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Some Applications of a Local Minimum Result

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This article is concerned with a class of elliptic equations depending on one real parameter. Our approach is based on variational methods. More precisely, we establish the existence of at least two weak solutions for the problems discussed under the celebrated Ambrosetti-Rabinowitz condition.

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1. Introduction

As pointed out in [8] a direct consequence of a Pucci-Serrin result (see [7, Theorem 1]) and of a local minimum theorem for smooth functionals due to Ricceri (see [9, Theorem 2.5]) reads as follows.

Theorem 1.1. *Let $(E, \|\cdot\|)$ be a reflexive real Banach space, and let $\Phi, \Psi : E \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous and coercive, i.e.*

$$\lim_{\|u\|\to+\infty} \Phi(u) = +\infty.$$

Further, assume that Ψ is sequentially weakly upper semicontinuous and that, for each $\mu > 0$, the functional $J_\mu := \mu\Phi - \Psi$ satisfies the classical compactness Palais-Smale (briefly (PS)) condition. Then, for each $\varrho > \inf_E \Phi$ and each

$$\mu > \inf_{u\in\Phi^{-1}\left((-\infty,\varrho)\right)} \frac{\displaystyle\sup_{v\in\Phi^{-1}\left((-\infty,\varrho)\right)} \Psi(v) - \Psi(u)}{\varrho - \Phi(u)},$$

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the following alternative holds: either the functional J_μ *has a strict global minimum which lies in* $\Phi^{-1}\big((-\infty,\varrho)\big)$, *or* J_μ *has at least two critical points, one of which lies in* $\Phi^{-1}\big((-\infty,\varrho)\big)$.

We notice that Theorem 1.1 above is a suitable form of [8, Theorem 6] that we can write thanks to the regular form of [1, Theorem 2.1].

The aim of this short note is to present some direct consequences of Theorem 1.1. More precisely, we establish the existence of at least two weak solutions for some classes of variational problems respectively of fractional and subelliptic type (see Sections 2 and 3). By using our abstract tool the classical Ambrosetti-Rabinowitz allows us to prove our main results without additional assumptions at zero on the nonlinear term f.

For instance, in the fractional setting, on the contrary of [11, Theorems 1 and 2], in Theorem 2.1 below we don't require

$$\lim_{t\to 0}\frac{f(x,t)}{t}=0, \tag{1}$$

uniformly in Ω.

Finally, we emphasize that Theorem 1.1 has been previously used in [8, Theorem 4] proving the existence of weak solutions for some elliptic equations involving the classical Laplacian operator.

We suggest the recent book [6] as general reference for the variational methods used along the paper.

2. Fractional problems

For our goal we briefly recall the definition of the functional space X_0, firstly introduced in [10, 11]. The reader familiar with this topic may skip this section and go directly to the next one. Fix $s\in(0,1)$ and let $K:\mathbb{R}^n\setminus\{0\}\to(0,+\infty)$ be a kernel function with the properties that:

(k_1) $\gamma K\in L^1(\mathbb{R}^n)$, *where* $\gamma(x):=\min\{|x|^2,1\}$;

(k_2) *there exists* $\theta>0$ *such that*

$$K(x)\geq\theta|x|^{-(n+2s)},$$

for any $x\in\mathbb{R}^n\setminus\{0\}$.

A typical example of the kernel K is given by $K(x):=|x|^{-(n+2s)}$. In this case $\mathcal{L}_K$ is the fractional Laplace operator defined as

$$-(-\Delta)^s u(x):=\int_{\mathbb{R}^n}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy,\quad x\in\mathbb{R}^n.$$

Now, let Ω be a bounded domain in $(\mathbb{R}^n,|\cdot|)$, where $n>2s$, with continuous boundary $\partial\Omega$.

The functional space X denotes here the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function g in X belongs to $L^2(\Omega)$ and

$$((x,y) \mapsto (g(x)-g(y))\sqrt{K(x-y)}) \in L^2\big((\mathbb{R}^n \times \mathbb{R}^n) \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega), dxdy\big),$$

where $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$. We denote by X_0 the following linear subspace of X

$$X_0 := \big\{g \in X : g = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\big\}.$$

We remark that X and X_0 are non-empty, since $C_0^2(\Omega) \subseteq X_0$ by [10, Lemma 5.1].

