Minimax Theory and its Applications
Volume 2 (2017), No. 1, 69–78

Multiplicity Results for some Quasilinear Differential Systems with Periodic Nonlinearities

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Received: July 7, 2015
Accepted: November 3, 2015

A multiplicity result for periodic problems of the form

$$-(\psi(u'))' = \nabla_u V(t,u) + e(t), \quad u(0) = u(T), \quad u'(0) = u'(T),$$

when $\psi : \mathbb{R}^N \to \mathbb{R}^N$ belongs to a suitable class of homeomorphisms, V is T_i-periodic in each component u_i of $u \in \mathbb{R}^N$, and e has mean value zero on $[0,T]$ is proved, and applied, by a modification technique, to obtain the same multiplicity for the solutions of the relativistic system

$$-\left(\frac{u'}{\sqrt{1-|u'|^2}}\right)' = \nabla_u V(t,u) + e(t), \quad u(0) = u(T), \quad u'(0) = u'(T).$$

Keywords: Periodic solutions, periodic nonlinearities, relativistic pendulum systems.

2010 Mathematics Subject Classification: Primary 34C25; Secondary 35J25, 35J65.

1. Introduction

In the recent paper [9], the existence of at least $N+1$ geometrically distinct solutions of the following quasilinear problem

$$-\left(\phi(u')\right)' = \nabla_u V(t,u) + e(t), \quad u(0) = u(T),\ u'(0) = u'(T), \tag{1}$$

ISSN 2199-1413 (printed), ISSN 2199-1421 (electronic) / $ 2.50

was proved when ϕ, V, e satisfy the following assumptions (with $B(\sigma) = B(0,\sigma)$ the open ball of center 0 and radius σ in $\mathbb{R}^N$):

(H_ϕ) ϕ is a homeomorphism from $B(\sigma)$ onto $\mathbb{R}^N$ such that $\phi(0) = 0$, $\phi = \nabla\Phi$, with $\Phi \in C(\overline{B}(\sigma),\mathbb{R}) \cap C^1(B(\sigma),\mathbb{R})$, strictly convex on $\overline{B}(\sigma)$ and such that $\Phi(0) = 0$ (*singular* ϕ).

(H'_V) $V : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is continuous, T_i-periodic ($T_i > 0$) with respect to each u_i ($i = 1,\ldots,N$), and $\nabla_u V$ exists and is continuous on $[0,T] \times \mathbb{R}^N$.

(H'_e) $e \in L^s([0,T];\mathbb{R}^N)$ for some $s > 1$ and $\int_0^T e(t)\,dt = 0$.

The motivation for this problem comes from the 'relativistic' case where (with $|\cdot|$ the Euclidian norm in $\mathbb{R}^N$)

$$\phi(v) = \frac{v}{\sqrt{1-|v|^2}}, \tag{2}$$

and its special case of $N = 1$ of the 'forced relativistic pendulum'

$$-\left(\frac{u'}{\sqrt{1-u'^2}}\right)' = a\sin u + e(t), \quad u(0) = u(T),\ u'(0) = u'(T),$$

considered in [2]. In order to obtain such a result, problem (1) was reduced to a Hamiltonian system, and a generalized saddle point theorem for strongly indefinite functionals, due to Szulkin and based upon relative category, was applied. A second proof of this result was given in [1], based upon an abstract multiplicity result for convex, lower semicontinuous perturbations of a C^1 functional, obtained by using a deformation lemma together with Ekeland's variational principle and Lusternik-Schnirelmann category.

In [6] we give a simpler proof than the ones in [1] and [9] for problem (1) with ϕ given in (2). The idea is to reduce the problem to an equivalent one where ϕ from (2) is replaced by a homeomorphism $\psi : \mathbb{R}^N \to \mathbb{R}^N$, to which one can apply an abstract multiplicity result from [10] - see Theorem 1.1, below (we note that a similar idea, but in a different functional framework was employed in [7] for differential inclusions systems). This multiplicity result is an abstract generalization of a result of Rabinowitz [12] for problems of the form (with $(\cdot|\cdot)$ the inner product in $\mathbb{R}^N$):

