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Perturbation Effects for a Singular Elliptic Problem with Lack of Compactness and Critical Exponent

Vicenţiu D. Rădulescu
Department of Mathematics, Faculty of Sciences, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
vincentiu.radulescu@imar.ro

Ionela-Loredana Stăncuţ
Department of Mathematics, University of Craiova, 200585 Craiova, Romania
stancutloredana@yahoo.com

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We study the existence of multiple weak entire solutions of the nonlinear elliptic equation

$$-\Delta u = V(x)|x|^{\alpha}|u|^{\frac{2(\alpha+2)}{N-2}}u + \lambda g(x) \quad \text{in } \mathbb{R}^N \ (N \geq 3),$$

where $V(x)$ is a positive potential, $\alpha \in (-2, 0)$, λ is a positive parameter, and g belongs to an appropriate weighted Sobolev space. We are concerned with the perturbation effects of the potential g and we establish the existence of some $\lambda_* > 0$ such that our problem has two solutions for all $\lambda \in (0, \lambda_*)$, hence for small perturbations of the right-hand side. A first solution is a local minimum near the origin, while the second solution is obtained as a mountain pass. The proof combines the Ekeland variational principle, the mountain pass theorem without the Palais-Smale condition, and a weighted version of the Brezis-Lieb lemma.

Keywords: Singular elliptic equation, Caffarelli-Kohn-Nirenberg inequality, perturbation, critical point, weighted Sobolev space.

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1. Introduction

Consider the nonlinear elliptic equation

$$-\Delta u = V(x)|x|^{\alpha}|u|^{\frac{2(\alpha+2)}{N-2}}u + \lambda g(x) \quad \text{in } \mathbb{R}^N, \tag{1}$$

where $N \geq 3$, $\alpha \in (-2, 0)$, λ is a positive parameter, g belongs to an appropriate weighted Sobolev space, and V is a positive potential.

The function V is supposed to satisfy the following hypotheses:

(V1) $V \in L^{\infty}(\mathbb{R}^N)$,

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(V2) $\operatorname*{ess\,lim}_{|x|\to 0} V(x) = \operatorname*{ess\,lim}_{|x|\to\infty} V(x) = V_0 \in (0,\infty)$ and $V(x) \geq V_0$ a.a. $x \in \mathbb{R}^N$,

(V3) $\operatorname{meas}(\{x \in \mathbb{R}^N : V(x) > V_0\}) > 0$.

We denote by $p_\alpha = \frac{2(N+\alpha)}{N-2} > 2$, and notice that $p_\alpha - 2 = \frac{2(\alpha+2)}{N-2}$. Also, p_α is assumed to fulfill $\alpha = -bp_\alpha$, with $\alpha \in (-2,0)$ and $b \in (0,1)$.

We now recall the following inequality due to Caffarelli, Kohn and Nirenberg [6]:

$$\left(\int_{\mathbb{R}^N} |x|^{-bp}|u|^p dx\right)^{\frac{1}{p}} \leq C(a,b)\left(\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2 dx\right)^{\frac{1}{2}}, \tag{2}$$

where, for $N \geq 3$, parameters a, b, and p satisfy the assumptions

$$-\infty < a < \frac{N-2}{2}, \quad a < b < a+1 \quad \text{and} \quad p = \frac{2N}{N-2(a+1-b)}\,. \tag{3}$$

The Caffarelli-Kohn-Nirenberg inequality (2) also holds for $a = b$, but in this case, the best Sobolev constant $C(a,b)$ is never reached as it is shown in [8]. The Sobolev inequality can be obtained taking $a = b = 0$ in (2), as well as the Hardy inequality derives by taking $a = 0$ and $b = 1$ in (2) (for more details, we refer to [4], [11], [18]).

A straightforward computation shows that $p_\alpha = \frac{2N}{N-2(1-b)}$, with $0 < b < 1$ such that $\alpha = -bp_\alpha$, meaning that relation (3) is fulfilled for $a = 0$ and $b \in (0,1)$. In other words, problem (1) contains the critical Caffarelli-Kohn-Nirenberg-type exponent $p_\alpha = \frac{2(N+\alpha)}{N-2}$. Thus, in relationship with our problem, inequality (2) can be written as

$$\left(\int_{\mathbb{R}^N} |x|^{-bp_\alpha}|u|^{p_\alpha} dx\right)^{\frac{1}{p_\alpha}} \leq C_b\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{\frac{1}{2}} \tag{4}$$

or equivalently

$$\left(\int_{\mathbb{R}^N} |x|^{\alpha}|u|^{\frac{2(N+\alpha)}{N-2}} dx\right)^{\frac{N-2}{N+\alpha}} \leq C_\alpha \int_{\mathbb{R}^N} |\nabla u|^2 dx\,, \tag{5}$$

for all $u \in C_0^\infty(\mathbb{R}^N)$, where α, p_α, and b are as above.

Since $p_\alpha - 1 = \frac{N+2+2\alpha}{N-2}$, our problem can also be seen as a perturbed version of the nonlinear critical elliptic equation

$$-\Delta u = (N+\alpha)(N-2)|x|^\alpha u^{\frac{N+2+2\alpha}{N-2}} \quad \text{in } \mathbb{R}^N. \tag{6}$$

Problem (6) was treated in [15] in the framework of positive solutions, with $\alpha > 0$ and $N \geq 3$.

We refer to [10], where it is considered a degenerate perturbed semilinear elliptic problem with subcritical Sobolev exponent on the whole space. In [17] it is studied a singular elliptic equation involving nonautonomous perturbation and with the critical exponent $2^*_\alpha = \frac{2N}{N-2+\alpha}$, for $\alpha \in (0,2)$, in unbounded domain. We also refer to [2], where it is analyzed the critical degenerate and singular quasilinear elliptic problem

$$-\text{div}\left[|x|^{-ap}|\nabla u|^{p-2}\nabla u\right] + \lambda|u|^{-(a+1)p}|u|^{p-2}u = |x|^{-bq}|u|^{q-2}u + f \quad \text{in } \mathbb{R}^N,$$

under certain conditions on p, q, a, and b such that the Caffarelli-Kohn-Nirenberg inequality [6] holds. The authors first consider the existence of positive solutions when the perturbing term f is identically zero, and then they prove the existence of at least two positive solutions if $\lambda = 0$.

Existence and nonexistence, as well as qualitative properties, of nontrivial nonnegative solutions for elliptic equations involving the Laplace operator and singular coefficients are studied in many papers; regarding bounded domains, we cite the papers [1], [7], [9], [13], [19], while for unbounded domains we refer to [14], [16], [19].

The purpose of this work is to analyze the existence and the multiplicity of nontrivial weak solutions of problem (1). More precisely, we prove the existence of at least two weak solutions of problem (1), provided that $\lambda > 0$ is sufficiently small, and $g \neq 0$ belongs to an appropriate space, in a framework with lack of compactness in which the Palais-Smale condition is not assumed. To this end, we use the Ekeland variational principle [12], the mountain pass theorem without the Palais-Smale condition (see [5]), and a weighted version of Brezis-Lieb lemma [3].

2. The main result

Consider the function space $H^1(\mathbb{R}^N)$, which is a Hilbert space with respect to the inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v \, dx, \quad \forall u, v \in H^1(\mathbb{R}^N).$$

The space $H^1(\mathbb{R}^N)$ is defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{\frac{1}{2}}.$$

In particular, the inequality (5) holds for all $u \in H^1(\mathbb{R}^N)$.

Let $\|\cdot\|_{-1}$ denote the norm in the dual space $H^{-1}(\mathbb{R}^N)$ of $H^1(\mathbb{R}^N)$. Throughout this paper we consider $g \in H^{-1}(\mathbb{R}^N) \setminus \{0\}$.

