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Convex Functions over the Whole Space Locally Satisfying Fractional Equations

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We investigate the structure of convex functions over the whole space which satisfy in some convex domain an equation involving the fractional Laplacian. Roughly speaking, it turns out that such solutions are either strictly convex in the given domain, or degenerate in the sense that their graph is a ruled hypersurface. We also consider regular solutions, that some fractional equations admit, and show that the convexity of the datum is transmitted to the solution through its regularity. The results are obtained by means of a fractional form of the celebrated *convexity maximum principle* devised by Korevaar in the 80's. More precisely, we construct an anisotropic, degenerate, fractional operator that nevertheless satisfies a maximum principle, and we apply such an operator to the concavity function associated to the solution. An explicit, two-dimensional example is also constructed.

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1. Introduction

This paper deals with convexity properties of continuous functions $u\colon \mathbb{R}^N \to \mathbb{R}$, $N \geq 1$, satisfying the equation

$$(-\Delta)^s\, u(x) = f(u) \tag{1}$$

in a domain (= open, connected subset) $\Omega \subset \mathbb{R}^N$. The operator $(-\Delta)^s$, $s \in (0,1)$, is the *fractional Laplacian*

$$\begin{aligned}(-\Delta)^s\, u(x_0) &= c_{N,s}\ \mathrm{P.V.}\int_{\mathbb{R}^N} \frac{u(x_0)-u(x)}{|x_0-x|^{N+2s}}\, dx \qquad (2)\\ &= c_{N,s} \lim_{\varepsilon\to 0^+} \int_{|x_0-x|>\varepsilon} \frac{u(x_0)-u(x)}{|x_0-x|^{N+2s}}\, dx.\end{aligned}$$

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Here P.V. stands for *principal value*, and the constant $c_{N,s}$ (which is found, for instance, in [1, Remark 3.11]) is given by

$$c_{N,s} = \frac{4^s\, s\, \Gamma(\frac{N}{2} + s)}{\pi^{\frac{N}{2}}\, \Gamma(1 - s)}.$$

A function $u \in C^0(\mathbb{R}^N)$ is a solution of (1) if the integral in (2) converges for all $x_0 \in \Omega$, and if the equation in (1) is satisfied pointwise.

Three results are obtained. In order to introduce the first one, recall that the so-called *layer solutions*, i.e., solutions u of (1) in $\Omega = \mathbb{R}^N$ that are monotone with respect to one variable and bounded, have been investigated in [2]. In Theorem 1.1, instead, we allow (but not require) Ω to be a proper subset of $\mathbb{R}^N$ and we concentrate on solutions that are *convex* in the whole space with respect to one variable. Roughly speaking, the result asserts that such solutions are either *strictly* convex in that variable over the domain Ω, or degenerate in the sense that their graph is a ruled hypersurface. A precise statement is the following:

Theorem 1.1. *Let $u \in C^0(\mathbb{R}^N)$ be convex over the whole space in the direction of some $\xi_0 \in \mathbb{R}^N \setminus \{0\}$ and satisfying equation* (1) *in a domain Ω, also convex in the direction ξ_0. Suppose that the function f is convex. If there exist two distinct points $x_0, y_0 \in \Omega$ such that the difference $x_0 - y_0$ is proportional to ξ_0 and $u(\frac{x_0+y_0}{2}) = \frac{u(x_0)+u(y_0)}{2}$, then the graph of u over $\mathbb{R}^N$ is made up of straight lines whose projections onto $\mathbb{R}^N$ are parallel to ξ_0.*

The convexity condition on f can be relaxed into *harmonic concavity*, which is recalled in Definition 4.2. Here we just observe that if f is negative and convex then $-1/f$ is also convex. The next result allows f to satisfy the last condition, provided that the solution u and the domain Ω are convex (in every direction) and $s \in [\frac{1}{2}, 1)$. The theorem also gives information on the level sets of the composite function $f(u(x))$, $x \in \Omega$.

Theorem 1.2. *Let $u \in C^0(\mathbb{R}^N)$ be convex over the whole space and satisfying equation* (1) *in a convex domain Ω. Suppose that either*

(a) the function f is convex, or

(b) $s \in [\frac{1}{2}, 1)$, and the function $g = -f$ is non-negative and harmonic concave.

If there exist two distinct points $x_0, y_0 \in \Omega$ such that $u(\frac{x_0+y_0}{2}) = \frac{u(x_0)+u(y_0)}{2}$, then

(i) the graph of u is made up of parallel straight lines whose projections onto $\mathbb{R}^N$ are parallel to $x_0 - y_0$;

(ii) the function $f(u(x))$ is constant along every such projection as long as $x \in \Omega$.

In dimension $N = 1$, Theorem 1.2 continues to hold and includes Theorem 1.1 as a special case:

Corollary 1.3. *Let $u \in C^0(\mathbb{R})$ be convex over the real line and satisfying equation* (1) *in an open interval Ω. Suppose that either condition* (*a*) *or condition* (*b*) *of Theorem* 1.2 *holds true. If there exist two distinct points* $x_0, y_0 \in \Omega$ *such that* $u(\frac{x_0+y_0}{2}) = \frac{u(x_0)+u(y_0)}{2}$*, then* u *is an affine function, and consequently* $f(u(x)) = 0$ *for all* $x \in \Omega$*.*

The third result deals with the convexity over the whole space of sufficiently smooth solutions of the problem

$$\begin{cases} (-\Delta)^s\, u = f(u) & \text{in } \Omega; \\ u = g & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3}$$

where Ω is a convex, bounded domain, and g is a convex function. General regularity results ensure that any solution u of (3) is Hölder continuous in $\overline{\Omega}$ provided that f, g and Ω satisfy suitable assumptions (see, for instance, [15]). Nevertheless, it may well happen that problem (3) admits a (nontrivial) solution u which is regular in a neighborhood of the boundary $\partial\Omega$: an explicit example is exhibited in Section 7. The following theorem shows that when u is of class C^2 in a neighborhood of $\partial\Omega$, it inherits the convexity of the datum g, provided that g is strictly convex near $\partial\Omega$. The result deals with the more general case of *directional* convexity:

Theorem 1.4 (Convexity of regular solutions). *Let* Ω *be a bounded domain in* $\mathbb{R}^N$*, convex in the direction of some* $\xi_0 \in \mathbb{R}^N \setminus \{0\}$*. Let* f *be monotone nonincreasing and convex, and let* $g\colon \mathbb{R}^N \to \mathbb{R}$ *be convex in the direction of* ξ_0*. Suppose that the boundary* $\partial\Omega$ *is included in an open subset* $\mathcal{U} \subset \mathbb{R}^N$ *such that* $g \in C^2(\mathcal{U})$*, and the directional derivative* $d^2g/d\xi_0^2$ *is positive in* $\mathcal{U}$*. If a solution* u *of* (3) *belongs to* $C^2(\mathcal{U})$*, then* u *is convex in the direction of* ξ_0 *over the whole space.*

The proof of the three theorems is based on an extension of the celebrated *convexity maximum principle* to the fractional Laplacian. The convexity maximum principle was proved by Nick Korevaar [11, 12] to answer a question posed by his advisor, prof. Robert Finn, concerning convexity of capillary surfaces in convex pipes. Korevaar's idea gave birth to a number of subsequent contributions, especially due to Kawohl [6, 7] and Kennington [8, 9, 10]. To be more specific, in order to prove the convexity of a continuous function $u(x)$ in a convex domain Ω, the *concavity function*

$$C(x,y) = 2\,u(\tfrac{x+y}{2}) - u(x) - u(y), \quad x, y \in \Omega \tag{4}$$

was introduced (see [7, p. 113, (3.30)]). One may also deal with the function $c(x,y,\lambda) = (1-\lambda)\,u(x) + \lambda\,u(y) - u((1-\lambda)\,x + \lambda y)$ for $x, y \in \Omega$ and $\lambda \in [0,1]$ as in [10, p. 687], but we prefer to keep $\lambda = \frac{1}{2}$ for simplicity. This is enough because $u(x)$ is continuous. Using the equation satisfied by u, it was shown that $C(x,y)$ cannot have an interior, positive maximum in the set $\Omega^2 = \Omega \times \Omega$. Then, using the boundary conditions, the possibility for a positive maximum to occur at the boundary of Ω^2 was also excluded.

