

On a Minimax Theorem: an Improvement, a New Proof and an Overview of its Applications

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Theorem 1 of [14], a minimax result for functions $f : X \times Y \rightarrow \mathbf{R}$, where Y is a real interval, was partially extended by the author to the case where Y is a convex set in a Hausdorff topological vector space ([15], Theorem 3.2). As a key tool in the proof a partial extension of the same result to the case where Y is a convex set in \mathbf{R}^n ([7], Theorem 4.2) was used. In the present paper, we first obtain a full extension of the result in [14] by means of a new proof fully based on the use of the result itself via an inductive argument. Then, we present an overview of the various and numerous applications of these results.

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1. Introduction

In [14], we established the following result:

Theorem 1.A ([14], Theorem 1). *Let X be a topological space, $Y \subseteq \mathbf{R}$ an interval and $f : X \times Y \rightarrow \mathbf{R}$ a function satisfying the following conditions:*

- (a) *for each $y \in Y$, the function $f(\cdot, y)$ is lower semicontinuous and inf-compact;*
- (b) *for each $x \in X$, the function $f(x, \cdot)$ is continuous and quasi-concave.*

Then, at least one of the following assertions holds:

- (i) *there exists $\hat{y} \in Y$ such that the function $f(\cdot, \hat{y})$ has at least two global minima;*
- (ii) *one has $\sup_Y \inf_X f = \inf_X \sup_Y f$.*

Actually, in [14], Y is assumed to be open. However, the same identical proof works for any interval Y (see Remark 2.1 below). Later, in [7], S. J. N. Mosconi obtained

Theorem 1.B ([7], Theorem 4.2). *Let X be a topological space, $Y \subseteq \mathbf{R}^n$ a non-empty convex set and $f : X \times Y \rightarrow \mathbf{R}$ a function satisfying the following conditions:*

- (a) *for each $y \in Y$, the function $f(\cdot, y)$ is lower semicontinuous and inf-compact;*
- (b) *for each $x \in X$, the function $f(x, \cdot)$ is upper semicontinuous and concave.*

Then, the conclusion of Theorem 1.A holds.

Finally, in [15], using Theorem 1.B, we obtained

Theorem 1.C ([15], Theorem 3.2). *Let X be a topological space, E a Hausdorff topological vector space, $Y \subseteq E$ a non-empty convex set and $f : X \times Y \rightarrow \mathbf{R}$ a function satisfying the following conditions:*

- (a) *for each $y \in Y$, the function $f(\cdot, y)$ is lower semicontinuous and inf-compact;*
- (b) *for each $x \in X$, the function $f(x, \cdot)$ is upper semicontinuous and concave.*

Then, the conclusion of Theorem 1.A holds.

In comparing the above results, two natural questions arise: does Theorem 1.A hold if "continuous" is relaxed to "upper semicontinuous"?; does Theorem 1.A hold if Y is any non-empty convex set in a Hausdorff topological vector space?

The answer to the first question is negative. In this connection, consider the following

Example 1.1. Let $X = \{x_0, x_1\}$ (with $x_0 \neq x_1$ and X equipped with the discrete topology) and let $f : X \times [0, 1] \rightarrow \mathbf{R}$ be defined by

$$f(x_i, y) = \begin{cases} f(x_i, y) = y & \text{if } i = 0, y \in [0, 1] \\ f(x_i, y) = -y & \text{if } i = 1, y \in]0, 1] \\ f(x_i, 0) = 1 & \text{if } i = 1. \end{cases}$$

Of course, x_1 is the only global minimum of $f(\cdot, y)$ for all $y \in]0, 1]$, while x_0 is the only global minimum of $f(\cdot, 0)$. Moreover, $f(x_i, \cdot)$ is upper semicontinuous and quasi-concave for $i = 0, 1$. However, we have

$$\sup_{[0,1]} \inf_X f = 0 < 1 = \inf_X \sup_{[0,1]} f .$$

To the contrary, the answer to the second question is positive. Indeed, we will prove

Theorem 1.1. *Let X be a topological space, E a topological vector space, $Y \subseteq E$ a non-empty convex set and $f : X \times Y \rightarrow \mathbf{R}$ a function satisfying the following conditions:*

- (a) *for each $y \in Y$, the function $f(\cdot, y)$ is lower semicontinuous and inf-compact;*
- (b) *for each $x \in X$, the function $f(x, \cdot)$ is continuous and quasi-concave.*

Then, the conclusion of Theorem 1.A holds.

The aim of the present paper is twofold. On the one hand, we just wish to prove Theorem 1.1. We stress that our proof of Theorem 1.1 is fully based on the use of Theorem 1.A, via an inductive argument. In turn, using Theorem 1.1, we obtain

Theorem 1.2. *Let X be a topological space, E a vector space, $Y \subseteq E$ a non-empty convex set and $f : X \times Y \rightarrow \mathbf{R}$ a function satisfying the following conditions:*

- (a) *for each $y \in Y$, the function $f(\cdot, y)$ is lower semicontinuous and inf-compact;*
- (b) *for each $x \in X$, the function $f(x, \cdot)$ is concave.*

Then, the conclusion of Theorem 1.A holds.

Hence, Theorem 1.2 is an improvement of Theorem 1.C obtained without resorting to Mosconi's result. On the other hand, we wish to offer an overview of the several and various applications of Theorem 1.A (with its "sequential" version) and Theorem 1.C known up to now ([12]–[21]).

2. Proofs of Theorems 1.1 and 1.2

As usual, a generic real-valued function φ on a topological space X is said to be *inf-compact* (resp. *inf-sequentially compact*) if, for each $r \in \mathbf{R}$, the set $\varphi^{-1}(] - \infty, r])$ (called sub-level set) is compact (resp. sequentially compact). If φ is defined on a convex set of a vector space, it is said to be quasi-concave if, for each $r \in \mathbf{R}$, the set $\varphi^{-1}([r, +\infty[)$ is convex.

For each $n \in \mathbf{N}$, we put

$$S_n = \{(\lambda_1, \dots, \lambda_n) \in ([0, +\infty[)^n : \lambda_1 + \dots + \lambda_n = 1\} .$$

The core of our proof of Theorem 1.1 is to prove it first in the case where $Y = S_n$:

Lemma 2.1. *Let X be a topological space and let $f : X \times S_n \rightarrow \mathbf{R}$ be a function satisfying the following conditions:*

- (a) *for each $y \in S_n$, the function $f(\cdot, y)$ is lower semicontinuous, inf-compact and has a unique global minimum ;*
- (b) *for each $x \in X$, the function $f(x, \cdot)$ is continuous and quasi-concave.*

Then, one has

$$\sup_{S_n} \inf_X f = \inf_X \sup_{S_n} f .$$

Proof. We prove the theorem by induction on n . Clearly, it (trivially) holds for $n = 1$. Now, assume that it is true for $n = k$ ($k \geq 2$). We are going to prove that it is true for $n = k + 1$. So, we are assuming that $f : X \times S_{k+1} \rightarrow \mathbf{R}$ is a function satisfying (a) and (b) with $n = k + 1$. Let $\psi : S_k \times [0, 1] \rightarrow S_{k+1}$ be the continuous function defined by

$$\psi(\lambda_1, \dots, \lambda_k, \mu) = (\mu\lambda_1, \dots, \mu\lambda_k, 1 - \mu)$$

for all $(\lambda_1, \dots, \lambda_k, \mu) \in S_k \times [0, 1]$. Now, consider the function $\tilde{f} : X \times S_k \times [0, 1] \rightarrow \mathbf{R}$ defined by

$$\tilde{f}(x, \lambda_1, \dots, \lambda_k, \mu) = f(x, \psi(\lambda_1, \dots, \lambda_k, \mu))$$

for all $(x, \lambda_1, \dots, \lambda_k, \mu) \in X \times S_k \times [0, 1]$. For each $\mu \in [0, 1]$ and for each $x \in X$, since $f(x, \cdot)$ is quasi-concave in S_{k+1} and $\psi(\cdot, \mu)$ is affine in S_k , it clearly follows that $\tilde{f}(x, \cdot, \mu)$ is quasi-concave in S_k . Therefore, by the induction assumption, we have

$$\sup_{y \in S_k} \inf_{x \in X} \tilde{f}(x, y, \mu) = \inf_{x \in X} \sup_{y \in S_k} \tilde{f}(x, y, \mu) . \quad (2.1)$$

From (2.1), we then infer

$$\begin{aligned} \sup_{(y, \mu) \in S_k \times [0, 1]} \inf_{x \in X} \tilde{f}(x, y, \mu) &= \sup_{\mu \in [0, 1]} \sup_{y \in S_k} \inf_{x \in X} \tilde{f}(x, y, \mu) \\ &= \sup_{\mu \in [0, 1]} \inf_{x \in X} \sup_{y \in S_k} \tilde{f}(x, y, \mu) . \end{aligned} \quad (2.2)$$

Now, consider the function $g : X \times [0, 1] \rightarrow \mathbf{R}$ defined by

$$g(x, \mu) = \sup_{y \in S_k} \tilde{f}(x, y, \mu)$$

for all $(x, \mu) \in X \times [0, 1]$. Fix $\mu \in [0, 1]$. From (2.1), by compactness and semicontinuity, we infer the existence of a point $(\hat{x}, \hat{y}) \in X \times S_k$ such that

$$\sup_{y \in S_k} \tilde{f}(\hat{x}, y, \mu) = \tilde{f}(\hat{x}, \hat{y}, \mu) = \inf_{x \in X} \tilde{f}(x, \hat{y}, \mu) . \quad (2.3)$$

Now, let $x \in X$, with $x \neq \hat{x}$. By (a) and (2.3), we have

$$g(\hat{x}, \mu) = \tilde{f}(\hat{x}, \hat{y}, \mu) < \tilde{f}(x, \hat{y}, \mu) \leq g(x, \mu) .$$

In other words, \hat{x} is the only global minimum of the function $g(\cdot, \mu)$ which is also lower semicontinuous and inf-compact. Now, fix $x \in X$ and $r \in \mathbf{R}$. Set

$$C = \{u \in S_{k+1} : f(x, u) \geq r\} .$$

Of course, we have

$$\{\mu \in [0, 1] : g(x, \mu) \geq r\} = \bigcup_{y \in S_k} \{\mu \in [0, 1] : \tilde{f}(x, y, \mu) \geq r\} . \quad (2.4)$$

Note that the right-hand side of (2.4) is equal to the projection of the set $\psi^{-1}(C)$ on $[0, 1]$. But, for each $(\lambda_1, \dots, \lambda_{k+1}) \in S_{k+1}$, we have

$$\psi^{-1}(\lambda_1, \dots, \lambda_{k+1}) = \begin{cases} \left\{ \left(\frac{\lambda_1}{1-\lambda_{k+1}}, \dots, \frac{\lambda_k}{1-\lambda_{k+1}}, 1 - \lambda_{k+1} \right) \right\} & \text{if } \lambda_{k+1} \neq 1 \\ S_k \times \{0\} & \text{if } \lambda_{k+1} = 1 . \end{cases}$$

Hence, ψ is onto S_{k+1} and, by a classical result, for each compact and connected set $D \subseteq S_{k+1}$, the set $\psi^{-1}(D)$ is compact and connected. So, since C is compact and connected (being convex), the set $\psi^{-1}(C)$ is connected and hence so is its projection on $[0, 1]$. Therefore, in view of (2.4), the set $\{\mu \in [0, 1] : g(x, \mu) \geq r\}$ is compact and connected. In other words, the function $g(x, \cdot)$ is upper semicontinuous and quasi-concave in $[0, 1]$. At this point, we can apply Theorem 1.A to g . So, we obtain

$$\sup_{\mu \in [0,1]} \inf_{x \in X} g(x, \mu) = \inf_{x \in X} \sup_{\mu \in [0,1]} g(x, \mu) .$$

Hence

$$\begin{aligned} \sup_{\mu \in [0,1]} \inf_{x \in X} \sup_{y \in S_k} \tilde{f}(x, y, \mu) &= \inf_{x \in X} \sup_{\mu \in [0,1]} \sup_{y \in S_k} \tilde{f}(x, y, \mu) \\ &= \inf_{x \in X} \sup_{(y, \mu) \in S_k \times [0,1]} \tilde{f}(x, y, \mu) . \end{aligned} \tag{2.5}$$

Then, from (2.2) and (2.5), we get

$$\sup_{(y, \mu) \in S_k \times [0,1]} \inf_{x \in X} \tilde{f}(x, y, \mu) = \inf_{x \in X} \sup_{(y, \mu) \in S_k \times [0,1]} \tilde{f}(x, y, \mu) . \tag{2.6}$$

On the other hand, since $\psi(S_k \times [0, 1]) = S_{k+1}$, we have

$$\sup_{(y, \mu) \in S_k \times [0,1]} \inf_{x \in X} \tilde{f}(x, y, \mu) = \sup_{S_{k+1}} \inf_X f$$

as well as

$$\inf_{x \in X} \sup_{(y, \mu) \in S_k \times [0,1]} \tilde{f}(x, y, \mu) = \inf_X \sup_{S_{k+1}} f$$

and so (2.6) gives

$$\sup_{S_{k+1}} \inf_X f = \inf_X \sup_{S_{k+1}} f ,$$

as claimed. □

A family of sets \mathcal{C} is said to be filtering if for each pair $C_1, C_2 \in \mathcal{C}$ there is $C_3 \in \mathcal{C}$ such that $C_1 \cup C_2 \subseteq C_3$. Now, we establish the following