We say that $u \in H^1(\mathbb{R}^N)$ is a *weak solution* of problem (1) if

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla v \, dx - \int_{\mathbb{R}^N} V(x)|x|^\alpha |u|^{\frac{2(\alpha+2)}{N-2}} uv \, dx - \lambda \int_{\mathbb{R}^N} g(x) v \, dx = 0,$$

for all $v \in C_0^\infty(\mathbb{R}^N)$.

Let $I_\lambda : H^1(\mathbb{R}^N) \to \mathbb{R}$ be the *energy functional* defined by

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{N-2}{2(N+\alpha)}\int_{\mathbb{R}^N} V(x)|x|^\alpha |u|^{\frac{2(N+\alpha)}{N-2}} dx - \lambda \int_{\mathbb{R}^N} g(x) u \, dx.$$

Thus, the weak solutions of problem (1) coincide with the *critical points* of I_λ.

We also consider the functionals $I, J : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{N-2}{2(N+\alpha)}\int_{\mathbb{R}^N} V(x)|x|^\alpha |u|^{\frac{2(N+\alpha)}{N-2}} dx,$$

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{N-2}{2(N+\alpha)}\int_{\mathbb{R}^N} V_0 |x|^\alpha |u|^{\frac{2(N+\alpha)}{N-2}} dx.$$

Using the Caffarelli-Kohn-Nirenberg inequality (5) in relationship with hypotheses (V1) and (V2), we deduce that the functionals I_λ, I, and J are well defined and of class $C^1(H^1(\mathbb{R}^N), \mathbb{R})$. Moreover, for all $v \in H^1(\mathbb{R}^N)$

$$\langle I_\lambda'(u), v\rangle = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v \, dx - \int_{\mathbb{R}^N} V(x)|x|^\alpha |u|^{\frac{2(\alpha+2)}{N-2}} uv \, dx - \lambda \int_{\mathbb{R}^N} g(x) v \, dx,$$

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v \, dx - \int_{\mathbb{R}^N} V(x)|x|^\alpha |u|^{\frac{2(\alpha+2)}{N-2}} uv \, dx,$$

$$\langle J'(u), v\rangle = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v \, dx - \int_{\mathbb{R}^N} V_0 |x|^\alpha |u|^{\frac{2(\alpha+2)}{N-2}} uv \, dx.$$

The main result in this paper establishes the existence of at least two solutions, provided that the perturbation given by the potential g is small enough.

Theorem 2.1. *Assume that hypotheses (V1), (V2), (V3) are satisfied, and let* $g \in H^{-1}(\mathbb{R}^N) \setminus \{0\}$ *be fixed. Then there exists* $\lambda_0 > 0$ *such that for all* $\lambda \in (0, \lambda_0)$*, problem* (1) *has at least two weak solutions.*

Remark 2.2. Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set. We define the weighted Lebesgue space $L_\alpha^{\frac{2(N+\alpha)}{N-2}}(\Omega)$ as the space of measurable real functions u defined on Ω so that $\int_\Omega |x|^\alpha |u|^{\frac{2(N+\alpha)}{N-2}} dx < \infty$. Considering inequality (5), we deduce that the embedding $H^1(\Omega) \hookrightarrow L_\alpha^{\frac{2(N+\alpha)}{N-2}}(\Omega)$ is continuous.

We note that since the embedding $H^1(\mathbb{R}^N) \hookrightarrow L_\alpha^{\frac{2(N+\alpha)}{N-2}}(\mathbb{R}^N)$ is not compact, then the energy functional I_λ does not satisfy the Palais-Smale condition. Also, given that $g \not\equiv 0$, it follows that 0 is not a solution of problem (1), and for this reason can not directly apply the mountain pass theorem. Although this occurs, by using the Ekeland variational principle, we can achieve the first solution of problem (1) by applying some similar ideas to those developed in [20]. On the other hand, we can obtain the second weak solution of problem (1) applying the mountain pass theorem without the Palais-Smale condition, which provides us a bounded Palais-Smale sequence whose limit corresponds to a critical point of energy functional I_λ. Finally, we will see that, indeed, these two weak solutions are different.

Remark 2.3. Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded set, and $0 \notin \overline{\Omega}$. Using the Sobolev inequality, we obtain

$$\left(\int_\Omega |x|^\alpha |u|^{\frac{2(N+\alpha)}{N-2}} dx\right)^{\frac{N-2}{2(N+\alpha)}} \leq c_1 \left(\int_\Omega |u|^{\frac{2(N+\alpha)}{N-2}} dx\right)^{\frac{N-2}{2(N+\alpha)}} \leq c_2 \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}},$$

for any $u \in H^1(\Omega)$. Hence, we deduce that the embedding $H^1(\Omega) \hookrightarrow L_\alpha^{\frac{2(N+\alpha)}{N-2}}(\Omega)$ is compact.

Remark 2.4. If a sequence $\{u_n\}$ converges weakly to some u_0 in $H^1(\mathbb{R}^N)$, inequality (5) involves that $\{u_n\}$ is bounded in $L_\alpha^{\frac{2(N+\alpha)}{N-2}}(\mathbb{R}^N)$. Passing to a subsequence, labeled again $\{u_n\}$, we can assume that

$$u_n \rightharpoonup u_0 \text{ in } L_{\alpha,\,\mathrm{loc}}^{\frac{2(N+\alpha)}{N-2}}(\mathbb{R}^N \setminus \{0\}) \text{ and } u_n \to u_0 \text{ a.a. } x \in \mathbb{R}^N. \tag{7}$$

3. Auxiliary results

In this section we provide some technical results, which are useful in the proof of Theorem 2.1.

Definition 3.1. Let E be a Banach space , $\Phi : E \to \mathbb{R}$ be a functional of class C^1, and c be a real number. We say that a sequence $\{u_n\} \subset E$ is a $(PS)_c$ sequence of Φ if $\Phi(u_n) \to c$ and $\|\Phi'(u_n)\|_{E^*} \to 0$.

Lemma 3.2. *Consider a sequence* $\{v_n\}$ *that converges weakly to* 0 *in* $H^1(\mathbb{R}^N)$. *Then the following properties hold:*

$$I_\lambda(v_n) - J(v_n) \underset{n\to\infty}{\longrightarrow} 0, \tag{8}$$

$$\langle I'_\lambda(v_n), v_n\rangle - \langle J'(v_n), v_n\rangle \underset{n\to\infty}{\longrightarrow} 0. \tag{9}$$

Proof. We first observe that

$$I_\lambda(v_n) = J(v_n) - \frac{N-2}{2(N+\alpha)} \int_{\mathbb{R}^N} (V(x) - V_0)|x|^\alpha |v_n|^{\frac{2(N+\alpha)}{N-2}} dx - \lambda \int_{\mathbb{R}^N} g(x) v_n \, dx,$$

$$\langle I'_\lambda(v_n), v_n\rangle = \langle J'(v_n), v_n\rangle - \int_{\mathbb{R}^N} (V(x) - V_0)|x|^\alpha |v_n|^{\frac{2(N+\alpha)}{N-2}} dx - \lambda \int_{\mathbb{R}^N} g(x) v_n \, dx.$$

Therefore, considering that $v_n \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$, it is sufficient to demonstrate that

$$\int_{\mathbb{R}^N} (V(x) - V_0)|x|^\alpha |v_n|^{\frac{2(N+\alpha)}{N-2}} dx \underset{n\to\infty}{\longrightarrow} 0. \tag{10}$$