Here, since we deal with functions u defined over the whole space $\mathbb{R}^N$, the concavity function in (4) is extended from Ω^2 to the whole $\mathbb{R}^{2N}$. Furthermore, $C(x,y)$ is well defined for all $(x,y) \in \mathbb{R}^{2N}$ even if Ω is not convex. Of course, the convexity of $u(x)$ in $\mathbb{R}^N$ is equivalent to the inequality $C(x,y) \leq 0$ in $\mathbb{R}^{2N}$. Theorem 1.1 and Theorem 1.2 are proved by showing that $C(x,y)$ satisfies a degenerate inequality (see Section 4). The inequality is constructed by introducing in Section 2 a convenient degenerate operator, denoted by $(-\Delta_A)^s$, which is proved to satisfy the maximum principle. The computation of $(-\Delta_A)^s\, C$ in terms of $(-\Delta)^s\, u$ is done in Section 3. The present method extends the one in [4], where local operators were considered. In Section 5 we develop a slightly different approach, suitable for proving Theorem 1.4. The three theorems are proved in Section 6. In the final Section 7 we explicitly construct a two-dimensional example which fulfills the hypotheses of the theorems.

By a convenient adaptation of the assumptions, all theorems may be turned into statements dealing with the *concavity* of the solution u. However, the corresponding results do not apply to the solution of

$$\begin{cases} (-\Delta)^s\, u = 1 & \text{in } \Omega; \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5}$$

where Ω is a smooth, convex, bounded domain. The method fails because the solution u of (5), which is positive in Ω and vanishes outside, is *never* concave in the whole space, and it is not even differentiable along the boundary. The concavity of *the restriction* of u to the domain Ω is investigated in [13].

Equations posed in *the exterior* of a convex body K have also been considered in the literature: confining ourselves to fractional operators, we mention that the solution of

$$\begin{cases} (-\Delta)^{\frac{1}{2}}\, u = 0 & \text{in } \mathbb{R}^N \setminus K; \\ u = 1 & \text{in } K; \\ \lim\limits_{|x| \to +\infty} u(x) = 0 \end{cases}$$

is shown to have *convex level sets* in [14].

2. Degenerate anisotropic fractional Laplacian

The following definition introduces a linear, non-local operator $(-\Delta_A)^s$, which includes the fractional Laplacian as a special case. Apart from being non-local, such an operator may also be degenerate due to the fact that the domain of integration A indicated in (6) is allowed to have lower dimension than the whole space. Accordingly, a *degenerate* maximum principle holds (see Theorem 2.2).

Definition 2.1. (Degenerate anisotropic fractional Laplacian) Let $w \colon \mathbb{R}^m \to \mathbb{R}$ be an upper semicontinuous function over $\mathbb{R}^m$, $m \geq 1$. Take $\boldsymbol{x}_0 \in \mathbb{R}^m$, and choose an affine subspace $A = A(\boldsymbol{x}_0) \subset \mathbb{R}^m$ of positive dimension $k \leq m$ passing through $\boldsymbol{x}_0$. The operator $(-\Delta_A)^s$ is defined as follows:

$$(-\Delta_A)^s\, w(\boldsymbol{x}_0) = c_{k,s}\ \text{P.V.}\!\int_A \frac{w(\boldsymbol{x}_0) - w(\boldsymbol{x})}{|\boldsymbol{x}_0 - \boldsymbol{x}|^{k+2s}}\, d\mathcal{H}^k(\boldsymbol{x}) \tag{6}$$

provided that the integral in the right-hand side is well defined. The notation $d\mathcal{H}^k$ represents the k-dimensional Hausdorff measure.

When $k = m$, i.e. when $A = \mathbb{R}^m$, the operator $(-\Delta_A)^s$ is non-degenerate and coincides with the usual fractional Laplacian $(-\Delta)^s$. In such a case, a strong minimum principle is found in [5].

Theorem 2.2 (Sectional maximum principle). *Let $w\colon \mathbb{R}^m \to \mathbb{R}$ be an upper semicontinuous function satisfying*

$$(-\Delta_A)^s\, w(\boldsymbol{x}_0) \le b(\boldsymbol{x}_0)\, w(\boldsymbol{x}_0) \tag{7}$$

at some point $\boldsymbol{x}_0 \in \mathbb{R}^m$, where $s \in (0,1)$, $A \subset \mathbb{R}^m$ is an affine subspace of dimension $k > 0$ passing through $\boldsymbol{x}_0$, and $b(\boldsymbol{x}_0)$ is a number. Suppose, further,

$$w(\boldsymbol{x}_0) = \max_{\boldsymbol{x}\in A} w(\boldsymbol{x}).$$

- *(i) Assume that $b(\boldsymbol{x}_0) \le 0$, and that $\boldsymbol{x}_0$ belongs to a bounded open set $G \subset \mathbb{R}^m$ such that $w \le 0$ in the difference $A \setminus G$. Then $w(\boldsymbol{x}_0) \le 0$.*
- *(ii) If $w(\boldsymbol{x}_0) = 0$, then $w(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in A$.*

Remarks. (1) Claim (*ii*) holds irrespectively for the sign of $b(\boldsymbol{x}_0)$.

(2) The degeneracy of the operator $(-\Delta_A)^s$ for $k < m$ reflects on Claim (*ii*): indeed, even assuming $w \le 0$ in all of $\mathbb{R}^m$, from the equality $w(\boldsymbol{x}_0) = 0$ it is not possible to deduce $w = 0$ in all of $\mathbb{R}^m$ as in the non-degenerate case $k = m$.

(3) The non-local character of the operator $(-\Delta_A)^s$ also appears. Indeed, both claims are derived under the assumption that inequality (7) holds *just at the point* where the maximum is attained. By contrast, if we let $A = \mathbb{R}^m$, and if we replace the operator $(-\Delta_A)^s$ with $-\Delta$, we end up with a false statement.

Proof of Theorem 2.2. Claim (*i*). We argue by contradiction. If $w(\boldsymbol{x}_0) > 0$ then $b(\boldsymbol{x}_0)\, w(\boldsymbol{x}_0) \le 0$ because $b(\boldsymbol{x}_0) \le 0$ by assumption. Taking (7) into account, and since $w \le 0$ in $A \setminus G$, we may write

$$\begin{aligned} 0 \ \ge\ (-\Delta_A)^s\, w(\boldsymbol{x}_0) \ &\ge\ c_{k,s}\ \text{P.V.}\!\int_{G\cap A} \frac{w(\boldsymbol{x}_0) - w(\boldsymbol{x})}{|\boldsymbol{x}_0 - \boldsymbol{x}|^{k+2s}}\, d\mathcal{H}^k(\boldsymbol{x}) \\ &+\ c_{k,s} \int_{A\setminus G} \frac{w(\boldsymbol{x}_0)}{|\boldsymbol{x}_0 - \boldsymbol{x}|^{k+2s}}\, d\mathcal{H}^k(\boldsymbol{x}). \end{aligned}$$

The first integral is non-negative because $w(\boldsymbol{x}_0) = \max_A w$. The second integral, where we have omitted P.V. because $\boldsymbol{x}_0$ is interior to G, is positive: indeed, we are assuming $w(\boldsymbol{x}_0) > 0$ and the domain of integration $A \setminus G$ has an infinite k-dimensional measure, being G bounded. Since we have reached a contradiction, we must conclude that $w(\boldsymbol{x}_0) \leq 0$, as claimed.