Moreover, the space X is endowed with the norm defined as

$$\|g\|_X := \|g\|_{L^2(\Omega)} + \Big(\int_Q |g(x)-g(y)|^2 K(x-y)dxdy\Big)^{1/2},$$

where $Q := (\mathbb{R}^n \times \mathbb{R}^n) \setminus \mathcal{O}$ and $\mathcal{O} := \mathcal{C}\Omega \times \mathcal{C}\Omega \subset \mathbb{R}^n \times \mathbb{R}^n$. It is easily seen that $\|\cdot\|_X$ is a norm on X; see [11].

By [11, Lemmas 6 and 7] in the sequel we can take the function

$$X_0 \ni v \mapsto \|v\|_{X_0} := \left(\int_Q |v(x)-v(y)|^2 K(x-y)dxdy\right)^{1/2} \tag{2}$$

as a norm on X_0.

Also $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space with scalar product

$$\langle u,v\rangle_{X_0} := \int_Q \big(u(x)-u(y)\big)\big(v(x)-v(y)\big)\, K(x-y)dxdy,$$

see [11, Lemma 7].

Note that in (2) (and in the related scalar product) the integral can be extended to all $\mathbb{R}^n \times \mathbb{R}^n$, since $v \in X_0$ (and so $v = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$).

While for a general kernel K satisfying conditions from (k_1) to (k_2) we have that $X_0 \subset H^s(\mathbb{R}^n)$, in the model case $K(x) := |x|^{-(n+2s)}$ the space X_0 consists of all the functions of the usual fractional Sobolev space $H^s(\mathbb{R}^n)$ which vanish a.e. outside Ω.

Here $H^s(\mathbb{R}^n)$ denotes the usual fractional Sobolev space endowed with the norm (the so-called *Gagliardo norm*)

$$\|g\|_{H^s(\mathbb{R}^n)} = \|g\|_{L^2(\mathbb{R}^n)} + \Big(\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|g(x)-g(y)|^2}{|x-y|^{n+2s}}\, dxdy\Big)^{1/2}.$$

Before concluding this subsection, we recall the embedding properties of X_0 into the usual Lebesgue spaces; see [11, Lemma 8].

The embedding $j : X_0 \hookrightarrow L^\nu(\mathbb{R}^n)$ is continuous for any $\nu \in [1, 2^*]$, while it is compact whenever $\nu \in [1, 2^*_s)$, where $2^*_s := 2n/(n-2s)$ denotes the *fractional critical Sobolev exponent.*

For further details on the fractional Sobolev spaces we refer to [3, 6] and to the references therein.

In this setting, exploiting Theorem 1.1, our main result reads as follows.

Theorem 2.1. *Let $s \in (0,1)$, $n > 2s$ and Ω be an open bounded set of $\mathbb{R}^n$ with continuous boundary. Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ be a function satisfying conditions (k_1)–(k_2) and let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ such that*

$$f \text{ is continuous in } \overline{\Omega} \times \mathbb{R}\,, \tag{3}$$

$$\begin{gathered} \text{there exist } a_1, a_2 > 0 \text{ and } q \in (2, 2^*_s), \text{ such that} \\ |f(x,t)| \leq a_1 + a_2|t|^{q-1}\ \forall x \in \Omega,\ t \in \mathbb{R}\,, \end{gathered} \tag{4}$$

and

$$\begin{gathered} \text{there exist } \nu > 2 \text{ and } r_0 > 0 \text{ such that} \\ 0 < \nu F(x,t) \leq t f(x,t) \text{ for any } x \in \Omega, \text{ and} |t| \geq r_0, \end{gathered} \tag{5}$$

where the function F is the primitive of f with respect to its second variable. Then, for every $\varrho > 0$ and each

$$\alpha > \inf_{u \in B_\varrho} \frac{\displaystyle\sup_{v \in B_\varrho} \int_\Omega F(x, v(x))dx - \int_\Omega F(x, u(x))dx}{\varrho - \|u\|^2_{X_0}},$$

where

$$B_\varrho := \left\{ u \in X_0 : \|u\|^2_{X_0} < \varrho \right\},$$

the following problem

$$\begin{cases} -\mathcal{L}_K u = \dfrac{1}{2\alpha} f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases} \tag{6}$$

admits at least two weak solutions, one of which lies in B_ϱ.