For this purpose, we fix $\varepsilon \in \mathbb{R}_+^*$. By (V1) and (V2), we can choose R_ε and r_ε, with $0 < r_\varepsilon < R_\varepsilon$, so that

$$|V(x) - V_0| = V(x) - V_0 < \varepsilon \quad \text{for a.a. } x \in \mathbb{R}^N \setminus \Omega_\varepsilon,$$

where $\Omega_\varepsilon = \overline{B(0, R_\varepsilon)} \setminus B(0, r_\varepsilon)$, that is $\Omega_\varepsilon \cup \{x \in \mathbb{R}^N : |x| < r_\varepsilon\} \cup \{x \in \mathbb{R}^N : |x| > R_\varepsilon\} = \mathbb{R}^N$. It follows that

$$\begin{aligned}
&\int_{\mathbb{R}^N} (V(x) - V_0)|x|^\alpha |v_n|^{\frac{2(N+\alpha)}{N-2}} dx \\
&\quad = \int_{\mathbb{R}^N\setminus\Omega_\varepsilon} (V(x) - V_0)|x|^\alpha |v_n|^{\frac{2(N+\alpha)}{N-2}} dx + \int_{\Omega_\varepsilon} (V(x) - V_0)|x|^\alpha |v_n|^{\frac{2(N+\alpha)}{N-2}} dx \\
&\quad \le \varepsilon \int_{\mathbb{R}^N\setminus\Omega_\varepsilon} |x|^\alpha |v_n|^{\frac{2(N+\alpha)}{N-2}} dx + (\|V\|_\infty - V_0) \int_{\Omega_\varepsilon} |x|^\alpha |v_n|^{\frac{2(N+\alpha)}{N-2}} dx \\
&\quad \le \varepsilon \int_{\mathbb{R}^N} |x|^\alpha |v_n|^{\frac{2(N+\alpha)}{N-2}} dx + (\|V\|_\infty - V_0) \int_{\Omega_\varepsilon} |x|^\alpha |v_n|^{\frac{2(N+\alpha)}{N-2}} dx.
\end{aligned}$$

By virtue of $v_n \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$, and the Caffarelli-Kohn-Nirenberg-type inequality (5) we infer that $\{v_n\}$ is bounded in $L_\alpha^{\frac{2(N+\alpha)}{N-2}}(\mathbb{R}^N)$, and therefore $v_n \to 0$ in $L_{\alpha,\mathrm{loc}}^{\frac{2(N+\alpha)}{N-2}}(\mathbb{R}^N \setminus \{0\})$ (as a consequence of (7)). It follows that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} (V(x) - V_0)|x|^\alpha |v_n|^{\frac{2(N+\alpha)}{N-2}} dx \le c\varepsilon,$$

where the positive constant c is independent of ε and n. Since $\varepsilon \in \mathbb{R}_+^*$ is arbitrary, we get the relation (10). Thus, the proof of Lemma 3.2 is complete. ☐

The following auxiliary result asserts that the weak limit of a $(PS)_c$ sequence of I_λ is solution of problem (1).

Lemma 3.3. *Let $\{u_n\} \subset H^1(\mathbb{R}^N)$ be a $(PS)_c$ sequence of I_λ for some real number c. Assume that $u_n \rightharpoonup u_0$ in $H^1(\mathbb{R}^N)$. Then u_0 is a weak solution of problem* (1).

Proof. We consider an arbitrary function $\varphi \in C_0^\infty(\mathbb{R}^N\setminus\{0\})$. Set $\Omega := \operatorname{supp}\varphi$. In virtue of Definition 3.1, hypothesis of lemma leads us to $I'_\lambda(u_n) \to 0$ in $H^{-1}(\mathbb{R}^N)$. Hence, $\langle I'_\lambda(u_n), \varphi\rangle \underset{n\to\infty}{\longrightarrow} 0$, more precisely,

$$\int_\Omega \nabla u_n \cdot \nabla\varphi \, dx - \int_\Omega V(x)|x|^\alpha |u_n|^{\frac{2(\alpha+2)}{N-2}} u_n\varphi \, dx - \lambda \int_\Omega g(x)\varphi \, dx \underset{n\to\infty}{\longrightarrow} 0. \tag{11}$$

At the same time, since $u_n \rightharpoonup u_0$ in $H^1(\mathbb{R}^N)$, we deduce that

$$\int_\Omega \nabla u_n \cdot \nabla \varphi \, dx \underset{n\to\infty}{\longrightarrow} \int_\Omega \nabla u_0 \cdot \nabla \varphi \, dx. \tag{12}$$

By $u_n \rightharpoonup u_0$ in $H^1(\mathbb{R}^N)$ we also derive the fact that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. This together with the Caffarelli-Kohn-Nirenberg-type inequality (5) implies that $\{|u_n|^{\frac{2(\alpha+2)}{N-2}} u_n\}$ is bounded in $L_\alpha^{\frac{2(N+\alpha)}{N+2+2\alpha}}(\mathbb{R}^N)$. Therefore, by relation (7) we deduce at once that $|u_n|^{\frac{2(\alpha+2)}{N-2}} u_n \to |u_0|^{\frac{2(\alpha+2)}{N-2}} u_0$ a.a. $x \in \mathbb{R}^N$, and $|u_n|^{\frac{2(\alpha+2)}{N-2}} u_n \rightharpoonup |u_0|^{\frac{2(\alpha+2)}{N-2}} u_0$ in $L_\alpha^{\frac{2(N+\alpha)}{N+2+2\alpha}}(\Omega)$. Hence, we have

$$\int_\Omega V(x)|x|^\alpha |u_n|^{\frac{2(\alpha+2)}{N-2}} u_n \varphi \, dx \underset{n\to\infty}{\longrightarrow} \int_\Omega V(x)|x|^\alpha |u_0|^{\frac{2(\alpha+2)}{N-2}} u_0 \varphi \, dx. \tag{13}$$

Accordingly, by relations (11), (12), and (13) we infer that

$$\int_\Omega \nabla u_0 \cdot \nabla \varphi \, dx - \int_\Omega V(x)|x|^\alpha |u_0|^{\frac{2(\alpha+2)}{N-2}} u_0 \varphi \, dx - \lambda \int_\Omega g(x)\varphi \, dx = 0.$$

In fact, the identity established above holds true for any $\varphi \in H^1(\mathbb{R}^N)$, that is, $I'_\lambda(u_0) = 0$. In other words, u_0 is a weak solution of problem (1). This concludes the proof of Lemma 3.3. □

Further, we establish a weighted version of the Brezis-Lieb lemma [3]. We point out that the Brezis-Lieb lemma is a refinement of the Fatou lemma that plays an important role in the analysis of partial differential equations.

Lemma 3.4. *Consider a sequence $\{u_n\}$ that converges weakly to u_0 in $H^1(\mathbb{R}^N)$. Setting $q := 2(N+\alpha)/(N-2)$, we assert that*

$$\int_{\mathbb{R}^N} V(x)|x|^\alpha \left(|u_n|^q - |u_n - u_0|^q\right) dx \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^N} V(x)|x|^\alpha |u_0|^q dx. \tag{14}$$

Proof. Because $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$, the Caffarelli-Kohn-Nirenberg-type inequality (5) implies that $\{u_n\}$ is bounded in $L^q(\mathbb{R}^N)$. Next, we consider $\varepsilon \in \mathbb{R}_+^*$. If we take into account the assumptions (V1) and (V2), we can deduce that there are R_ε, r_ε such that $0 < r_\varepsilon < R_\varepsilon$ satisfying

$$\int_{|x|<r_\varepsilon} V(x)|x|^\alpha |u_0|^q dx < \varepsilon, \tag{15}$$

$$\int_{|x|>R_\varepsilon} V(x)|x|^\alpha |u_0|^q dx < \varepsilon. \tag{16}$$

Set $\Omega_\varepsilon = \overline{B(0, R_\varepsilon)} \setminus B(0, r_\varepsilon)$. The following inequality holds true:

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} V(x)|x|^\alpha \left(|u_n|^q - |u_0|^q - |u_n - u_0|^q\right) dx \right| \\
&\qquad \leq \left| \int_{\Omega_\varepsilon} V(x)|x|^\alpha \left(|u_n|^q - |u_0|^q\right) dx \right| + \int_{\Omega_\varepsilon} V(x)|x|^\alpha |u_n - u_0|^q dx \\
&\qquad\quad + \int_{|x|<r_\varepsilon} V(x)|x|^\alpha |u_0|^q dx + \left| \int_{|x|<r_\varepsilon} V(x)|x|^\alpha \left(|u_n|^q - |u_n - u_0|^q\right) dx \right| \\
&\qquad\quad + \int_{|x|>R_\varepsilon} V(x)|x|^\alpha |u_0|^q dx + \left| \int_{|x|>R_\varepsilon} V(x)|x|^\alpha \left(|u_n|^q - |u_n - u_0|^q\right) dx \right| . \qquad (17)
\end{aligned}$$