Claim (ii). Suppose, contrary to the claim, that there exists $\boldsymbol{x}_1 \in A$ such that $w(\boldsymbol{x}_1) < 0$. Then, by upper semicontinuity, there exists $\varepsilon_1 > 0$ such that $-w(\boldsymbol{x}) \geq \varepsilon_1$ for all $\boldsymbol{x}$ in the ball $B_1 = B(\boldsymbol{x}_1, \varepsilon_1)$. By reducing ε_1 if necessary, we may assume that $\boldsymbol{x}_0 \notin \overline{B}_1$, thus avoiding singularities in the second integral below. Recalling that $w(\boldsymbol{x}_0) = 0$, we may write

$$\begin{aligned}(-\Delta_A)^s\, w(\boldsymbol{x}_0) \quad &\geq \quad c_{k,s}\ \text{P.V.}\int_{A\setminus B_1} \frac{-w(\boldsymbol{x})}{|\boldsymbol{x}_0 - \boldsymbol{x}|^{k+2s}}\, d\mathcal{H}^k(\boldsymbol{x}) \\ &+ \quad c_{k,s}\int_{B_1\cap A} \frac{\varepsilon_1}{|\boldsymbol{x}_0 - \boldsymbol{x}|^{k+2s}}\, d\mathcal{H}^k(\boldsymbol{x}).\end{aligned}$$

As before, the first integral non-negative because now $w \leq 0$ in A. Furthermore, the second integral is positive because the intersection $B_1 \cap A$ has a positive k-dimensional measure. Hence we get $(-\Delta_A)^s\, w(\boldsymbol{x}_0) > 0$. However, since $b(\boldsymbol{x}_0)\, w(\boldsymbol{x}_0) = 0$, a contradiction with (7) is reached. Thus, we must have $w(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in A$, and the proof is complete. ☐

3. Fundamental expansion

In order to investigate the properties of convex functions $u \in C^0(\mathbb{R}^N)$ satisfying (1) in a domain Ω, we apply the operator $(-\Delta_A)^s$ introduced in (6) to the concavity function $w(\boldsymbol{x}) = C(x, y)$ associated to u. To this aim we let $m = 2N$ and $k = N$. In the present section we suitably choose the subspace A and give an expression of $(-\Delta_A)^s\, C$ in terms of $(-\Delta)^s\, u$. We start from a spectral analysis of the matrix M in (9), which was used in [4] as the characteristic matrix of a local operator to be applied to $C(x, y)$. It is worth recalling that the idea of a rotation of the coordinate frame in order to give a PDE a more convenient form goes back to d'Alembert, who investigated the wave equation (see [3, p. 216]).

Proposition 3.1 (Spectral analysis). *Let I be the $N \times N$ unit matrix, $N \geq 1$, and let σ, τ be two real numbers such that $\sigma^2 + \tau^2 > 0$. Furthermore, let $\omega \in [0, 2\pi)$ be the angle determined uniquely by*

$$\cos\omega = \frac{\sigma}{\sqrt{\sigma^2 + \tau^2}}, \qquad \sin\omega = \frac{\tau}{\sqrt{\sigma^2 + \tau^2}}. \tag{8}$$

Then, the $2N \times 2N$ symmetric matrix $M = M(\sigma, \tau)$ given by

$$M = \begin{pmatrix} \sigma^2\, I & \sigma\tau\, I \\ \sigma\tau\, I & \tau^2\, I \end{pmatrix} \tag{9}$$

is transformed into a diagonal matrix by means of the orthogonal, symmetric matrix P_ω defined as follows:

$$P_\omega = \begin{pmatrix} (\cos\omega)\, I & (\sin\omega)\, I \\ (\sin\omega)\, I & (-\cos\omega)\, I \end{pmatrix}. \tag{10}$$

More precisely, we have

$$P_\omega^T\, M\, P_\omega = (\sigma^2+\tau^2)\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

where the exponent ^{T} *denotes transposition, and* 0 *is the* $N \times N$ *null matrix. The matrix* $M(\sigma,\tau)$ *has two distinct eigenvalues: the eigenvalue* $\lambda_0 = 0$ *and the eigenvalue* $\lambda_1 = \sigma^2+\tau^2$*, each one of multiplicity* N*. The corresponding eigenspaces* $V_0(\omega)$ *and* $V_1(\omega)$ *are given by*

$$V_0(\omega) = \left\{ (x,y) \in \mathbb{R}^{2N} \;\middle|\; \tbinom{x}{y} = P_\omega \tbinom{0}{\eta},\ \eta \in \mathbb{R}^N \setminus \{0\} \right\};$$

$$V_1(\omega) = \left\{ (x,y) \in \mathbb{R}^{2N} \;\middle|\; \tbinom{x}{y} = P_\omega \tbinom{\xi}{0},\ \xi \in \mathbb{R}^N \setminus \{0\} \right\}.$$

Proof. Because of (8), all claims are easily verified by computation. ☐

We can now prove the following fundamental lemma, which gives an expansion of $(-\Delta_A)^s\, C$ in terms of $(-\Delta)^s\, u$ provided that A is defined as in (11).

Lemma 3.2 (Fundamental expansion). *Let* $u\colon \mathbb{R}^N \to \mathbb{R}$ *be a continuous function such that the fractional Laplacian* $(-\Delta)^s\, u(x)$ *is well defined at two points* $x_0, y_0 \in \mathbb{R}^N$*, as well as at the middle point* $z_0 = (x_0+y_0)/2$*. Choose an angle* $\omega \in [0, 2\pi)$ *and define the* N*-dimensional affine subspace* $A \subset \mathbb{R}^{2N}$ *as follows:*

$$\begin{aligned} A &= (x_0,y_0) + \Big(\{\,0\,\} \cup V_1(\omega)\Big) \qquad (11)\\ &= \left\{ (x,y) \in \mathbb{R}^{2N} \;\middle|\; \tbinom{x}{y} = \tbinom{x_0}{y_0} + P_\omega \tbinom{\xi}{0},\ \xi \in \mathbb{R}^N \right\} \end{aligned}$$

where the matrix P_ω *is as in* (10) *Then*

$$\begin{aligned} (-\Delta_A)^s\, C(x_0,y_0) &= 2\left(\tfrac{|\cos\omega+\sin\omega|}{2}\right)^{2s} (-\Delta)^s\, u(z_0) \qquad (12)\\ &\quad -|\cos\omega|^{2s}\,(-\Delta)^s\, u(x_0) - |\sin\omega|^{2s}\,(-\Delta)^s\, u(y_0). \end{aligned}$$

