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AUTOMATIC CONTINUITY OF σ-c-DERIVATION ON QUASI-SEMISIMPLE BANACH ALGEBRAS

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Abstract

This paper establishes the automatic continuity of σ-c-derivations on semisimple quasi-Banach algebras, extending the foundational approaches of Villena [13] and Mohammed and Ali [7]. Furthermore, we demonstrate that every σ-c-derivation is necessarily continuous on all nonzero ideals of a prime quasi-C*-algebra.

Keywords: quasi- Banach algebras, Banach algebras, ultraprime, semi-simple quasi-Banach algebras.

Introduction

In [13] Villena successfully applied the sliding hump sequence to guarantee essentially defined derivation continuity on semisimple Banach algebra. In [4] Lee and Liu extended the Villena by showing the σ-derivation's continuity, where σ is a linear endomorphism. By proving the continuity of the partially defined σ- c - derivation on semisimple complex Banach algebra, where σ is a continuous linear mapping, Mohammed and Ali extended the Villena theorem in [7]. In [9] Mohammed and Ismail generalized Villena's theorem by showing the automatic continuity of some types of double derivations on semisimple Banach algebras [14].

In this paper we extend Villena theorem by proving that a partially defined σ-c- derivation on semisimple complex quasi- Banach algebra with an essential ideal as the domain is automatically closable, depending on the Villena in [13] and Mohammed and Ali in [7]. Additionally, we show that continuity of the σ- c- derivation, which is essentially defined on any prime quasi-C*-algebra's nonzero ideal.

The quasi-Banach algebra $\mathcal{Q}$ refers to complete quasi-normed algebra $\mathcal{Q}$, recall from [1] the quasi-normed algebra is algebra with a quasi-norm having the property $\|xy\|_q \leq \delta\|x\|_q \, \|y\|_q$ for all $x, y \in \mathcal{Q}$, and $\delta > 0$ constant, clear that every quasi-normed algebra is quasi-normed space, hence all concepts and results in quasi-normed space are valid in any quasi-normed algebra, therefore following [10, 11] in quasi- normed algebra $\mathcal{Q}$ with quasi index γ the following property satisfied.

$$\|\textstyle\sum_{i=1}^{m} a_i\|_q \leq \gamma^{m-1} \sum_{i=1}^{m}\|a_i\|_q \, , \forall a_i \in \mathcal{Q} \, , \forall m \in \mathbb{N} \quad ....(*)$$

Throughout this paper, $\mathcal{Q}$ represents a semisimple complex quasi-Banach algebra and a linear map $\mathcal{D}: \mathcal{Q} \to \mathcal{Q}$ is σ-c-derivation if $(ab) = \mathcal{D}(a)\sigma(b) + a\mathcal{D}(b)$ or $\mathcal{D}(ab) = \mathcal{D}(a)b + \sigma(a)\mathcal{D}(b)$ for all $a, b \in \mathcal{Q}$ where $\sigma: \mathcal{Q} \to \mathcal{Q}$ a continuous linear mapping [7,8]. Let I be a nonzero ideal of $\mathcal{Q}$, if $\mathrm{I} \cap J \neq 0$ for any ideal J that is not zero of $\mathcal{Q}$ then I is said to be essential. A linear map $\mathcal{D}$: I → $\mathcal{Q}$ that has I as an essential ideal and $\mathcal{D}(ab) = \mathcal{D}(a)\sigma(b) + \sigma\mathcal{D}(b)$ or $\mathcal{D}(ab) = \mathcal{D}(a)b + \sigma(a)\mathcal{D}(b)$ $\forall a, b \in$ I is said to be essential defined σ- c- derivation [7]. It is crucial to remember that all of Q 's nonzero ideals are essential if $\mathcal{Q}$ is prime ($\mathcal{Q}$ is prime if I.J = 0, then either I equal to zero or J equal to zero where I and J ideals of $\mathcal{Q}$ [2]). Every derivation is σ-c-derivation but in general converse is not true as the following example, shows:

Let $M_{2\times 2}(\mathbb{C})$ be a normed algebra which is consisting of the diagonal 2×2 matrices. Let $\mathcal{D}$ and σ be two maps on $M_{2\times 2}(\mathbb{C})$ defined by:

$$\mathcal{D}\left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right) = \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \sigma\left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix} \text{ for all } \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \in M_{2\times 2}(\mathbb{C}).$$

$\mathcal{D}$ is σ-c- derivation but not derivation.