Taking into consideration that $u_n \rightharpoonup u_0$ in $H^1(\mathbb{R}^N)$, by (7) we obtain the following two relations:

$$\int_{\Omega_\varepsilon} V(x)|x|^\alpha \left(|u_n|^{\frac{2(N+\alpha)}{N-2}} - |u_0|^{\frac{2(N+\alpha)}{N-2}}\right) dx \underset{n\to\infty}{\longrightarrow} 0\,, \qquad (18)$$

and

$$\int_{\Omega_\varepsilon} V(x)|x|^\alpha |u_n - u_0|^{\frac{2(N+\alpha)}{N-2}} dx \underset{n\to\infty}{\longrightarrow} 0\,. \qquad (19)$$

Applying the Lagrange mean value theorem to the function $f \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$, $f(v) = |v|^{\frac{2(N+\alpha)}{N-2}}$, we find an element $\kappa = \tau u_0 + (u_n - u_0)$ between $u_n - u_0$ and u_n, with $\tau(x) \in (0,1)$, such that

$$\begin{aligned}
&\int_{|x|<r_\varepsilon} V(x)|x|^\alpha \left(|u_n|^{\frac{2(N+\alpha)}{N-2}} - |u_n - u_0|^{\frac{2(N+\alpha)}{N-2}}\right) dx \\
&\qquad = \frac{2(N+\alpha)}{N-2} \int_{|x|<r_\varepsilon} V(x)|x|^\alpha |\tau u_0 + (u_n - u_0)|^{\frac{N+2+2\alpha}{N-2}} |u_0|\, dx. \qquad (20)
\end{aligned}$$

To proceed further, we need the following relation: for every $s > 0$, there is a constant $c = c(s)$ so that

$$(y+z)^s \leq c(y^s + z^s) \quad \text{for any } y, z \in (0, \infty).$$

Thus, the Hölder inequality and relation (15) yield

$$
\begin{aligned}
&\int_{|x|<r_\varepsilon} V(x)|x|^\alpha |\tau u_0 + (u_n - u_0)|^{\frac{N+2+2\alpha}{N-2}} |u_0| dx \\
&\quad \le \int_{|x|<r_\varepsilon} V(x)|x|^\alpha \left(|\tau u_0| + |u_n - u_0|\right)^{\frac{N+2+2\alpha}{N-2}} |u_0| dx \\
&\quad \le c \int_{|x|<r_\varepsilon} V(x)|x|^\alpha \left(|\tau u_0|^{\frac{N+2+2\alpha}{N-2}} + |u_n - u_0|^{\frac{N+2+2\alpha}{N-2}}\right) |u_0| dx \\
&\quad \le c \int_{|x|<r_\varepsilon} V(x)|x|^\alpha \left(|u_0|^{\frac{2(N+\alpha)}{N-2}} + |u_n - u_0|^{\frac{N+2+2\alpha}{N-2}} |u_0|\right) dx \\
&\quad = c \int_{|x|<r_\varepsilon} V(x)|x|^\alpha |u_0|^{\frac{2(N+\alpha)}{N-2}} dx + c \int_{|x|<r_\varepsilon} V(x)|x|^\alpha |u_n - u_0|^{\frac{N+2+2\alpha}{N-2}} |u_0| dx \\
&\quad \le c\varepsilon + c \left(\int_{|x|<r_\varepsilon} V(x)|x|^\alpha |u_n - u_0|^q dx\right)^{\frac{N+2+2\alpha}{2(N+\alpha)}} \left(\int_{|x|<r_\varepsilon} V(x)|x|^\alpha |u_0|^q dx\right)^{\frac{N-2}{2(N+\alpha)}} \\
&\quad \le \bar{c}\left(\varepsilon + \varepsilon^{\frac{N-2}{2(N+\alpha)}}\right),
\end{aligned}
$$

where $\bar{c}$ is a constant that does not depend on ε and n. This together with equality (20) imply that

$$
\begin{aligned}
&\int_{|x|<r_\varepsilon} V(x)|x|^\alpha |u_0|^{\frac{2(N+\alpha)}{N-2}} dx + \left|\int_{|x|<r_\varepsilon} V(x)|x|^\alpha \left(|u_n|^{\frac{2(N+\alpha)}{N-2}} - |u_n - u_0|^{\frac{2(N+\alpha)}{N-2}}\right) dx\right| \\
&\quad \le \varepsilon + \frac{2\bar{c}(N+\alpha)}{N-2}\left(\varepsilon + \varepsilon^{\frac{N-2}{2(N+\alpha)}}\right) \\
&\quad \le \frac{2c_1(N+\alpha)}{N-2}\left(\varepsilon + \varepsilon^{\frac{N-2}{2(N+\alpha)}}\right). \qquad (21)
\end{aligned}
$$

Similarly, we deduce that

$$
\begin{aligned}
&\int_{|x|>R_\varepsilon} V(x)|x|^\alpha |u_0|^{\frac{2(N+\alpha)}{N-2}} dx + \left|\int_{|x|>R_\varepsilon} V(x)|x|^\alpha \left(|u_n|^{\frac{2(N+\alpha)}{N-2}} - |u_n - u_0|^{\frac{2(N+\alpha)}{N-2}}\right) dx\right| \\
&\quad \le \frac{2c_2(N+\alpha)}{N-2}\left(\varepsilon + \varepsilon^{\frac{N-2}{2(N+\alpha)}}\right). \qquad (22)
\end{aligned}
$$

Finally, putting together relations (17), (18), (19), (21), and (22) we obtain

$$
\begin{aligned}
&\limsup_{n\to\infty} \left|\int_{\mathbb{R}^N} V(x)|x|^\alpha \left(|u_n|^{\frac{2(N+\alpha)}{N-2}} - |u_0|^{\frac{2(N+\alpha)}{N-2}} - |u_n - u_0|^{\frac{2(N+\alpha)}{N-2}}\right) dx\right| \\
&\qquad \le \left(\frac{2C(N+\alpha)}{N-2} + 1\right)\left(\varepsilon + \varepsilon^{\frac{N-2}{2(N+\alpha)}}\right).
\end{aligned}
$$

But we considered $\varepsilon \in \mathbb{R}_+^*$ to be arbitrary, therefore the limit (14) holds true, and the proof of Lemma 3.4 is concluded. □

Lemma 3.5. *There exist* $\lambda^* > 0$ *and* $R = R(\lambda^*) > 0$ *such that for any* $\lambda \in (0, \lambda^*)$, *the energy functional* I_λ *admits a* $(PS)_{c^*}$ *sequence, where* $c^* = c^*(R) = \inf_{u \in \overline{B}_R} I_\lambda(u)$. *In addition,* c^* *is achieved by an element* $u_0 \in H^1(\mathbb{R}^N)$ *satisfying* $I'_\lambda(u_0) = 0$.