Proof. Since the operator $(-\Delta_A)^s$ is linear, and by (4), we start by computing $(-\Delta_A)^s\, w(x_0,y_0)$, where $w(x,y) = u(z)$ and $z = \frac{x+y}{2}$. By means of the matrix P_ω we perform the change of variables $\binom{x}{y} = \binom{x_0}{y_0} + P_\omega \binom{\xi}{0}$ and find

$$\begin{aligned} (-\Delta_A)^s\, u(\tfrac{x+y}{2})_{|(x_0,y_0)} &= c_{N,s}\ \mathrm{P.V.}\!\int_A \frac{u(z_0)-u(z)}{|(x_0,y_0)-(x,y)|^{N+2s}}\, d\mathcal{H}^N(x,y)\\ &= c_{N,s}\ \mathrm{P.V.}\!\int_{\mathbb{R}^N} \frac{u(z_0)-u(z_0+\frac{\cos\omega+\sin\omega}{2}\,\xi)}{|\xi|^{N+2s}}\, d\xi. \end{aligned}$$

We define $d := \cos\omega + \sin\omega$. In the case when $d = 0$, we immediately obtain $(-\Delta_A)^s\, u(\frac{x+y}{2})_{|(x_0,y_0)} = 0$. Otherwise we take $z = z_0 + d\xi/2$ as the new variable of integration. Since $dz = (|d|/2)^N\, d\xi$, we arrive at

$$\begin{aligned}(-\Delta_A)^s\, u(\tfrac{x+y}{2})_{|(x_0,y_0)} &= \Big(\frac{|d|}{2}\Big)^{2s} c_{N,s}\ \text{P.V.}\!\int_{\mathbb{R}^N} \frac{u(z_0)-u(z)}{|z_0-z|^{N+2s}}\, dz \\ &= \Big(\frac{|d|}{2}\Big)^{2s} (-\Delta)^s\, u(z_0).\end{aligned}$$

Note that the last equality collects the case $\cos\omega + \sin\omega = 0$ as well. To proceed further, let us compute $(-\Delta_A)^s\, w(x_0, y_0)$ where the function w, differently from before, is given by $w(x, y) = u(x)$. Denote by $\pi_1(x, y) = x$ the canonical projection over $\mathbb{R}^N$. Using again the change of variables $\binom{x}{y} = \binom{x_0}{y_0} + P_\omega \binom{\xi}{0}$ we find

$$\begin{aligned}(-\Delta_A)^s\, u(\pi_1(x,y))_{|(x_0,y_0)} &= c_{N,s}\ \text{P.V.}\!\int_A \frac{u(x_0)-u(x)}{|(x_0,y_0)-(x,y)|^{N+2s}}\, d\mathcal{H}^N(x,y) \\ &= c_{N,s}\ \text{P.V.}\!\int_{\mathbb{R}^N} \frac{u(x_0)-u(x_0+(\cos\omega)\,\xi)}{|\xi|^{N+2s}}\, d\xi.\end{aligned}$$

If $\cos\omega = 0$ we immediately obtain $(-\Delta_A)^s\, u(\pi_1(x, y))_{|(x_0,y_0)} = 0$. Otherwise we use $x = x_0 + (\cos\omega)\,\xi$ as the new variable of integration. Since $dx = |\cos\omega|^N\, d\xi$, we arrive at

$$\begin{aligned}(-\Delta_A)^s\, u(\pi_1(x,y))_{|(x_0,y_0)} &= |\cos\omega|^{2s}\, c_{N,s}\ \text{P.V.}\!\int_{\mathbb{R}^N} \frac{u(x_0)-u(x)}{|x_0-x|^{N+2s}}\, dx \\ &= |\cos\omega|^{2s}\, (-\Delta)^s\, u(x_0).\end{aligned}$$

The equality above continues to hold when $\cos\omega = 0$. Finally, a similar computation shows that $(-\Delta_A)^s\, u(\pi_2(x, y))_{|(x_0,y_0)} = |\sin\omega|^{2s}\, (-\Delta)^s\, u(y_0)$, where $\pi_2(x, y) = y$ is the second canonical projection over $\mathbb{R}^N$. The lemma follows. □

4. A non-local inequality

We establish a non-local inequality of the form (7) satisfied by the function $w(\boldsymbol{x}) = C(x, y)$. More precisely, we give conditions on the function f in (1) sufficient to obtain such an inequality through the expansion (12). We state first the general assumption (13), then we discuss some special cases where such an assumption holds.

Lemma 4.1 (Non-local inequality). *Let $u \in C^0(\Omega)$ be a solution of (1) in a domain $\Omega \subset \mathbb{R}^N$, and pick $x_0, y_0 \in \Omega$ such that also $\frac{x_0+y_0}{2} \in \Omega$. Denote by $U = \{\, t \in \mathbb{R} \mid t = u(x)$ for some $x \in \Omega \,\}$ the interval described by $u(x)$ as x ranges in Ω. Suppose that for every couple of real numbers $t_1, t_2 \in U$ there exists an angle $\omega = \omega(t_1, t_2) \in [0, 2\pi)$ such that*

$$2\left(\frac{|d|}{2}\right)^{2s} f\Big(\frac{t_1+t_2}{2}\Big) - |\cos\omega|^{2s} f(t_1) - |\sin\omega|^{2s} f(t_2) \le 0\,, \tag{13}$$

with $d := \cos\omega + \sin\omega$*. Then we may define the* N*-dimensional affine subspace* $A \subset \mathbb{R}^{2N}$ *by letting* $\omega = \omega(u(x_0), u(y_0))$ *in* (11)*, and we have*

$$(-\Delta_A)^s\, C(x_0,y_0) \le b(x_0,y_0)\, C(x_0,y_0) \tag{14}$$

where the coefficient $b(x_0,y_0)$ *is given by*

$$b(x_0,y_0) = \begin{cases} 2\left(\frac{|d|}{2}\right)^{2s} \dfrac{f(u(\frac{x_0+y_0}{2})) - f(\frac{u(x_0)+u(y_0)}{2})}{C(x_0,y_0)}, & C(x_0,y_0) \ne 0; \\ 0, & C(x_0,y_0) = 0. \end{cases}$$

Proof. The conclusion follows from Lemma 3.2 by using assumption (13) and the identity

$$2\left(\frac{|d|}{2}\right)^{2s} f(u(z_0)) = 2\left(\frac{|d|}{2}\right)^{2s} f\Big(\frac{u(x_0)+u(y_0)}{2}\Big) + b(x_0,y_0)\, C(x_0,y_0). \quad \square$$

Remarks. (1) Assumption (13) is satisfied if $f(t) \ge 0$ for all $t \in U$. This is readily seen by letting $\omega(t_1,t_2) = \frac{3}{4}\pi$ for all $t_1, t_2 \in U$, so that $\cos\omega + \sin\omega = 0$.

(2) If f is a convex function (hence, in particular, if f is constant) then assumption (13) holds with $\omega(t_1,t_2) \equiv \frac{5}{4}\pi$. This choice of ω is preferable for consistency in the proofs of Theorem 1.1 and Theorem 1.2.

A further condition implying (13) involves the *harmonic concavity* of the function $g = -f$. For the present purposes, it is convenient to adopt the following definition:

Definition 4.2. (Harmonic concavity) A non-negative function g defined in an interval $U \subset \mathbb{R}$ is *harmonic concave* if for every $t_1, t_2 \in U$ we have

$$\tfrac{g(t_1)+g(t_2)}{2}\, g(\tfrac{t_1+t_2}{2}) \ge g(t_1)\, g(t_2).$$

In comparison to the definition in [4, 9, 10], the present one is restricted to the case $g \ge 0$: this because we will consider the power function g^{2s-1} in the proof of the following proposition. In the realm of continuous, non-negative functions, all the mentioned definitions coincide. Continuity enters in this equivalence because the definition here (as well as in [4]) involves just the middle point $\frac{t_1+t_2}{2}$ instead of the whole interval $\lambda\, t_1 + (1-\lambda)\, t_2$, $\lambda \in (0,1)$ as in [9, 10]. Finally, it is worth observing that (*i*) if $g \ge 0$ is concave, then it satisfies Definition 4.2 (cf. [10, p. 688]); (*ii*) a *positive* continuous function g satisfies Definition 4.2 if and only if $1/g$ is convex.