Proposition 2.1. *Let X be a topological space, Y a non-empty set, $y_0 \in Y$ and $f : X \times Y \rightarrow \mathbf{R}$ a function such that $f(\cdot, y)$ is lower semicontinuous for all $y \in Y$ and inf-compact for $y = y_0$. Assume also that there is a filtering cover \mathcal{C} of Y such that*

$$\sup_C \inf_X f = \inf_X \sup_C f$$

for all $C \in \mathcal{C}$. Then, one has

$$\sup_Y \inf_X f = \inf_X \sup_Y f .$$

Proof. Denote by \mathcal{C}_0 the family of all $C \in \mathcal{C}$ containing y_0 . Clearly, \mathcal{C}_0 is a filtering cover of Y . Arguing by contradiction, suppose that

$$\sup_Y \inf_X f < \inf_X \sup_Y f .$$

Fix r satisfying

$$\sup_Y \inf_X f < r < \inf_X \sup_Y f$$

and, for each $C \in \mathcal{C}_0$, put

$$A_C = \left\{ x \in X : \sup_{y \in C} f(x, y) \leq r \right\} .$$

Notice that $A_C \neq \emptyset$ since, otherwise, we would have

$$r \leq \inf_X \sup_C f = \sup_C \inf_X f \leq \sup_Y \inf_X f ,$$

against the choice of r . Now, observe that, if C_1, \dots, C_k are finitely many members of \mathcal{C}_0 , since \mathcal{C}_0 is filtering, there is $\tilde{C} \in \mathcal{C}_0$ such that

$$\bigcup_{i=1}^k C_i \subseteq \tilde{C} .$$

This implies that

$$A_{\tilde{C}} \subseteq \bigcap_{i=1}^k A_{C_i}$$

and so $\bigcap_{i=1}^k A_{C_i}$ is non-empty. Therefore, $\{A_C\}_{C \in \mathcal{C}_0}$ is a family of closed subsets of the compact set $\{x \in X : f(x, y_0) \leq r\}$ possessing the finite intersection property. As a consequence, there would be $\tilde{x} \in \bigcap_{C \in \mathcal{C}_0} A_C$. So, since \mathcal{C}_0 is a cover of Y , we would have

$$\sup_Y \inf_X f = \sup_{C \in \mathcal{C}_0} \sup_C \inf_X f \leq \sup_{C \in \mathcal{C}_0} \sup_{y \in C} f(\tilde{x}, y) \leq r ,$$

against the choice of r . □

The “sequential” version of Proposition 2.1 is as follows:

Proposition 2.2. *Let X be a topological space, Y a non-empty set, $y_0 \in Y$ and $f : X \times Y \rightarrow \mathbf{R}$ a function such that $f(\cdot, y)$ is sequentially lower semicontinuous for all $y \in Y$ and inf-sequentially compact for $y = y_0$. Assume also that there is an at most countable filtering cover \mathcal{C} of Y such that*

$$\sup_C \inf_X f = \inf_X \sup_C f$$

for all $C \in \mathcal{C}$. Then, one has

$$\sup_Y \inf_X f = \inf_X \sup_Y f .$$

Proof. Keep the notations of the above proof. An obvious inductive argument shows that there is a non-decreasing sequence $\{C_k\}$ in \mathcal{C}_0 such that $\cup_{k \in \mathbb{N}} C_k = X$. So, $\{A_{C_k}\}$ turns out to be a non-increasing sequence of non-empty sequentially closed subsets of the sequentially compact set $\{x \in X : f(x, y_0) \leq r\}$. As a consequence, $\cap_{k \in \mathbb{N}} A_{C_k} \neq \emptyset$, and the proof goes as before. \square

Proof of Theorem 1.1. Denote by \mathcal{P} the family of all convex polytopes of Y . Of course, \mathcal{P} is a filtering cover of Y . Fix $P \in \mathcal{P}$. Let $x_1, \dots, x_n \in P$ be such that

$$P = \text{conv}(\{x_1, \dots, x_n\}) .$$

Consider the function $\eta : S_n \rightarrow P$ defined by

$$\eta(\lambda_1, \dots, \lambda_n) = \lambda_1 x_1 + \dots + \lambda_n x_n$$

for all $(\lambda_1, \dots, \lambda_n) \in S_n$. Plainly, the function $(x, \lambda_1, \dots, \lambda_n) \rightarrow f(x, \eta(\lambda_1, \dots, \lambda_n))$ satisfies in $X \times S_n$ the assumptions of Lemma 2.1, and so

$$\sup_{(\lambda_1, \dots, \lambda_n) \in S_n} \inf_{x \in X} f(x, \eta(\lambda_1, \dots, \lambda_n)) = \inf_{x \in X} \sup_{(\lambda_1, \dots, \lambda_n) \in S_n} f(x, \eta(\lambda_1, \dots, \lambda_n)) .$$

Since $\eta(S_n) = P$, we then have

$$\sup_P \inf_X f = \inf_X \sup_P f .$$

Now, the conclusion follows from Proposition 2.1. \square

Proof of Theorem 1.2. Denote by \mathcal{C} the family of all finite-dimensional convex subsets of Y . Fix $C \in \mathcal{C}$. Denote by L the linear span of C . Consider L with the Euclidean topology. Since C is convex, the relative interior of C (say A) is non-empty. By (b), for each $x \in X$, the function $f(x, \cdot)|_A$ is continuous in A and one has

$$\sup_{y \in A} f(x, y) = \sup_{y \in C} f(x, y) .$$

By Theorem 1.1, we have

$$\sup_A \inf_X f = \inf_X \sup_A f .$$

Therefore

$$\sup_A \inf_X f \leq \sup_C \inf_X f \leq \inf_X \sup_C f = \inf_X \sup_A f$$

and so

$$\sup_C \inf_X f = \inf_X \sup_C f .$$

Now, the conclusion follows from Proposition 2.1 \square

Remark 2.1. As we said at the beginning, the proof of Theorem 1.A given in [14] holds for any interval Y . Actually, with the notation of [14], to get the

lower semicontinuity of Ψ in $X \times I$ it is enough to apply Lemma 5 of [24] (which holds also when f is quasi-concave in I). Furthermore, Theorem 1.A is still true if, instead of (a), we assume that, for each $y \in Y$, the function $f(\cdot, y)$ is sequentially lower semicontinuous and inf-sequentially compact. In this case, to get the sequential lower semicontinuity of Ψ , it is enough to apply Lemma 5 of [24] again, this time considering on X the topology whose members are the sequentially open subsets of X .

3. A well-posedness theory

In this section, we present a well-posedness theory for constrained minimization problems which is based on the use of Theorem 1.A in its “sequential” version (Remark 2.1).

In the sequel, X is a topological space, J, Φ are two real-valued functions defined in X , and a, b are two numbers in $[-\infty, +\infty]$, with $a < b$. If $a \in \mathbf{R}$ (resp. $b \in \mathbf{R}$), we denote by M_a (resp. M_b) the set of all global minima of the function $J + a\Phi$ (resp. $J + b\Phi$), while if $a = -\infty$ (resp. $b = +\infty$), M_a (resp. M_b) stands for the empty set. We adopt the conventions $\inf \emptyset = +\infty$, $\sup \emptyset = -\infty$.

We also set

$$\alpha := \max \left\{ \inf_X \Phi, \sup_{M_b} \Phi \right\}, \quad \text{and} \quad \beta := \min \left\{ \sup_X \Phi, \inf_{M_a} \Phi \right\}.$$

Note that, by the next proposition, one has $\alpha \leq \beta$.

Proposition 3.1. *Let Y be a nonempty set, $f, g : Y \rightarrow \mathbf{R}$ two functions, and λ, μ two real numbers, with $\lambda < \mu$. Let \hat{y}_λ be a global minimum of the function $f + \lambda g$ and let \hat{y}_μ be a global minimum of the function $f + \mu g$.*

Then, one has $g(\hat{y}_\mu) \leq g(\hat{y}_\lambda)$. If either \hat{y}_λ or \hat{y}_μ is strict and $\hat{y}_\lambda \neq \hat{y}_\mu$, then $g(\hat{y}_\mu) < g(\hat{y}_\lambda)$.

Proof. We have

$$f(\hat{y}_\lambda) + \lambda g(\hat{y}_\lambda) \leq f(\hat{y}_\mu) + \lambda g(\hat{y}_\mu)$$

as well as

$$f(\hat{y}_\mu) + \mu g(\hat{y}_\mu) \leq f(\hat{y}_\lambda) + \mu g(\hat{y}_\lambda).$$

Summing up, we get

$$\lambda g(\hat{y}_\lambda) + \mu g(\hat{y}_\mu) \leq \lambda g(\hat{y}_\mu) + \mu g(\hat{y}_\lambda)$$

and so

$$(\lambda - \mu)g(\hat{y}_\lambda) \leq (\lambda - \mu)g(\hat{y}_\mu)$$

from which the first conclusion follows. If either \hat{y}_λ or \hat{y}_μ is strict and $\hat{y}_\lambda \neq \hat{y}_\mu$, then one of the first two inequalities is strict and hence so is the third one. \square

As usual, given a function $f : X \rightarrow \mathbf{R}$ and a set $C \subseteq X$, we say that the problem of minimizing f over C is *well-posed* if the following two conditions hold:

- the restriction of f to C has a unique global minimum, say \hat{x} ; and
- every sequence $\{x_n\}$ in C such that $\lim_{n \rightarrow \infty} f(x_n) = \inf_C f$, converges to \hat{x} .

Clearly, when f is inf-sequentially compact, the problem of minimizing f over a sequentially closed set C is well-posed if and only if $f|_C$ has a unique global minimum. The basic result is as follows:

Theorem 3.1. *Assume that $\alpha < \beta$ and that, for each $\lambda \in]\alpha, \beta[$, the function $J + \lambda\Phi$ is sequentially lower semicontinuous, inf-sequentially compact and admits a unique global minimum in X .*

Then, for each $r \in]\alpha, \beta[$, the problem of minimizing J over $\Phi^{-1}(r)$ is well-posed. Moreover, if we denote by \hat{x}_r the unique global minimum of $J|_{\Phi^{-1}(r)}$ ($r \in]\alpha, \beta[$), the functions $r \rightarrow \hat{x}_r$ and $r \rightarrow J(\hat{x}_r)$ are continuous in $]\alpha, \beta[$.

Proof. Fix $r \in]\alpha, \beta[$ and consider the function $f : X \times \mathbf{R} \rightarrow \mathbf{R}$ defined by

$$f(x, \lambda) = J(x) + \lambda(\Phi(x) - r)$$

for all $(x, \lambda) \in X \times \mathbf{R}$. Clearly, the restriction of the function f to $X \times]a, b[$ satisfies all the assumptions of the variant of Theorem 1.A pointed out in Remark 2.1. Consequently, since (i) does not hold, we have

$$\sup_{\lambda \in]a, b[} \inf_{x \in X} (J(x) + \lambda(\Phi(x) - r)) = \inf_{x \in X} \sup_{\lambda \in]a, b[} (J(x) + \lambda(\Phi(x) - r)). \quad (3.1)$$

Note that

$$\begin{aligned} \sup_{\lambda \in]a, b[} \inf_{x \in X} f(x, \lambda) &\leq \sup_{\lambda \in [a, b] \cap \mathbf{R}} \inf_{x \in X} f(x, \lambda) \\ &\leq \inf_{x \in X} \sup_{\lambda \in [a, b] \cap \mathbf{R}} f(x, \lambda) = \inf_{x \in X} \sup_{\lambda \in]a, b[} f(x, \lambda), \end{aligned}$$

and so from (3.1) it follows

$$\sup_{\lambda \in [a, b] \cap \mathbf{R}} \inf_{x \in X} (J(x) + \lambda(\Phi(x) - r)) = \inf_{x \in X} \sup_{\lambda \in [a, b] \cap \mathbf{R}} (J(x) + \lambda(\Phi(x) - r)). \quad (3.2)$$

Now, observe that the function $\inf_{x \in X} f(x, \cdot)$ is upper semicontinuous in $[a, b] \cap \mathbf{R}$ and that

$$\lim_{\lambda \rightarrow +\infty} \inf_{x \in X} f(x, \lambda) = -\infty$$

if $b = +\infty$ (since $r > \inf_X \Phi$), and

$$\lim_{\lambda \rightarrow -\infty} \inf_{x \in X} f(x, \lambda) = -\infty$$

if $a = -\infty$ (since $r < \sup_X \Phi$). From this, it clearly follows that there exists $\hat{\lambda}_r \in [a, b] \cap \mathbf{R}$ such that

$$\inf_{x \in X} f(x, \hat{\lambda}_r) = \sup_{\lambda \in [a, b] \cap \mathbf{R}} \inf_{x \in X} f(x, \lambda).$$

Since

$$\sup_{\lambda \in [a, b] \cap \mathbf{R}} f(x, \lambda) = \sup_{\lambda \in]a, b[} f(x, \lambda)$$

for all $x \in X$, the sub-level sets of the function $\sup_{\lambda \in [a, b] \cap \mathbf{R}} f(\cdot, \lambda)$ are sequentially compact. Hence, there exists $\hat{x}_r \in X$ such that

$$\sup_{\lambda \in [a, b] \cap \mathbf{R}} f(\hat{x}_r, \lambda) = \inf_{x \in X} \sup_{\lambda \in [a, b] \cap \mathbf{R}} f(x, \lambda).$$

Then, thanks to (3.2), $(\hat{x}_r, \hat{\lambda}_r)$ is a saddle-point of f , that is

$$\begin{aligned} J(\hat{x}_r) + \hat{\lambda}_r(\Phi(\hat{x}_r) - r) &= \inf_{x \in X} (J(x) + \hat{\lambda}_r(\Phi(x) - r)) \\ &= J(\hat{x}_r) + \sup_{\lambda \in [a, b] \cap \mathbf{R}} \lambda(\Phi(\hat{x}_r) - r). \end{aligned} \quad (3.3)$$

First of all, from (3.3) it follows that \hat{x}_r is a global minimum of $J + \hat{\lambda}_r \Phi$. We now show that $\Phi(\hat{x}_r) = r$. We distinguish four cases.

Case 1: $a = -\infty$ and $b = \infty$. In this case, the equality $\Phi(\hat{x}_r) = r$ follows from the fact that $\sup_{\lambda \in \mathbf{R}} \lambda(\Phi(\hat{x}_r) - r)$ is finite.

Case 2: $a > -\infty$ and $b = +\infty$. In this case, the finiteness of $\sup_{\lambda \in [a, +\infty[} \lambda(\Phi(\hat{x}_r) - r)$ implies that $\Phi(\hat{x}_r) \leq r$. But, if $\Phi(\hat{x}_r) < r$, from (3.3), we would infer that $\hat{\lambda}_r = a$ and so $\hat{x}_r \in M_a$. This would imply $\inf_{M_a} \Phi < r$, contrary to the choice of r .

Case 3: $a = -\infty$ and $b < +\infty$. In this case, the finiteness of $\sup_{\lambda \in]-\infty, b]} \lambda(\Phi(\hat{x}_r) - r)$ implies that $\Phi(\hat{x}_r) \geq r$. But, if $\Phi(\hat{x}_r) > r$, from (3.3) again, we would infer $\hat{\lambda}_r = b$, and so $\hat{x}_r \in M_b$. Therefore, $\sup_{M_b} \Phi > r$, contrary to the choice of r .