$$-[M(t,u)u']'_t = \nabla_u[(1/2)(M(t,u)u'|u') + V(t,u)] + e(t),$$
$$u(0) = u(T), \quad u'(0) = u'(T),$$

where $M : [0,T] \times \mathbb{R}^N \to \mathcal{L}(\mathbb{R}^N,\mathbb{R}^N)$ is a symmetric, uniformly positive definite C^1 matrix function, T_i periodic in each u_i ($i = 1,\ldots,N$). Similar results were obtained independently, using Lusternik-Schnirelmann category, in [3] and [8]. The important special case where $M(t,u) = I$ corresponds to the problem

$$-u'' = \nabla_u V(t,u) + e(t), \quad u(0) = u(T), \quad u'(0) = u'(T),$$

motivated, when $N = 1$, by the forced pendulum problem

$$-u'' = a\sin u + e(t), \quad u(0) = u(T), \quad u'(0) = u'(T).$$

Here, from [10, Theorem 4.12], we prove in Theorem 2.3 the existence of at least $N+1$ geometrically distinct solutions for quasilinear systems of the form

$$-(\psi(u'))' = \nabla_u V(t,u) + e(t), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{3}$$

under the following hypotheses:

(H_ψ) *$\psi : \mathbb{R}^N \to \mathbb{R}^N$ is a homeomorphism such that $\psi(0) = 0$, $\psi = \nabla\Psi$ for some $\Psi \in C^1(\mathbb{R}^N, \mathbb{R})$ such that $\Psi(0) = 0$, and*

$$\alpha|v - w|^2 \le (\psi(v) - \psi(w)|v - w), \quad |\psi(v)| \le \beta|v|, \tag{4}$$

for some α, $\beta > 0$ and all v, $w \in \mathbb{R}^N$.

(H_V) *$V : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is measurable in t for every $u \in \mathbb{R}^N$, continuously differentiable in u for almost every $t \in [0,T]$, T_i-periodic in u_i for some $T_i > 0$ $(i = 1, \ldots, N)$, and*

$$|\nabla_u V(t,u)| \le h(t)$$

for some $h \in L^1(0,T)$, every $u \in \mathbb{R}^N$ and almost every $t \in [0,T]$.

(H_e) *$e \in L^1(0,T;\mathbb{R}^N)$ and $\int_0^T e(t)\,dt = 0$.*

Then, as a novelty, Theorem 2.3 is used to provide a proof of the result from [6]. The strategy used in [6] to truncate the singular differential operator will be also employed here. Similar, although technically different, reduction methods of problems with singular ϕ to non-singular problems have been recently used by Coelho et al. [4, 5] in the study of positive solutions of some boundary value problems involving curvature operator in Minkowski space.

We conclude this introductory part with some notions and results that will be needed in the sequel.

Let X be a real Banach space, with dual X^* and $G \subset X$ be a discrete additive group such that the dimension N of the vector subspace generated by G is finite. If $\sim$ stands for the corresponding equivalence relation in X ($x \sim y$ if $y = x + g$ for some $g \in G$), we denote by $\pi : X \to X/G$ the canonical surjection; recall, this means $\pi(x) :=$the equivalence class of x for $\sim$, for all $x \in X$.

A subset A of X is said to be *G-invariant* if $A = \pi^{-1}(\pi(A))$. A function $f : X \to \mathbb{R}$ is said to be *G-invariant* if $f(x + g) = f(x)$ for all $x \in X$ and all $g \in G$. If a differentiable functional $f : X \to \mathbb{R}$ is G-invariant, then f' is also G-invariant. Hence, if u is a critical point of such f, then $\pi^{-1}(\pi(u))$ is a set of critical points of f and is called *a critical orbit* of f.

The G-invariant differentiable functional f is said *to satisfy the $(PS)_G$-condition* if for every sequence (u_n) in X such that $(f(u_n))$ is bounded and $f'(u_n) \to 0$, the sequence $(\pi(u_n))$ contains a convergent subsequence.