Proof. We fix $\lambda \in (0,1)$. Given the relation (V1), the Caffarelli-Kohn-Nirenberg-type inequality (5), and the elementary inequality $yz \leq \frac{y^2+z^2}{2}$, we infer that

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\|u\|^2 - \frac{N-2}{2(N+\alpha)}\int_{\mathbb{R}^N} V(x)|x|^\alpha |u|^{\frac{2(N+\alpha)}{N-2}}dx - \lambda\int_{\mathbb{R}^N} g(x)u\,dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{N-2}{2(N+\alpha)}\|V\|_\infty C_\alpha^{\frac{2(N+\alpha)}{N-2}}\|u\|^{\frac{2(N+\alpha)}{N-2}} - \lambda\|g\|_{-1}\|u\| \\ &\geq \frac{1-\lambda}{2}\|u\|^2 - \frac{N-2}{2(N+\alpha)}\|V\|_\infty C_\alpha^{\frac{2(N+\alpha)}{N-2}}\|u\|^{\frac{2(N+\alpha)}{N-2}} - \frac{\lambda}{2}\|g\|_{-1}^2, \end{aligned}$$

for any $u \in H^1(\mathbb{R}^N)$. Taking into consideration that the function $f : [0,\infty) \to \mathbb{R}$ defined by

$$f(t) = \frac{1-\lambda}{2}t^2 - \frac{N-2}{2(N+\alpha)}\|V\|_\infty C_\alpha^{\frac{2(N+\alpha)}{N-2}} t^{\frac{2(N+\alpha)}{N-2}} - \frac{\lambda}{2}\|g\|_{-1}^2$$

is decreasing, we can find $\lambda^* > 0$, $R = R(\lambda^*) > 0$, and $\delta = \delta(\lambda^*) > 0$ so that we have

$$I_\lambda(u) \geq -\frac{\lambda}{2}\|g\|_{-1}^2, \quad \text{for every } u \in \overline{B}_R \text{ and } \lambda \in (0,\lambda^*), \tag{23}$$

$$I_\lambda(u) \geq \delta > 0, \quad \text{for every } u \in \partial B_R \text{ and } \lambda \in (0,\lambda^*). \tag{24}$$

For example, we choose

$$\lambda^* := \min\left\{\frac{1}{2}, \frac{\alpha+2}{2^{\frac{N+2+2\alpha}{\alpha+2}}(N+\alpha)\|g\|_{-1}^2}\left[\frac{1}{\|V\|_\infty C_\alpha^{\frac{2(N+\alpha)}{N-2}}}\right]^{\frac{N-2}{\alpha+2}}\right\},$$

$$R(\lambda^*) := \left[\frac{1-\lambda^*}{\|V\|_\infty C_\alpha^{\frac{2(N+\alpha)}{N-2}}}\right]^{\frac{N-2}{2(\alpha+2)}}, \text{ and } \delta(\lambda^*) := \frac{\lambda^*}{2}\|g\|_{-1}^2.$$

We define further, as in the statement of the lemma, $c^* := c^*(R) = \inf_{u\in\overline{B}_R} I_\lambda(u)$. It is obvious that $c^* \leq I_\lambda(0) = 0$. Also, we emphasize that $\overline{B}_R$ is a complete metric space with the metric induced by the norm

$$\operatorname{dist}(u,v) = \|u-v\|, \quad \text{for every } u,v \in \overline{B}_R.$$

Since the energy functional I_λ is lower semicontinuous and bounded from below on $\overline{B}_R$, applying the Ekeland variational principle (see [12]), there exists u_n such that

$$c^* \leq I_\lambda(u_n) \leq c^* + \frac{1}{n}, \tag{25}$$

$$I_\lambda(w) \geq I_\lambda(u_n) - \frac{1}{n}\|u_n - w\|, \quad \text{for every } w \in \overline{B}_R. \tag{26}$$

We intend to show that $\|u_n\| < R$ for n sufficiently large. To this end, we argue indirectly. Thus, suppose that $\|u_n\| = R$ for infinitely many n. Passing eventually to a subsequence, labeled again $\{u_n\}$, one can assume that $\|u_n\| = R$ for every $n \geq 1$. So, by relation (24) we infer that $I_\lambda(u_n) \geq \delta > 0$. This together with relation (25), by letting $n \to \infty$, lead us to $0 < \delta \leq c^* \leq 0$, which is obviously a contradiction.

Fix $u \in H^1(\mathbb{R}^N)$, with $\|u\| = 1$. Also, we set the sequence $w_n = u_n + tu$ which, for some fixed n, fulfills $\|w_n\| \leq \|u_n\| + |t| < R$ for $|t|$ sufficiently small. First we take the case when $t > 0$. Keeping in mind inequality (26) we get

$$I_\lambda(w_n) \geq I_\lambda(u_n) - \frac{1}{n}\|u_n - w_n\|$$

or equivalently

$$\frac{I_\lambda(u_n + tu) - I_\lambda(u_n)}{t} \geq -\frac{1}{n}.$$

By letting $t \to 0$, we obtain $\langle I'_\lambda(u_n), u\rangle \geq -\frac{1}{n}$. Also, for the case $t < 0$, we can show in a similar fashion that $\langle I'_\lambda(u_n), u\rangle \leq \frac{1}{n}$ when $t \to 0$. Due to the fact that $u \in H^1(\mathbb{R}^N)$ was taken arbitrarily, such that to satisfy $\|u\| = 1$, we find that

$$\|I'_\lambda(u_n)\| = \sup_{\substack{u \in H^1(\mathbb{R}^N) \\ \|u\|=1}} |\langle I'_\lambda(u_n), u\rangle| \leq \frac{1}{n} \to 0 \quad \text{as } n \to \infty.$$

For that reason we actually achieve $I'_\lambda(u_n) \to 0$ in $H^{-1}(\mathbb{R}^N)$. At the same time, relation (25) assures us that $I_\lambda(u_n) \to c^*$. These two things put together means that $\{u_n\} \subset H^1(\mathbb{R}^N)$ is a $(PS)_{c^*}$ sequence of I_λ. On the other hand, $\|u_n\| < R$ ensures that there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, which converges weakly in $H^1(\mathbb{R}^N)$ to some u_0. Thus, relying on what we pointed above, Remark 2.3 and Remark 2.4 we deduce that

$$u_n \to u_0 \;\text{ a.a. } x \in \mathbb{R}^N \quad \text{and} \quad I'_\lambda(u_0) = 0.$$

The last aim is to show that $I_\lambda(u_0) = c^*$. First, we observe that

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 dx - \int_{\mathbb{R}^N} V(x)|x|^\alpha |u_n|^{\frac{2(N+\alpha)}{N-2}} dx - \lambda \int_{\mathbb{R}^N} g(x) u_n \, dx = \langle I'_\lambda(u_n), u_n\rangle = o(1).$$

A simple computation yields

$$I_\lambda(u_n) = \frac{\alpha+2}{2(N+\alpha)} \int_{\mathbb{R}^N} V(x)|x|^\alpha |u_n|^{\frac{2(N+\alpha)}{N-2}} dx - \frac{\lambda}{2} \int_{\mathbb{R}^N} g(x) u_n \, dx + o(1),$$

and hence

$$I_\lambda(u_0) = \frac{\alpha+2}{2(N+\alpha)} \int_{\mathbb{R}^N} V(x)|x|^\alpha |u_0|^{\frac{2(N+\alpha)}{N-2}} dx - \frac{\lambda}{2} \int_{\mathbb{R}^N} g(x) u_0 \, dx + o(1).$$

Employing the Fatou lemma, it follows that

$$\begin{aligned} c^* &= \liminf_{n\to\infty} I_\lambda(u_n) \\ &\geq \frac{\alpha+2}{2(N+\alpha)} \int_{\mathbb{R}^N} V(x)|x|^\alpha |u_0|^{\frac{2(N+\alpha)}{N-2}} dx - \frac{\lambda}{2} \int_{\mathbb{R}^N} g(x) u_0 \, dx + o(1) \\ &= I_\lambda(u_0). \end{aligned}$$

Since $u_0 \in \overline{B}_R$, we also have $c^* \leq I_\lambda(u_0)$. We conclude that $I_\lambda(u_0) = c^*$, and the proof of Lemma 3.5 is completed. ☐

4. Proof of the main result

First of all we define

$$\mathcal{J} = \{u \in H^1(\mathbb{R}^N) \setminus \{0\};\ \langle J'(u), u\rangle = 0\}.$$

We want to justify that $\mathcal{J} \neq \emptyset$. Fix $u \in H^1(\mathbb{R}^N) \setminus \{0\}$. For every $\lambda > 0$, we also set

$$\Phi(\lambda) = \langle J'(\lambda u), \lambda u\rangle = \lambda^2 \int_{\mathbb{R}^N} |\nabla u|^2 dx - \lambda^{\frac{2(N+\alpha)}{N-2}} \int_{\mathbb{R}^N} V_0 |x|^\alpha |u|^{\frac{2(N+\alpha)}{N-2}} dx.$$

It is clear that $\Phi(\lambda) > 0$ for λ small enough and $\Phi(\lambda) < 0$ for λ sufficiently large. From here we infer that there is $\lambda > 0$ which meets $\Phi(\lambda) = 0$, and thereby $\lambda u \in \mathcal{J}$.