Case 4: $-\infty < a$ and $b < +\infty$. In this case, if $\Phi(\hat{x}_r) \neq r$, as we have just seen, we would have either $\inf_{M_a} \Phi < r$ or $\sup_{M_b} \Phi > r$, contrary to the choice of r .

Having proved that $\Phi(\hat{x}_r) = r$, we also get that $\hat{\lambda}_r \in]a, b[$. Indeed, if $\hat{\lambda}_r \in \{a, b\}$, we would have either $\hat{x}_r \in M_a$ or $\hat{x}_r \in M_b$ and so either $\inf_{M_a} \Phi \leq r$ or $\sup_{M_b} \Phi \geq r$, contrary to the choice of r . From (3.3) once again, we furthermore infer that any global minimum of $J|_{\Phi^{-1}(r)}$ (and \hat{x}_r is so) is a global minimum of $J + \hat{\lambda}_r \Phi$ in X . But, since $\hat{\lambda}_r \in]a, b[$, $J + \hat{\lambda}_r \Phi$ has exactly one global minimum in X which, therefore, coincides with \hat{x}_r . Since the sub-level sets of $J + \hat{\lambda}_r \Phi$ are sequentially compact, we then conclude that any minimizing sequence in X for $J + \hat{\lambda}_r \Phi$ converges to \hat{x}_r . But any minimizing sequence in $\Phi^{-1}(r)$ for J is a minimizing sequence for $J + \hat{\lambda}_r \Phi$, and so it converges to \hat{x}_r . Consequently, the problem of minimizing J over $\Phi^{-1}(r)$ is well-posed, as claimed.

Now, let us prove the other assertions made in thesis. By Proposition 3.1, it clearly follows that the function $\lambda \rightarrow \Phi(\hat{y}_\lambda)$ is non-increasing in $]a, b[$ and that its range is contained in $[\alpha, \beta]$. On the other hand, by the first assertion of the thesis, this range contains $]\alpha, \beta[$. Of course, from this it follows that the

function $\lambda \rightarrow \Phi(\hat{y}_\lambda)$ is continuous in $]a, b[$. Now, observe that the function $\lambda \rightarrow \inf_{x \in X} (J(x) + \lambda \Phi(x))$ is concave and hence continuous in $]a, b[$. This, in particular, implies that the function $\lambda \rightarrow J(\hat{y}_\lambda)$ is continuous in $]a, b[$. Now, for each $r \in]\alpha, \beta[$, put

$$\Lambda_r = \{ \lambda \in]a, b[: \Phi(\hat{y}_\lambda) = r \} .$$

Let us prove that the multifunction $r \rightarrow \Lambda_r$ is upper semicontinuous in $] \alpha, \beta [$. Of course, it is enough to show that the restriction of the multifunction to any bounded open sub-interval of $] \alpha, \beta [$ is upper semicontinuous. So, let $s, t \in] \alpha, \beta [$, with $s < t$. Let $\mu, \nu \in]a, b[$ be such that $\Phi(\hat{y}_\mu) = t, \Phi(\hat{y}_\nu) = s$. By Proposition 3.1, we have

$$\bigcup_{r \in]s, t[} \Lambda_r \subseteq [\mu, \nu] .$$

Then, to show that the restriction of multifunction $r \rightarrow \Lambda_r$ to $]s, t[$ is upper semicontinuous, it is enough to prove that its graph is closed in $]s, t[\times [\mu, \nu]$ ([6], Theorem 7.1.16). But, this latter fact follows immediately from the continuity of the function $\lambda \rightarrow \Phi(\hat{y}_\lambda)$. At this point, we observe that, for each $r \in]\alpha, \beta[$, the function $\lambda \rightarrow \hat{y}_\lambda$ is constant in Λ_r . Indeed, let $\lambda, \mu \in \Lambda_r$ with $\lambda \neq \mu$. If it was $\hat{y}_\lambda \neq \hat{y}_\mu$, by Proposition 3.1 it would follow

$$r = \Phi(\hat{y}_\lambda) \neq \Phi(\hat{y}_\mu) = r ,$$

which is absurd. Hence, the function $r \rightarrow \hat{x}_r$, as composition of the upper semicontinuous multifunction $r \rightarrow \Lambda_r$ and the continuous function $\lambda \rightarrow \hat{y}_\lambda$, is continuous. Analogously, the continuity of the function $r \rightarrow J(\hat{x}_r)$ follows observing that it is the composition of $r \rightarrow \Lambda_r$ and the continuous function $\lambda \rightarrow J(\hat{y}_\lambda)$. The proof is complete. \square

Remark 3.1. It is important to remark that, under the assumptions of Theorem 3.1, we have actually proved that, for each $r \in]\alpha, \beta[$, there exists $\hat{\lambda}_r \in]a, b[$ such that the unique global minimum of $J + \hat{\lambda}_r \Phi$ belongs to $\Phi^{-1}(r)$.

When $a \geq 0$, we can obtain a conclusion dual to that of Theorem 3.1, under the same key assumption.

Theorem 3.2. *Let $a \geq 0$. Assume that, for each $\lambda \in]a, b[$, the function $J + \lambda \Phi$ is sequentially lower semicontinuous, inf-sequentially compact and admits a unique global minimum in X . Set*

$$\gamma := \max \left\{ \inf_X J, \sup_{\hat{M}_a} J \right\} , \quad \text{and} \quad \delta := \min \left\{ \sup_X J, \inf_{\hat{M}_b} J \right\} ,$$

where

$$\hat{M}_a = \begin{cases} M_a & \text{if } a > 0 \\ \emptyset & \text{if } a = 0 \end{cases} \quad \text{and} \quad \hat{M}_b = \begin{cases} M_b & \text{if } b < +\infty \\ \inf_X \Phi & \text{if } b = +\infty . \end{cases}$$

Assume that $\gamma < \delta$. Then, for each $r \in]\gamma, \delta[$, the problem of minimizing Φ over $J^{-1}(r)$ is well-posed. Moreover, if we denote by \tilde{x}_r the unique global minimum of $\Phi|_{J^{-1}(r)}$ ($r \in]\gamma, \delta[$), the functions $r \rightarrow \tilde{x}_r$ and $r \rightarrow \Phi(\tilde{x}_r)$ are continuous in $] \gamma, \delta[$.

Proof. Let $\mu \in]b^{-1}, a^{-1}[$. Then, since $\mu^{-1} \in]a, b[$ and

$$\Phi + \mu J = \mu(J + \mu^{-1}\Phi) ,$$

we clearly have that the function $J + \mu\Phi$ has sequentially compact sub-level sets and admits a unique global minimum. At this point, the conclusion follows applying Theorem 3.1 with the roles of J and Φ interchanged. \square

We now state the version of Theorem 3.1 obtained in the setting of a reflexive Banach space endowed with the weak topology.

Theorem 3.3. *Let X be a sequentially weakly closed set in a reflexive real Banach space. Assume that $\alpha < \beta$ and that, for each $\lambda \in]a, b[$, the function $J + \lambda\Phi$ is sequentially weakly lower semicontinuous, has bounded sub-level sets and has a unique global minimum in X .*

Then, for each $r \in]\alpha, \beta[$, the problem of minimizing J over $\Phi^{-1}(r)$ is well-posed in the weak topology. Moreover, if we denote by \hat{x}_r the unique global minimum of $J|_{\Phi^{-1}(r)}$ ($r \in]\alpha, \beta[$), the functions $r \rightarrow \hat{x}_r$ and $r \rightarrow J(\hat{x}_r)$ are continuous in $] \alpha, \beta[$, the first one in the weak topology.

Proof. Our assumptions clearly imply that, for each $\lambda \in]a, b[$, the sub-level sets of $J + \lambda\Phi$ are sequentially weakly compact, by the Eberlein-Smulyan theorem. Hence, considering X with the relative weak topology, we are allowed to apply Theorem 3.1, from which the conclusion directly follows. \square

Analogously, from Theorem 3.2 we get

Theorem 3.4. *Let $a \geq 0$ and let X be a sequentially weakly closed set in a reflexive real Banach space. Assume that, for each $\lambda \in]a, b[$, the function $J + \lambda\Phi$ is sequentially weakly lower semicontinuous, has bounded sub-level sets and has a unique global minimum in X . Assume also that $\gamma < \delta$, where γ, δ are defined as in Theorem 3.2.*

Then, for each $r \in]\gamma, \delta[$, the problem of minimizing Φ over $J^{-1}(r)$ is well-posed in the weak topology. Moreover, if we denote by \tilde{x}_r the unique global minimum of $\Phi|_{J^{-1}(r)}$ ($r \in]\gamma, \delta[$), the functions $r \rightarrow \tilde{x}_r$ and $r \rightarrow \Phi(\tilde{x}_r)$ are continuous in $] \gamma, \delta[$, the first one in the weak topology.

Finally, it is worth noticing that Theorem 3.1 also offers the perspective of a novel way of seeing whether a given function possesses a global minimum. Let us formalize this using Remark 3.1.

Theorem 3.5. *Assume that $b > 0$ and that, for each $\lambda \in]0, b[$, the function $J + \lambda\Phi$ has sequentially compact sub-level sets and admits a unique global minimum, say \hat{y}_λ . Assume also that*

$$\lim_{\lambda \rightarrow 0^+} \Phi(\hat{y}_\lambda) < \sup_X \Phi . \quad (3.4)$$

Then, one has

$$\lim_{\lambda \rightarrow 0^+} \Phi(\hat{y}_\lambda) = \inf_M \Phi ,$$

where M is the set of all global minima of J in X .

Proof. We already know that the function $\lambda \rightarrow \Phi(\hat{y}_\lambda)$ is non-increasing in $]a, b[$ and that its range is contained in $[\alpha, \beta]$. We claim that

$$\beta = \lim_{\lambda \rightarrow 0^+} \Phi(\hat{y}_\lambda) .$$

Assume the contrary. Let us apply Theorem 3.1, with $a = 0$ (so, $M_0 = M$), using the conclusion pointed out in Remark 3.1. Choose r satisfying

$$\lim_{\lambda \rightarrow 0^+} \Phi(\hat{y}_\lambda) < r < \beta .$$

Then, (since also $\alpha < r$) it would exist $\hat{\lambda}_r \in]0, b[$ such that $\Phi(\hat{y}_{\hat{\lambda}_r}) = r$, contrary to the choice of r . At this point, the conclusion follows directly from (3.4). \square

For the remainder of this section, X is an infinite-dimensional real Hilbert space and $\Psi : X \rightarrow \mathbf{R}$ is a sequentially weakly continuous C^1 functional, with $\Psi(0) = 0$.

For each $r > 0$, set

$$S_r = \{x \in X : \|x\|^2 = r\}$$

as well as

$$\gamma(r) = \sup_{x \in S_r} \Psi(x) .$$

Also, set

$$r^* = \inf\{r > 0 : \gamma(r) > 0\} .$$

In [25], M. Schechter and K. Tintarev developed a very elegant, transparent and precise theory which can be summarized in the following result:

Theorem 3.A. *Assume that Ψ has no local maximum in $X \setminus \{0\}$. Moreover, let $I \subseteq]r^*, +\infty[$ be an open interval such that, for each $r \in I$, there exists a unique $\hat{x}_r \in S_r$ satisfying $\Psi(\hat{x}_r) = \gamma(r)$. Then, the following conclusions hold:*

- (i₁) *the function $r \rightarrow \hat{x}_r$ is continuous in I ;*
- (i₂) *the function γ is C^1 and increasing in I ;*
- (i₃) *one has $\Psi'(\hat{x}_r) = 2\gamma'(r)\hat{x}_r$ for all $r \in I$.*

The next result can be regarded as the most complete fruit of a joint application of Theorems 3.A and 3.1.

Theorem 3.6. *Set*

$$\rho = \limsup_{\|x\| \rightarrow +\infty} \frac{\Psi(x)}{\|x\|^2} \quad \text{and} \quad \sigma = \sup_{x \in X \setminus \{0\}} \frac{\Psi(x)}{\|x\|^2} .$$

Let a, b satisfy $\max\{0, \rho\} \leq a < b \leq \sigma$. Assume that Ψ has no local maximum in $X \setminus \{0\}$, and that, for each $\lambda \in]a, b[$, the functional $x \rightarrow \lambda\|x\|^2 - \Psi(x)$ has a unique global minimum, say \hat{y}_λ . Let M_a (resp. M_b if $b < +\infty$ or $M_b = \emptyset$ if $b = +\infty$) be the set of all global minima of the functional $x \rightarrow a\|x\|^2 - \Psi(x)$ (resp. $x \rightarrow b\|x\|^2 - \Psi(x)$ if $b < +\infty$). Set

$$\alpha = \max \left\{ 0, \sup_{x \in M_b} \|x\|^2 \right\} \quad \text{and} \quad \beta = \inf_{x \in M_a} \|x\|^2 .$$

Then, the following assertions hold:

- (a₁) one has $r^* \leq \alpha < \beta$;
- (a₂) the function $\lambda \rightarrow g(\lambda) := \|\hat{y}_\lambda\|^2$ is decreasing in $]a, b[$ and its range is $] \alpha, \beta[$;
- (a₃) for each $r \in] \alpha, \beta[$, the point $\hat{x}_r := \hat{y}_{g^{-1}(r)}$ is the unique global maximum of $\Psi|_{S_r}$ towards which every maximizing sequence in S_r converges;
- (a₄) the function $r \rightarrow \hat{x}_r$ is continuous in $] \alpha, \beta[$;
- (a₅) the function γ is C^1 , increasing and strictly concave in $] \alpha, \beta[$;
- (a₆) one has $\Psi'(\hat{x}_r) = 2\gamma'(r)\hat{x}_r$ for all $r \in] \alpha, \beta[$;
- (a₇) one has $\gamma'(r) = g^{-1}(r)$ for all $r \in] \alpha, \beta[$.