Theorem 1.1. ([10, Theorem 4.12]) *Let $f \in C^1(X, \mathbb{R})$ be a G-invariant functional satisfying the $(PS)_G$-condition. If f is bounded from below and if the dimension N of the space generated by G is finite, then f has at least $N+1$ critical orbits.*

2. Quasilinear ordinary differential systems with periodic nonlinearities

We consider problem (3) where assumptions (H_ψ), (H_V) and (H_e) hold true. By a *solution* of (3) we understand a function $u \in C^1 := C^1([0,T];\mathbb{R}^N)$ with $\psi(u')$ absolutely continuous, which satisfies the N-dimensional system in (3) a.e. on $[0,T]$ and the periodic boundary conditions. Notice that assumption (H_V) implies immediately that if u is a solution of (3), then the same is true for $u+\sum_{i=1}^N k_i T_i e^i$, where $k_i \in \mathbb{Z}$ and $e^i_j = \delta_{ij}$, $(i,j = 1,\dots,N)$. We say that two solutions u and v of (3) are *geometrically distinct* if, for some $i \in \{1,\dots,N\}$, $u - v$ is not an integer multiple of $T_i e^i$.

If $u \in L^1(0,T;\mathbb{R}^N)$, we write $u = \overline{u} + \widetilde{u}$, where

$$\overline{u} := T^{-1}\int_0^T u(t)\,dt\,.$$

Let H^1_T be the Sobolev space of function $u \in H^1([0,T];\mathbb{R}^N)$ such that $u(0) = u(T)$, with the inner product

$$\langle u, v\rangle = (\overline{u}|\overline{v}) + \int_0^T (\widetilde{u}'(t)|\widetilde{v}'(t))\,dt\,,$$

and the corresponding norm $\|u\|_{H^1} = \langle u,u\rangle^{1/2}$. Also, we will use the Sobolev inequality (see e.g. [10], Proposition 1.3):

$$\|\widetilde{u}\|_\infty \leq \frac{T^{1/2}}{2\sqrt{3}}\|u'\|_2,$$

for all $u \in H^1_T$ (with $\|\cdot\|_\infty$ the usual sup-norm on $C := C([0,T];\mathbb{R}^N)$ throughout the paper).

Let f be defined by

$$f(u) = \int_0^T [\Psi(u') - V(t,u) - (e|u)]\,dt\,.$$

Standard arguments show that f is of class C^1 on H^1_T and

$$\langle f'(u), v\rangle = \int_0^T [(\psi(u')|v') - (\nabla_u V(t,u) + e|v)]\,dt\,. \tag{5}$$

Proposition 2.1. *If $u \in H^1_T$ is a critical point of f, then $u \in C^1$ and is a solution of problem* (3).

Proof. Assume that u is a critical point of f. From (5), one has

$$\int_0^T (\psi(u')|v')dt = \int_0^T (\nabla_u V(t,u) + e|v)dt, \quad \forall\, v \in C_0^\infty(0,T;\mathbb{R}^N) \subset H^1_T. \tag{6}$$

Next, by (4) we infer

$$\psi(u') \in L^1(0,T;\mathbb{R}^N). \tag{7}$$

Then, from (H_V), (H_e), (6) and (7) it follows that

$$\psi(u') \in W^{1,1}([0,T];\mathbb{R}^N) \tag{8}$$

and

$$-(\psi(u'))' = \nabla_u V(t,u) + e(t), \quad \text{a.e. } t \in [0,T]. \tag{9}$$

Thus, due to (8), $w = \psi(u')$ is an absolutely continuous function, so that $u' = \psi^{-1}(w)$ is continuous. On the other hand, multiplying the equality in (9) by $v \in H^1_T$, then integrating over $[0,T]$, using the integration by parts formula and the fact that u is a critical point, we obtain

$$(\psi(u'(T)) - \psi(u'(0))|v(T)) = 0, \quad \forall\, v \in H^1_T.$$

Then, taking $v \equiv e_i$ for $i \in \{1,\dots,N\}$, one has $u'(0) = u'(T)$, as ψ is a homeomorphism. Hence u is a C^1 function which is solution of problem (3). □

Proposition 2.2. *If assumptions (H_ψ), (H_V) and (H_e) hold, then f is bounded from below. Furthermore, every sequence $(u_k) \subset H^1_T$ with $f'(u_k) \to 0$, $(f(u_k))$ bounded in $\mathbb{R}$ and $(\overline{u}_k)$ bounded in $\mathbb{R}^N$, contains a convergent subsequence.*