We highlight that for any $u \in \mathcal{J}$, the following identity occurs:

$$\frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\displaystyle\int_{\mathbb{R}^N} V_0 |x|^\alpha |u|^{\frac{2(N+\alpha)}{N-2}} dx} = 1. \tag{27}$$

Proposition 4.1. *Let $J_\infty = \inf_{u\in\mathcal{J}} J(u)$. Then there is $\bar{u} \in H^1(\mathbb{R}^N)$ satisfying*

$$J_\infty = J(\bar{u}) = \sup_{t\geq 0} J(t\bar{u}). \tag{28}$$

Proof. Fix $\phi \in H^1(\mathbb{R}^N) \setminus \{0\}$. We consider

$$h(t) = J(t\phi) = \frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla\phi|^2 dx - \frac{(N-2)V_0}{2(N+\alpha)}\, t^{\frac{2(N+\alpha)}{N-2}} \int_{\mathbb{R}^N} |x|^\alpha |\phi|^{\frac{2(N+\alpha)}{N-2}} dx.$$

It is evident that

$$h'(t) = t\int_{\mathbb{R}^N} |\nabla\phi|^2 dx - V_0\, t^{\frac{N+2+2\alpha}{N-2}} \int_{\mathbb{R}^N} |x|^\alpha |\phi|^{\frac{2(N+\alpha)}{N-2}} dx.$$

We point out that h achieves its positive global maximum

$$\begin{aligned} h(t_0) = J(t_0\phi) &= \sup_{t\geq 0} J(t\phi) \\ &= \frac{\alpha+2}{2(N+\alpha)} \left\{ \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla\phi|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^N} V_0 |x|^\alpha |\phi|^{\frac{2(N+\alpha)}{N-2}} dx\right)^{\frac{N-2}{N+\alpha}}} \right\}^{\frac{N+\alpha}{\alpha+2}}, \end{aligned} \tag{29}$$

where

$$t_0 = t_0(\phi) = \left\{ \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla\phi|^2 dx}{\displaystyle\int_{\mathbb{R}^N} V_0 |x|^\alpha |\phi|^{\frac{2(N+\alpha)}{N-2}} dx} \right\}^{\frac{N-2}{2(\alpha+2)}} > 0. \tag{30}$$

For simplicity, we define

$$Q_\alpha := \inf_{\substack{\phi\in H^1(\mathbb{R}^N)\\ \phi\neq 0}} \left\{ \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla\phi|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^N} V_0 |x|^\alpha |\phi|^{\frac{2(N+\alpha)}{N-2}} dx\right)^{\frac{N-2}{N+\alpha}}} \right\}. \tag{31}$$

So we have

$$\inf_{\substack{\phi\in H^1(\mathbb{R}^N)\\ \phi\neq 0}} \sup_{t\geq 0} J(t\phi) = \frac{\alpha+2}{2(N+\alpha)} Q_\alpha^{\frac{N+\alpha}{\alpha+2}}. \tag{32}$$

By (27) and (30) we obviously obtain that, for every $u \in \mathcal{J}$, $t_0(u) = 1$. Therefore, using (29) we get

$$J(u) = \sup_{t\geq 0} J(tu), \quad \text{for any } u \in \mathcal{J}. \tag{33}$$

Keeping in mind the Theorems 1.2, 7.2, and 7.6 in [8], we can deduce that the infimum in (31) is attained by a function $U \in H^1(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N} V_0 |x|^\alpha |U|^{\frac{2(N+\alpha)}{N-2}} = 1.$$

We consider $\bar{u} = Q_\alpha^{\frac{N-2}{2(\alpha+2)}} U$. By a simple calculation we obtain $\langle J'(\bar{u}), \bar{u}\rangle = 0$, that is $\bar{u} \in \mathcal{J}$. We also have

$$J(\bar{u}) = \frac{\alpha+2}{2(N+\alpha)} Q_\alpha^{\frac{N+\alpha}{\alpha+2}}. \tag{34}$$

Finally, using the relations (32)-(34) we conclude

$$\begin{aligned} J_\infty &= \inf_{u\in\mathcal{J}} J(u) = \inf_{u\in\mathcal{J}} \sup_{t\geq 0} J(tu) \\ &\geq \inf_{\substack{u\in H^1(\mathbb{R}^N)\\ u\neq 0}} \sup_{t\geq 0} J(tu) = \frac{\alpha+2}{2(N+\alpha)} Q_\alpha^{\frac{N+\alpha}{\alpha+2}} = J(\bar{u}) = \sup_{t\geq 0} J(t\bar{u}), \end{aligned}$$

and this completes the proof of Proposition 4.1. □

Proposition 4.2. *Let* $\{u_n\}$ *be a* $(PS)_c$ *sequence of* I_λ *such that* $u_n \rightharpoonup u_0$ *in* $H^1(\mathbb{R}^N)$*, and let* J_∞ *given in Proposition 4.1. Then the following alternative holds: either* $u_n \to u_0$ *in* $H^1(\mathbb{R}^N)$*, or* $I_\lambda(u_0) + J_\infty \leq c$*.*

Proof. Thanks to the fact that $\{u_n\}$ is a $(PS)_c$ sequence of I_λ and $\{u_n\}$ is weakly convergent in $H^1(\mathbb{R}^N)$ to u_0, we immediately deduce that

$$I_\lambda(u_n) = c + o(1), \tag{35}$$

$$\langle I'_\lambda(u_n), u_n\rangle = o(1). \tag{36}$$

Let $v_n = u_n - u_0$. Clearly we have $v_n \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$. Therefore, as a consequence of (7) the following relations hold true:

$$\int_{\mathbb{R}^N} g(x) v_n \, dx \underset{n\to\infty}{\longrightarrow} 0, \tag{37}$$

$$\int_{\mathbb{R}^N} \nabla v_n \cdot \nabla u_0 \, dx \underset{n\to\infty}{\longrightarrow} 0. \tag{38}$$

Bearing in mind the definition of I_λ and I, we can easily derive that

$$I_\lambda(v_n) = I(v_n) - \lambda \int_{\mathbb{R}^N} g(x) v_n \, dx,$$

and considering (37) it follows that

$$I_\lambda(v_n) = I(v_n) + o(1). \tag{39}$$

Also, it is true that

$$\|u_n\|^2 = \|v_n\|^2 + \|u_0\|^2 + 2\int_{\mathbb{R}^N} \nabla v_n \cdot \nabla u_0 \, dx$$

which together with relation (38) show that

$$\|u_n\|^2 = \|v_n\|^2 + \|u_0\|^2 + o(1). \tag{40}$$

Further, applying Lemma 3.4 for $u_n = v_n + u_0$ we can see that

$$\int_{\mathbb{R}^N} V(x)|x|^\alpha |v_n + u_0|^{\frac{2(N+\alpha)}{N-2}} dx =$$
$$= \int_{\mathbb{R}^N} V(x)|x|^\alpha |v_n|^{\frac{2(N+\alpha)}{N-2}} dx + \int_{\mathbb{R}^N} V(x)|x|^\alpha |u_0|^{\frac{2(N+\alpha)}{N-2}} dx + o(1).$$