Proof. First of all, observe that, by Proposition 3.1, the function g is non-increasing in $]a, b[$ and $g(]a, b[) \subseteq] \alpha, \beta[$. Now, let $I \subset]a, b[$ be a non-degenerate interval. If g was constant in I , then, by Proposition 3.1 again, the function $\lambda \rightarrow \hat{y}_\lambda$ would be constant in I . Let y^* be its unique value. Then, y^* would be a critical point of the functional $x \rightarrow \lambda\|x\|^2 - \Psi(x)$ for all $\lambda \in I$. That is to say

$$2\lambda y^* = \Psi'(y^*)$$

for all $\lambda \in I$. This would imply that $y^* = 0$, and so (since $\Psi(0) = 0$) we would have $\inf_{x \in X} (\lambda\|x\|^2 - \Psi(x)) = 0$ for all $\lambda \in I$, against the fact that $\inf_{x \in X} (\lambda\|x\|^2 - \Psi(x)) < 0$ for all $\lambda < \sigma$. Consequently, g is decreasing in $]a, b[$, and so, in particular, $\alpha < \beta$. Next, observe that

$$\lim_{\|x\| \rightarrow +\infty} (\lambda\|x\|^2 - \Psi(x)) = +\infty$$

for each $\lambda > \max\{0, \rho\}$. From this, recalling that Ψ is sequentially weakly continuous, it clearly follows that we can apply Theorem 3.1, taking $J = -\Psi$ and $\Phi(\cdot) = \|\cdot\|^2$. Consequently (see Remark 3.1), for every $r \in] \alpha, \beta[$, there exists $\lambda_r \in]a, b[$ such that $\|\hat{y}_{\lambda_r}\|^2 = r$. Therefore, by the strict monotonicity of g , we have $g(]a, b[) =] \alpha, \beta[$. Now, let us prove (a₃). Fix $r \in] \alpha, \beta[$. Clearly, we have $\|\hat{x}_r\|^2 = r$. Since

$$g^{-1}(r)\|\hat{x}_r\|^2 - \Psi(\hat{x}_r) \leq g^{-1}(r)\|x\|^2 - \Psi(x)$$

for all $x \in X$, we then have $\Psi(x) \leq \Psi(\hat{x}_r)$ for all $x \in S_r$. Hence, \hat{x}_r is a global maximum of $\Psi|_{S_r}$. On the other hand, if v is a global maximum of $\Psi|_{S_r}$, then

$$g^{-1}(r)\|v\|^2 - \Psi(v) = g^{-1}(r)\|\hat{x}_r\|^2 - \Psi(\hat{x}_r)$$

and hence, since

$$\inf_{x \in X} (g^{-1}(r)\|x\|^2 - \Psi(x)) = g^{-1}(r)\|\hat{x}_r\|^2 - \Psi(\hat{x}_r) ,$$

we have $v = \hat{x}_r$. In other words, \hat{x}_r is the unique global maximum of $\Psi|_{S_r}$. Since the sub-level sets of the functional $x \rightarrow g^{-1}(r)\|x\|^2 - \Psi(x)$ are sequentially weakly compact, any minimizing sequence of this functional in X converges weakly to \hat{x}_r . Now, let $\{w_n\}$ be any sequence in S_r such that $\lim_{n \rightarrow \infty} \Psi(w_n) = \gamma(r)$. Then, we have

$$\lim_{n \rightarrow \infty} g^{-1}(r)\|w_n\|^2 - \Psi(w_n) = \inf_{x \in X} (g^{-1}(r)\|x\|^2 - \Psi(x))$$

and so $\{w_n\}$ converges weakly to \hat{x}_r . But then, since $\lim_{n \rightarrow \infty} \|w_n\| = \|\hat{x}_r\|$ and X is a Hilbert space, we have $\lim_{n \rightarrow \infty} \|w_n - \hat{x}_r\| = 0$ by a classical result. Let us prove that $r^* \leq \alpha$. Arguing by contradiction, assume that $\alpha < r^*$. Choose $r \in]\alpha, \min\{r^*, \beta\}[$. Then, since γ is non-decreasing in $]0, +\infty[$ (see Lemma 2.1 of [25]) and Ψ is continuous, we would have $\gamma(r) = 0$, and so $\Psi(\hat{x}_r) = 0$, and this would contradict the fact that $\inf_{x \in X} (g^{-1}(r)\|x\|^2 - \Psi(x)) < 0$ since $g^{-1}(r) < \sigma$. At this point, we are allowed to apply Theorem 3.A taking $I =]\alpha, \beta[$. Consequently, the function γ is C^1 and increasing in $]\alpha, \beta[$, and (a_4) , (a_6) come directly from (i_1) , (i_3) respectively. Fix $r \in]\alpha, \beta[$ again. Since \hat{x}_r is a critical point of the functional $x \rightarrow g^{-1}(r)\|x\|^2 - \Psi(x)$, we have

$$2g^{-1}(r)\hat{x}_r = \Psi'(\hat{x}_r)$$

and then (a_7) follows from a comparison with (a_6) . Finally, from (a_7) , since g^{-1} is decreasing in $]\alpha, \beta[$, it follows that γ is strictly concave there, and the proof is complete. \square

Remark 3.2. If the derivative of Ψ is compact and if, for some $\lambda > \rho$, the functional $x \rightarrow \lambda\|x\|^2 - \Psi(x)$ has at most two critical points in X , then the same functional has a unique global minimum in X . Indeed, if this functional had at least two global minima, taken into account that it satisfies the classical Palais-Smale condition ([29], Example 38.25), it would have at least three critical points by Corollary 1 of [9].

4. A strict minimax inequality theory

In order to use the results of Section 1 to get the multiplicity of global minima, we need to know that, in the considered case, the strict minimax inequality holds. The present section is devoted to a theory on exactly this matter. To state our results in a more compact form, we now fix some notations.

Throughout this section, X is a non-empty set, Λ, Y are two topological spaces, y_0 is a point in Y . A family \mathcal{N} of non-empty subsets of X is said to be a *weakly filtering cover* of X if for each $x_1, x_2 \in X$ there is $A \in \mathcal{N}$ such that $x_1, x_2 \in A$.

We denote by \mathcal{G} the family of all lower semicontinuous functions $\varphi : Y \rightarrow [0, +\infty[$, with $\varphi^{-1}(0) = \{y_0\}$, such that, for each neighbourhood V of y_0 , one has

$$\inf_{Y \setminus V} \varphi > 0 . \tag{4.1}$$

Moreover, we denote by \mathcal{H} the family of all functions $\Psi : X \times \Lambda \rightarrow Y$ such that, for each $x \in X$, $\Psi(x, \cdot)$ is continuous, injective, open, takes the value y_0 at a point λ_x and the function $x \rightarrow \lambda_x$ is not constant. Furthermore, we denote by \mathcal{M} the family of all functions $J : X \rightarrow \mathbf{R}$ whose set of all global minima (noted by M_J) is non-empty.

Finally, for each $\varphi \in \mathcal{G}$, $\Psi \in \mathcal{H}$ and $J \in \mathcal{M}$, we put

$$\theta(\varphi, \Psi, J) = \inf \left\{ \frac{J(x) - J(u)}{\varphi(\Psi(x, \lambda_u))} : (u, x) \in M_J \times X \text{ with } \lambda_x \neq \lambda_u \right\} .$$

With such notations, our theory is summarized in the following result:

Theorem 4.1. *Let $\varphi \in \mathcal{G}$, $\Psi \in \mathcal{H}$ and $J \in \mathcal{M}$. Then, for each $\mu > \theta(\varphi, \Psi, J)$ and each weakly filtering cover \mathcal{N} of X , there exists $A \in \mathcal{N}$ such that*

$$\sup_{\lambda \in \Lambda} \inf_{x \in A} (J(x) - \mu\varphi(\Psi(x, \lambda))) < \inf_{x \in A} \sup_{z \in A} (J(x) - \mu\varphi(\Psi(x, \lambda_z))) .$$

Proof. Let $\mu > \theta(\varphi, \Psi, J)$ and let \mathcal{N} be a weakly filtering cover of X . Choose $u \in M_J$ and $x_1 \in X$, with $\lambda_{x_1} \neq \lambda_u$, such that

$$J(x_1) - \mu\varphi(\Psi(x_1, \lambda_u)) < J(u) .$$

Let $A \in \mathcal{N}$ be such that $u, x_1 \in A$. We have

$$0 \leq \inf_{z \in A} \varphi(\Psi(x, \lambda_z)) \leq \varphi(\Psi(x, \lambda_x)) = 0$$

for all $x \in A$, and so, since u is a global minimum of J , it follows that

$$\begin{aligned} \inf_{x \in A} \sup_{z \in A} (J(x) - \mu\varphi(\Psi(x, \lambda_z))) &= \inf_{x \in A} \left(J(x) - \mu \inf_{z \in A} \varphi(\Psi(x, \lambda_z)) \right) \\ &= \inf_X J = J(u) . \end{aligned} \tag{4.2}$$

Since the function $\varphi(\Psi(x_1, \cdot))$ is lower semicontinuous at λ_u , there are $\epsilon > 0$ and a neighbourhood U of λ_u such that

$$J(x_1) - \mu\varphi(\Psi(x_1, \lambda)) < J(u) - \epsilon$$

for all $\lambda \in U$. So, we have

$$\sup_{\lambda \in U} \inf_{x \in A} (J(x) - \mu\varphi(\Psi(x, \lambda))) \leq \sup_{\lambda \in U} (J(x_1) - \mu\varphi(\Psi(x_1, \lambda))) \leq J(u) - \epsilon . \tag{4.3}$$

Since $\Psi(u, \cdot)$ is open, the set $\Psi(u, U)$ is a neighbourhood of y_0 . Hence, by (4.1), we have

$$\nu := \inf_{y \in Y \setminus \Psi(u, U)} \varphi(y) > 0 . \tag{4.4}$$

Moreover, since $\Psi(u, \cdot)$ is injective, if $\lambda \notin U$ then $\Psi(u, \lambda) \notin \Psi(u, U)$. So, from (4.4), it follows that

$$\sup_{\lambda \in \Lambda \setminus U} \inf_{x \in A} (J(x) - \mu\varphi(\Psi(x, \lambda))) \leq J(u) - \mu \inf_{\lambda \in \Lambda \setminus U} \varphi(\Psi(u, \lambda)) \leq J(u) - \mu\nu. \quad (4.5)$$

Now, the conclusion follows directly from (4.2), (4.3), (4.4) and (4.5). \square

Remark 4.1. From the conclusion of Theorem 4.1 it clearly follows that, for any set $D \subseteq \Lambda$ with $\lambda_x \in D$ for all $x \in A$, one has

$$\sup_{\lambda \in D} \inf_{x \in A} (J(x) - \mu\varphi(\Psi(x, \lambda))) < \inf_{x \in A} \sup_{\lambda \in D} (J(x) - \mu\varphi(\Psi(x, \lambda))).$$

Remark 4.2. From the definition of $\theta(\varphi, \Psi, J)$, it clearly follows that $u \in M_J$ if and only if u is a global minimum of the function $x \rightarrow J(x) - \theta(\varphi, \Psi, J)\varphi(\Psi(x, \lambda_u))$. So, when $\theta(\varphi, \Psi, J) > 0$, from knowing that

$$J(u) \leq J(x)$$

for all $x \in X$, we automatically get

$$J(u) \leq J(x) - \theta(\varphi, \Psi, J)\varphi(\Psi(x, \lambda_u))$$

for all $x \in X$, which is a much better inequality since $\varphi(y) > 0$ for all $y \in Y \setminus \{y_0\}$.

Remark 4.3. It is likewise important to observe that if $\theta(\varphi, \Psi, J) > 0$, then the function $x \rightarrow \lambda_x$ is constant in M_J . As a consequence, if $\theta(\varphi, \Psi, J) > 0$ and the function $x \rightarrow \lambda_x$ is injective, then J has a unique global minimum. In particular, note that $x \rightarrow \lambda_x$ is injective when $\Psi(\cdot, \lambda)$ is injective for all $\lambda \in \Lambda$.

Remark 4.4. Remarks 4.2 and 4.3 show the importance of knowing when $\theta(\varphi, \Psi, J) > 0$. Theorem 4.1 can also be useful for this. Indeed, if for some $\mu > 0$, there is a weakly filtering cover \mathcal{N} of X such that

$$\sup_{\lambda \in \Lambda} \inf_{x \in A} (J(x) - \mu\varphi(\Psi(x, \lambda))) \geq \inf_{x \in A} \sup_{z \in A} (J(x) - \mu\varphi(\Psi(x, \lambda_z)))$$

for all $A \in \mathcal{N}$, then $\theta(\varphi, \Psi, J) \geq \mu$.

Notice the following consequence of Theorem 4.1:

Theorem 4.2. *Let Y be a inner product space, and let $I : X \rightarrow \mathbf{R}$, $\Phi : X \rightarrow Y$ and $\mu > 0$ be such that the function $x \rightarrow I(x) + \mu\|\Phi(x)\|^2$ has a global minimum.*

Then, at least one of the following assertions holds:

(a) *for each weakly filtering cover \mathcal{N} of X , there exists $A \in \mathcal{N}$ such that*

$$\sup_{\lambda \in Y} \inf_{x \in A} (I(x) + \mu(2\langle \Phi(x), \lambda \rangle - \|\lambda\|^2)) < \inf_{x \in A} \sup_{\lambda \in \Phi(A)} (I(x) + \mu(2\langle \Phi(x), \lambda \rangle - \|\lambda\|^2));$$

(b) *for each global minimum u of $x \rightarrow I(x) + \mu\|\Phi(x)\|^2$, one has for all $x \in X$*

$$I(u) \leq I(x) + 2\mu(\langle \Phi(x), \Phi(u) \rangle - \|\Phi(u)\|^2).$$

Proof. Take $\Lambda = Y$, $y_0 = 0$. For each $x \in X$, $y, \lambda \in Y$, set

$$\varphi(y) = \|y\|^2, \quad \Psi(x, \lambda) = \Phi(x) - \lambda \quad \text{and} \quad J(x) = I(x) + \mu\|\Phi(x)\|^2,$$

so that

$$J(x) - \mu\varphi(\Psi(x, \lambda)) = I(x) + \mu(2\langle\Phi(x), \lambda\rangle - \|\lambda\|^2).$$

With these choices, (b) is equivalent to the inequality $\mu \leq \theta(\varphi, \Psi, J)$. Now, the conclusion is a direct consequence of Theorem 4.1. \square

In turn, from Theorem 4.2, we get

Theorem 4.3. *Let X be a non-empty set, $x_0 \in X$, Y a real inner product space, $I : X \rightarrow \mathbf{R}$, $\Phi : X \rightarrow Y$, with $I(x_0) = 0$, $\Phi(x_0) = 0$, and $\mu > 0$. Assume that*

$$\inf_{x \in X} I(x) < 0 \leq \inf_{x \in X} (I(x) + \mu\|\Phi(x)\|^2).$$

Then, for each weakly filtering cover \mathcal{N} of X , there exists $A \in \mathcal{N}$ such that

$$\sup_{y \in Y} \inf_{x \in A} (I(x) + \mu(2\langle\Phi(x), y\rangle - \|y\|^2)) < \inf_{x \in A} \sup_{y \in \Phi(A)} (I(x) + \mu(2\langle\Phi(x), y\rangle - \|y\|^2)).$$

Proof. The assumptions imply that x_0 is a global minimum of $x \rightarrow I(x) + \mu\|\Phi(x)\|^2$. But, at the same time, since $\inf_X I < 0$, x_0 is not a global minimum of I . Hence, (b) of Theorem 4.2 does not hold and so (a) holds. \square

5. Multiplicity of global minima

In this section, we apply the results stated in Section 1 to obtain multiple global minima.

