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On The Periodicity of Alternating Links

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Abstract

Purpose: This study aims to provide empirical and theoretical support for the Jakuba and Naik conjecture, which posits that the odd prime period of a periodic alternating link divides its crossing number. It explores the conjecture through polynomial invariants and periodic graph theory.

Methods: A mixed-methods approach is employed, combining computational analysis of alternating knots (up to 15 crossings) with theoretical investigation of the Dubravonic version of the Kauffman polynomial. The study extends the analysis to knotted trivalent graphs, using the substitution $z = \omega(a - a^{-1})$ to simplify periodicity criteria.

Results: The computational data from alternating knots supports the Jakuba and Naik conjecture. Analysis of the Kauffman polynomial reveals degree properties and coefficients that further underpin the conjecture's resolution. Applying the substitution $z = \omega(a - a^{-1})$ to knotted trivalent graphs simplifies periodicity criteria.

Conclusion: This study offers empirical and theoretical evidence supporting the Jakuba and Naik conjecture. By integrating polynomial invariants, combinatorial knot theory, and periodic graph structures, this study provides new insights into link periodicity and advances the understanding of relationships within knot theory.

Keywords: Knot Theory, Alternating Links, Jakuba-Naik Conjecture, Polynomial Invariants, Kauffman Polynomial, Periodic Links, Knotted Trivalent Graphs, Link Periodicity.

Introduction

Knot Theory, an active area of study in topology, is interested in the mathematical idealization of knots (Picione, 2023). Mathematical knots differ from ordinary knots one creates in a piece of rope: mathematical knots are closed curves contained in three-dimensional space, taken as equivalent if one can be deformed into the other by continuous deformation without cutting (Constant, 2021). Knot Theory is fundamentally concerned with the classification and differentiation of these knots based on invariants that don't change when they're deformed (Grant, 2023). These invariants usually presented as polynomials or algebraic objects, give a strong framework for describing the topological properties of knots and links (Sossinsky, 2023). Aside from its beauty as a theory, Knot Theory has surprisingly been applied to a wide variety of fields, including molecular biology, where it helps explain the structure of DNA as its molecules frequently form knotted and linked configurations during vital biological processes such as replication and recombination; chemistry, where it assists in the description of intricate molecular architectures; and even theoretical physics, especially in the research on string theory (Levitt et al., 2022).

Within Knot Theory, a specific class of knots known as alternating links is of particular interest (Adams et al., 2021). Alternating links are those that possess a diagram with crossings that alternate between over and under as one travels along the link (Crowell, 1959; Menasco & Thistlethwaite, 1993). These links have unique properties related to their polynomial invariants and topological characteristics, making them essential objects of study in the broader field.

One of the key problems in knot theory is developing criteria to determine whether links are periodic or not (Przytycki et al., 2024). Following previous research (Aboufattoum et al., 2018), this study explored the Jakuba and Naik conjecture. New developments have returned focus to this conjecture, with notable advances through analytic and algebraic techniques. The previous research by Stoimenow (2020) established a framework for proving this conjecture with an accent on the relationship between link periodicity and polynomial invariants

(Stoimenow, 2020). The current study enhances the evidence toward proving the validity of the conjecture through combining evidence from combinatorial knot theory, polynomial invariants, and periodic graph structures.

Section 1 consolidates empirical evidence for the Jakuba and Naik conjecture using computational data from alternating knots with up to 15 crossings. Sections 2 and 3 revisit the Dubravonic version of the Kauffman Polynomial $F_L(a, z)$, analyzing its degree properties and coefficients for alternating links, while advancing arguments that underpin the conjecture's resolution. Section 4 extends this analysis to the three-variable Kauffman polynomial for knotted trivalent graphs, introducing the substitution $z = \omega(a - a^{-1})$ that simplifies periodicity criteria. Section 5 examines periodic knotted trivalent plane graphs, deriving properties that can be generalized to links. Finally, Section 6 synthesizes these results to propose a unified solution for periodic prime alternating links.

