(u_{n_k})\| + \|M(u_{n_k}) - M(\tilde{t})\| \leq \|u_{n_k} - M(u_{n_k})\| + \|u_{n_k} - \tilde{t}\|.$$

It follows from (4.8) that

$$\liminf_{k\to\infty} \|u_{n_k} - M(\tilde{t})\| \leq \liminf_{k\to\infty} \|u_{n_k} - \tilde{t}\|.$$

Leveraging the Opial property of the Hilbert space H (as established in Lemma 2.2)), we find that $M(\tilde{t}) = \tilde{t}$, which in turn confirms that $\tilde{t} \in F(M)$. Since $\tilde{t} \in \omega_w(t_n)$, it is therefore established that $\omega_w(t_n) \subset \Omega$. Our next step is to demonstrate that

$$\limsup_{k\to\infty} \langle h(\tilde{t}) - \tilde{t}, t_{n_k+1} - \tilde{t}\rangle \leq 0.$$

Consider the subsequence $\{t_{n_{k_j}}\}$ of $\{t_{n_k}\}$ that converges weakly to some $\hat{t} \in \Omega$, and which is chosen such that

$$\lim_{j\to\infty} \langle h(\tilde{t}) - \tilde{t}, t_{n_{k_j}} - \tilde{t}\rangle = \limsup_{k\to\infty} \langle h(\tilde{t}) - \tilde{t}, t_{n_k} - \tilde{t}\rangle.$$

Given that the the subsequence $\{t_{n_{k_j}}\}$ converges weakly to $\hat{t} \in \Omega$ and $\tilde{t} = P_\Omega[h(\tilde{t})]$, we can deduce that

$$\limsup_{k\to\infty} \langle h(\tilde{t}) - \tilde{t}, t_{n_k+1} - \tilde{t}\rangle = \limsup_{k\to\infty} \langle h(\tilde{t}) - \tilde{t}, t_{n_k} - \tilde{t}\rangle = \langle h(\tilde{t}) - \tilde{t}, \hat{t} - \tilde{t}\rangle \leq 0. \quad (4.17)$$

Using (2.5), we have

$$\begin{aligned}
\|t_{n_k+1} - \tilde{t}\|^2 &= \Big\| \delta_{n_k}(h(d_{n_k}) - h(\tilde{t})) + \beta_{n_k}(u_{n_k} - \tilde{t}) + \lambda_{n_k}\left(M(u_{n_k}) - \tilde{t}\right) \\
&\quad + \delta_{n_k}(h(\tilde{t}) - \tilde{t}) \Big\|^2 \\
&\leq \Big\| \delta_{n_k}(h(d_{n_k}) - h(\tilde{t})) + \beta_{n_k}(u_{n_k} - \tilde{t}) + \lambda_{n_k}(M(u_{n_k}) - \tilde{t}) \Big\|^2 \\
&\quad + 2\delta_{n_k}\langle h(\tilde{t}) - \tilde{t}, t_{n_k+1} - \tilde{t}\rangle \\
&\leq \delta_{n_k}\|h(d_{n_k}) - h(\tilde{t})\|^2 + \beta_{n_k}\|u_{n_k} - \tilde{t}\|^2 + \lambda_{n_k}\left(\|M(u_{n_k}) - M(\tilde{t})\|\right)^2 \\
&\quad + 2\delta_{n_k}\langle h(\tilde{t}) - \tilde{t}, t_{n_k+1} - \tilde{t}\rangle \\
&\leq \delta_{n_k} k\|d_{n_k} - \tilde{t}\|^2 + (1 - \delta_{n_k})\|u_{n_k} - \tilde{t}\|^2 + 2\delta_{n_k}\langle h(\tilde{t}) - \tilde{t}, t_{n_k+1} - \tilde{t}\rangle.
\end{aligned} \quad (4.18)$$

Applying Lemma 3.2, (4.1), (4.3) and (4.18), we get

$$\begin{aligned}
\|t_{n_k+1} - \tilde{t}\|^2 &\leq \Big(1 - (1-k)\delta_{n_k}\Big)\|t_{n_k} - \tilde{t}\|^2 + \delta_{n_k}\Big(kpp_{n_k} + (1 - \delta_{n_k})qq_{n_k}\Big) + 2\delta_{n_k}\langle h(\tilde{t}) - \tilde{t}, t_{n_k+1} - \tilde{t}\rangle \\
&\leq \Big(1 - (1-k)\delta_{n_k}\Big)\|t_{n_k} - \tilde{t}\|^2 + (1-k)\delta_{n_k}\left(\frac{pp_{n_k} + qq_{n_k}}{1-k} + \frac{2\langle h(\tilde{t}) - \tilde{t}, t_{n_k+1} - \tilde{t}\rangle}{1-k}\right) \\
&= (1 - \sigma_{n_k})\|t_{n_k} - \tilde{t}\|^2 + \sigma_{n_k}\left(\frac{pp_{n_k} + qq_{n_k}}{1-k} + \frac{2\langle h(\tilde{t}) - \tilde{t}, t_{n_k+1} - \tilde{t}\rangle}{1-k}\right)
\end{aligned}$$