Theorem 5.1. *Let X be a topological space and $J, \Phi : X \rightarrow \mathbf{R}$ two functions satisfying the following conditions:*

- (a₁) *for each $\lambda > 0$, the function $J + \lambda\Phi$ has compact and closed sub-level sets;*
- (b₁) *there exist $\rho \in]\inf_X \Phi, \sup_X \Phi[$ and $u_1, u_2 \in X$ such that*

$$\Phi(u_1) < \rho < \Phi(u_2)$$

and

$$\frac{J(u_1) - \inf_{\Phi^{-1}(]-\infty, \rho])} J}{\rho - \Phi(u_1)} < \frac{J(u_2) - \inf_{\Phi^{-1}(]-\infty, \rho])} J}{\rho - \Phi(u_2)}.$$

Under these hypotheses, there exists $\lambda^ > 0$ such that the function $J + \lambda^*\Phi$ has at least two global minima.*

Proof. Observe that, in view of Theorem 1 of [1], condition (b₁) is equivalent to the inequality

$$\sup_{\lambda \geq 0} \inf_{x \in X} (J(x) + \lambda(\Phi(x) - \rho)) < \inf_{x \in X} \sup_{\lambda \geq 0} (J(x) + \lambda(\Phi(x) - \rho)).$$

On the other hand, since the function $\lambda \rightarrow \inf_{x \in X} (J(x) + \lambda(\Phi(x) - \rho))$ is concave (and real-valued) in $]0, +\infty[$, it is lower semicontinuous in $]0, +\infty[$ and so

$$\sup_{\lambda \geq 0} \inf_{x \in X} (J(x) + \lambda(\Phi(x) - \rho)) = \sup_{\lambda > 0} \inf_{x \in X} (J(x) + \lambda(\Phi(x) - \rho)).$$

Consequently, condition (b_1) is equivalent to the inequality

$$\sup_{\lambda > 0} \inf_{x \in X} (J(x) + \lambda(\Phi(x) - \rho)) < \inf_{x \in X} \sup_{\lambda > 0} (J(x) + \lambda(\Phi(x) - \rho)).$$

Now, we can apply Theorem 1.A taking $I =]0, +\infty[$ and

$$\Psi(x, \lambda) = J(x) + \lambda(\Phi(x) - \rho),$$

and the conclusion follows. □

A suitable application of Theorem 5.1 gives the following result:

Theorem 5.2. *Let S be a topological space and $F, \Phi : S \rightarrow \mathbf{R}$ two lower semicontinuous functions satisfying the following conditions:*

(a₂) the function Φ is inf-compact ;

(b₂) for some $a > 0$, one has $\inf_{x \in \Phi^{-1}(]a, +\infty[)} \frac{F(x)}{\Phi(x)} = -\infty$.

Under these hypotheses, for each ρ large enough, there exists $\lambda_\rho^ > 0$ such that the restriction of the function $F + \lambda_\rho^* \Phi$ to $\Phi^{-1}(]-\infty, \rho])$ has at least two global minima.*

Proof. Fix $\rho_0 > \inf_X \Phi$, $x_0 \in \Phi^{-1}(]-\infty, \rho_0])$ and λ satisfying

$$\lambda > \frac{F(x_0) - \inf_{\Phi^{-1}(]-\infty, \rho_0])} F}{\rho_0 - \Phi(x_0)} .$$

Hence, one has

$$F(x_0) + \lambda \Phi(x_0) < \lambda \rho_0 + \inf_{\Phi^{-1}(]-\infty, \rho_0])} F . \tag{5.1}$$

Since $\Phi^{-1}(]-\infty, \rho_0])$ is compact, lower semicontinuity implies the existence of $\hat{x} \in \Phi^{-1}(]-\infty, \rho_0])$ such that

$$F(\hat{x}) + \lambda \Phi(\hat{x}) = \inf_{x \in \Phi^{-1}(]-\infty, \rho_0])} (F(x) + \lambda \Phi(x)) . \tag{5.2}$$

We claim that $\Phi(\hat{x}) < \rho_0$. Arguing by contradiction, assume that $\Phi(\hat{x}) \geq \rho_0$. Then, in view of (5.1), we would have

$$F(x_0) + \lambda \Phi(x_0) < F(\hat{x}) + \lambda \Phi(\hat{x})$$

against (5.2). By (b_2) , there is a sequence $\{u_n\}$ in $\Phi^{-1}(]a, +\infty[)$ such that

$$\lim_{n \rightarrow \infty} \frac{F(u_n)}{\Phi(u_n)} = -\infty .$$

Now, set

$$\gamma = \min \left\{ 0, \inf_{x \in \Phi^{-1}(]-\infty, \rho_0])} (F(x) + \lambda\Phi(x)) \right\}$$

and fix $\hat{n} \in \mathbf{N}$ so that

$$\frac{F(u_{\hat{n}})}{\Phi(u_{\hat{n}})} < -\lambda + \frac{\gamma}{a} .$$

We then have

$$F(u_{\hat{n}}) + \lambda\Phi(u_{\hat{n}}) < \frac{\gamma}{a}\Phi(u_{\hat{n}}) \leq \gamma .$$

Hence, if we put $\rho^* = \Phi(u_{\hat{n}})$, we have

$$\inf_{x \in \Phi^{-1}(]-\infty, \rho^*])} (F(x) + \lambda\Phi(x)) < \inf_{x \in \Phi^{-1}(]-\infty, \rho_0])} (F(x) + \lambda\Phi(x)) .$$

At this point, for each $\rho \geq \rho^*$, we realize that it is possible to apply Theorem 5.1 taking $X = \Phi^{-1}(]-\infty, \rho])$ and $J = F + \lambda\Phi$. Indeed, with these choices and taking $u_1 = \hat{x}$, $u_2 = u_{\hat{n}}$, the left-hand side of the last inequality in (b_1) is zero, while the right-hand side is positive. Consequently, there exists $\hat{\lambda}_\rho > 0$ such that the restriction of the function $F + \lambda\Phi + \hat{\lambda}_\rho\Phi$ to $\Phi^{-1}(]-\infty, \rho])$ has at least two global minima. So, the conclusion follows taking $\lambda_\rho^* = \lambda + \hat{\lambda}_\rho$. \square

It is worth noticing the following consequence of Theorem 5.2.

Theorem 5.3. *Let S be a cone in a real vector space equipped with a (not necessarily vector) topology and let $F, \Phi : S \rightarrow \mathbf{R}$ be two lower semicontinuous functions satisfying the following conditions:*

- (a_3) *the function Φ is positively homogeneous of degree α and inf-compact;*
- (b_3) *the function F is positively homogeneous of degree $\beta > \alpha$ and there is $\tilde{x} \in S$ such that $F(\tilde{x}) < 0 < \Phi(\tilde{x})$.*

Under these hypotheses, there exists $\rho^ > \inf_S \Phi$ such that the restriction of the function $F + \Phi$ to $\Phi^{-1}(]-\infty, \rho^*])$ has at least two global minima.*

Proof. Clearly, we have

$$\lim_{\lambda \rightarrow +\infty} \frac{F(\lambda\tilde{x})}{\Phi(\lambda\tilde{x})} = \lim_{\lambda \rightarrow +\infty} \frac{F(\tilde{x})}{\Phi(\tilde{x})} \lambda^{\beta-\alpha} = -\infty .$$

So, the hypotheses of Theorem 5.2 are satisfied and hence there exist $\rho > \inf_S \Phi$ and $\lambda > 0$ such that the restriction of the function $F + \lambda\Phi$ to $\Phi^{-1}(]-\infty, \rho])$ has at least two global minima, say v_1, v_2 . Now, observe that

$$\lambda^{\frac{\beta}{\alpha-\beta}} (F(x) + \lambda\Phi(x)) = F(\lambda^{\frac{1}{\alpha-\beta}} x) + \Phi(\lambda^{\frac{1}{\alpha-\beta}} x)$$

for all $x \in S$. From this, it easily follows that the points $\lambda^{\frac{1}{\alpha-\beta}} v_1$ and $\lambda^{\frac{1}{\alpha-\beta}} v_2$ are two global minima of the restriction of the function $F + \Phi$ to $\Phi^{-1}(]-\infty, \lambda^{\frac{\alpha}{\alpha-\beta}} \rho])$, that is the conclusion. \square

Remark 5.1. We also remark that the number ρ^* in the conclusion of Theorem 5.3 can be unique. In this connection, a very simple example is provided by taking $S = \mathbf{R}$, $\Phi(x) = x^2$ and $F(x) = -x^3$. Actually, it is seen at once that, if $r > 0$, the restriction of the function $x \rightarrow x^2 - x^3$ to $[-r, r]$ has a unique global minimum when $r \neq 1$ and exactly two global minima when $r = 1$.

With the notations of Section 4, a joint application of Theorem 1.2 and Theorem 4.1 gives

Theorem 5.4. *Let $\varphi \in \mathcal{G}$, $\Psi \in \mathcal{H}$ and $J \in \mathcal{M}$. Moreover, assume that X is a topological space, that Λ is a real vector space and that $\varphi(\Psi(x, \cdot))$ is convex for each $x \in X$. Finally, let $\mu > \theta(\varphi, \Psi, J)$ and \mathcal{N} be a weakly filtering cover of X such that, for each $A \in \mathcal{N}$, the function $x \rightarrow J(x) - \mu\varphi(\Psi(x, \lambda))$ is lower semicontinuous and inf-compact in A for all $\lambda \in \text{conv}(\{\lambda_x : x \in X\})$.*

Under these hypotheses, there exist $A \in \mathcal{N}$ and $\lambda^ \in \text{conv}(\{\lambda_x : x \in A\})$ such that the restriction of the function $x \rightarrow J(x) - \mu\varphi(\Psi(x, \lambda^*))$ to A has at least two global minima.*

Proof. For each $(x, \lambda) \in X \times \Lambda$, put $f(x, \lambda) = J(x) - \mu\varphi(\Psi(x, \lambda))$. By Theorem 4.1, there exists $A \in \mathcal{N}$ such that

$$\sup_{\lambda \in D} \inf_{x \in A} f(x, \lambda) < \inf_{x \in A} \sup_{\lambda \in D} f(x, \lambda),$$

where $D = \text{conv}(\{\lambda_x : x \in A\})$. Now, the conclusion follows directly applying Theorem 1.2 to the restriction of f to $A \times D$. □

A non-empty set C in a normed space S is said to be *uniquely remotal* with respect to a set $D \subseteq S$ if, for each $y \in D$, there exists a unique $x \in C$ such that

$$\|x - y\| = \sup_{u \in C} \|u - y\| .$$

The main problem in the theory of such sets is to know if they are singletons.

If E, F are two real vector spaces and D is a convex subset of E , we say that an operator $\Phi : D \rightarrow F$ is *affine* if

$$\Phi(\lambda x + (1 - \lambda)y) = \lambda\Phi(x) + (1 - \lambda)\Phi(y)$$

for all $x, y \in D$, $\lambda \in [0, 1]$.

The next three results are applications of Theorem 5.4.

Theorem 5.5. *Let Y be a real normed space and let $X \subseteq Y$ be a non-empty compact uniquely remotal set with respect to $\text{conv}(X)$. Then, X is a singleton.*

Proof. Arguing by contradiction, assume that X contains at least two points. Now, apply Theorem 5.4 taking: $\Lambda = Y$, $y_0 = 0$, $\varphi(x) = \|x\|$, $\Psi(x, \lambda) = x - \lambda$, $J = 0$ and $\mathcal{N} = \{X\}$. Note that we are allowed to apply Theorem 5.4 since $x \rightarrow \lambda_x$ is not constant. Then, it would exist $\lambda^* \in \text{conv}(X)$ such that the function $x \rightarrow -\|x - \lambda^*\|$ has at least two global minima in X , against the hypotheses. □

Remark 5.2. Observe that Theorem 5.5 improves a classical result by V. L. Klee ([5]) under two aspects: Y does not need to be complete and $\overline{\text{conv}}(X)$ is replaced by $\text{conv}(X)$. Note also that our proof is completely different from that of Klee which is based on the Schauder fixed point theorem.

Theorem 5.6. *Let X be a finite-dimensional real Hilbert space and $J : X \rightarrow \mathbf{R}$ a C^1 function. Set*

$$\eta = \liminf_{\|x\| \rightarrow +\infty} \frac{J(x)}{\|x\|^2}$$

and

$$\theta = \inf \left\{ \frac{J(x) - J(u)}{\|x - u\|^2} : (u, x) \in M_J \times X \text{ with } x \neq u \right\},$$

where M_J denotes the set of all global minima of J . Assume that $\theta < \eta$. Then, for each $\mu \in]2\theta, 2\eta[$, there exists $y_\mu \in X$ such that the equation

$$J'(x) - \mu x = y_\mu$$

has at least three solutions.