The primary aims of this study are to provide empirical and theoretical support for the Jakuba and Naik conjecture through polynomial invariants and periodic graph theory, analyze the Kauffman polynomial $\Lambda_D(a, z)$for alternating links with a focus on degree-breadth relationships tied to crossing numbers, and introduce substitutions and algebraic techniques to reduce the conjecture's complexity, particularly via $z = \omega(a - a^{-1})$. Additional objective includes the extension of periodicity requirements to knotted trivalent graphs, drawing comparisons with alternating links, and coalescing such methods to give a proof methodology applicable to alternating prime knots that have few crossings. Through pursuing these objectives, this study solved one of the persistent issues in knot periodicity as well as demonstrated the applicability of polynomial invariants in combinatorial topology.

2 Methodology

2.1 Study Design

This study utilizes a mixed-methods strategy where computational analysis is used alongside theoretical investigation to study the Jakuba and Naik conjecture. The first part entails a scan of previous literature and computational information regarding knots and links, especially alternating links with up to 15 crossings. This is then complemented by a detailed study of the Dubravonic variant of the Kauffman polynomial, including its characteristics and response to certain transformations. The study then proceeds to the analysis of knotted trivalent graphs and their related polynomial invariants.

2.1.1Section 1. Empirical Evidence for the Jakuba and Naik Conjecture

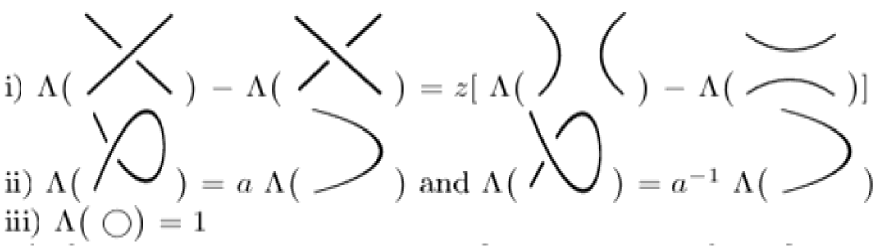
Previous study by Jabuka & Naik, (2016), found that from all knots up to 15 crossings, the knots 3_1 , 8_{19} , 9_1, 9_{35}, 9_{40}, 9_{41} , 9_{47} , 9_{49} , 12a503 , 12a561 , 12a615 , 12a1022 , 12a1202 are 3−periodic, the knots 5_2 , 10_{123} , 10_{124} , 15a64035 , 15a84903 , 15a85262 , 15a85263 are 5−periodic, the knots 7_2 , 14a19470 are 7−periodic, and the knot 11a367 is 11−periodic (Jabuka & Naik, 2016). All these examples support the Jakuba and Naik conjecture, even the knot 819 which can be incorrectly understood that it violates the conjecture, indeed, this knot is not alternating. So, it supports the conjecture.

2.1.2 Section 2. Two-variable Kauffman Polynomial

In Kauffman, (1990), Kauffman introduced a polynomial, which is called the Dubravonic version of the Kauffman polynomial as in the following definition (Kauffman, 1990).

Definition 1.

a) For an unoriented link diagram D, the polynomial $\Lambda_D(a, z)$can be defined so that it satisfies:



iv) If D_1 and D_2 are related by a finite sequence of the first and the second Reidemeister moves then $\Lambda_{D_1} = \Lambda_{D_2}$.

b) For any oriented link L with a diagram D, the Kauffman polynomial $F_L(a,z)$ is defined as $F_L(a,z) = a^{-w(D)}\Lambda_D(a,z)$, where w(D) is the writhe of D.