where $\sigma_{n_k} = (1-k)\delta_{n_k}$. Let $\phi_{n_k} = \frac{pp_{n_k}+qq_{n_k}}{1-k} + \frac{2\langle h(\tilde{t})-\tilde{t}, t_{n_k+1}-\tilde{t}\rangle}{1-k}$, since

$$\sum_{n_k=1}^{\infty} \delta_{n_k} = \infty, \lim_{k\to\infty} \delta_{n_k} = 0,$$

It is clear that

$$\sum_{n_k=1}^{\infty} \sigma_{n_k} = \infty, \lim_{k\to\infty} \sigma_{n_k} = 0$$

and from (4.5), (4.17), we obtain

$$\limsup_{k\to\infty} \phi_{n_k} \leq 0.$$

Thus, condition (4.7) ensures that all requirements of Lemma 2.5 are met. This allows us to deduce that $\lim_{n\to\infty} \|t_n - \tilde{t}\|^2 = 0$. We immediately have $\lim_{n\to\infty} \|t_n - \tilde{t}\| = 0$, which by definition means t_n converges strongly to $\tilde{t}$. The proof is now complete. □

5 Numerical examples

In this section, we present numerical experiments to illustrate the convergence of our proposed algorithm. Furthermore, by setting $\psi = 0$ and $T = 0$, we compare our method against related methods found in the literature.
The parameters utilized in all subsequent numerical experiments were selected as follows:

- In our approach, we use $r_1 = 0.35, \chi = 0.4, \theta = 0.65, \varpi = 0.22, \tau_n = \frac{1}{(n+1)^{1.1}}, \epsilon_n = \xi_n = \frac{1}{(2n+1)^3}, \rho_n = \frac{1}{(n+1)}, \delta_n = \frac{1}{n}, \beta_n = \lambda_n = \frac{n-1}{n}, k = 0.5, M(x) = A(x) = D(x) = x, e_n = \|z_n - c_n\|$.

- For the method proposed by Yang [49], $\lambda_1 = 0.8, \mu = 0.9, \delta_n = 0.4, e_n = \|w_n - y_n\|$.

- For the method proposed by Shehu et al. [36], $\lambda_n = 0.7, \mu = 0.9, \delta_n = 0.4, \tau_1 = 0.8, e_n = \|w_n - y_n\|$.

- For the method proposed by Thong et al. [43], $\lambda_n = 0.8, \mu = 0.9, \nu_n = 0.4, \tau_1 = 0.8, e_n = \|w_n - y_n\|$.

Example 5.1: Consider the affine operator $G : \mathbb{R}^n \to \mathbb{R}^n$ defined as $G(x) = Bx + q$, with $q \in \mathbb{R}^n$. The matrix B is constructed as $B = N^T N + U + D$, where D is a non-negative diagonal matrix, U is skew-symmetric matrix, and $N \in \mathbb{R}^{n\times n}$. Note that the structure of B ensures its positive definiteness. The feasible set is defined as $C = \{x \in \mathbb{R}^n \mid -5 \leq x_i \leq 5,\ i = 1, \ldots, n\}$. Ideally, G is monotone and uniformly continuous.

The convergence of the algorithms in [49] and [43] relies on the mapping G being monotone and L-Lipschitz continuous with $L = \|B\|$.
In our experiments, we set q to the zero vector, while the entries of N, U, and D are generated randomly. The initial values $t_1 = ones(n, 1), t_0 = 2t_1$ and the stopping criterion is defined as $e_n < 10^{-4}$. Table 5.1 presents the number of iterations (denoted as No. It.) and the computational time for different problem dimensions. The experimental results, including a comparison of the proposed method to methods in [36], [43] and [49], are detailed in Table 5.1.