Proof. Let $\mu \in]2\theta, 2\eta[$. We clearly have

$$\lim_{\|x\| \rightarrow +\infty} \left(J(x) - \frac{\mu}{2} \|x - \lambda\|^2 \right) = +\infty \quad (5.3)$$

for all $\lambda \in X$. So, since X is finite-dimensional, the function $x \rightarrow J(x) - \frac{\mu}{2} \|x - \lambda\|^2$ is continuous and inf-compact for all $\lambda \in X$. Therefore, we can apply Theorem 5.4 taking: $X = Y = \Lambda$, $y_0 = 0$, $\varphi(y) = \|y\|^2$, $\Psi(x, \lambda) = x - \lambda$ and $\mathcal{N} = \{X\}$. Consequently, there exists $\lambda_\mu^* \in X$, such that the function $x \rightarrow J(x) - \frac{\mu}{2} \|x - \lambda_\mu^*\|^2$ has at least two global minima. By (5.3) and the finite-dimensionality of X again, the same function satisfies the Palais-Smale condition, and so it admits at least three critical points, thanks to Corollary 1 of [9]. Of course, this gives the conclusion, taking $y_\mu = \lambda_\mu^*$. \square

Remark 5.3. Clearly, there are two situations in which Theorem 5.6 can immediately be applied: when $\eta = +\infty$, and when $\eta > 0$ and $\theta = 0$. Note that one has $\theta = 0$ if, in particular, J possesses at least two global minima.

smallskip

Remark 5.4. It is also clear that under the same assumptions as those of Theorem 5.6 but the finite-dimensionality of X , the conclusion is still true for every $\mu \in]2\theta, 2\eta[$ such that, for each $\lambda \in X$, the functional $x \rightarrow J(x) - \frac{\mu}{2} \|x - \lambda\|^2$ is weakly lower semicontinuous and satisfies the Palais-Smale condition.

Theorem 5.7. *Let Y be a finite-dimensional real Hilbert space, $J : Y \rightarrow \mathbf{R}$ a C^1 function with locally Lipschitzian derivative, and $\varphi : Y \rightarrow [0, +\infty[$ a C^1 convex function with locally Lipschitzian derivative at 0 and $\varphi^{-1}(0) = \{0\}$.*

Then, for each $x_0 \in Y$ for which $J'(x_0) \neq 0$, there exists $\delta > 0$ such that, for each $r \in]0, \delta[$, the restriction of J to $B(x_0, r)$ has a unique global minimum u_r which satisfies

$$J(u_r) \leq J(x) - \varphi(x - u_r)$$

for all $x \in B(x_0, r)$, where $B(x_0, r) = \{x \in Y : \|x - x_0\| \leq r\}$.

Proof. First of all, observe that $\varphi \in \mathcal{G}$, with $y_0 = 0$. Indeed, let $V \subset Y$ be a neighbourhood of 0 and let $s > 0$ be such that $B(0, s) \subseteq V$. Set

$$\alpha = \inf_{\|x\|=s} \varphi(x) .$$

Since $\dim(Y) < \infty$, $\partial B(0, s)$ is compact and so $\alpha > 0$. Let $x \in Y$ with $\|x\| > s$. Let S be the segment joining 0 and x . By convexity, we have $\varphi(z) \leq \varphi(x)$ for all $z \in S$. Since S meets $\partial B(0, s)$, we infer that $\alpha \leq \varphi(x)$. Hence, we have

$$\alpha \leq \inf_{\|x\|>s} \varphi(x) \leq \inf_{x \in X \setminus V} \varphi(x) .$$

Now, fix $x_0 \in Y$ with $J'(x_0) \neq 0$. Taking into account that $\varphi'(0) = 0$, by continuity, we can choose $\sigma > 0$ so that $\|\varphi'(\lambda)\| < \|J'(x)\|$ for all $(x, \lambda) \in B(x_0, \sigma) \times B(0, \sigma)$. For each $(x, \lambda) \in Y \times Y$, put

$$f(x, \lambda) = J(x) - \varphi(x - x_0 - \lambda) .$$

Of course, we have $f'_x(x, \lambda) \neq 0$ for all $(x, \lambda) \in B(x_0, \frac{\sigma}{2}) \times B(0, \frac{\sigma}{2})$. Next, since J' is locally Lipschitzian at x_0 and φ' is locally Lipschitzian at 0, there are $\rho \in]0, \frac{\sigma}{2}]$ and $L > 0$ such that

$$\|f'_x(x, \lambda) - f'_x(y, \lambda)\| \leq L\|x - y\|$$

for all $x, y \in B(x_0, \rho)$, $\lambda \in B(0, \rho)$. Now, fix $\lambda \in B(0, \rho)$. Denote by Γ_λ the set of all global minima of the restriction of the function $x \rightarrow f(x, \lambda) + \frac{L}{2}\|x - x_0\|^2$ to $B(x_0, \rho)$. Note that $x_0 \notin \Gamma_\lambda$ (since $f'_x(x_0, \lambda) \neq 0$). As f is continuous, the multifunction $\lambda \rightarrow \Gamma_\lambda$ is upper semicontinuous and so the function $\lambda \rightarrow \text{dist}(x_0, \Gamma_\lambda)$ is lower semicontinuous. As a consequence, by compactness, we have

$$\delta := \inf_{\lambda \in B(0, \rho)} \text{dist}(x_0, \Gamma_\lambda) > 0 .$$

At this point, from the proof of Theorem 1 of [11] it follows that, for each $\lambda \in B(0, \rho)$ and each $r \in]0, \delta[$, the restriction of function $f(\cdot, \lambda)$ to $B(x_0, r)$ has a unique global minimum. Fix $r \in]0, \delta[$. Apply Theorem 5.4 with $X = B(x_0, r)$, $\Lambda = Y$, $\mathcal{N} = \{B(x_0, r)\}$ and $\Psi(x, \lambda) = x - x_0 - \lambda$. With such choices, its conclusion does not hold with $\mu = 1$ (recall, in particular, that $r < \rho$). This implies that $1 \leq \theta(\varphi, \Psi, J)$ since the other assumptions are satisfied. But the above inequality is just equivalent to

$$J(u_r) \leq J(x) - \varphi(x - u_r)$$

for all $x \in B(x_0, r)$, where u_r is the unique global minimum of $J|_{B(x_0, r)}$, and the proof is complete. \square

A joint application of Theorems 1.1 and 4.1 gives

Theorem 5.8. *Let $\varphi \in \mathcal{G}$, $\Psi \in \mathcal{H}$ and $J \in \mathcal{M}$. Moreover, assume that X is a topological space, that Λ is a real topological vector space and that $\varphi(\Psi(x, \cdot))$ is quasi-convex and continuous for each $x \in X$. Finally, assume $\mu > \theta(\varphi, \Psi, J)$ and let $C \subseteq \Lambda$ be a convex set, with $\{\lambda_x : x \in X\} \subseteq \overline{C}$, such that the function $x \rightarrow J(x) - \mu\varphi(\Psi(x, \lambda))$ is lower semicontinuous and inf-compact in X for all $\lambda \in C$.*

Under these hypotheses, there exists $\lambda^ \in C$ such that the function $x \rightarrow J(x) - \mu\varphi(\Psi(x, \lambda^*))$ has at least two global minima in X .*

Proof. Set $D = \{\lambda_x : x \in X\}$ and, for each $(x, \lambda) \in X \times \Lambda$, put

$$f(x, \lambda) = J(x) - \mu\varphi(\Psi(x, \lambda)) .$$

Theorem 4.1 ensures that

$$\sup_{\lambda \in \Lambda} \inf_{x \in X} f(x, \lambda) < \inf_{x \in X} \sup_{\lambda \in D} f(x, \lambda) . \tag{5.4}$$

But, since $f(x, \cdot)$ is continuous and $D \subseteq \overline{C}$, we have

$$\sup_{\lambda \in D} f(x, \lambda) = \sup_{\lambda \in \overline{D}} f(x, \lambda) \leq \sup_{\lambda \in \overline{C}} f(x, \lambda) = \sup_{\lambda \in C} f(x, \lambda)$$

for all $x \in X$, and hence, from (5.4), it follows that

$$\sup_{\lambda \in C} \inf_{x \in X} f(x, \lambda) < \inf_{x \in X} \sup_{\lambda \in D} f(x, \lambda) \leq \inf_{x \in X} \sup_{\lambda \in C} f(x, \lambda) .$$

At this point, the conclusion follows applying Theorem 1.1 to the restriction of the function f to $X \times C$. \square

The next result comes from a joint application of Theorems 4.3 and 1.C.

Theorem 5.9. *Let X be a real inner product space and let τ be a topology on X . Moreover, let $J : X \rightarrow \mathbf{R}$ be a functional such that*

$$J(0) = 0 < \sup_X J \quad \text{and} \quad \beta^* := \sup_{x \in X \setminus \{0\}} \frac{J(x)}{\|x\|^2} < +\infty . \tag{5.5}$$

Finally, let $\lambda > \frac{1}{\beta^}$ and let \mathcal{N} be a weakly filtering cover of X such that, for each $A \in \mathcal{N}$ and each $y \in X$, the restriction to A of the functional $x \rightarrow \|x\|^2 - \lambda J(x) + \langle x, y \rangle$ is τ -lower semicontinuous and inf- τ -compact.*

Then, there exists $\tilde{A} \in \mathcal{N}$ with the following property: for every convex set $C \subseteq X$ whose closure (in the strong topology) contains \tilde{A} , there exists $\tilde{y} \in C$ such that the restriction to \tilde{A} of the functional $x \rightarrow \|x\|^2 - \lambda J(x) + \langle x, 2(\beta^ \lambda - 1)\tilde{y} \rangle$ has at least two global minima.*

Proof. In view of (5.5), we have $\inf_{x \in X} (\|x\|^2 - \lambda J(x)) < 0$, as well as

$$\inf_{x \in X} (\|x\|^2 - \lambda J(x) + (\beta^* \lambda - 1) \|x\|^2) \geq 0 .$$

So, we can apply Theorem 4.3 taking $Y = X$, $\mu = \beta^* \lambda - 1$, $I(x) = \|x\|^2 - \lambda J(x)$ and $\Phi(x) = x$. Therefore, there exists $\tilde{A} \in \mathcal{N}$ such that

$$\begin{aligned} \sup_{y \in Y} \inf_{x \in \tilde{A}} (\|x\|^2 - \lambda J(x) + (\beta^* \lambda - 1)(2\langle x, y \rangle - \|y\|^2)) < \\ < \inf_{x \in \tilde{A}} \sup_{y \in \tilde{A}} (\|x\|^2 - \lambda J(x) + (\beta^* \lambda - 1)(2\langle x, y \rangle - \|y\|^2)) . \end{aligned} \quad (5.6)$$

Now, consider the function $f : X \times X \rightarrow \mathbf{R}$ defined by

$$f(x, y) = \|x\|^2 - \lambda J(x) + (\beta^* \lambda - 1)(2\langle x, y \rangle - \|y\|^2)$$

for all $(x, y) \in X \times X$. Since $f(x, \cdot)$ is continuous and $\tilde{A} \subseteq \overline{C}$, we have

$$\sup_{y \in \tilde{A}} f(x, y) = \sup_{v \in \tilde{A}} f(x, v) \leq \sup_{v \in \overline{C}} f(x, v) = \sup_{y \in C} f(x, y)$$

for all $x \in X$, and hence, taking (5.6) into account, it follows that

$$\sup_{y \in C} \inf_{x \in \tilde{A}} f(x, y) < \inf_{x \in \tilde{A}} \sup_{y \in \tilde{A}} f(x, y) \leq \inf_{x \in \tilde{A}} \sup_{y \in C} f(x, y) . \quad (5.7)$$

Now, in view of (5.7), taking into account that $f|_{\tilde{A} \times C}$ is τ -lower semicontinuous and inf- τ -compact in \tilde{A} , and continuous and concave in C , we can apply Theorem 1.C to $f|_{\tilde{A} \times C}$. Consequently, there exists $\tilde{y} \in C$ such that $f|_{\tilde{A}}(\cdot, \tilde{y})$ has at least two global minima, and the proof is complete. \square

In turn, from Theorem 5.9, we get

Theorem 5.10. *Let X be a real Hilbert space and let $J : X \rightarrow \mathbf{R}$ be a C^1 functional, with compact derivative, such that*

$$\alpha^* := \max \left\{ 0, \limsup_{\|x\| \rightarrow +\infty} \frac{J(x)}{\|x\|^2} \right\} < \beta^* := \sup_{x \in X \setminus \{0\}} \frac{J(x)}{\|x\|^2} < +\infty .$$

Then, for every $\lambda \in \left] \frac{1}{2\beta^}, \frac{1}{2\alpha^*} \right[$ and for every convex set $C \subseteq X$ dense in X , there exists $\tilde{y} \in C$ such that the equation*

$$x = \lambda J'(x) + \tilde{y}$$

has at least three solutions, two of which are global minima of the functional $x \rightarrow \frac{1}{2} \|x\|^2 - \lambda J(x) - \langle x, \tilde{y} \rangle$.

Proof. Fix $\lambda \in \left] \frac{1}{2\beta^*}, \frac{1}{2\alpha^*} \right[$ and a convex set $C \subseteq X$ dense in X . For each $y \in X$, we have

$$\liminf_{\|x\| \rightarrow +\infty} \left(1 - 2\lambda \frac{J(x)}{\|x\|^2} - \frac{\langle x, y \rangle}{\|x\|^2} \right) = 1 - 2\lambda \limsup_{\|x\| \rightarrow +\infty} \frac{J(x)}{\|x\|^2} > 0 .$$

So, from the identity

$$\|x\|^2 - 2\lambda J(x) - \langle x, y \rangle = \|x\|^2 \left(1 - 2\lambda \frac{J(x)}{\|x\|^2} - \frac{\langle x, y \rangle}{\|x\|^2} \right)$$

it follows that

$$\lim_{\|x\| \rightarrow +\infty} (\|x\|^2 - 2\lambda J(x) - \langle x, y \rangle) = +\infty . \tag{5.8}$$

Since J' is compact, J is sequentially weakly continuous ([29], Corollary 41.9). Then, in view of (5.8) and of the Eberlein-Smulyan theorem, for each $y \in X$, the functional $x \rightarrow \|x\|^2 - 2\lambda J(x) + \langle x, y \rangle$ is inf-weakly compact in X . So, we can apply Theorem 5.9 taking the weak topology as τ and $\mathcal{N} = \{X\}$. Consequently, since the set $\frac{1}{1-2\beta^*\lambda}C$ is convex and dense in X , there exists $\hat{y} \in \frac{1}{1-2\beta^*\lambda}C$ such that the functional $x \rightarrow \|x\|^2 - 2\lambda J(x) + \langle x, 2(2\beta^*\lambda - 1)\hat{y} \rangle$ has at least two global minima in X which are two of its critical points. Since the same functional satisfies the Palais-Smale condition ([29], Example 38.25), it has a third critical point in view of Corollary 1 of [9]. The conclusion follows taking $\tilde{y} = (1 - 2\beta^*\lambda)\hat{y}$. \square

We conclude this section with a further consequence of Theorem 1.2.