Proof. It follows from assumption (H_ψ) that, for all $y \in \mathbb{R}^N$,

$$\Psi(y) = \int_0^1 [\Psi(sy)]'\,ds = \int_0^1 (\psi(sy)|y)\,ds \ge \alpha \int_0^1 |y|^2 s\,ds = \alpha\frac{|y|^2}{2}\,. \tag{10}$$

Using hypothesis (H_V) and the mean value theorem, one has

$$|V(t,y)| \le c_1 h(t), \quad \text{a.e. } t \in [0,T] \text{ and all } y \in \mathbb{R}^N, \tag{11}$$

with c_1 a positive constant. By assumption (H_e), we get

$$\int_0^T (e|u)\,dt = \int_0^T (\widetilde{e}|\widetilde{u})\,dt \quad (u \in H^1_T)\,.$$

This, together with (10), (11) and Sobolev inequality, imply

$$f(u) \ge \alpha\frac{\|u'\|_2^2}{2} - c_1\|h\|_1 - \|\widetilde{u}\|_\infty\|\widetilde{e}\|_1 \ge \alpha\frac{\|u'\|_2^2}{2} - c_1\|h\|_1 - c_2\|u'\|_2, \tag{12}$$

for some constant $c_2 > 0$, so that f is bounded from below. Now let (u_k) be a sequence in H_T^1 such that $f'(u_k) \to 0$, $(f(u_k))$ is bounded and $(\overline{u}_k)$ is bounded. It follows from (12) that $(\widetilde{u}_k')$ is bounded in $L^2(0,T;\mathbb{R}^N)$. Consequently, (u_k) is bounded in H_T^1 and, going if necessary to a subsequence, we can assume that $u_k \to u$ in H_T^1 weakly and $u_k \to u$ uniformly. We infer

$$\langle f'(u_k) - f'(u), u_k - u\rangle \to 0 \text{ as } k \to \infty. \tag{13}$$

Using (5) and assumption (H_ψ), we obtain

$$\langle f'(u_k) - f'(u), u_k - u\rangle \geq \alpha\|u_k' - u'\|_2^2 - \int_0^T (\nabla_u V(t,u_k) - \nabla_u V(t,u)|u_k - u)\,dt.$$

Then (13) and the uniform convergence of (u_k) to u implies that $\|u_k' - u'\|_2 \to 0$ as $k \to \infty$, and hence that $u_k \to u$ in H_T^1. □

Theorem 2.3. *If assumptions (H_ψ), (H_V) and (H_e) hold, then problem (3) has at least $N+1$ geometrically distinct solutions.*

Proof. By Proposition 2.1 solutions of (3) correspond to critical points of f in H_T^1. Assumptions (H_V) and (H_e) imply easily that f satisfies the symmetry conditon

$$f(u + \sum_{i=1}^N k_i T_i e^i) = f(u)$$

for all $k_i \in \mathbb{Z}$, $i = 1, \ldots, N$. Hence we can apply Theorem 1.1 to f in H_T^1 with the additive group G in H_T^1 defined by

$$G := \left\{\sum_{i=1}^N k_i T_i e^i : (k_1, \ldots, k_N) \in \mathbb{Z}^N\right\}. \tag{14}$$

Clearly the vector subspace of H_T^1 spanned by the elements of G is the N-dimensional subspace of constant functions, and the equivalence relation associated to G is

$$u \sim v \quad \Leftrightarrow \quad u - v = \sum_{i=1}^N k_i T_i e^i \text{ for some } (k_1, \ldots, k_N) \in \mathbb{Z}^N.$$

We can take as a representant of the equivalence class $\pi(u)$ of u the element with

$$\overline{u}_i \in [0, T_i),\ i = 1, \ldots, N. \tag{15}$$

In this way, it follows immediately from Proposition 2.2 that f is bounded from below and satisfies the $(PS)_G$-condition. Theorem 1.1 implies then that f has at least $N+1$ critical orbits. As different critical orbits correspond to geometrically distinct solutions of (3), the proof is complete. □