Proposition 4.3. *Suppose* $s \in [\frac{1}{2}, 1)$*. If* $f(t) \le 0$ *for all* $t \in U$*, and if the function* $g = -f$ *is harmonic concave, then* (13) *holds.*

Proof. Fix $t_1, t_2 \in U$. If $f(t_1) = f(t_2) = 0$, we may take ω arbitrarily and (13) holds because $f(\frac{t_1+t_2}{2}) \leq 0$. For later purposes we choose $\omega = \frac{5}{4}\pi$. If, instead, $f(t_1) + f(t_2) < 0$, then let $\omega \in [\pi, \frac{3}{4}\pi]$ be the angle determined by (8) with $\sigma = f(t_2)$ and $\tau = f(t_1)$. Since $|\sigma + \tau| = |\sigma| + |\tau|$, the target condition (13) may be rewritten as

$$\frac{|\sigma|^{2s} g(t_1) + |\tau|^{2s} g(t_2)}{2} \leq \left(\frac{|\sigma| + |\tau|}{2}\right)^{2s} g(\tfrac{t_1+t_2}{2}). \tag{15}$$

If either $|\sigma| = g(t_2) = 0$ or $|\tau| = g(t_1) = 0$, then (15) holds because $g(\frac{t_1+t_2}{2}) \geq 0$. Otherwise, since g is harmonic concave, in order to prove (15) it is enough to check that

$$\frac{(g(t_2))^{2s} g(t_1) + (g(t_1))^{2s} g(t_2)}{2} \leq \left(\frac{g(t_2) + g(t_1)}{2}\right)^{2s-1} g(t_1)\, g(t_2).$$

Since we are considering $g(t_1), g(t_2) > 0$, we may divide by $g(t_1)\, g(t_2)$ and get the equivalent inequality

$$\frac{(g(t_2))^{2s-1} + (g(t_1))^{2s-1}}{2} \leq \left(\frac{g(t_2) + g(t_1)}{2}\right)^{2s-1},$$

which holds true because the power function a^{2s-1} with $s \in [\frac{1}{2}, 1)$ is concave in the variable $a > 0$. $\square$

5. The *frozen* concavity function

In view of an application to the proof of Theorem 1.4, we now develop an alternative approach based on a somehow *frozen* concavity function. In comparison to the preceding technique, the present one needs more restrictive conditions on the function f in (3). The advantage is that the fixed vector ξ (see below), which is taken small enough in the application, detects the convexity of the solution u near the boundary $\partial\Omega$. Note that in order to prove convexity of $u \in C^0(\mathbb{R}^N)$ in the direction of ξ_0 in all of $\mathbb{R}^N$ it is enough that $C_{\lambda\xi_0}(x) \leq 0$ for every $x \in \mathbb{R}^N$ and for every *sufficiently small* $\lambda > 0$.

Definition 5.1. (*Frozen* concavity function) Let $u \colon \mathbb{R}^N \to \mathbb{R}$ be any real-valued function defined over the whole space, and choose $\xi \in \mathbb{R}^N \setminus \{0\}$. The frozen concavity function $C_\xi(x)$, $x \in \mathbb{R}^N$, is given by

$$\begin{aligned} C_\xi(x) &= C(x,\, x + \xi) \\ &= 2\, u(x + \tfrac{\xi}{2}) - u(x) - u(x + \xi). \end{aligned}$$

By applying the fractional Laplacian $(-\Delta)^s$ to the frozen concavity function, we get

Lemma 5.2 (Expansion of $(-\Delta)^s\, C_\xi$). *Let $u \colon \mathbb{R}^N \to \mathbb{R}$ be a continuous function such that the fractional Laplacian $(-\Delta)^s u(x)$ is well defined at some $x_0 \in \mathbb{R}^N$, as well as at $x_0 + \xi$ and at the middle point $x_0 + \frac{\xi}{2}$. Then*

$$(-\Delta)^s\, C_\xi(x_0) = 2\,(-\Delta)^s\, u(x_0 + \tfrac{\xi}{2}) - (-\Delta)^s\, u(x_0) - (-\Delta)^s\, u(x_0 + \xi). \tag{16}$$

Proof. The expansion is readily obtained by computation. □

Lemma 5.3 (Inequality for C_ξ). *Let* $u \in C^0(\Omega)$ *be a solution of* (1) *in a domain* $\Omega \subset \mathbb{R}^N$*. Pick* $x_0 \in \Omega$ *and* $\xi \in \mathbb{R}^N \setminus \{0\}$ *such that* $x_0 + \xi \in \Omega$ *and also* $x_0 + \frac{\xi}{2} \in \Omega$*. Denote by* $U = \{ t \in \mathbb{R} \mid t = u(x) \text{ for some } x \in \Omega \}$ *the interval described by* $u(x)$ *as* x *ranges in* Ω*. Suppose that the function* $f(t)$ *in* (1) *is convex over the interval* U*. Then we have*

$$(-\Delta)^s \, C_\xi(x_0) \le b(x_0) \, C_\xi(x_0), \tag{17}$$

where the coefficient $b(x_0)$ *is given by*

$$b(x_0) = \begin{cases} 2 \, \frac{f(u(x_0 + \frac{\xi}{2})) - f(\frac{u(x_0) + u(x_0 + \xi)}{2})}{C_\xi(x_0)}, & C_\xi(x_0) \ne 0; \\ 0, & C_\xi(x_0) = 0. \end{cases}$$

Proof. The lemma follows from (16) by using the convexity of f and the identity

$$2 \, f(u(x_0 + \tfrac{\xi}{2})) = 2 \, f(\tfrac{u(x_0) + u(x_0 + \xi)}{2}) + b(x_0) \, C_\xi(x_0).$$

□

Remark. If the function f is monotone non-increasing then $b(x_0) \le 0$. The last inequality is an assumption of Claim (i) of the maximum principle (Theorem 2.2).

6. Proofs of Theorems 1.1, 1.2 and 1.4

Let us prove the theorems stated in the Introduction. The proof of Theorem 1.1 contains the main ideas also used in the proof of Theorem 1.2.

Proof of Theorem 1.1. Observe, firstly, that since the solution u is convex by assumption in the direction of ξ_0, if $C(x, y) = 0$ at some couple $(x, y) \in \mathbb{R}^{2N}$ satisfying $x - y = \lambda \, \xi_0$ for some $\lambda \ne 0$ then the graph of u contains the whole line segment whose endpoints are $(x, u(x)), (y, u(y)) \in \mathbb{R}^{N+1}$. This will be repeatedly used in the sequel.