We put the following table to check some claims about $\Lambda(a,z)$), or alternatively $F(a,z)$for some chosen few crossing knots:

Table 1. Kauffman polynomial of some knots

The knot	Its Kauffman polynomial
3_1	$a^5z + a^4z^2 - a^4 + a^3z + a^2z^2 - 2a^2$
4_1	$a^2z^2 + a^{-2}z^2 - a^2 - a^{-2} + az^3 + a^{-1}z^3 - az - a^{-1}z + 2z^2 - 1$
5_2	$a^7z^3 - 2a^7z + a^6z^4 - 2a^6z^2 + a^6 + 2a^5z^3 - 2a^5z + a^4z^4 - a^4z^2 + a^4 +$ $a^3z^3 + a^2z^2 - a^2$
8_{19}	$a^{-6}z^6 + a^{-8}z^6 + a^{-7}z^5 + a^{-9}z^5 - 6a^{-6}z^4 - 6a^{-8}z^4 - 5a^{-7}z^3 - 5a^{-9}z^3$ $+10a^{-6}z^2 + 10a^{-8}z^2 + 5a^{-7}b + 5a^{-9}z - 5a^{-6} - 5a^{-8} - a^{-10}$
9_1	$a^{17}z + a^{16}z^2 + a^{15}z^3 - a^{15}z + a^{14}z^4 - 2a^{14}z^2 + a^{13}z^5 - 3a^{13}z^3 + a^{13}z +$ $a^{12}z^6 - 4a^{12}z^4 + 3a^{12}z^2 + a^{11}z^7 - 5a^{11}z^5 + 6a^{11}z^3 - a^{11}z + a^{10}z^8 - 7a^{10}z^6 +$ $16a^{10}z^4 - 14a^{10}z^2 + 4a^{10} + a^9z^7 - 6a^9z^5 + 10a^9z^3 - 4a^9z +$ $a^8z^8 - 8a^8z^6 + 21a^8z^4 - 20a^8z^2 + 5a^8$

2.2 Section 3. Kauffman Polynomial of Alternating Links

In Yokota, (1995) connected any reduced connected alternating link diagram with its Tait graph to introduce the following property (Yokota, 1995).

Proposition 3.1. If D is a reduced connected alternating diagram of a link L, the breadth of $F_L(a,z)$ with respect to a coincides with the crossing number of L.

Thistlethwaite treated the coefficients of $\Lambda_D(a,z)$ and their relations with the powers of a and z (Thistlethwaite, 1988).

Theorem 3.2. Let D be an n-crossing link diagram, which is connected sum of link diagrams $D_1, D_2, \ldots, D_k$. *Let* $\Lambda_D(a,z) = \sum u_{ij}a^iz^j$, *and let* $b_1, b_2, \ldots, b_k$ be the lengths of the longest bridges of $D_1, D_2, \ldots, D_k$, respectively. Then, for each non-zero coefficients u_{ij}, $|i| + j \leq n$, $and j \leq n - (b_1 + b_2 + \cdots + b_k)$. In other words, the non-zero coefficients of $\Lambda_D(a,z)$ lie in the region in the ij-plane illustrated in Figure 1.

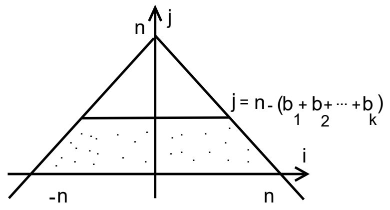


Figure 1. The ij-plane region

Now it is known that we always can normalize $\Lambda_D(a,z)$ $as \sum_{j\geq 0,i} u_{ij} a^i z^j$. Some graphical approach can be used to show that the extreme values of the target function that is the degree of $\Lambda_D(a,z)$ as a polynomial of two variables a and z on the bounded region shown in Figure 1 occur on the line segment of the line $i + j = n$ and the corner point $(-n, 0)$. Then the highest degree is n and the lowest degree is $-n$. Is this always the case? Or is this the case even if it is allowed to z to take negative powers?

In Thistlethwaite (1988), it is shown that for adequate links and therefore for alternating links the outermost coefficients u_{ij}, $i + j = n$ $and -i + j = n$ are not zeros and n is the crossing number of the link L.