Note that a *weak solution* for the fractional problem (6), is a function $u : \mathbb{R}^n \to \mathbb{R}$ such that

$$\begin{cases} \displaystyle\int_{\mathbb{R}^n \times \mathbb{R}^n} (u(x) - u(y))(\varphi(x) - \varphi(y))K(x-y)dxdy \\ \qquad\qquad = \displaystyle\frac{1}{2\alpha} \int_\Omega f(x, u(x))\varphi(x)dx \quad \forall\, \varphi \in X_0 \\ u \in X_0. \end{cases}$$

We observe that problem (6) has a variational structure, indeed it is the Euler-Lagrange equation of the functional $\mathcal{J}_{K,\alpha} : X_0 \to \mathbb{R}$ defined by

$$\mathcal{J}_{K,\alpha}(u) := \frac{1}{2}\|u\|^2_{X_0} - \frac{1}{2\alpha}\int_\Omega F(x,u(x))dx, \quad \forall\, u \in X_0\,. \tag{7}$$

Now, the functional $\mathcal{J}_{K,\alpha}$ is Fréchet differentiable in $u \in X_0$ and one has

$$\begin{aligned}\langle \mathcal{J}'_{K,\alpha}(u), \varphi\rangle = &\int_{\mathbb{R}^n\times\mathbb{R}^n} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dxdy \\ &- \frac{1}{2\alpha}\int_\Omega f(x,u(x))\varphi(x)dx, \quad \text{for every } \varphi \in X_0.\end{aligned}$$

Thus, the critical points of $\mathcal{J}_{K,\alpha}$ are exactly the weak solutions to problem (6).

Remark 2.2. Let us fix $\varrho > 0$. A concrete bound for the parameter

$$\chi(\varrho) := \inf_{u\in B_\varrho} \frac{\sup_{v\in B_\varrho}\int_\Omega F(x,v(x))dx - \int_\Omega F(x,u(x))dx}{\varrho - \|u\|^2_{X_0}}$$

can be obtained. Indeed, since $0 \in B_\varrho$, it follows that

$$\chi(\varrho) \le \frac{\sup_{v\in B_\varrho}\int_\Omega F(x,v(x))dx}{\varrho}. \tag{8}$$

On the other hand, by using the growth condition (4), it is easy to see that

$$\frac{\sup_{v\in B_\varrho}\int_\Omega F(x,v(x))dx}{\varrho} \le \frac{c_1}{\sqrt{\varrho}}a_1 + \frac{c_q^q a_2}{q}\varrho^{\frac{q}{2}-1}, \tag{9}$$

where

$$c_s := \sup_{u\in X_0\setminus\{0\}} \frac{\|u\|_{L^s(\Omega)}}{\|u\|_{X_0}}, \quad s \in \{1,q\}.$$

Conditions (8) and (9) immediately yield

$$\chi(\varrho) \le \frac{c_1}{\sqrt{\varrho}}a_1 + \frac{c_q^q a_2}{q}\varrho^{\frac{q}{2}-1}.$$

Then, Theorem 2.1 ensures that, for every $\varrho > 0$ and each

$$\alpha > \frac{c_1}{\sqrt{\varrho}}a_1 + \frac{c_q^q a_2}{q}\varrho^{\frac{q}{2}-1},$$

problem (6) admits at least two weak solutions one of which lies in B_ϱ.

3. Elliptic equations on Carnot groups

The results contained in this section are recently proved in [5]. We refer the interested reader to the cited paper for a detailed proof of our results.

Let $(\mathbb{R}^n, \circ)$ a Lie Group equipped with a family of group automorphisms, namely *dilatations*, $\mathfrak{F} := \{\delta_\eta\}_{\eta>0}$ such that, for every $\eta > 0$, the map

$$\delta_\eta : \prod_{k=1}^{r} \mathbb{R}^{n_k} \to \prod_{k=1}^{r} \mathbb{R}^{n_k}$$

is given by

$$\delta_\eta(\xi^{(1)}, ..., \xi^{(r)}) := (\eta\xi^{(1)}, \eta^2\xi^{(2)}, ..., \eta^r\xi^{(r)}),$$

where $\xi^{(k)} \in \mathbb{R}^{n_k}$ for every $k \in \{1, ..., r\}$ and $\sum_{k=1}^{r} n_k = n$.