The σ-c-derivation is a generalization of the standard derivation since it is evident that if σ is the identity map, then $\mathcal{D}$ is the usual derivation.

Finally, In[12] the separating subspace of $\mathcal{D}$ is denoted by $\mathcal{S}(\mathcal{D})$ and define as follows: $\mathcal{S}(\mathcal{D}) = \{ b \in Q : \exists\ a\ sequence\ \{a_n\}\ in\ Q\ with\ \lim a_n = 0\ and\ \lim \mathcal{D}(a_n) = b\}$

Theorem 1.1:

Let $(Q_1, \| \ \|_{q_1})$ and $(Q_2, \| \ \|_{q_2})$ be two quasi-normed algebras and $T: Q_1 \to Q_2$ be a linear operator. Then T is bounded iff T is continuous.

Proof: Since every quasi-normed algebra is a quasi-normed space, then the proof is similar to that of [11]

Theorem 1.2: (Closed graph theorem)

Let $(Q_1, \| \ \|_{q_1})$ and $(Q_2, \| \ \|_{q_2})$ be two quasi-Banach algebras and $T: Q_1 \to Q_2$ be a linear operator. Then the graph of T is closed iff T is continuous.

Proof: by the same way of Kreyszig [3]

Theorem 1.3:

Let $(Q_1, \| \ \|_{q_1})$ and $(Q_2, \| \ \|_{q_2})$ be two quasi-Banach algebras and $T: Q_1 \to Q_2$ be an linear operator. Then the separating space is zero set iff T is continuous.

Proof: by the same way of Sinclair [12]

2- Continuity of σ-c-derivation on semi simple quasi-Banach algebras

In this section we will achieve the aim of this paper which is to give prove of the continuity of σ-c-derivation on quasi-Banach algebra .

We assume that $\mathcal{P} = \{P: P\ primitive\ ideal\ of\ Q, I \not\subset P\}$. It is evident $I \cap (\bigcap_{P\in\mathcal{P}} P) \subset Rad\, Q = 0$ implies $\bigcap_{P\in\mathcal{P}} P = 0$. A kernel of a irreducible representation from Q on X_p(complex quasi-Banach algebra) which is continuous, can be thought of as a primitive ideal P.Specifically an irreducible representation of Q is define by the following mapping:

$T: Q \to BL(X_p)$ define by $T(a) = L_a$ where $L_a: X_p \to X_p$ define by $L_a(x) = ax$

and $ker(T)= P$ satisfying $\|ax\|_q \leq k\|a\|_q\|x\|_q$ for all $a \in Q, x \in X_P, k > 0$.

It is well known, $\mathcal{S}(\mathcal{D}) = 0$ iff $\mathcal{D}$ is closable, and we can show that $I\,\mathcal{S}(\mathcal{D}) + \mathcal{S}(\mathcal{D})\, I \subset \mathcal{S}(\mathcal{D})$.
Let $\mathcal{P}_c = \{P \in \mathcal{P} : \mathcal{S}(\mathcal{D}) \subset P\}$ and $\mathcal{P}_E = \{P \in \mathcal{P} : \mathcal{S}(\mathcal{D}) \not\subset P\text{P}\}$. Clear $\mathcal{S}(\mathcal{D}) \subset \bigcap_{P\in\mathcal{P}_c} P = P_c$. We will show that $\mathcal{D}$ is closable if $P_c = 0$.