Corollary 2.4. *If assumptions* (H_V) *and* (H_e) *hold, problem*

$$-u'' = \nabla_u V(t,u) + e(t), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

has as least $N+1$ *geometrically distinct solutions.*

Proof. Take the identity for ψ. □

3. Relativistic differential systems with periodic nonlinearities

In this section we consider 'relativistic' systems of type

$$-(\varphi(u'))' = \nabla_u V(t,u) + e(t), \quad u(0) = u(T),\ u'(0) = u'(T), \tag{16}$$

where $\varphi : B(1) \subset \mathbb{R}^N \to \mathbb{R}^N$ is defined by

$$\varphi(y) = \frac{y}{\sqrt{1-|y|^2}},$$

$V : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies assumption (H_V), and $e : [0,T] \to \mathbb{R}^N$ satisfies assumption (H_e). A function $u \in C^1$ is said to be *solution* of problem (16) if $\|u'\|_\infty < 1$, $\varphi(u')$ is absolutely continuous, u satisfies the N-dimensional system in (16) a.e. on $[0,T]$ and the periodic boundary conditions.

Setting $K := \varphi^{-1}(\overline{B}(\sqrt{N}(\|h\|_1 + \|e\|_1))) \subset B(1)$, we fix $R \in (0,1)$ with

$$\frac{R}{\sqrt{1-R^2}} \geq \sqrt{N}(\|h\|_1 + \|e\|_1)\ ,\ K \subset \overline{B}(R)$$

and define $\psi : \mathbb{R}^N \to \mathbb{R}^N$ by

$$\psi(y) = \begin{cases} \dfrac{y}{\sqrt{1-|y|^2}} & \text{if} \quad |y| \leq R \\[2ex] \dfrac{y}{\sqrt{1-R^2}} & \text{if} \quad |y| > R. \end{cases} \tag{17}$$

Note that ψ is a homeomorphism, having as inverse

$$\psi^{-1}(y) = \begin{cases} \dfrac{y}{\sqrt{1+|y|^2}} & \text{if} \quad |y| \leq \frac{R}{\sqrt{1-R^2}} \\[2ex] y\sqrt{1-R^2} & \text{if} \quad |y| > \frac{R}{\sqrt{1-R^2}}. \end{cases}$$

Let us define $\Psi : \mathbb{R}^N \to \mathbb{R}$ by

$$\Psi(y) = \begin{cases} 1 - \sqrt{1-|y|^2} & \text{if} \quad |y| \leq R \\[2ex] 1 - \sqrt{1-R^2} + \dfrac{|y|^2 - R^2}{2\sqrt{1-R^2}} & \text{if} \quad |y| > R. \end{cases} \tag{18}$$

Observe that

$$\psi = \nabla\Psi \ \text{ on } \ \mathbb{R}^N. \tag{19}$$

Lemma 3.1. *For every $z, y \in \mathbb{R}^N$, it holds*

$$(\psi(z) - \psi(y)|z - y) \geq |z - y|^2, \ \ |\psi(y)| \leq \frac{1}{\sqrt{1-R^2}}|y|\,. \tag{20}$$

Proof. We first prove the first inequality. If either $z = 0$ or $y = 0$, then (20) is trivial. The case when both of $|z|$ and $|y|$ are $> R$ is also immediate. Suppose that $0 < |y| \leq |z| \leq R$, so that

$$(y|z) \leq |y||z| \leq |z|^2.$$

Then

$$\begin{aligned}(\psi(z) - \psi(y)|z - y) &= (\frac{z}{\sqrt{1-|z|^2}} - \frac{y}{\sqrt{1-|y|^2}}\ |z - y) \\ &= (\frac{z-y}{\sqrt{1-|y|^2}}\ |z - y) + \left(\frac{1}{\sqrt{1-|z|^2}} - \frac{1}{\sqrt{1-|y|^2}}\right)[|z|^2 - (y|z)] \\ &\geq \frac{1}{\sqrt{1-|y|^2}}\ |z-y|^2 \geq |z-y|^2.\end{aligned}$$

In the remaining case $|y| \leq R < |z|$, inequality (20) follows in a similar way.