In order to prove the theorem, let us apply the fractional convexity maximum principle. Since Ω is convex in the direction of ξ_0, the middle point $\frac{x_0 + y_0}{2}$ also belongs to Ω. Since f is convex, (13) holds with $\omega = \frac{5}{4} \pi$, and therefore $C(x, y)$ satisfies inequality (14). The assumptions on u imply that $C(x_0, y_0) = 0$, and $C(x, y) \le 0$ whenever $x - y$ is proportional to ξ_0. By applying Claim (ii) of Theorem 2.2 we obtain

$$C(x, y) = 0 \text{ for all } (x, y) \in A = A(x_0, y_0) \tag{18}$$

where the N-dimensional affine subspace $A \subset \mathbb{R}^{2N}$ is given by (11) with $\omega = \frac{5}{4} \pi$. Hence, by (18) we may write $C(x, y) = 0$ for all $x, y \in \mathbb{R}^N$ given by

$$x = x_0 - \xi \quad \text{and} \quad y = y_0 - \xi$$

as ξ ranges in $\mathbb{R}^N$. By restricting our attention to $\xi = \lambda\,(x_0 - y_0)$ for $\lambda \in \mathbb{R}$, and recalling the initial observation, we deduce that the graph of u contains not only the line segment whose endpoints are $(x_0, u(x_0))$ and $(y_0, u(y_0))$, but also the whole straight line containing such a segment.

To complete the proof, let us consider any point $x_1 \in \mathbb{R}^N$, and define $y_1 = y_0 + x_1 - x_0$. Thus, $C(x_1, y_1) = 0$. Since the solution u is convex by assumption in the direction of $x_1 - y_1$, which is the same direction of $x_0 - y_0$, arguing as before we get that the graph of u contains the whole straight line passing through $(x_1, u(x_1))$ and $(y_1, u(y_1))$. Since x_1 is arbitrary, the conclusion follows. ☐

Proof of Theorem 1.2. Claim (*i*). Under assumption (*a*), by Theorem 1.1 the graph of u is made up of straight lines whose projections onto $\mathbb{R}^N$ are parallel to $x_0 - y_0$. Since u is convex by assumption, its graph must be made up of parallel lines, as claimed. Assuming (*b*), let us apply the fractional convexity maximum principle. Since Ω is convex, the middle point $\frac{x_0+y_0}{2}$ also belongs to Ω. By Proposition 4.3, assumption (*b*) implies that (13) holds, and therefore $C(x, y)$ satisfies inequality (14). The assumptions on u imply that $C(x_0, y_0) = 0$, and $C(x, y) \leq 0$ for all $(x, y) \in \mathbb{R}^{2N}$. By applying Claim (*ii*) of Theorem 2.2 we arrive again at (18). Now the N-dimensional affine subspace $A \subset \mathbb{R}^{2N}$ is given by (11) with the angle $\omega \in [\pi, \frac{3}{2}\pi]$ chosen as in the proof of Proposition 4.3: if $f(u(x_0)) = f(u(y_0)) = 0$ then $\omega = \frac{5}{4}\pi$, otherwise ω is determined by (8) with $\sigma = f(u(y_0))$ and $\tau = f(u(x_0))$. In conclusion, by (18) we may write $C(x, y) = 0$ for all $x, y \in \mathbb{R}^N$ given by

$$x = x_0 + (\cos\omega)\,\xi \quad \text{and} \quad y = y_0 + (\sin\omega)\,\xi$$

as ξ ranges in $\mathbb{R}^N$. By restricting our attention to $\xi = \lambda\,(x_0 - y_0)$ for $\lambda \in \mathbb{R}$, as in the proof of Theorem 1.1, we see that the graph of u contains the whole straight line passing through $(x_0, u(x_0))$ and $(y_0, u(y_0))$. To complete the proof of Claim (*i*), since $\cos\omega$ and $\sin\omega$ cannot vanish simultaneously, without loss of generality suppose $\cos\omega \neq 0$. Every $x_1 \in \mathbb{R}^N$ is given by $x_1 = x_0 + (\cos\omega)\,\xi_1$ for a convenient ξ_1, which in its turn defines a particular point $y_1 = y_0 + (\sin\omega)\,\xi_1$ such that $C(x_1, y_1) = 0$. Taking ξ_1 linearly independent from $x_0 - y_0$, we have $y_1 \neq x_1$ and therefore the line through x_1, y_1 is uniquely determined. By assumption, u is convex in every direction, hence its graph contains the whole straight line passing through $(x_1, u(x_1))$ and $(y_1, u(y_1))$. Let us check that

$$\omega = (5/4)\,\pi. \tag{19}$$

If, contrary to the claim, we suppose $\omega \neq \frac{5}{4}\pi$, then an elementary computation shows that the straight line passing through x_1, y_1 intersects the line passing through x_0, y_0 at the point $\overline{x} = x_0 + \overline{\lambda}\,(x_0 - y_0)$, where $\overline{\lambda} = (\cos\omega)/(\sin\omega - \cos\omega)$. Hence the graph of u is made up of straight lines passing through the point $(\overline{x}, u(\overline{x})) \in \mathbb{R}^{N+1}$. Being u convex over $\mathbb{R}^N$ by assumption, u must be the affine function $u(x) = a_0 \cdot (x - \overline{x}) + u(\overline{x})$ for some $a_0 \in \mathbb{R}^N$. But then $(-\Delta)^s\, u \equiv 0$, hence $f(u(x_0)) = 0 = f(u(y_0))$, which in its turn imply $\omega = \frac{5}{4}\pi$. Thus (19) holds true, as claimed, and therefore $\sin\omega = \cos\omega$. Consequently, the line through x_1, y_1 is parallel to $x_0 - y_0$. Since x_1 is arbitrary, and by the convexity of u, Claim (i) follows. To prove Claim (ii), observe that (19) implies $f(u(x_0)) = f(u(y_0))$. By replacing x_0, y_0 with any couple of points $x, y \in \Omega$ such that $x - y$ is parallel to $x_0 - y_0$, since $C(x, y) = 0$ we may repeat the preceding argument and get $f(u(x)) = f(u(y))$. The proof is complete. $\square$

Next we prove Theorem 1.4. Now monotonicity of f is needed because Claim (i) of Theorem 2.2 requires $b \leq 0$.

Proof of Theorem 1.4. Since the directional derivative $d^2g/d\xi_0^2$ is positive by assumption in a neighborhood $\mathcal{U}$ of the (bounded) boundary $\partial\Omega$, and since $u \in C^2(\mathcal{U})$, there exists a positive number $r_0 > 0$ such that $C_\xi(x) < 0$ for all $x \in \mathbb{R}^N$ satisfying $\operatorname{dist}(x, \partial\Omega) < r_0$ and all ξ proportional to ξ_0 and such that $0 < |\xi| < r_0$. Let us verify that $C_\xi(x) \leq 0$ for all $x \in \mathbb{R}^N$ and for all ξ as above.

Consider, for instance, the case when x, $x + \xi \notin \Omega$. If also the middle point $x + \frac{\xi}{2}$ is not in Ω, then $C_\xi(x) \leq 0$ because u coincides with the convex function g outside Ω. If, instead, $x + \frac{\xi}{2} \in \Omega$, then $\operatorname{dist}(x, \partial\Omega) < r_0$, and therefore we have $C_\xi(x) < 0$. The same conclusion is also reached under the assumption that $x \notin \Omega$ and $x + \xi \in \Omega$, as well as in the case when $x \in \Omega$ and $x + \xi \notin \Omega$.

It remains to deal with the case when both x and $x + \xi$ are in Ω. Now, the convexity maximum principle comes into play. Indeed, since Ω is convex in the direction of ξ, the middle point $x_0 + \frac{\xi}{2}$ is interior to Ω. Furthermore, since f is convex, the frozen concavity function satisfies inequality (17), where $b \leq 0$ because f is non-increasing. Recall that Theorem 2.2 also holds for the usual fractional Laplacian in $\mathbb{R}^N$, which corresponds to the case $k = m = N$. We have seen before that $C_\xi(x) \leq 0$ for all x outside the bounded domain $G = \{\, x \in \Omega \mid x + \xi \in \Omega \,\}$.