2.2 Section 4. Kauffman Polynomial of Knotted Trivalent Graphs

Definition 4.1. A knotted trivalent graph is a graph embedded in $\mathbb{R}^3$ so that its vertices are trivalent, and the graph is knotted as in Figure 2.

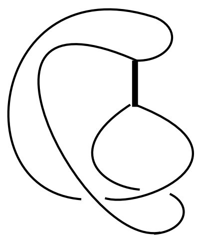


Figure 2. A knotted trivalent graph

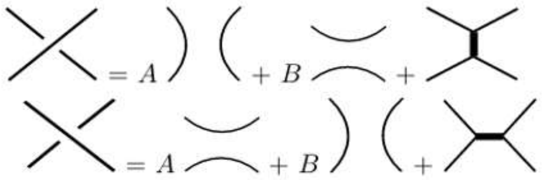
The last definition assumes that the knotted trivalent graph has some vertices, some edges and some crossings. Thus, the link graph is a specialization of the knotted trivalent graph.

Definition 4.2. A knotted trivalent graph is p-periodic if it has a planar projection G such that the rotation of $\mathbb{R}^3$ by an angle $\frac{2\pi}{p}$ leaves G fixed set wise. G and the axis of rotation are assumed to be disjoint.

Definition 4.3. The quotient of a periodic knotted trivalent graph is the closure of the copy of the sub graph that has been repeated so that no extra crossings are introduced.

In Caprau & Tipton, (2015), the authors constructed a three-variable rational function, which is an invariant of knotted unoriented trivalent plane graphs whose restriction to links yields the Dubravonic version of the Kauffman polynomial (Caprau & Tipton, 2015). The following is a quick overview of that construction:

Let Y be a planar projection of a knotted trivalent graph and associate to each crossing of Y a formal linear combination of planar trivalent graphs as in the following two equations, where A and B are commuting variables.

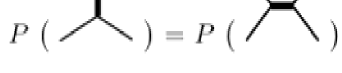


(4.1)

(4.2)

After finitely many steps of applying Equation 4.1 or Equation 4.2, we obtain the states of Y, denoted by Γ. A polynomial $P(\Gamma) \in \mathbb{Z}[a^{\pm 1}, A, B, (A-B)^{\pm 1}]$ is assigned to each state Γ so that it takes the value 1 for the unknot and satisfies six more relations, we write the first three:

$$P(\Gamma \cup \bigcirc) = \alpha\, P(\Gamma) \tag{4.3}$$



(4.4)

(4.5),

where $\alpha = \frac{a - a^{-1}}{A-B} + 1,\ \beta = \frac{Aa^{-1} - Ba}{A-B} - A - B$.

Now after resolving all crossings of the knotted trivalent plane graph, we express it as finite linear combination of the states whose coefficients are monomials in the commuting variables A and B. We associate to the planar knotted trivalent graph Y the Kauffman polynomial $\mathbb{P}_Y = \sum_\Gamma A^m B^n P(\Gamma)$, where m, n is the number of times of A-resolution, B-resolution respectively in obtaining the graph state Γ.

2.2 Section 5. Kauffman Polynomial of Periodic Knotted Trivalent Graphs

Knotted trivalent graphs generalize links by incorporating vertices alongside edges and crossings. A three-variable rational function invariant was constructed for these graphs, which reduces to the Dubravonic version of the Kauffman polynomial when restricted to links.

Theorem 5.1. For any p-periodic knotted trivalent plane graph Y, we have $Y,\ we\ have\ \mathbb{P}_Y(a, A, B) = (\mathbb{P}_{Y_*}(a, A, B))^p\ mod\ (p, \frac{a-a^{-1}}{A-B} - \omega)$, where Y∗ is the quotient knotted trivalent graph of Y and ω is a primitive $(p-1)$-th root of unity, p is an odd prime number.