Table 5.1: Numerical results for Example 5.1

	The method in [49]		The method in [36]		The method in [43]		The proposed method	
n	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)
$n = 10$	274	2.996	262	2.829	196	2.067	125	1.322
$n = 20$	768	9.338	737	8.055	554	5.701	336	3.481
$n = 30$	1021	12.689	991	11.754	764	10.051	385	4.628
$n = 50$	1334	17.584	1320	15.954	1059	13.114	388	5.032

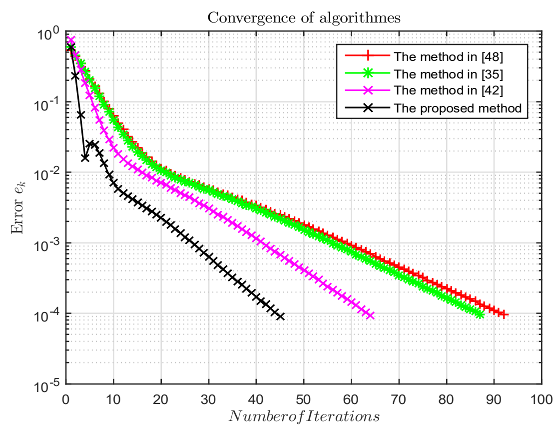

Figure 5.1: $n = 5$

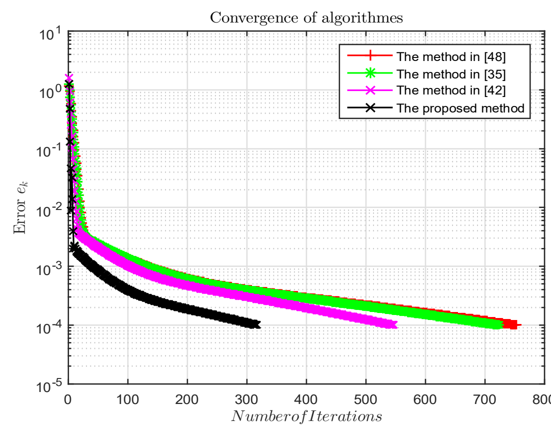

Figure 5.2: $n = 20$

Example 5.2: Let H be the Hilbert space $L^2([0, 1])$ equipped with the standard inner product $\langle x, y\rangle = \int_0^1 x(t)y(t)\, dt$ and the induced norm $\|x\| = \sqrt{\langle x, x\rangle}$. The feasible set C is defined as the unit ball in H:

$$C = \{x \in L^2([0, 1]) : \|x\| \leq 1\}.$$

Let $G : H \to H$ be the operator defined by:

$$G(x)(t) = \int_0^1 \Big(x(t) - E(t, s)f(x(s))\Big)\, ds + n(t),$$

where E, f, and n are given functions defined as follows

$$E(t,s) = \frac{2tse^{(t+s)}}{e\sqrt{e^2-1}}, \quad f(x) = cosx, \quad n(t) = \frac{2te^t}{e\sqrt{e^2-1}}.$$

One can easily show that G is monotone and L-Lipschitz continuous with constant $L = 1$. The initial values $t_1 = t_0 = sin(x)$ and the stopping criterion is defined as $e_n < 10^{-4}$. This example is studied in [45]. Table 5.2 reports the results of this experiment, to assess the efficiency of our algorithm, we compare it with the methods found in [36], [43] and [49].

Table 5.2: Numerical results for Example 5.2

	The method in [49]		The method in [36]		The method in [43]		The proposed method	
n	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)
$n = 50$	48	0.994	36	0.211	29	0.097	19	0.078
$n = 100$	42	3.092	37	1.232	30	0.287	19	0.179
$n = 150$	48	6.507	37	2.124	30	0.299	19	0.125
$n = 200$	48	14.361	36	3.425	31	0.563	19	0.365

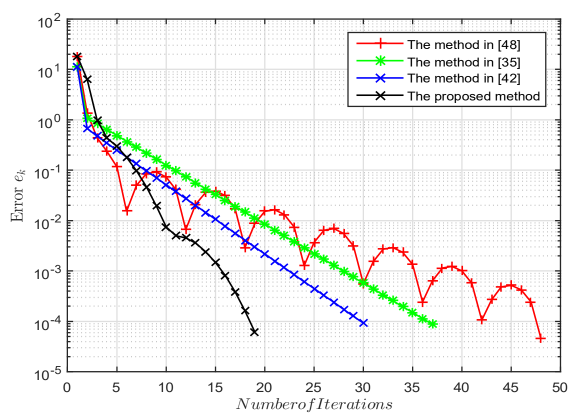

Figure 5.3: $n = 150$

Remark 5.4 *The results displayed in Figures 5.1-5.3 and Tables 5.1-5.2 confirm the efficiency of our approach.*

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