The structure $\mathbb{G} := (\mathbb{R}^n, \circ, \mathfrak{F})$ is called an *homogeneous* group with *homogeneous dimension*

$$\dim_h \mathbb{G} := \sum_{k=1}^{r} k n_k. \tag{10}$$

Note that the number $\dim_h \mathbb{G}$ is naturally associated to the family $\mathfrak{F}$ since, for every $\eta > 0$, the Jacobian of the map

$$\xi \mapsto \delta_\eta(\xi), \quad \forall\, \xi \in \mathbb{R}^n$$

equals $\eta^{\dim_h \mathbb{G}}$. Now, let $\mathfrak{g}$ be the Lie algebra of left invariant vector fields on $\mathbb{G}$ and assume that $\mathfrak{g}$ is *stratified*, that is:

$$\mathfrak{g} = \bigoplus_{k=1}^{r} V_k,$$

where, V_k is a linear subspace of $\mathfrak{g}$, for every $k \in \{1, ..., r\}$, such that

$\dim V_k = n_k$, for every $k \in \{1, ..., r\}$;

$[V_1, V_k] = V_{k+1}$, for $1 \leq k \leq r - 1$, and $[V_1, V_r] = \{0\}$.

In this setting the symbol $[V_1, V_k]$ denotes the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$, where $X \in V_1$ and $Y \in V_k$.

A *Carnot group* is a homogeneous group $\mathbb{G}$ such that the Lie algebra $\mathfrak{g}$, associated to $\mathbb{G}$, is stratified. Moreover, the *sub-Laplacian* operator on $\mathbb{G}$ is the second-order differential operator, given by

$$\Delta_{\mathbb{G}} := \sum_{k=1}^{n_1} X_k^2,$$

where $\{X_1, ..., X_{n_1}\}$ is a basis of V_1. We shall denote by

$$\nabla_{\mathbb{G}} := (X_1, ..., X_{n_1})$$

the related *sub-elliptic gradient.* We also recall that a *homogeneous norm* on $\mathbb{G}$ is a continuous (with respect to the Euclidean topology) function $|\cdot|_{\mathbb{G}} : \mathbb{G} \to [0, +\infty)$ smooth away from the origin such that

$|\delta_\lambda(\xi)|_{\mathbb{G}} = \lambda |\xi|_{\mathbb{G}}$ for every $\xi \in \mathbb{G}$ and every $\lambda > 0$;
$|\xi^{-1}|_{\mathbb{G}} = |\xi|_{\mathbb{G}}$ for every $\xi \in \mathbb{G}$;
$|\xi|_{\mathbb{G}} = 0$ iff $\xi = 0$;
for every $\xi, \eta \in \mathbb{G}$ one has $|\xi \circ \eta|_{\mathbb{G}} \leq \beta(|\xi|_{\mathbb{G}} + |\eta|_{\mathbb{G}})$ for some $\beta > 0$.

A homogeneous norm naturally induces the *pseudo-distance* on $\mathbb{G}$ as follows:

$$\mathrm{dist}_{\mathbb{G}}(\xi, \eta) := |\eta^{-1} \circ \xi|_{\mathbb{G}}, \quad \forall\, \xi, \eta \in \mathbb{G}.$$

Finally, a crucial role in the functional analysis on Carnot groups is played by the following Sobolev-type inequality

$$\int_{\mathbb{G}} |u(\xi)|^{2^*} d\xi \leq C \int_{\Omega} |\nabla_{\mathbb{G}} u(\xi)|^2 d\xi \quad \forall\, u \in C_0^\infty(\mathbb{G}) \tag{11}$$

proved by Folland. In the above expression C is a positive constant (independent on u) and

$$2^* := \frac{2\mathrm{dim}_h \mathbb{G}}{\mathrm{dim}_h \mathbb{G} - 2},$$

is the *critical Sobolev exponent.* Note that from now on we assume that $\mathrm{dim}_h \mathbb{G} \geq 3$. Indeed, if $\mathrm{dim}_h \mathbb{G} < 3$, then the Carnot group $\mathbb{G}$ is the ordinary Euclidean space. Inequality (11) ensures that if Ω is a bounded subset of $\mathbb{G}$, then the function

$$u \mapsto \|u\|_{S_0^1(\Omega)} := \left(\int_{\Omega} |\nabla_{\mathbb{G}} u(\xi)|^2 d\xi\right)^{1/2}$$

is a norm in $C_0^\infty(\Omega)$.

We shall denote by $S_0^1(\Omega)$ the *Folland-Stein space* defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{S_0^1(\Omega)}$.

The exponent 2^* is critical for $\Delta_{\mathbb{G}}$ since, as in the classical Laplacian setting, the embedding $S_0^1(\Omega) \hookrightarrow L^q(\Omega)$ is compact when $1 \leq q < 2^*$, while it is only continuous if $q = 2^*$.

See the monograph [2] for basic definitions and general facts on Carnot groups and subelliptic structures.

In this setting our result reads as follows.