In this study we would like to clarify why the last condition is put in the statement of the theorem. Let us consider a case of an unlink of two components in its periodic form. The unlink itself is a quotient link of the part)). Since the Kauffman polynomial of an unlink of two components is $\frac{a-a^{-1}}{A-B} + 1$, and since the theorem proposed that $P_Y = (P_{Y_*})^p\ then\ (\frac{a-a^{-1}}{A-B} + 1)^p = \frac{a-a^{-1}}{A-B} + 1,\ or\ (\frac{a-a^{-1}}{A-B})^p + 1 = \frac{a-a^{-1}}{A-B} + 1\ mod\ p.$

Then $\frac{a-a^{-1}}{A-B} = \sqrt[n-1]{1}\ mod\ p$. So, the condition $mod\ \left(p, \frac{a-a^{-1}}{A-B} - \omega\right)$ is chosen.

Furthermore, we will introduce a very useful substitution, which is completely obtained from the last condition that is related only to periodic links. Since during the last discussion we have

$\frac{a-a^{-1}}{z} = \sqrt[n-1]{1} = \omega \Rightarrow a - a^{-1} = z\omega, or\ z = \omega(a - a^{-1}).$

Results

Now this study supports the Jakuba and Naik conjecture again by the aid of Table 2. Here, each p–periodic prime knot of few crossings up to 10 crossings known so far with terms of its $F_L(a, z)$, where the highest and the lowest powers of the variable a appear after applying $z = \omega(a - a^{-1})$.

Table 2. Specific terms of Kauffman polynomials

Prime knot	Period	Alternating?	The terms
3_1	3	yes	a^4z^2, a^2z^2
5_2	5	yes	a^6z^4, a^4z^4
7_2	7	yes	a^8z^6, a^6z^6
9_1	3	yes	$a^{10}z^8$, a^8z^8
9_{35}	3	yes	$a^{10}z^8$, a^8z^8
9_{40}	3	yes	$4a^4z^8$, $4a^2z^8$
9_{41}	3	yes	$2a^4z^8$, $2a^2z^8$
9_{47}	3	no	not pass
9_{49}	3	no	not pass
10_{123}	5	yes	$4az^9$, $4a^{-1}z^9$
10_{124}	5	no	not pass

3.1 Interpretations Based on Reduced Alternating Diagrams

Returning to the nature of computing $\Lambda_D(a, z)$ specially of reduced alternating link diagrams and with the aid of all computed Kauffman polynomials, see Table 2. It is reasonable to claim that the chance of the variable z to appear is greater than the variable a, which means the highest power of z is greater than or equal to the highest power of a and since the breadth of $\Lambda_a(a, z)$equals the crossing number, which means the lowest power of a is controlled by the highest power of a and the crossing number. This implies that applying $z = \omega(a - a^{-1})$ will guarantee that either the highest degree or the lowest degree of a in $\Lambda(a, z)$, or alternatively of $F(a, z)$ occurs at some term where z takes its highest power, which is $c - 1$ as it is already shown and since the coefficient of $z^{c-1}\ in\ \Lambda(a, z)\ is\ k(a + a^{-1})$, for some non-negative integer k in the case of the prime reduced alternating diagrams. Then the breadth of $\Lambda(a, z),\ or\ F(a, z)\ is\ (1 + c - 1) - (-1 - c + 1) = 2c.$

3.2 Section 3. Analysis

In the case of the reduced connected link L it can be shown that $deg_z F(a, z) \leq c - d$, where c is the crossing number of L and d is the longest bridge of L. Automatically, if L is a reduced connected alternating link then the maximum degree of z in $F_z(a,z)$ $is\ c-1$, as shown in Table 1.