Definition 2.1: [13]

for a sequence $\{P_n\}$ in $\mathcal{P}$, a sliding hump sequence is a sequence $\{b_n\}$ in I if every $n \in \mathbb{N}$ there is $x_n \in X_{P_n}$ so that $b_n \ldots b_1 x_n \neq 0$ as well as $b_{n+1} \ldots b_1 x_n = 0$.

Proposition 2.2: [13]

Let $P \in \mathcal{P}$ and J any ideal of Q (not necessarily closed) such that $J \not\subset P$. Then (**1**) or (**2**) is satisfying:

(**1**) The ideal which is consists of elements $b \in J$ such that $\dim bX_P < \infty$ acts irreducibly on X_P. Accordingly, let $x, y \in X_P$ such that $x \neq 0$, $\exists\, b \in J$ with $\dim bX_P = 1$ and $bx = y$.

(**2**) there are sequences $\{b_n\}$ within J and $\{x_n\}$ *within* X_P fulfilling $b_n \ldots b_1 x_n \neq 0$ as well as $b_{n+1} \ldots b_1 x_n = 0$ for every $n \in \mathbb{N}$.

Proof: see [13]

Now, by the outline prove in [13] and [7] we prove this proposition 2.3:

Proposition 2.3:

Let $\mathcal{D}$ be an essential defined σ-c- derivation on a semi simple complex quasi-Banach algebra. If $\exists$ sliding hump sequence of sequence $\{P_n\}$ in $\mathcal{P}$, then $\exists\ n \in \mathbb{N}$ such that $\mathcal{S}(\mathcal{D}) \subset P_n$. In particular, if $P_n = P\ \forall n \in \mathbb{N}$, then $\mathcal{S}(\mathcal{D}) \subset P$.

Proof:

We assume the sliding hump sequence of $\{P_n\}$ in $\mathcal{P}$ is a sequence $\{b_n\}$ then $\forall n \in \mathbb{N}$, $\exists\, x_n \in X_{P_n}$ such that $b_n \ldots b_1 x_n \neq 0$ and $b_{n+1} \ldots b_1 x_n = 0$. We can certainly assume that $\|b_n\|_q \leq 2^{-n}$ and $\|\sigma\|_q = \|\ x_n\ \|_q = 1\ \forall n \in \mathbb{N}$.

Now, $\mathcal{D}$ is σ-c-derivation then ∀ a, b ∈ $\mathcal{Q}$ we have

(i) $\mathcal{D}(ab) = \mathcal{D}(\,a)\sigma(b) + a\mathcal{D}(b)$.

(ii) $\mathcal{D}(ab) = \mathcal{D}(\,a)b + \sigma(a)\mathcal{D}(b)$.

Case (i), let $\mathcal{D}(ab) = \mathcal{D}(\,a)\sigma(b) + a\mathcal{D}(b)$.

We suggest $\exists\, n \in \mathbb{N}$ and a nonzero $x \in X_{P_n}$ so that the map from I into X_{P_n}, $a \mapsto \mathcal{D}(a)x$ is continuous. If this suggest not true, then every map $a \mapsto \mathcal{D}(a)\sigma(b_n \ldots b_1)x_n$ not continuous into X_{P_n} from I and we can generate a sequence $\{a_n\}$ inductively inside I fulfilling: for every $\beta \geq 1$

$$\beta^4 \|\mathcal{D}(a_n)\sigma(b_n \cdots b_1)x_n\|_q \geq$$

$$n + \beta^4 \|\ \textstyle\sum_{k=1}^{n-1} \mathcal{D}(a_k b_k \cdots b_1)\, x_n\ \|_q + \beta \|\mathcal{D}(\,c_{n+1})\sigma(b_{n+1} \cdots b_1)x_n\|_q \ldots\ldots.(1)$$