The above identity and relation (38) provide that

$$I_\lambda(u_n) = I_\lambda(v_n + u_0) = I_\lambda(v_n) + I_\lambda(u_0) + o(1).$$

Hence, by (8), (35), and the above obtained identity we arrive at

$$o(1) + c = I_\lambda(u_n) = I_\lambda(v_n) + I_\lambda(u_0) + o(1) = I_\lambda(u_0) + J(v_n) + o(1). \tag{41}$$

Next, considering again (38) it results that

$$\langle I'_\lambda(u_n), u_n\rangle = \langle I'_\lambda(v_n), v_n\rangle + \langle I'_\lambda(u_0), u_0\rangle + o(1). \tag{42}$$

Thus, Lemma 3.3, relations (9), (36), and (42) yield

$$\begin{aligned} o(1) = \langle I'_\lambda(u_n), u_n\rangle &= \langle I'_\lambda(v_n), v_n\rangle + \langle I'_\lambda(u_0), u_0\rangle + o(1) \\ &= \langle J'(v_n), v_n\rangle + o(1). \end{aligned} \tag{43}$$

If $v_n \to 0$ in $H^1(\mathbb{R}^N)$, then $u_n \to u_0$ in $H^1(\mathbb{R}^N)$. Thus, $I_\lambda(u_n) \underset{n\to\infty}{\longrightarrow} I_\lambda(u_0)$.

Else if $v_n \not\to 0$ in $H^1(\mathbb{R}^N)$, since $v_n \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$, we can assume that $\|v_n\| \to \ell > 0$. Due to the fact that (41) occurs, to show that $c \geq I_\lambda(u_0) + J_\infty$, it is enough to prove that $J(v_n) \geq J_\infty + o(1)$. We intend to show that there is a sequence $\{t_n\} \subset (0, \infty)$ satisfying $t_n \to 1$ and $\langle J'(t_n v_n), t_n v_n\rangle = 0$, which would mean that $t_n v_n \in \mathcal{J}$. That would result in

$$\begin{aligned} J(v_n) &= J(t_n v_n) + \frac{1 - t_n^2}{2}\|v_n\|^2 - \frac{V_0(N-2)}{2(N+\alpha)}\left[1 - t_n^{\frac{2(N+\alpha)}{N-2}}\right] \int_{\mathbb{R}^N} |x|^\alpha |v_n|^{\frac{2(N+\alpha)}{N-2}} dx \\ &= J(t_n v_n) + o(1) \geq J_\infty + o(1), \end{aligned}$$

and the conclusion follows. To this end, we make the following notations:

$$\zeta_n = \int_{\mathbb{R}^N} |\nabla v_n|^2 dx = \|v_n\|^2 \geq 0,$$

$$\xi_n = V_0 \int_{\mathbb{R}^N} |x|^\alpha |v_n|^{\frac{2(N+\alpha)}{N-2}} dx \geq 0, \quad \mu_n = \zeta_n - \xi_n.$$

Considering the relation (43) we deduce that $\mu_n = \langle J'(v_n), v_n\rangle \underset{n\to\infty}{\longrightarrow} 0$. If $\mu_n = 0$, then we take $t_n = 1$. We will assume that $\mu_n \neq 0$. Let $t > 0$. We have

$$\langle J'(tv_n), tv_n\rangle = t^2 \int_{\mathbb{R}^N} |\nabla v_n|^2 dx - t^{\frac{2(N+\alpha)}{N-2}} V_0 \int_{\mathbb{R}^N} |x|^\alpha |v_n|^{\frac{2(N+\alpha)}{N-2}} dx.$$

Taking $\delta \in \mathbb{R}$ such that $|\delta| > 0$ is small enough and $t = 1+\delta$, we have

$$\begin{aligned}\langle J'(tv_n), tv_n\rangle &= (1+\delta)^2\zeta_n - (1+\delta)^{\frac{2(N+\alpha)}{N-2}}\xi_n = (1+\delta)^2\zeta_n - (1+\delta)^{\frac{2(N+\alpha)}{N-2}}(\zeta_n - \mu_n)\\ &= -\frac{2(\alpha+2)}{N-2}\delta\zeta_n + o(\delta)\zeta_n + (1+\delta)^{\frac{2(N+\alpha)}{N-2}}\mu_n.\end{aligned}$$

We know that $\zeta_n = \|v_n\|^2 \to \ell^2 > 0$ and $\mu_n \to 0$. Hence, for n sufficiently large, can be defined the sequences $\delta_n^+ = \frac{(N-2)|\mu_n|}{(\alpha+2)\zeta_n} \searrow 0$ and $\delta_n^- = -\frac{(N-2)|\mu_n|}{(\alpha+2)\zeta_n} \nearrow 0$. The following occur:

$$\langle J'((1+\delta_n^+)v_n), (1+\delta_n^+)v_n\rangle < 0 \quad\text{and}\quad \langle J'((1+\delta_n^-)v_n), (1+\delta_n^-)v_n\rangle > 0.$$

Therefore, we can infer that there is a sequence $t_n \in (1+\delta_n^-, 1+\delta_n^+)$ satisfying $t_n \to 1$ and $\langle J'(t_nv_n), t_nv_n\rangle = 0$, and this concludes the proof of Proposition 4.2. □

Further, we fix $\bar{u} \in H^1(\mathbb{R}^N)$ so that relation (28) holds true. View of the fact that $\alpha > -2$, we can derive the existence of some $\bar{t}$ so that

$$J(t\bar{u}) < 0 \quad\text{for any}\quad t > \bar{t},$$

$$I_\lambda(t\bar{u}) < 0 \quad\text{for any}\quad t > \bar{t} \quad\text{and}\quad \lambda > 0.$$

We set now

$$\Gamma := \{\gamma \in C([0,1], H^1(\mathbb{R}^N));\ \gamma(0) = 0,\ \gamma(1) = \bar{t}\bar{u}\}, \tag{44}$$

$$c_g = \inf_{\gamma\in\Gamma}\sup_{u\in\gamma([0,1])} I_\lambda(u). \tag{45}$$

Proposition 4.3. *We assert that there are $\lambda_* > 0$, $R_* = R_*(\lambda_*) > 0$, $\delta_* = \delta_*(\lambda_*) > 0$ so that $I_\lambda|_{\partial B_{R_*}} \geq \delta_*$ and $c_g < J_\infty + c_*$ for any $\lambda \in (0, \lambda_*)$, with $c_* = \inf_{u\in\overline{B}_{R_*}} I_\lambda(u)$.*

Proof. As a consequence of hypothesis (V3) it can be assumed that

$$I(t\bar{u}) < J(t\bar{u}) \quad\text{for any}\quad t > 0.$$

By a basic calculation we find some $t_0 \in (0, \bar{t})$ so that it happens

$$\sup_{t\geq 0} I(t\bar{u}) = I(t_0\bar{u}) < J(t_0\bar{u}) \leq \sup_{t\geq 0} J(t\bar{u}) = J_\infty,$$

and so can be chosen $\epsilon \in (0,1)$ such that

$$\sup_{t\geq 0} I(t\bar{u}) < J_\infty - \epsilon. \tag{46}$$

We set

$$\lambda_* := \min\left\{\lambda^*, \frac{\epsilon}{2\bar{t}\|\bar{u}\|\|g\|_{-1}}, \frac{\epsilon}{2\|g\|_{-1}^2}\right\}. \tag{47}$$

In view of Lemma 3.5, one can derive that there is $R_* = R_*(\lambda_*) > 0$ so that for every $\lambda \in (0, \lambda_*)$, the conclusion of Lemma 3.5 holds true. In the proof of that lemma we have also shown that there is $\delta_* = \delta_*(\lambda_*) > 0$ so that $I_\lambda|_{\partial \overline{B}_{R_*}} \geq \delta_*$. Taking into account (23) and (47) it results that

$$c_* = \inf_{u \in \overline{B}_{R_*}} I_\lambda(u) \geq -\frac{\lambda}{2}\|g\|_{-1}^2 > -\frac{\epsilon}{2}, \text{ for any } \lambda \in (0, \lambda_*). \tag{48}$$