To prove the permanent presence of the term z^{c-1} $in\ \Lambda_z(a,z)$, as it is introduced in Thistlethwaite, (1988), from $\Lambda_D = \sum u_{ij} a^i z^j$, if we consider the following truncated polynomial so that $|i| + j = n$ or $j \geq n-2$, $\bar{\Lambda}_D(a,z) = \sum_{i=1,2}(u_{-i,c-i}a^{-i} + u_{i,c-i}\, a^i)\, z^{c-1}$. By connecting $\Lambda_D(a,z)$ with the Tutte polynomial of its Tait graph $\psi_G(x,y)$, it is shown that each of the following coefficients is not zero; $u_{-1,u-1}, u_{1,u-1}, u_{-2,n-2}, u_{0,n-2}, and\ u_{2,n-2}$, and consequently if L admitting a prime reduced alternating diagram D and c is its crossing number then the coefficient of z^{c-1} $in\ \Lambda_D(a,z)$ $is\, k(a+a^{-1})$, which implies that the coefficient of z^{c-1} $in\, F_z(a,z)$ $is\ ka^{-w(D)}(a+a^{-1})$, k is non negative integer. Therefore, the term z^{c-1} cannot be canceled in $\Lambda_z(a,z)$ $and\, F_z(a,z)$. We used Barnatan and Morrison (The knot Atlas) to check, for example, for 3_1, 5_2, 7_2, and 9_{35}, $k=1$. For 9_{40}, 10_{123}, $k=4$. For 9_{41}, $k=2$.

3.3 Section 4. Analysis

The polynomial $\mathbb{P}_Y$ restricted to links gives the Kauffman polynomial $\Lambda_D(a,z)$ using the following formula that first appeared in Caprau and Tipton, (2015):

$$\Lambda_D(a,z) = \Lambda_D(a, A-B) = \mathbb{P}_D(a,A,B)$$

It is not hard to show that if L is an unlink of n components then

$$\mathbb{P}_L(a,A,B) = (\frac{a-a^{-1}}{A-B} + 1)^{n-1}$$

Proposition 4.1. For any knotted trivalent plane graph Y, we have $\mathbb{P}_Y(a,A,B) \in \mathbb{Z}[\omega, a^{\pm}, A, B, (A-B)^{\pm}]$. In particular $\Lambda(a,A,B) \in \mathbb{Z}[\omega, a^{\pm}, A, B, (A-B)^{\pm}]$, where ω is a primitive $(p-1)$-th root of unity, p is an odd prime number.

3.4 Section 5. Analysis

Corollary 5.1. For any p-periodic link L of periodic diagram D, we have $\Lambda_D\big(a, \omega(a-a^{-1})\big) = (\Lambda_{D_*}\big(a, \omega(a-a^{-1})\big))^p\ mod\ p$. This implies $F_L\big(a, \omega(a-a^{-1})\big) = (F_{L_*}\big(a, \omega(a-a^{-1})\big))^p\ mod\ p$, where L_* is the quotient link with quotient diagram D_*.

Theorem 5.2. If L is a p-periodic link of crossing number c, then p divides the breadth of $F_L\big(a, \omega(a-a^{-1})\big)\ mod\ p$.

Proof. We assume that D is a p-periodic diagram of the link L. According to Corollary 5.1, all terms in $\Lambda_D\big(a, \omega(a-a^{-1})\big)\ mod\ p$ have powers multiple of p. Since the writhe of D is the sum of all signs of all crossings of D, i.e., writhe$(D) = \#(+ve\ signs) - \#(-ve\ signs)$, and since each part of the right-hand side of the last equation is a multiple of p, then the writhe of D is also a multiple of p. This implies that all terms of $F_L(a, \omega(a-a^{-1}))$ also have powers multiple of p. Then the breadth of $F_L\big(a, \omega(a-a^{-1})\big) mod\ p$ is a multiple of p. Therefore, p divides the breadth of $F_L(a, \omega(a-a^{-1}))$.