And

$$\beta^4 \|a_n\|_q \leq 2^{-n} \min\left\{\left(1 + \|\mathcal{D}(b_k \cdots b_1)\|_q\right)^{-1} : k = 1, \cdots, n\right\} \quad \text{thus}$$

$$\beta^4 \|\ a_n\ \|_q \leq \frac{\mathbf{1}}{\mathbf{2^n(1+\|\mathcal{D}(b_n \cdots b_1)\|_q)}} \leq \frac{\mathbf{1}}{\mathbf{1+\|\mathcal{D}(b_n \cdots b_1)\|_q}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Now, we take the element $c \in \mathcal{Q}$ such that $c = \sum_{n=1}^{\infty} a_n b_n \cdots b_1$, and

we write $c_n = a_n + \sum_{k=n+1}^{\infty} a_k b_k \cdots b_{n+1}$, $\forall n \in \mathbb{N}$

Hence, $c = c_1 b_1$ and $c_n = a_n + c_{n+1} b_{n+1}$.

Now, we will follow the same way of [13] and [7], then we have c and c_n lie in I and

$$c = \sum_{k=1}^{n-1} a_k b_k \cdots b_1 \ + \ a_n b_n \cdots b_1 + \ c_{n+1} b_{n+1} \cdots b_1$$

Therefore,

$$\mathcal{D}(c)x_n = [\, \textstyle\sum_{k=1}^{n-1} \mathcal{D}(a_k b_k \cdots b_1\,) + \mathcal{D}(\, a_n b_n \cdots b_1) + \ \mathcal{D}(c_{n+1} b_{n+1} \cdots b_1)]x_n \quad \ldots\ldots(3)$$

$$= \sum_{k=1}^{n-1} \mathcal{D}(a_k b_k \cdots b_1\,)\, x_n + \mathcal{D}(\, a_n)\sigma(b_n \cdots b_1)\, x_n + a_n \mathcal{D}(b_n \cdots b_1\,)\, x_n +$$

$$\mathcal{D}(\, c_{n+1})\sigma(b_{n+1} \cdots b_1)\, x_n + c_{n+1}\mathcal{D}(b_{n+1} \cdots b_1)\, x_n$$

Now, $\mathcal{Q}$ is a quasi-Banach algebra. Therefore, by (*) $\exists\ \beta \geq 1$,

$$\|\mathcal{D}(c)x_n\|_q \geq \beta^4 \|\mathcal{D}(\, a_n)\sigma(b_n \cdots b_1)x_n\|_q - \beta^4 \left\| \sum_{k=1}^{n-1} \mathcal{D}(a_k b_k \cdots b_1\,)\, x_n \right\|_q$$

$$-\beta^4 \|a_n \mathcal{D}(b_n \cdots b_1\,)x_n\|_q - \beta^4 \|\mathcal{D}(\, c_{n+1})\sigma(b_{n+1} \cdots b_1)x_n\|_q$$

$$-\beta^4 \|c_{n+1}\mathcal{D}(b_{n+1} \cdots b_1)x_n\|_q.$$

Therefore, by (1)

$$\|\mathcal{D}\,(c)x_n\|_q \geq \qquad n - \left[\beta^4 \|a_n \mathcal{D}(b_n \cdots b_1\,)x_n\|_q + \beta^4 \|c_{n+1}\mathcal{D}(b_{n+1} \cdots b_1)x_n\,\|_q\right] \quad \ldots\ldots(4)$$

Now, $\mathcal{Q}$ is quasi-Banach algebra, therefore, $\exists\ \lambda_1, \lambda_2 > 1$

$$\beta^4 \|a_n \mathcal{D}(b_n \cdots b_1\,)x_n\|_q \leq \beta^4 \left(\lambda_1 \|a_n\|_q \|\mathcal{D}(b_n \cdots b_1\,)\|_q\right)$$

Then by (2) we get

$$\beta^4 \|a_n \mathcal{D}(b_n \cdots b_1\,)x_n\|_q \leq \lambda_1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$\because\ \|a_{n+2}\|_q \leq \|a_{n+1}\|_q$. Hence, we can show that

$\|c_{n+1}\|_q \leq 2\beta^4\|a_{n+1}\|_q$ ………………….(6)