Let us introduce the following notations. We denote by \mathbf{R}^X the space of all functionals $\varphi : X \rightarrow \mathbf{R}$. For each $I \in \mathbf{R}^X$ and for each of non-empty subset A of X , we denote by $E_{I,A}$ the set of all $\varphi \in \mathbf{R}^X$ such that $I + \varphi$ is sequentially weakly lower semicontinuous and coercive, and $\inf_A \varphi \leq 0$.

Theorem 5.11. *Let $I : X \rightarrow \mathbf{R}$ be a functional and A, B two non-empty subsets of X such that*

$$\sup_A I < \inf_B I . \tag{5.9}$$

Then, for every convex set $Y \subseteq E_{I,A}$ such that

$$\inf_{x \in B} \sup_{\varphi \in Y} \varphi(x) \geq 0 \text{ and } \inf_{x \in X \setminus B} \sup_{\varphi \in Y} \varphi(x) = +\infty , \tag{5.10}$$

there exists $\tilde{\varphi} \in Y$ such that the functional $I + \tilde{\varphi}$ has at least two global minima.

Proof. Consider the function $f : X \times \mathbf{R}^X \rightarrow \mathbf{R}$ defined by

$$f(x, \varphi) = I(x) + \varphi(x)$$

for all $x \in X$ and $\varphi \in \mathbf{R}^X$. Fix $\varphi \in Y$. In view of (5.9), we can also fix an $\epsilon \in]0, \inf_B I - \sup_A I[$. Since $\inf_A \varphi \leq 0$, there is $\bar{x} \in A$ such that $\varphi(\bar{x}) < \epsilon$. Hence, we have

$$\inf_{x \in X} (I(x) + \varphi(x)) \leq I(\bar{x}) + \varphi(\bar{x}) < \sup_A I + \epsilon ,$$

from which it follows that

$$\sup_{\varphi \in Y} \inf_{x \in X} (I(x) + \varphi(x)) \leq \sup_A I + \epsilon < \inf_B I . \tag{5.11}$$

On the other hand, in view of (5.10), one has

$$\begin{aligned} \inf_B I &\leq \inf_{x \in B} (I(x) + \sup_{\varphi \in Y} \varphi(x)) = \inf_{x \in B} \sup_{\varphi \in Y} (I(x) + \varphi(x)) \\ &= \inf_{x \in X} \sup_{\varphi \in Y} (I(x) + \varphi(x)). \end{aligned} \tag{5.12}$$

Finally, from (5.11) and (5.12), it follows that

$$\sup_{\varphi \in Y} \inf_{x \in X} f(x, \varphi) < \inf_{x \in X} \sup_{\varphi \in Y} f(x, \varphi).$$

Therefore, the function f satisfies the assumptions of Theorem 1.2, and the conclusion follows. □

Notice the following remarkable corollary of Theorem 5.11:

Corollary 5.1. *Let $I : X \rightarrow \mathbf{R}$ be a sequentially weakly lower semicontinuous, non-convex functional such that $I + \varphi$ is coercive for all $\varphi \in X^*$. Then, for every convex set $Y \subseteq X^*$, dense in X^* , there exists $\tilde{\varphi} \in Y$ such that the functional $I + \tilde{\varphi}$ has at least two global minima.*

Proof. Since I is not convex, there exist $x_1, x_2 \in X$ and $\lambda \in]0, 1[$ such that

$$\lambda I(x_1) + (1 - \lambda)I(x_2) < I(x_3)$$

where $x_3 = \lambda x_1 + (1 - \lambda)x_2$. Fix $\psi \in X^*$ so that

$$\psi(x_1) - \psi(x_2) = I(x_1) - I(x_2)$$

and put

$$\tilde{I}(x) = I(x_3 - x) - \psi(x_3 - x)$$

for all $x \in X$. It is easy to check that

$$\tilde{I}(\lambda(x_1 - x_2)) = \tilde{I}((1 - \lambda)(x_2 - x_1)) < \tilde{I}(0). \tag{5.13}$$

Fix a convex set $Y \subseteq X^*$ dense in X^* and put $\tilde{Y} = -Y - \psi$. Hence, \tilde{Y} is convex and dense in X^* too. Now, set

$$A = \{\lambda(x_1 - x_2), (1 - \lambda)(x_2 - x_1)\}.$$

Clearly, we have

$$X^* \subset E_{\tilde{I}, A}. \tag{5.14}$$

Since \tilde{Y} is dense in X^* , we have

$$\sup_{\varphi \in \tilde{Y}} \varphi(x) = +\infty$$

for all $x \in X \setminus \{0\}$. Hence, in view of (5.13) and (5.14), we can apply Theorem 5.11 with $B = \{0\}$, $I = \tilde{I}$, $Y = \tilde{Y}$. Accordingly, there exists $\tilde{\varphi} \in Y$ such that the functional $\tilde{I} - \tilde{\varphi} - \psi$ has two global minima in X , say u_1, u_2 . At this point, it is clear that $x_3 - u_1, x_3 - u_2$ are two global minima of the functional $I + \tilde{\varphi}$, and the proof is complete. □

6. A range property for non-expansive potential operators

In this section, $(X, \langle \cdot, \cdot \rangle)$ is an infinite-dimensional real Hilbert space and $T : X \rightarrow X$ is a non-expansive potential operator. This means that

$$\|T(x) - T(y)\| \leq \|x - y\|$$

for all $x, y \in X$ and that T is the Gâteaux derivative of a functional $J : X \rightarrow \mathbf{R}$. For instance, any continuous symmetric linear operator from X into itself, with norm less than or equal to 1, is a non-expansive potential operator.

Another classical example of such operators is as follows. Let $f : X \rightarrow \mathbf{R}$ be a convex continuous function and, for each $x \in X$, let $\partial f(x)$ denote the sub-differential of f at x , i. e.

$$\partial f(x) = \left\{ z \in X : \inf_{y \in X} (f(y) - \langle z, y \rangle) \geq f(x) - \langle z, x \rangle \right\} .$$

Then $x \rightarrow x - (\text{id} + \partial f)^{-1}(x)$ is a non-expansive potential operator.

Now let $\Phi : X \rightarrow X$ be the operator defined by $\Phi(x) = x + T(x)$ for all $x \in X$.

The following result does highlight a range property of the operator Φ . The proof is based on combining some ideas from [11] with Theorem 1.C.

Theorem 6.1. *If the functional J is sequentially weakly lower semicontinuous, then there exists a closed ball B in X such that $\Phi(B)$ intersects each convex and dense subset of X .*

Before proving Theorem 6.1, some remarks are in order.

When T is a contraction, Theorem 6.1 is immediate. Actually, in that case, thanks to the Banach fixed point principle, the operator Φ turns out to be a homeomorphism between X and itself. Hence, the really interesting case is when the Lipschitz constant of T is exactly 1.

Theorem 6.1 is no longer true if J is not sequentially weakly lower semicontinuous. In this connection, the simplest example is provided by $T(x) = -x$. Actually, since $\dim(X) = \infty$, the norm is not sequentially weakly upper semicontinuous.

A further remark is that, under the assumptions of Theorem 6.1, it may happen that the set $\Phi(B)$ has an empty interior for every ball B in X . In this connection, consider the case where T is a compact, symmetric, negative linear operator with norm 1. In such a case, by classical results, J is sequentially weakly continuous and $\Phi(X) \neq X$. Since $\Phi(X)$ is a linear subspace, this clearly implies that $\text{int}(\Phi(X)) = \emptyset$.

Proof of Theorem 6.1. If the functional J is convex, then T is maximal monotone and, by a classical result of Minty, Φ turns out to be a homeomorphism between X and itself. So, in that case, we are done. Therefore, assume that J is not convex. As a consequence, there exist $x_1, x_2 \in X$ and $\lambda \in]0, 1[$ such that

$$\lambda J(x_1) + (1 - \lambda)J(x_2) < J(x_3)$$

where $x_3 = \lambda x_1 + (1 - \lambda)x_2$. Fix $z \in X$ so that

$$\langle x_1 - x_2, z \rangle = J(x_1) - J(x_2)$$

and put $\tilde{J}(x) = J(x_3 - x) - \langle x_3 - x, z \rangle$ for all $x \in X$. Note that

$$\tilde{J}(\lambda(x_1 - x_2)) = \tilde{J}((1 - \lambda)(x_2 - x_1)) < \tilde{J}(0) . \tag{6.1}$$

Now, put

$$r = \max\{\lambda, 1 - \lambda\} \|x_1 - x_2\|$$

and denote by C the closed ball in X of radius r centered at 0. Fix a convex and dense set $V \subseteq X$ and put $Y = V - x_3 - z$. Hence, Y is convex and dense too. Consider the function $f : X \times Y \rightarrow \mathbf{R}$ defined by

$$f(x, y) = \tilde{J}(x) + \langle x, y \rangle$$

for all $(x, y) \in X \times Y$. Observing that, for each $y \in Y$, one has

$$\min\{\langle \lambda(x_1 - x_2), y \rangle, \langle (1 - \lambda)(x_2 - x_1), y \rangle\} \leq 0 ,$$

in view of the equality in (6.1), it follows that

$$\sup_{y \in Y} \inf_{x \in C} f(x, y) \leq \tilde{J}(\lambda(x_1 - x_2)) . \tag{6.2}$$

On the other hand, by the density of Y , for each $x \in C \setminus \{0\}$, one has

$$\sup_{y \in Y} \langle x, y \rangle = +\infty$$

and hence

$$\inf_{x \in C} \sup_{y \in Y} f(x, y) = \tilde{J}(0) . \tag{6.3}$$

Thus, from (6.1), (6.2), (6.3) it follows that

$$\sup_{y \in Y} \inf_{x \in C} f(x, y) < \inf_{x \in C} \sup_{y \in Y} f(x, y) .$$

Then, since $f(\cdot, y)|_C$ is weakly lower semicontinuous in C (thanks to the Eberlein-Smulyan theorem) and $f(x, \cdot)$ is concave and continuous in Y , we can apply Theorem 1.C. Accordingly, there exists $\hat{y} \in Y$ such that $f(\cdot, \hat{y})|_C$ has at least two global minima u_1, u_2 in C . Now, consider the function $g : X \times [1, +\infty[\rightarrow \mathbf{R}$ defined by

$$g(x, \lambda) = \frac{\lambda}{2} (\|x\|^2 - r^2) + f(x, \hat{y})$$

for all $(x, \lambda) \in X \times [1, +\infty[$. Observe that the functional $g(\cdot, \lambda)$ (besides being continuous) is strictly convex and coercive if $\lambda > 1$, while it is convex if $\lambda = 1$. Indeed, let $\lambda \geq 1$. For each $x, y \in X$, we have

$$\begin{aligned} \langle \lambda x + T(x_3 - x) - \lambda y + T(x_3 - y), x - y \rangle &= \lambda \|x - y\|^2 - \langle T(x_3 - x) - T(x_3 - y), x - y \rangle \\ &\geq \lambda \|x - y\|^2 - \|T(x_3 - x) - T(x_3 - y)\| \|x - y\| \geq (\lambda - 1) \|x - y\|^2 . \end{aligned}$$

From this, it follows that the Gâteaux derivative of the functional $g(\cdot, \lambda)$ (that is, the operator $x \rightarrow \lambda x - T(x_3 - x) + z + \hat{y}$) is monotone and that it is uniformly monotone if $\lambda > 1$. Now the claim follows from classical results ([29], pp. 247-248). Furthermore, for each $x \in X$, the function $g(x, \cdot)$ is concave and continuous, and $\lim_{\lambda \rightarrow +\infty} g(0, \lambda) = -\infty$. So, we are allowed to apply a classical saddle-point theorem ([29], Theorem 49.A) to the function g . Accordingly, there exists $(\hat{x}, \hat{\lambda}) \in X \times [1, +\infty[$ such that

$$g(\hat{x}, \hat{\lambda}) = \inf_{x \in X} g(x, \hat{\lambda}) = \sup_{\lambda \geq 1} g(\hat{x}, \lambda) .$$

This implies that $\hat{x} \in C$,

$$\frac{\hat{\lambda}}{2} \|\hat{x}\|^2 + f(\hat{x}, \hat{y}) = \inf_{x \in X} \left(\frac{\hat{\lambda}}{2} \|x\|^2 + f(x, \hat{y}) \right) \tag{6.4}$$

and

$$\frac{\hat{\lambda}}{2} (\|\hat{x}\|^2 - r^2) = \frac{1}{2} (\|\hat{x}\|^2 - r^2) . \tag{6.5}$$

We claim that $\hat{\lambda} = 1$. If $\|\hat{x}\| < r$, this follows directly from (6.5). So, assume that $\|\hat{x}\| = r$. In this case, for $i = 1, 2$, we have

$$\frac{\hat{\lambda}}{2} \|u_i\|^2 + f(u_i, \hat{y}) \leq \frac{\hat{\lambda}}{2} \|\hat{x}\|^2 + f(\hat{x}, \hat{y})$$

and hence from (6.4) it follows that

$$\frac{\hat{\lambda}}{2} \|u_i\|^2 + f(u_i, \hat{y}) = \inf_{x \in X} \left(\frac{\hat{\lambda}}{2} \|x\|^2 + f(x, \hat{y}) \right) . \tag{6.6}$$

But, for $\lambda > 1$, the functional $x \rightarrow \frac{\lambda}{2} \|x\|^2 + f(x, \hat{y})$ has a unique global minimum in X because it is strictly convex. So, the equality $\hat{\lambda} = 1$ follows from (6.6). Therefore, by (6.4), the Gâteaux derivative of the functional $x \rightarrow \frac{1}{2} \|x\|^2 + f(x, \hat{y})$ vanishes at \hat{x} . This means that

$$T(x_3 - \hat{x}) - \hat{x} = z + \hat{y} .$$

Therefore, if B is the closed ball of radius r centered at x_3 , we have $x_3 - \hat{x} \in B$ and $\Phi(x_3 - \hat{x}) \in V$, and the proof is complete. □

7. Singular points of non-monotone potential operators

In this section, $(X, \|\cdot\|)$ is a reflexive real Banach space, with topological dual X^* , and $T : X \rightarrow X^*$ is a continuous potential operator. As a consequence, the functional

$$x \rightarrow J_T(x) := \int_0^1 T(sx)(x) ds$$

is of class C^1 in X and its Gâteaux derivative is equal to T . Let us recall a few classical definitions.