Set $\gamma_* = \{t\bar{t}\bar{u};\, t \in [0,1]\} \in \Gamma$. For $u \in \gamma_*$ we have the following:

$$\begin{aligned} |I_\lambda(u) - I(u)| &= \lambda \left| \int_{\mathbb{R}^N} g(x) u \, dx \right| = \lambda \left| \int_{\mathbb{R}^N} g(x) t\bar{t}\bar{u} \, dx \right| \leq \lambda \bar{t} \left| \int_{\mathbb{R}^N} g(x) \bar{u} \, dx \right| \\ &\leq \lambda \bar{t} \|\bar{u}\| \|g\|_{-1} \leq \frac{\epsilon}{2}, \text{ for any } \lambda \in (0, \lambda_*), \end{aligned}$$

meaning that

$$I_\lambda(u) \leq I(u) + \frac{\epsilon}{2}, \text{ for any } \lambda \in (0, \lambda_*). \tag{49}$$

Finally, on the basis of (45), (46), (48), and (49) we have

$$\begin{aligned} c_g &= \inf_{\gamma \in \Gamma} \sup_{u \in \gamma([0,1])} I_\lambda(u) \leq \sup_{u \in \gamma_*} I_\lambda(u) \leq \sup_{u \in \gamma_*} I(u) + \frac{\epsilon}{2} \\ &\leq \sup_{t \geq 0} I(t\bar{u}) + \frac{\epsilon}{2} < J_\infty - \frac{\epsilon}{2} < J_\infty + c_* \,. \end{aligned}$$

By this relation the proof of Proposition 4.3 is complete. □

We shall now complete the proof of Theorem 2.1. Let $R_* > 0$, and $\delta_* > 0$ be as in Proposition 4.3. Based on the proof of this proposition, we can infer that for every $\lambda \in (0, \lambda_*)$, the conclusion of Lemma 3.5 holds true. Hence, there exists u_0, solution of problem (1), satisfying $I_\lambda(u_0) = c_*$.

By way of the mountain pass theorem without the Palais-Smale condition (see Theorem 2.2 in [5]), we get that there is $\{u_n\}$, a $(PS)_{c_g}$ sequence of I_λ. This means that

$$I_\lambda(u_n) \to c_g \text{ and } I'_\lambda(u_n) \to 0 \text{ in } H^{-1}(\mathbb{R}^N),$$

and the following hold true:

$$\begin{aligned} c_g + o(1) + \frac{N-2}{2(N+\alpha)} \|I'_\lambda(u_n)\|_{-1} \|u_n\| &\geq I_\lambda(u_n) - \frac{N-2}{2(N+\alpha)} \langle I'_\lambda(u_n), u_n \rangle \\ &= \frac{\alpha+2}{2(N+\alpha)} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx - \frac{\lambda(N+2+2\alpha)}{2(N+\alpha)} \int_{\mathbb{R}^N} g(x) u_n dx \\ &\geq \frac{\alpha+2}{2(N+\alpha)} \|u_n\|^2 - \frac{\lambda(N+2+2\alpha)}{2(N+\alpha)} \|g\|_{-1} \|u_n\|. \end{aligned} \tag{50}$$

In order to prove that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$, we argue by contradiction that $\|u_n\| \to \infty$. Dividing by $\|u_n\|$ in (50) and passing to the limit as $n \to \infty$ we obtain a contradiction. Hence, by passing to a subsequence, we can assume that $u_n \rightharpoonup u_1$ in $H^1(\mathbb{R}^N)$. Using Lemma 3.3 yields that u_1 is a weak solution of problem (1).

Finally, we intend to show that $u_0 \neq u_1$. It should be noticed that by Proposition 4.2, we have the alternative:

either $u_n \to u_1$ in $H^1(\mathbb{R}^N)$, providing

$$I_\lambda(u_1) = \lim_{n\to\infty} I_\lambda(u_n) = c_g > 0 \geq c_* = I_\lambda(u_0),$$

which means that $u_0 \neq u_1$;

or

$$c_g = \lim_{n\to\infty} I_\lambda(u_n) \geq I_\lambda(u_1) + J_\infty.$$

In this latter case, if we assume that $u_1 = u_0$, then $I_\lambda(u_1) = I_\lambda(u_0) = c_*$. This involves $c_g \geq c_* + J_\infty$, which is in contradiction to Proposition 4.3. This completes the proof of Theorem 2.1. □

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References

[1] Adimurthi, N. Chaudhuri, M. Ramaswamy: *An improved Hardy-Sobolev inequality and its application*, Proc. Amer. Math. Soc. 130 (2001) 489–505.

[2] R. B. Assunção, P. C. Carrião, O. H. Miyagaki: *Critical singular problems via concentration-compactness lemma*, J. Math. Anal. Appl. 326 (2007) 137–154.

[3] H. Brezis, E. H. Lieb: *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. 88 (1983) 486–490.

[4] H. Brezis, M. Marcus: *Hardy's inequalities revisited*, Ann. Sc. Norm. Super. Pisa, Cl. Sci. 25 (1997) 217–237.

[5] H. Brezis, L. Nirenberg: *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponent*, Commun. Pure Appl. Math. 36 (1983) 486–490.

[6] L. Caffarelli, R. Kohn, L. Nirenberg: *First order interpolation inequalities with weights*, Compositio Math. 53 (1984) 259–275.

[7] P. Caldiroli, A. Malchiodi: *Singular elliptic problems with critical growth*, Commun. Partial Differential Equations 27 (2002) 847–876.

[8] F. Catrina, Z.-Q. Wang: *On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence) and symmetry of extremal functions*, Commun. Pure Appl. Math. 54 (2001) 229–258.

[9] N. Chaudhuri, M. Ramaswamy: *Existence of positive solutions of some semilinear elliptic equations with singular coefficients*, Proc. Royal Soc. Edinburgh Sect. A Math. 131 (2001) 1275–1295.

[10] F. Cîrstea, V. Rădulescu: *Multiple solutions of degenerate perturbed elliptic problems involving a subcritical Sobolev exponent*, Topol. Meth. Nonlin. Anal. 15 (2000) 281–298.

[11] E. B. Davies: *A review of Hardy inequalities*, in: The Mazýa Anniversary Collection, vol. 2 (Rostock, 1998), 55–67, Oper. Theory Adv. Appl. 110, Birkhäuser, Basel (1999).

[12] I. Ekeland: *Nonconvex minimization problems*, Bull. Amer. Math. Soc. 1 (1979) 443–473.

[13] A. Ferrero, F. Gazzola: *Existence of solutions for singular critical growth semilinear elliptic equations*, J. Differential Equations 177 (2001) 494–522.

[14] M. Ghergu, V. Rădulescu: *Singular elliptic problems with lack of compactness*, Annali di Matematica Pura ed Applicata 185 (2006) 63–79.

[15] F. Gladiali, M. Grossi, S. Neves: *Nonradial solutions for the Hénon equation in* $\mathbb{R}^N$, Advances in Mathematics 249 (2013) 1–36.

[16] J. Li: *Existence of solution for a singular elliptic equation with critical Sobolev-Hardy exponents*, Int. J. Math. Math. Sci. 20 (2005) 3213–3223.

[17] V. Rădulescu, D. Smets: *Critical singular problems on infinite cones*, Nonlinear Anal. 54 (2003) 1153–1164.

[18] V. Rădulescu, D. Smets, M. Willem: *Hardy-Sobolev inequalities with remainder terms*, Topol. Methods Nonlinear Anal. 20 (2002) 145–149.

[19] D. Ruiz, M. Willem: *Elliptic problems with critical exponents and Hardy potentials*, J. Differential Equations 190 (2003) 524–538.

[20] G. Tarantello: *On nonhomogeneous elliptic equations involving critical Sobolev exponents*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 9 (1992) 281–304.