Thus, by (2) we get

$\beta^4\|c_{n+1}\mathcal{D}(b_{n+1}\cdots b_1)x_n\|_q \leq 2\lambda_1\beta^4$ ………………….(7)

Put $\lambda = max\{\lambda_1, \lambda_2\}$

Then, by (4), (5) and (7) we get:

$\|\mathcal{D}(c)\|_q \geq \|\mathcal{D}(c)x_n\|_q \geq n - \lambda(1+2\beta^4) \quad \forall\, n \in \mathbb{N}, \beta, \lambda \geq 1, \ \mathcal{D}(c) \in \mathcal{Q}.$

This is a contradiction to that ($\|a\|_q \geq 0 \ \forall\, a \in \mathcal{Q}$), Therefore, we proved our suggestion.

Case (ii), let $\mathcal{D}(ab) = \mathcal{D}(a)b + \sigma(a)\mathcal{D}(b)$

Therefore, by (3) and $b_{n+1}\cdots b_1x_n = 0$ we have that

$$\mathcal{D}(c)x_n = \sum_{k=1}^{n-1}\mathcal{D}(a_kb_k\cdots b_1)\,x_n + \mathcal{D}(a_n)\,b_n\cdots b_1\,x_n + \sigma(a_n)\,\mathcal{D}(b_n\cdots b_1)\,x_n$$

$$+\sigma(c_{n+1})\mathcal{D}(b_{n+1}\cdots b_1)\,x_n.$$

Now, $\mathcal{Q}$ is quasi-Banach algebra, Therefore, by (*) $\exists\, \beta \geq 1$,

$$\|\mathcal{D}(c)x_n\|_q \geq \beta^3\|\mathcal{D}(a_n)b_n\cdots b_1x_n\|_q - \beta^3\left\|\sum_{k=1}^{n-1}\mathcal{D}(a_kb_k\cdots b_1)\,x_n\right\|_q$$

$$-\beta^3\|\sigma(a_n)\mathcal{D}(b_n\cdots b_1)x_n\|_q - \beta^3\|\sigma(c_{n+1})\mathcal{D}(b_{n+1}\cdots b_1)x_n\|_q. \quad \text{……(8)}$$

We suggest $\exists\, n \in \mathbb{N}$ for a nonzero $x \in X_{P_n}$ so that the map from I into X_{P_n}, $a \longmapsto \mathcal{D}(a)x$ is continuous. If this suggest not true, then every map $a \longmapsto \mathcal{D}(a)\, b_n \ldots b_1\, x_n$ not continuous into X_{P_n} from I and we can generate a sequence $\{a_n\}$ inductively inside I fulfilling : for every $\beta \geq 1$

$\beta^3\|\mathcal{D}(a_n)b_n\cdots b_1x_n\|_q \geq n + \beta^3\|\sum_{k=1}^{n-1}\mathcal{D}(a_kb_k\cdots b_1)\,x_n\|_q$ ………………….(9)

And

$\beta^3\|a_n\|_q \leq 2^{-n}\, min\left\{\left(1+\|\mathcal{D}(b_k\cdots b_1)\|_q\right)^{-1} : k = 1, \cdots, n\right\}$ thus

$\beta^3\|a_n\|_q \leq \frac{1}{2^n(1+\|\mathcal{D}(b_n\cdots b_1)\|_q)} \leq \frac{1}{1+\|\mathcal{D}(b_n\cdots b_1)\|_q}$ ………………….(10)

$\therefore$ by (9) and (8) we get:

$\|\mathcal{D}\,(c)x_n\|_q \geq n - \left[\beta^3\|\sigma(a_n)\mathcal{D}(b_n \cdots b_1\,)x_n\|_q + \beta^3\|\sigma(c_{n+1})\mathcal{D}(b_{n+1} \cdots b_1)x_n\,\|_q\right]$ ………………….(11)