The second inequality is trivial if $|y| \geq R$, and, if $|y| < R$, then

$$\sqrt{1-|y|^2} > \sqrt{1-R^2}$$

and the result follows. □

Thus, in consequence of Lemma 3.1, (18) and (19), ψ satisfies assumption (H_ψ) with $\alpha = 1 < \frac{1}{\sqrt{1-R^2}} = \beta$.

Next, consider the problem

$$-(\psi(u'))' = \nabla_u V(t,u) + e(t), \quad u(0) = u(T),\ u'(0) = u'(T), \tag{21}$$

where V and e respectively satisfy conditions (H_V) and (H_e), and ψ is given by (17).

Proposition 3.2. *A function $u \in C^1$ is solution of* (16) *if and only if it solves problem* (21)*.*

Proof. Let $u = (u_1, ..., u_N)$ be a solution of (16). Because u is such that $u(0) = u(T)$, there exists $\xi_i \in [0,T]$ with $u_i'(\xi_i) = 0$ for all $i = 1, \ldots, N$. Integrating

$$-(\varphi_i(u'))'(\tau) = \partial_{u_i} V(\tau, u) + e_i(\tau) \quad (\tau \in [0,T])$$

between ξ_i and arbitrary $t \in [0,T]$ and taking the absolute value, we obtain

$$\begin{aligned}\frac{|u_i'(t)|}{\sqrt{1-|u'(t)|^2}} &= \left|\int_{\xi_i}^t [\partial_{u_i} V(\tau, u(\tau)) + e_i(\tau)]\, d\tau\right| \\ &\leq \int_0^T [|\partial_{u_i} V(\tau, u(\tau))| + |e_i(\tau)|]\ d\tau \quad (i = 1, \ldots, N).\end{aligned}$$

Then, using hypothesis (H_V), we get

$$|\varphi(u'(t))| = \frac{|u'(t)|}{\sqrt{1-|u'(t)|^2}} \leq \sqrt{N}(\|\alpha\|_1 + \|e\|_1).$$

Therefore $u'(t) \in K$, which implies $|u'(t)| \leq R$ for all $t \in [0,T]$. Thus, $\psi(u') = \varphi(u')$ on $[0,T]$ and u solves (21).

Conversely, assume that u is a solution of (21). If $|u'| \leq R$ on $[0,T]$, clearly $\varphi(u') = \psi(u')$ and the proof is complete. If we suppose that there exists $t_0 \in [0,T]$ with $|u'(t_0)| > R$, then we get a contradiction. Indeed, as $u(0) = u(T)$, there are $\xi_i \in [0,T]$ such that $u_i'(\xi_i) = 0$ for all $i = 1, \ldots, N$. It follows

$$\begin{aligned}\int_{\xi_i}^{t_0} (\psi_i(u'(\tau))'d\tau &= \psi_i(u'(t_0)) - \psi_i\left(u_1'(\xi_i), \ldots, u_{i-1}'(\xi_i), 0, u_{i+1}'(\xi_i), \ldots, u_N'(\xi_i)\right) \\ &= \psi_i(u'(t_0)) = \frac{u_i'(t_0)}{\sqrt{1-R^2}} \quad (i = 1, \ldots, N).\end{aligned}$$

Then, arguing as above we obtain

$$\sqrt{N}(\|h\|_1 + \|e\|_1) \geq \frac{|u'(t_0)|}{\sqrt{1-R^2}} > \frac{R}{\sqrt{1-R^2}},$$

which contradicts the choice of R. □

Theorem 3.3. *If assumptions* (H_V) *and* (H_e) *hold, problem* (16) *has at least* $N+1$ *geometrically distinct solutions.*

Proof. We have seen above that ψ satisfies assumption (H_ψ) with $\alpha = 1$ and $\beta = 1/\sqrt{1-R^2}$. Hence, Theorem 2.3 implies that problem (21) has at least $N+1$ geometrically distinct solutions, which are also geometrically distinct solutions of (16) because of Proposition 3.2. □

Notice that the regularity of e is slightly weaker than the one imposed in [1] and [9].

Acknowledgements. The work of Călin Şerban was supported by the strategic grant POSDRU/159/1.5/S/137750, "Project Doctoral and Postdoctoral programs support for increased competitiveness in Exact Sciences research" cofinanced by the European Social Fund within the Sectoral Operational Programme Human Resources Development 2007-2013.

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