Discussion

This study explored the complex world of Knot Theory, specifically the Jakuba and Naik conjecture, which states that the odd prime period of a periodic alternating link divides its crossing number (Boden & Karimi, 2022). The study seeks to support this conjecture by using a multi-faceted approach, with the aid of polynomial invariants and periodic graph theory. It also aims to examine the Kauffman polynomial for alternating links, focusing on degree breadth and crossing numbers' relationships, and to make the conjecture simpler through careful algebraic substitutions, especially $z = \omega(a-a^{-1})$. Other goals are to generalize periodicity conditions to knotted trivalent graphs and compare them with alternating links, and to combine these techniques to establish a proof method applicable to alternating prime knots with minimal crossings.

The findings of this study are strong evidence supporting the Jakuba and Naik conjecture. By computations on alternating knots with at most 15 crossings, the evidence from empirical testing was found to agree with the conjecture. Also, from the computations of the Dubravonic variant of the Kauffman Polynomial $F_L(a,z)$, degree properties and coefficients have been found to support the validity of the conjecture. The study applies this analysis to the three-variable Kauffman polynomial for knotted trivalent graphs, with the substitution $z = \omega(a-a^{-1})$ simplifying periodicity conditions and offering a unified solution for periodic prime alternating links. These findings suggest a close connection between link periodicity and polynomial invariants, thus deepening the knowledge of Knot Theory.

Previous studies like Barkataki & Panagiotou, (2024), Borodzik & Politarczyk, (2021) and Chbili, (2022) have provided the foundation for investigating the Jakuba and Naik conjecture, specifically highlighting the relationship between link periodicity and polynomial invariants (Barkataki & Panagiotou, 2024; Borodzik & Politarczyk, 2021; Chbili, 2022). There are still gaps in literature. Some research has not adequately discussed the intricacies of using polynomial invariants to ascertain link periodicity, particularly for knotted trivalent graphs. Past studies have also not thoroughly investigated the consequences of certain algebraic substitutions, like $z = \omega(a - a^{-1})$, in streamlining the resolution of the conjecture.

This study addresses these gaps by presenting a detailed examination of the Kauffman polynomial and its variations, providing a more nuanced understanding of their properties and uses in knot theory. By expanding the examination to knotted trivalent graphs and implementing strategic algebraic substitutions, this research provides new tools and insights for exploring link periodicity. Unlike earlier research, this study combined empirical findings with theoretical approaches to recommend an inclusive solution for periodic prime alternating links, which pushes the subject further than the limitations of what is currently known.

One of the strongest aspects of this study is that it employed a mixed-methods design, combining computational study with theoretical inquiry. This provides strong testing of the Jakuba and Naik conjecture by anchoring theoretical results in empirical data. The study's investigation into the Kauffman polynomial and its variants, and algebraic substitutions, offers a deep and well-rounded understanding of link periodicity. In addition, the extension of periodicity conditions to knotted trivalent graphs expands the applications of knot theory, providing new areas for research and utilization (von Worlik, 2024).

The computational approach is only up to alternating knots with 15 crossings, and it might not be comprehensive enough to fully account for the larger knots' complexity (Vidulis et al., 2023). While this study strongly supports the Jakuba and Naik conjecture, a formal proof is still missing and needs further research. Lastly, the attention to certain polynomial invariants and algebraic substitutions might ignore other possible avenues to explore link periodicity.

This study makes a substantial contribution to the study of knot theory through the empirical and theoretical validation of the Jakuba and Naik conjecture. By means of an in-depth examination of polynomial invariants, knotted trivalent graphs, and algebraic substitutions, this study filled in gaps in current literature and presents new findings on link periodicity. Although there are limitations, the virtues of this current study open up the door to future research and further insight into the complex interplay within Knot Theory.

Conclusion

This study has successfully integrated computational analysis, polynomial invariants, and periodic graph structures to provide substantial evidence supporting the Jakuba and Naik conjecture. By extending the analysis to knotted trivalent graphs and employing the algebraic substitution $z = \omega(a - a^{-1})$, this study has not only addressed gaps in previous studies but has also proposed a unified approach for understanding periodic prime alternating links. The findings highlight the significant role of the Kauffman polynomial in determining link periodicity, offering new perspectives and tools for future investigations in Knot Theory and related fields.

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