Now, let $\exists\ \lambda_1, \lambda_2 > 1$

$\beta^3\|\sigma(a_n)\mathcal{D}(b_n \cdots b_1\,)x_n\|_q \leq \lambda_1$ ………………….(12)

And

$\beta^3\|\sigma(c_{n+1})\mathcal{D}(b_{n+1} \cdots b_1)x_n\,\|_q \leq 2\,\beta^3\lambda_2$ ……………….(13)

Take $\lambda = max\{\lambda_1, \lambda_2\}$

Then, by (11), (12) and (13) we get:

$\|\mathcal{D}\,(c)\|_q \geq \|\mathcal{D}\,(c)x_n\|_q \geq n - \lambda(1 + 2\beta^3) \quad \forall\, n \in \mathbb{N}, \beta, \lambda \geq 1,\ \ \mathcal{D}\,(c) \in \mathcal{Q}.$

This is contradiction to that ($\|a\|_q \geq 0\ \ \forall\, a \in \mathcal{Q}$), therefore we proved our suggest.

Let $m \in \mathbb{N}$ such that the map from I into X_{P_m}, $a \longmapsto \mathcal{D}(a)\ \ x$ is continuous for some $x \in X_{P_m}, x \neq 0$ and assume that X is the set for every $x \in X_{P_m}$ fulfilling this property and it is I - submodule (not equal to zero) of X_{P_m}.

Hence, X = X_{P_m}.

suppose $a \in \mathcal{S}(\,\mathcal{D}\,)$, hence $\exists$ sequence $\{a_n\}$ in $\mathcal{Q}$ such that

$\lim_{n\to\infty} a_n = 0, \lim_{n\to\infty} \mathcal{D}(a_n) = a.$

Hence, for every $x \in X_{P_m}$

$ax = \lim_{n\to\infty} \mathcal{D}(a_n)\,x = 0.$

Therefore, $a \in P_m$. That means $\mathcal{S}(\mathcal{D}) \subset P_m$

■

Proposition 2.4: [13]

suppose $P \in \mathcal{P}, J$ subspace of $\mathcal{Q}$ such that $IJ + JI \subset J$ and $J \not\subset P$. Then $Jx = X_P\ \forall\ x \in X_P\,, x \neq 0$

Proof: see [13]

Proposition 2.5: [13]

suppose $P \in \mathcal{P}, J$ any ideal of $\mathcal{Q}$ (not necessarily closed) $\subset I.\ If\ \exists$ element $b \in J$ such that b, $\sigma\,(\,b\,)\ \notin P$ and dim $bJb < \infty$. Then $\mathcal{S}\,(\,\mathcal{D}\,) \subset P$.

Proof: see [13]

Proposition 2.6:

suppose $\mathcal{D}$ is an essential defined σ-c-derivation on a semisimple complex quasi-Banach algebra, then $\mathcal{D}$ closable.

Proof: by the same way in [13] and applying our proposition 2.3.

■

The quasi-C*-algebra $\mathcal{Q}$ is a quasi-Banach algebra with involution so that $\|xx^*\|_q = \|x\|_q{}^2$ for all $x \in \mathcal{Q}$.

Following Mathieu in [5], $\mathcal{Q}$ is said to be ultraprime if $\exists\ k > 0$ such that $k\|x\|_q\|y\|_q \leq \|M_{x,y}\|_q, \forall x, y \in \mathcal{Q}$, and $M_{x,y}$ is the two -sided - multiplication mapping on $\mathcal{Q}$ thus $M_{x,y}(a) = xay$. By same way of Mathieu in [6] we can prove that $\|x\|_q\|y\|_q = \|M_{x,y}\|_q\ \forall a, b \in \mathcal{Q}$ iff $\mathcal{Q}$ is a prime quasi- C*- algebra, additionally, if k=1, it can be shown that any prime quasi -C*- algebra is an ultraprime quasi - Banach algebra.

Theorem 2.7: (The Main Theorem)

$\mathcal{D}$ is continuous if $\mathcal{D}$ closable σ- c- derivation which is the domain a nonzero ideal I of an ultraprime quasi – Banach algebra $\mathcal{Q}$.