If we assume, by contradiction, that $C_\xi(x)$ becomes positive in G, then it must attain a positive maximum at some interior point $x_0 \in G$. By applying Claim (i) of Theorem 2.2 to the function $w = C_\xi$ it follows that $C_\xi(x_0) \leq 0$, hence $C_\xi(x)$ cannot be positive in G. Since ξ is arbitrary, subject to $0 < |\xi| < r_0$ and parallel to ξ_0, the function u must be convex in the direction of ξ_0, as claimed. $\square$

7. Examples

It is natural to ask for conditions on the data implying the existence of a solution of (1) satisfying the assumptions of Theorems 1.1, 1.2 and 1.4, respectively. In this section we discuss some examples aimed at putting some aspects of the problem into evidence. In particular, the last example shows that the class where the results apply is not empty.

Example 7.1. A typical problem associated to the fractional Laplacian reads as follows:

$$\begin{cases} (-\Delta)^s\, u(x) = f(x,u) & \text{in } \Omega; \\ u(x) = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{20}$$

where Ω is a bounded domain. Observe that the null function $u_0(x) \equiv 0$ is the unique convex function $u\colon \mathbb{R}^N \to \mathbb{R}$ vanishing outside Ω. Consequently, if $f(x,0) = 0$ for all $x \in \Omega$ then the unique solution to problem (20) which is convex over the whole space is the trivial solution u_0. If, instead, $f(x,0) \neq 0$ for some $x \in \Omega$, then problem (20) does not possess any solution u convex in $\mathbb{R}^N$.

Example 7.2. Consider the simplest non-constant, convex function u over the whole space, which is the affine function $u_\xi(x) = u_\xi(0) + \xi \cdot x$ with parameters $u_\xi(0) \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^N \setminus \{\, 0\,\}$. An elementary computation shows that the integral

$$\int_{|x_0 - x| > \varepsilon} \frac{u_\xi(x_0) - u_\xi(x)}{|x_0 - x|^{N+2s}}\, dx$$

is well defined in the sense of Lebesgue for all $\varepsilon > 0$ if and only if $s > \frac{1}{2}$. Consequently, if $s > \frac{1}{2}$ then we have $(-\Delta)^s\, u_\xi = 0$ in $\mathbb{R}^N$. If, instead, $s \in (0, \frac{1}{2}]$, then the fractional Laplacian $(-\Delta)^s\, u_\xi$ is undefined. Note that the threshold $s = \frac{1}{2}$ also appears in Claim (*b*) of Theorem 1.2.

Example 7.3. Let $s \in (\frac{1}{2}, 1)$. The convex, symmetric function $u(x) = |x|$ over the real line satisfies equation (1) in every open interval (a,b) not containing the origin, provided that

$$f(u) = -\,\frac{c_{1,s}}{(2s-1)\, s\, u^{2s-1}}, \quad u > 0.$$

To see this, without loss of generality choose $x_0 > 0$ and observe that the principal value

$$\text{P.V.} \int_0^{2x_0} \frac{x_0 - x}{|x_0 - x|^{1+2s}}\, dx$$

vanishes by oddity. Then, the fractional Laplacian $(-\Delta)^s\, u(x_0) = (-\Delta)^s\, u(-x_0)$ is given by

$$\frac{(-\Delta)^s\, u(x_0)}{c_{1,s}} = \int_{-\infty}^{0} \frac{x_0 + x}{(x_0 - x)^{1+2s}}\, dx - \int_{2x_0}^{+\infty} \frac{dx}{(x - x_0)^{2s}}.$$

Writing $x_0 + x = 2x_0 - (x_0 - x)$ the assertion follows easily. In this example the graph of u is *not* a straight line although the equality $u(\frac{x_0+y_0}{2}) = \frac{u(x_0)+u(y_0)}{2}$ holds whenever $x_0\, y_0 \geq 0$. In fact, neither the (negative) function f nor $-1/f$ are convex, and the assumptions of the theorems in the Introduction are not satisfied.

Example 7.4. Let us exhibit an example where the theorems stated in the Introduction apply. The construction is based on the following proposition:

Proposition 7.5. *Define the function* $u_0 \in C^1(\mathbb{R}^2)$ *by letting*

$$u_0(x, y) = \begin{cases} 0, & x \in (-\infty, 0); \\ x^2, & x \in [0, 1]; \\ 2x - 1, & x \in (1, +\infty), \end{cases}$$

and denote by I *the positive constant given by* $I = \int_0^{\frac{\pi}{2}} (\cos \vartheta)^{\frac{3}{2}} \, d\vartheta$. *Then for all* $(x_0, y_0) \in \mathbb{R}^2$ *we have*

$$\frac{-3}{32\, c_{2,\frac{3}{4}}\, I} \, (-\Delta)^{\frac{3}{4}} \, u_0(x_0, y_0) = \begin{cases} \sqrt{1 - x_0} - \sqrt{|x_0|}\,, & x_0 \in (-\infty, 0); \\ \sqrt{x_0} + \sqrt{1 - x_0}\,, & x_0 \in [0, 1]; \\ \sqrt{x_0} - \sqrt{x_0 - 1}\,, & x_0 \in (1, +\infty). \end{cases}$$

Proof. Let us give details for $x_0 \in (0, \frac{1}{2}]$, for shortness. Without loss of generality let $y_0 = 0$. By definition we have

$$\begin{aligned} \frac{1}{c_{2,\frac{3}{4}}} \, (-\Delta)^{\frac{3}{4}} \, u_0(x_0, 0) \; &= \; \iint_{x<0} \frac{x_0^2}{\rho^{\frac{7}{2}}} \, dx \, dy + \text{P.V.} \iint_{0<x<1} \frac{x_0^2 - x^2}{\rho^{\frac{7}{2}}} \, dx \, dy \\ &+ \; \iint_{x>1} \frac{x_0^2 - 2x + 1}{\rho^{\frac{7}{2}}} \, dx \, dy. \end{aligned} \tag{21}$$

where $\rho = |(x_0, 0) - (x, y)|$. To manage with the first integral we pass to polar coordinates

$$\begin{cases} \rho \, \cos \vartheta = |x_0 - x|; \\ \rho \, \sin \vartheta = y. \end{cases} \tag{22}$$

Here and in the sequel, the values of ϑ always belong to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. By symmetry reasons, doubling the integral over $(0, \frac{\pi}{2})$ will suffice. After a short computation we obtain

$$\iint_{x<0} \frac{x_0^2}{\rho^{\frac{5}{2}}} \, dx \, dy = \frac{4}{3} \, I \, \sqrt{x_0}\,.$$

To manage the last integral in (21) it is convenient to use the identity $x_0^2 - 2x + 1 = (1 - x_0)^2 - 2\,(x - x_0)$. Passing to polar coordinates as in (22) we get

$$\begin{aligned} \iint_{x>1} \frac{x_0^2 - 2x + 1}{\rho^{\frac{7}{2}}} \, dx \, dy = 2\,(1 - x_0)^2 \int_0^{\frac{\pi}{2}} \left(\int_{\frac{1-x_0}{\cos\vartheta}}^{+\infty} \frac{d\rho}{\rho^{\frac{5}{2}}} \right) d\vartheta - \\ - \, 4 \int_0^{\frac{\pi}{2}} \left(\int_{\frac{1-x_0}{\cos\vartheta}}^{+\infty} \frac{d\rho}{\rho^{\frac{3}{2}}} \right) \cos\vartheta \, d\vartheta = -\frac{4}{3} \, I \, \frac{x_0 + 5}{\sqrt{1 - x_0}}\,. \end{aligned}$$