Theorem 3.1. *Let Ω be a smooth and bounded domain of the Carnot group $\mathbb{G}$ and $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

(f_1) *there exist $a_1, a_2 > 0$ and $q \in (2, 2^*)$ such that*

$$|f(t)| \leq a_1 + a_2 |t|^{q-1}, \quad \textit{for every } t \in \mathbb{R};$$

(f_2) *there are* $\nu > 2$ *and* $r_0 > 0$ *such that*

$$0 < \nu \int_0^t f(\tau)d\tau \leq tf(t), \quad \textit{for any } |t| \geq r_0.$$

Then, for every $\varrho > 0$ *and each*

$$0 < \lambda < \frac{q\sqrt{\varrho}}{2\left(a_1 c_1 q + a_2 c_q^q \varrho^{\frac{q-1}{2}}\right)},$$

the following problem

$$\begin{cases} -\Delta_{\mathbb{G}} u = \lambda f(u) & \text{in } \Omega \\ u|_{\partial\Omega} = 0, \end{cases} \tag{12}$$

admits at least two weak solutions, one of which lies in

$$\mathbb{B}_\varrho := \left\{ u \in S_0^1(\Omega) : \|u\|^2_{S_0^1(\Omega)} < \varrho \right\}.$$

Remark 3.2. We point out that, in Theorem 3.3, due to the presence of the parameter λ, on the contrary of [4, Theorem 3.1], no conditions at zero on the nonlinear term f is requested.

The simplest example of a Carnot group is provided by the *Heisenberg group* $\mathbb{H}^n := (\mathbb{R}^{2n+1}, \circ)$, where, for every

$$p := (p_1, ..., p_{2n}, p_{2n+1}) \quad \text{and} \quad q := (q_1, ..., q_{2n}, q_{2n+1}) \in \mathbb{H}^n,$$

the usual group operation $\circ : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}^n$ is given by

$$(p, q) \mapsto \left(p_1 + q_1, ..., p_{2n} + q_{2n}, p_{2n+1} + q_{2n+1} + \frac{1}{2}\sum_{k=1}^{2n}(p_k q_{k+n} - p_{k+n} q_k) \right)$$

and the family of dilatations has the following form

$$\delta_\lambda(p) := (\lambda p_1, ..., \lambda p_{2n}, \lambda^2 p_{2n+1}), \quad \forall\, \lambda > 0.$$

Thus, by (10) it follows that

$$\dim_h \mathbb{H}^n := 2n + 2.$$

The Lie algebra of left invariant vector fields on $\mathbb{H}^n$ is denoted by $\mathfrak{h}$ and its standard basis is given by

$$X_k := \partial_k - \frac{p_{n+k}}{2}\partial_{2n+1}, \quad k \in \{1, ..., n\}$$

$$Y_k := \partial_{n+k} - \frac{p_k}{2}\partial_{2n+1}, \quad k \in \{1, ..., n\}$$

$$T := \partial_{2n+1}\,.$$

In such a case, the only non-trivial commutators relations are

$$[X_k, Y_k] = T, \quad \forall\, k \in \{1, ..., n\}.$$

Finally, the stratification of $\mathfrak{h}$ is given by

$$\mathfrak{h} = V_1 \oplus V_2,$$

where

$$V_1 := \text{span}\{X_1, ..., X_n, Y_1, ..., Y_n\} \text{ and } V_2 := \text{span}\{T\}.$$

Taking into account the above remarks a special case of Theorem 3.1 reads as follows.

Corollary 3.3. *Let D be a smooth and bounded domain of the Heisenberg group $\mathbb{H}^n$ and $f : \mathbb{R} \to \mathbb{R}$ be a continuous function for which*

(f_1') *there exist $a_1, a_2 > 0$ and $q \in \left(2, 2\left(\frac{n+1}{n}\right)\right)$ such that*

$$|f(t)| \leq a_1 + a_2|t|^{q-1}, \quad \textit{for every } t \in \mathbb{R}\,;$$

(f_2') *there are $\nu > 2$ and $r_0 > 0$ such that*

$$0 < \nu \int_0^t f(\tau)d\tau \leq tf(t), \quad \textit{for any } |t| \geq r_0\,.$$

Then, there exists an open interval $\Lambda \subset (0, +\infty)$ such that, for every $\lambda \in \Lambda$, the following problem

$$\begin{cases} -\Delta_{\mathbb{H}^n} u = \lambda f(u) & \text{in } D \\ u|_{\partial D} = 0, \end{cases}$$

admits at least two (distinct) weak solutions in the Folland-Stein space $S^1_0(D)$.

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