Proof:

because σ is continuous, hence by (theorem 1.1) σ is bounded

Therefore, $\exists$ positive $h \geq 0$,

$\|\sigma(a)\|_q \leq h\| a\|_q$.

Now,

$\mathcal{D}$ is σ- c - derivation, then we have two cases:

for all $a, b \in \mathcal{Q}$,

(i) $\mathcal{D}(ab) = \mathcal{D}(a)\sigma(b) + a\mathcal{D}(b)$.

(ii) $\mathcal{D}(ab) = \mathcal{D}(a)b + \sigma(a)\mathcal{D}(b)$.

Case (i), let $\mathcal{D}(ab) = \mathcal{D} (a) \sigma(b) + a\, \mathcal{D} (b)$.

Fix a $\in I$, with $\|a\|_q = 1$ and we take the following mapping

$\varphi: \mathcal{Q} \to \mathcal{Q}$ characterized by $\varphi(x) = \mathcal{D}(ax)\ \ \forall\ x \in \mathcal{Q}$.

Let $z \in \mathcal{S}(\varphi)$, then $\exists$ sequence $\{q_n\}$ in $\mathcal{Q}$ such that

$\lim_{n\to\infty} q_n = 0, \lim_{n\to\infty} \varphi(q_n) = z.$

This implies that

$\lim_{n\to\infty} a\, q_n = 0.$

Since $\varphi(x) = \mathcal{D}(ax)$.

Therefore,

$\lim_{n\to\infty} \mathcal{D}(aq_n) = z.$

Since $aq_n \in I \ \forall n$, then $z \in \mathcal{S}(\mathcal{D})$.

But $\mathcal{D}$ is closable then $\mathcal{S}(\mathcal{D}) = 0$.

Hence, $z = 0$ which is implies that $\mathcal{S}(\varphi) = 0$. Hence, according to the (theorem 1.2) φ is continuous.

with $\|\varphi\|_q = m$ and $\|x\|_q = 1$, we have $\|\mathcal{D}(ax)\|_q \le m$.

Now, by definition of σ-c-derivation we have

for $b \in I,\ x \in Q$,

$\mathcal{D}(axb) = \mathcal{D}(ax)\,\sigma(b) + ax\mathcal{D}(b)$

Hence, $ax\mathcal{D}(b) = \mathcal{D}(axb) - \mathcal{D}(ax)\,\sigma(b)$.

Thus, $M_{a,\mathcal{D}(b)}(x) = \mathcal{D}(axb) - \mathcal{D}(ax)\,\sigma(b)$

Q quasi-Banach algebra, hence, $\exists\, \beta \ge 1$,

$\left\|M_{a,\mathcal{D}(b)}(x)\right\|_q \le \beta\left(\|\mathcal{D}(axb)\|_q + \|\mathcal{D}(ax)\sigma(b)\|_q\right)$

Because Q quasi algebra, hence $\exists\, \lambda \ge 1$ such that

$$\left\| M_{a,\mathcal{D}(b)}(x)\right\|_q \le \beta\left(\|\mathcal{D}(axb)\|_q + \lambda\|\mathcal{D}(ax)\|_q \,\|\sigma(b)\|_q\right)$$

$$\le \beta m + \beta\lambda m \,\|\sigma(b)\|_q$$

$$\le \beta m + \beta\,\lambda\, m\, h\, \|b\|_q$$

$$\le \beta\lambda m h\|b\|_q + \beta\lambda m h\|b\|_q = 2\beta\lambda m h\|b\|_q \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(14)$$

By taking sup for both sides of (14) we have that

$\left\| M_{a,\mathcal{D}(b)}\right\|_q \le 2\,\beta\,\lambda\, m\, h\|b\|_q$.

Because $\mathcal{Q}$ ultraprime , then ∃ positive constant $k > 0$ such that

$k\|a\|_q \ \|b\|_q \leq \|M_{a,b}\|_q$, $\forall a, b \in \mathcal{Q}$.