Finally, in order to compute the principal value in (21), we take advantage of the identity

$$x_0^2 - x^2 = -(x_0 - x)^2 - 2x_0\,(x - x_0). \tag{23}$$

As long as x ranges in the strip $0 < x < 2x_0$, which is symmetric with respect to the line $x = x_0$, the integral simplifies by oddity. More precisely, we may write:

$$\begin{aligned} \text{P.V.}\iint_{0<x<1} \frac{x_0^2 - x^2}{\rho^{\frac{7}{2}}}\,dx\,dy \;&=\; -\,\text{P.V.}\iint_{0<x<2x_0} \frac{(x_0 - x)^2}{\rho^{\frac{7}{2}}}\,dx\,dy \\ &\quad + \iint_{2x_0<x<1} \frac{x_0^2 - x^2}{\rho^{\frac{7}{2}}}\,dx\,dy. \end{aligned} \tag{24}$$

Passing to polar coordinates (22), and since the integrand is now even with respect to $x = x_0$, the first integral in the right-hand side of (24) becomes

$$\begin{aligned} -\,\text{P.V.}\iint_{0<x<2x_0} \frac{(x_0 - x)^2}{\rho^{\frac{7}{2}}}\,dx\,dy \;&=\; -4\int_0^{\frac{\pi}{2}} \left(\int_0^{\frac{x_0}{\cos\vartheta}} \frac{d\rho}{\sqrt{\rho}}\right)\cos^2\vartheta\,d\vartheta \\ &=\; -8I\sqrt{x_0}\,. \end{aligned}$$

Using again the identity (23), as well as polar coordinates (22), the last integral in (24) is reduced to

$$\begin{aligned} \iint_{2x_0<x<1} \frac{x_0^2 - x^2}{\rho^{\frac{7}{2}}}\,dx\,dy &= -2\int_0^{\frac{\pi}{2}} \left(\int_{\frac{x_0}{\cos\vartheta}}^{\frac{1-x_0}{\cos\vartheta}} \frac{d\rho}{\sqrt{\rho}}\right)\cos^2\vartheta\,d\vartheta - \\ - 4x_0\int_0^{\frac{\pi}{2}} \left(\int_{\frac{x_0}{\cos\vartheta}}^{\frac{1-x_0}{\cos\vartheta}} \frac{d\rho}{\rho^{\frac{3}{2}}}\right)\cos\vartheta\,d\vartheta &= 4I\,\frac{3x_0 - 1}{\sqrt{1 - x_0}} - 4I\sqrt{x_0}\;. \end{aligned}$$

Thus, the conclusion follows in the case when $x_0 \in (0, \frac{1}{2}]$. The computation in the remaining cases is similar. □

Let us discuss the three theorems stated in the Introduction. In view of Proposition 7.5, the function u_0 satisfies $(-\Delta)^{\frac{3}{4}}\,u_0 = \tilde{f}(u_0)$ in the strip $0 \le x \le 1$, with $\tilde{f}$ given by

$$\tilde{f}(t) = \,-\,c_{2,\frac{3}{4}}\,I\,\frac{32}{3}\left(\sqrt[4]{t} + \sqrt{1 - \sqrt{t}}\,\right), \qquad t \in [0, 1].$$

Concerning Theorem 1.1, as well as Theorem 1.2 under assumption (a), we cut and paste the function $\tilde{f}$ in order to define a convex function f over the whole real line. We also let f be decreasing in preparation for Theorem 1.4. Since $\tilde{f}''(t) > 0$ for $t \in (0, 1)$ and $\tilde{f}'(t) \to -\infty$ as $t \to 0^+$, there exist $0 < t_1 < t_2 < 1$ such that $\tilde{f}'(t) < 0 < \tilde{f}''(t)$ for all $t \in [t_1, t_2]$. Therefore we let

$$f(t) = \begin{cases} \tilde{f}'(t_1)\,(t - t_1) + \tilde{f}(t_1), & t \in (-\infty, t_1); \\ \tilde{f}(t), & t \in [t_1, t_2]; \\ \tilde{f}'(t_2)\,(t - t_2) + \tilde{f}(t_2), & t \in (t_2, +\infty). \end{cases} \tag{25}$$

Consequently, we have $(-\Delta)^{\frac{3}{4}} u_0 = f(u_0)$ in the strip S_{12} given by

$$S_{12} = \{ (x, y) \mid \sqrt{t_1} < x < \sqrt{t_2} \}.$$

Let Ω be a two-dimensional, convex domain, included in the strip S_{12}. Take two distinct points $P_1 = (x_0, y_1)$ and $P_2 = (x_0, y_2)$ in Ω having the same first coordinate $x_0 \in (\sqrt{t_1}, \sqrt{t_2})$. Observe that u_0 is convex, and $C(P_1, P_2) = 0$. Then, by Theorem 1.1 the graph of u_0 is made up of straight lines whose projections onto $\mathbb{R}^2$ are parallel to the y axis. The graph of u_0 is in fact a cylinder: this stronger conclusion, together with $f(u(x, y)) =$ constant along the lines $x =$ constant, follows from Theorem 1.2 because condition (a) holds true.

In order to satisfy assumption (b) of Theorem 1.2, instead, we cut and paste $\tilde{f}$ and define a negative function f over the whole real line such that $-1/f$ is convex. Since $\tilde{f}'(t) \to +\infty$ as $t \to 1^-$, there exists $t_3 \in (t_1, 1)$ such that $\tilde{f}'(t_3) > 0$. Let $h \colon \mathbb{R} \to \mathbb{R}$ be the positive, convex function defined as follows:

$$h(t) = \begin{cases} \frac{\tilde{f}'(t_1)}{\tilde{f}^2(t_1)}(t - t_1) - \frac{1}{\tilde{f}(t_1)}, & t \in (-\infty, t_1); \\ -1/\tilde{f}(t), & t \in [t_1, t_3]; \\ \frac{\tilde{f}'(t_3)}{\tilde{f}^2(t_3)}(t - t_3) - \frac{1}{\tilde{f}(t_3)}, & t \in (t_3, +\infty). \end{cases}$$

Accordingly, we replace (25) with the function $f(t) = -1/h(t)$ for $t \in \mathbb{R}$, which satisfies assumption (b) of Theorem 1.2. If Ω is a convex domain included in the strip $S_{13} = \{ (x, y) \mid \sqrt{t_1} < x < \sqrt{t_3} \}$, we may apply the theorem and reach the same conclusion as before.

Finally, consider a bounded domain $\Omega \subset S_{12}$, convex with respect to the variable x, and let $f(t)$ be given again as in (25). Define $g(x, y) = u_0(x, y)$ for all $(x, y) \in \mathbb{R}^2$, and observe that g is a convex function in the plane which is of class C^2 in the open strip $\mathcal{U} = \{ (x, y) \mid 0 < x < 1 \}$ containing the boundary $\partial\Omega$. Furthermore, we have $\partial^2 g/\partial x^2 > 0$ in $\mathcal{U}$. Hence the assumptions of Theorem 1.4 are satisfied. By construction, the function u_0 belongs to the class $C^2(\mathcal{U})$ and is a solution of problem (3) with $s = \frac{3}{4}$. By Theorem 1.4, u_0 has to be convex with respect to x over the whole plane, and this is in fact the case.

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