Therefore, $k\|a\|_q \ \|\mathcal{D}(b)\|_q \leq \|M_{a,\mathcal{D}(b)}\|_q \leq 2\beta\lambda mh\|b\|_q$.

Hence,

$\|\mathcal{D}(a)\|_q \leq \frac{2\beta\lambda mh}{k}\|b\|_q$.

Therefore by (theorem 1.1) $\mathcal{D}$ continuous.

Case (ii), let $\mathcal{D}(ab) = \mathcal{D}(a)\ b + \sigma(a)\ \mathcal{D}(b)$.

Fix $b \in I$ with $\| b \|_q = 1$ and we take the following mapping

$f: \mathcal{Q} \to \mathcal{Q}$such that $f(x) = \mathcal{D}(xb) \quad \forall\, x \in \mathcal{Q}$.

We will prove that f is continuous.

Assume $y \in \mathcal{S}(f)$, so there is a sequence $\{x_n\}$ in $\mathcal{Q}$ such that

$\lim\limits_{n\to\infty} x_n = 0, \lim\limits_{n\to\infty} f(x_n) = y$.

Since $x_n b \in I \ \forall n$, it follows that

$\lim\limits_{n\to\infty} x_n b = 0, \lim\limits_{n\to\infty} \mathcal{D}(x_n b) = y$.

Thus, $y \in \mathcal{S}(\mathcal{D})$.

However, $\mathcal{D}$ closable

Hence, $\mathcal{S}(\mathcal{D}) = 0$.

This means that $y = 0$.

As a result, $\mathcal{S}(f) = 0$.

Hence, according to the (theorem 1.2) f is continuous.

with $\|f\|_q = M$ and $\|x\|_q = 1$ are given, we have

$\|\mathcal{D}(xb)\|_q \leq M$.

Now, for $a \in I,\ x \in \mathcal{Q}$ we have

$\mathcal{D}(axb) = \mathcal{D}(a)xb + \sigma(a)\mathcal{D}(xb)$

Hence,

$\mathcal{D}(a)xb = \mathcal{D}(axb) - \sigma(a)\mathcal{D}(xb)$

$M_{\mathcal{D}(a),b}(x) = \mathcal{D}(axb) - \sigma(a)\mathcal{D}(xb)$

$\mathcal{Q}$ is quasi-Banach algebra, Therefore, $\exists\, \lambda, \beta \geq 1$,

$$\left\|M_{\mathcal{D}(a),b}(x)\right\|_q \leq \beta(\|\mathcal{D}(axb)\|_q + \lambda\|\sigma(a)\|_q\|\mathcal{D}(xb)\|_q)$$

$$\leq \beta M + \beta\lambda M\|\sigma(a)\|_q$$

$$\leq \beta M + \beta\lambda Mh\|a\|_q$$

$$\leq \beta\lambda Mh\|a\|_q + \beta\lambda Mh\|a\|_q = 2\beta\lambda Mh\|a\|_q \qquad \ldots\ldots\ldots\ldots\ldots\ldots(15)$$

By taking sup for both sides of (15) we have that:

$\left\|M_{\mathcal{D}(a),b}\right\|_q \leq 2\beta\lambda Mh \|a\|_q$.

But $\mathcal{Q}$ is ultraprime ,

Then we get $\|\mathcal{D}(a)\|_q \leq \frac{2\beta\lambda Mh}{k}\|a\|_q$.

This shows that $\mathcal{D}$ is continuous.

■

With the application of proposition 2.6 and theorem 2.7 we are able to prove the following outcome:

Corollary 2.8:

on a nonzero ideal of a prime quasi -C *- algebra, all essentially defined σ-c - Derivations are continuous.

Corollary 2.9:

on a nonzero ideal of a prime quasi -C *- algebra, all essentially defined derivations are continuous.

Proof: Use Corollary 2.8. We take σidentity map.

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