T is said to be *monotone* if $(T(x) - T(y))(x - y) \geq 0$ for all $x, y \in X$. This is equivalent to the fact that the functional J_T is convex.

T is said to be *closed* if for each closed set $C \subseteq X$, the set $T(C)$ is closed in X^* .

T is said to be *compact* if for each bounded set $B \subset X$, the set $\overline{T(B)}$ is compact in X^* .

T is said to be *proper* if for each compact set $K \subset X^*$, the set $T^{-1}(K)$ is compact in X .

T is said to be a *local homeomorphism at a point* $x_0 \in X$ if there are a neighbourhood U of x_0 and a neighbourhood V of $T(x_0)$ such that the restriction of T to U is a homeomorphism between U and V . If T is not a local homeomorphism at x_0 , we say that x_0 is a *singular point* of T . We denote by S_T the set of all singular points of T . Clearly, the set T is closed.

Assume that the restriction of T to some open set $A \subseteq X$ is of class C^1 . We then denote by $\tilde{S}_{T|_A}$ the set of all $x_0 \in A$ such that the operator $T'(x_0)$ is not invertible. Since the set of all invertible operators belonging to $\mathcal{L}(X, X^*)$ is open in $\mathcal{L}(X, X^*)$, by the continuity of T' , the set $\tilde{S}_{T|_A}$ is closed in A .

T is said to be a *Fredholm operator of index zero* in A if, for each $x \in A$, the codimension of $T'(x)(X)$ and the dimension of $(T'(x))^{-1}(0)$ are finite and equal.

A set in a topological space is said to be *σ -compact* if it is the union of an at most countable family of compact sets.

A functional $I : X \rightarrow \mathbf{R}$ is said to be *coercive* if $\lim_{\|x\| \rightarrow +\infty} I(x) = +\infty$.

Let us recall the two following results:

Theorem 7.A. ([23], Theorem 2.1) *If X is infinite-dimensional, if T is closed and if S_T is σ -compact, then the restriction of T to $X \setminus S_T$ is a homeomorphism between $X \setminus S_T$ and $X \setminus T(S_T)$.*

Theorem 7.B. ([8], Theorem 5) *If $\dim(X) \geq 3$, if T is a C^1 proper Fredholm operator of index zero and if \tilde{S}_T is discrete, then T is a homeomorphism between X and X^* .*

We wish to show that a joint application of these results with Corollary 5.1 gives the following ones:

Theorem 7.1. *If X is infinite-dimensional, if T is closed and non-monotone, if J_T is sequentially weakly lower semicontinuous and $J_T + \varphi$ is coercive for all $\varphi \in X^*$, then both S_T and $T(S_T)$ are not σ -compact.*

Theorem 7.2. *In addition to the assumptions of Theorem 1, suppose that there exists a closed, σ -compact set $B \subset X$ such that the restriction of T to $X \setminus B$ is of class C^1 . Then, both $\tilde{S}_{T|(X \setminus B)}$ and $T(\tilde{S}_{T|(X \setminus B)})$ are not σ -compact.*

Theorem 7.3. Assume that $(X, \langle \cdot, \cdot \rangle)$ is a Hilbert space, with $\dim(X) \geq 3$, and that T is compact and of class C^1 with

$$\liminf_{\|x\| \rightarrow +\infty} \frac{J_T(x)}{\|x\|^2} \geq 0 \tag{7.1}$$

and, for some $\lambda_0 \geq 0$ and for all $\lambda > \lambda_0$

$$\lim_{\|x\| \rightarrow +\infty} \|x + \lambda T(x)\| = +\infty . \tag{7.2}$$

Set $\Gamma = \{(x, y) \in X \times X : \langle T'(x)(y), y \rangle < 0\}$ and, for each $\mu \in \mathbf{R}$,

$$A_\mu = \{x \in X : T'(x)(y) = \mu y \text{ for some } y \in X \setminus \{0\}\} .$$

When $\Gamma \neq \emptyset$, set also $\tilde{\mu} = \max \left\{ -\frac{1}{\lambda_0}, \inf_{(x,y) \in \Gamma} \frac{\langle T'(x)(y), y \rangle}{\|y\|^2} \right\}$.

Then, the following assertions are equivalent:

- (i) the operator T is not monotone;
- (ii) there exists $\mu < 0$ such that $A_\mu \neq \emptyset$;
- (iii) $\Gamma \neq \emptyset$ and, for each $\mu \in]\tilde{\mu}, 0[$, the set A_μ contains an accumulation point.

Remark 7.1. Of course, Theorem 7.2 is meaningful only when X and X^* are linearly isomorphic. Indeed, if not, the fact that $\tilde{S}_{T|(X \setminus B)}$ is not σ -compact follows directly from the equality $\tilde{S}_{T|(X \setminus B)} = X \setminus B$.

We now establish the following technical proposition:

Proposition 7.1. If V is an infinite-dimensional real Banach space and if $U \subset V$ is a σ -compact set, then there exists a convex cone $C \subset V$, dense in V , such that $U \cap C = \emptyset$.

Proof. We distinguish two cases. First, assume that V is separable. Fix a countable base $\{A_n\}$ of open sets in V . We claim that there exists a sequence $\{x_n\}$ in X such that, for each $n \in \mathbf{N}$, $x_n \in A_n$ and $U \cap C_{(x_1, \dots, x_n)} = \emptyset$ where

$$C_{(x_1, \dots, x_n)} = \left\{ \sum_{i=1}^n \lambda_i x_i : \lambda_i \geq 0, \sum_{i=1}^n \lambda_i > 0 \right\} .$$

We proceed by induction on n . Clearly, the set $\cup_{\lambda > 0} \lambda U$ is σ -compact and so, since X is infinite-dimensional, it does not contain A_1 . Thus, if we take $x_1 \in A_1 \setminus \cup_{\lambda > 0} \lambda U$, we have $U \cap C_{(x_1)} = \emptyset$. Now, assume that x_1, \dots, x_n , with the desired properties, have been constructed. Consider the set $\cup_{\mu > 0} \mu(U - \overline{C_{(x_1, \dots, x_n)}})$. One readily sees that it is σ -compact, and so it does not contain A_{n+1} . Choose $x_{n+1} \in A_{n+1} \setminus \cup_{\mu > 0} \mu(U - \overline{C_{(x_1, \dots, x_n)}})$. Then, one has

$$U \cap C_{(x_1, \dots, x_{n+1})} = \emptyset .$$

Indeed, if there was $\hat{x} \in U \cap C_{(x_1, \dots, x_{n+1})}$, we would have $\hat{x} = \sum_{i=1}^{n+1} \lambda_i x_i$, with $\lambda_i \geq 0$ and $\sum_{i=1}^{n+1} \lambda_i > 0$. In particular, $\lambda_{n+1} > 0$, since $U \cap C_{(x_1, \dots, x_n)} = \emptyset$. Consequently, we would have

$$x_{n+1} = \frac{1}{\lambda_{n+1}} \left(\hat{x} - \sum_{i=1}^n \lambda_i x_i \right)$$

and so $x_{n+1} \in \cup_{\mu > 0} \mu(U - \overline{C_{(x_1, \dots, x_n)}})$, against our choice. Thus, the claimed sequence $\{x_n\}$ does exist. Now, put

$$C = \bigcup_{n=1}^{\infty} C_{(x_1, \dots, x_n)} .$$

It is clear that C is a convex cone which does not meet U . Moreover, C is dense in V since it meets each set A_n . Now, assume that V is not separable. Let $\{x_\gamma\}_{\gamma \in \Gamma}$ be a Hamel basis of V . Set

$$\Lambda = \{\gamma \in \Gamma : x_\gamma \notin \text{span}(U)\} \quad \text{and} \quad L = \text{span}(\{x_\gamma : \gamma \in \Lambda\}) .$$

Clearly, $\text{span}(U)$ is separable since U is so. Hence, Λ is infinite. Introduce in Λ a total order \leq with no greatest element. Next, for each $\gamma \in \Lambda$, let $\psi_\gamma : L \rightarrow \mathbf{R}$ be a linear functional such that

$$\psi_\gamma(x_\alpha) = \begin{cases} 1 & \text{if } \gamma = \alpha, \\ 0 & \text{if } \gamma \neq \alpha. \end{cases}$$

Now, set $D = \{x \in L : \exists \beta \in \Lambda : \psi_\beta(x) > 0 \text{ and } \psi_\gamma(x) = 0 \forall \gamma > \beta\}$. Of course, D is a convex cone. Fix $x \in L$. So, there is a finite set $I \subset \Lambda$ such that $x = \sum_{\gamma \in I} \psi_\gamma(x) x_\gamma$. Now, fix $\beta \in \Lambda$ so that $\beta > \max I$. For each $n \in \mathbf{N}$, put

$$y_n = x + \frac{1}{n} x_\beta .$$

Clearly, $\psi_\beta(y_n) = \frac{1}{n}$ and $\psi_\gamma(y_n) = 0$ for all $\gamma > \beta$. Hence, $y_n \in D$. Since $\lim_{n \rightarrow \infty} y_n = x$, we infer that D is dense in L . It is now immediate to see that the set $D + \text{span}(U)$ is a convex cone, dense in V , which does not meet U . \square

Proof of Theorem 7.1. Let us prove that S_T is not σ -compact. Arguing by contradiction, assume the contrary. Then, by Theorem 7.A, for each $\varphi \in X^* \setminus T(S_T)$, the equation $T(x) = \varphi$ has a unique solution in X . Moreover, since T is continuous, $T(S_T)$ is σ -compact, too. Therefore, in view of Proposition 1, there is a convex set $Y \subset X^*$, dense in X^* , such that $T(S_T) \cap Y = \emptyset$. On the other hand, thanks to Corollary 5.1, there is $\tilde{\varphi} \in Y$ such that the functional $J_T - \tilde{\varphi}$ has at least two global minima in X which are therefore solutions of the equation $T(x) = \tilde{\varphi}$, a contradiction.

We now prove that $T(S_T)$ is not σ -compact. Arguing by contradiction, assume the contrary. Consequently, since T is proper ([23], Theorem 1), $T^{-1}(T(S_T))$

would be σ -compact. But then, since S_T is closed and $S_T \subseteq T^{-1}(T(S_T))$, S_T would be σ -compact – a contradiction, and the proof is complete. \square

Proof of Theorem 7.2. By Theorem 7.1, the set S_T is not σ -compact. Now, observe that if $x \in X \setminus (\tilde{S}_{T|(X \setminus B)} \cup B)$, then, by the inverse function theorem, T is a local homeomorphism at x , and so $x \notin S_T$. Hence, we have

$$S_T \subseteq \tilde{S}_{T|(X \setminus B)} \cup B.$$

We then infer that $\tilde{S}_{T|(X \setminus B)}$ is not σ -compact since, otherwise, $\tilde{S}_{T|(X \setminus B)} \cup B$ would be so, and hence also S_T would be σ -compact being closed. Finally, the fact that $T(\tilde{S}_{T|(X \setminus B)})$ is not σ -compact follows as in the final part of the proof of Theorem 7.1, taking into account that $\tilde{S}_{T|(X \setminus B)}$ is closed in $X \setminus B$. \square

Proof of Theorem 7.3. Clearly, since X is a Hilbert space, we are identifying X^* with X . Let us prove that (i) \rightarrow (iii). So, assume (i). Since J_T is not convex, by a classical characterization ([27], Theorem 2.1.11), the set Γ is non-empty. Fix $\mu \in]\tilde{\mu}, 0[$. For each $x \in X$, put

$$I_\mu(x) := \frac{1}{2}\|x\|^2 - \frac{1}{\mu}J_T(x).$$

Clearly, for some $(x, y) \in \Gamma$, we have

$$\left\langle y - \frac{1}{\mu}T'(x)(y), y \right\rangle < 0$$

and so, since

$$I_\mu''(x)(y) = y - \frac{1}{\mu}T'(x)(y),$$

the above recalled characterization implies that the functional I_μ is not convex. Since T is compact, on the one hand, J_T is sequentially weakly continuous ([29], Corollary 41.9) and, on the other hand, in view of (7.2) the operator I_μ' (recall that $-\frac{1}{\mu} > \lambda_0$) is proper ([28], Example 4.43). The compactness of T also implies that, for each $x \in X$, the operator $T'(x)$ is compact ([28], Proposition 7.33) and so, for each $\lambda \in \mathbf{R}$, the operator $y \rightarrow y + \lambda T'(x)(y)$ is Fredholm of index zero ([28], Example 8.16). Therefore, the operator I_μ' is non-monotone, proper and Fredholm of index zero. Clearly, by (7.1), the functional $x \rightarrow I_\mu(x) + \langle z, x \rangle$ is coercive for all $z \in X$. Then, in view of Corollary 5.1, the operator I_μ' is not injective. At this point, we can apply Theorem 7.B to infer that the set $\tilde{S}_{I_\mu'}$ contains an accumulation point. Finally, notice that

$$\tilde{S}_{I_\mu'} = A_\mu,$$

and (iii) follows. The implication (iii) \rightarrow (ii) is trivial. Finally, the implication (ii) \rightarrow (i) is provided by Theorem 2.1.11 of [27] again. \square

8. Integral functionals on L^p -spaces

In this section, we present an application of Theorem 3.1 to integral functionals on L^p -spaces. The main general result is Theorem 8.1 below from which, in turn, we derive a series of consequences.

In the sequel, (T, \mathcal{F}, μ) ($\mu(T) > 0$) is a σ -finite measure space, Y is a reflexive real Banach space and $\varphi, \psi : Y \rightarrow \mathbf{R}$ are two sequentially weakly lower semicontinuous functionals such that, for some $p > 0$,

$$\inf_{y \in Y} \frac{\min\{\varphi(y), \psi(y)\}}{1 + \|y\|^p} > -\infty. \tag{8.1}$$

For each $\lambda \in [0, \infty]$, we denote by M_λ the set of all global minima of $\varphi + \lambda\psi$ or the empty set according to whether $\lambda < +\infty$ or $\lambda = +\infty$. We adopt the conventions $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$.

Moreover, a, b are two fixed numbers in $[0, +\infty]$, with $a < b$, and α, β are the numbers so defined:

$$\alpha = \max \left\{ \inf_Y \psi, \sup_{M_b} \psi \right\}, \quad \beta = \min \left\{ \sup_Y \psi, \inf_{M_a} \psi \right\}.$$

As usual, $L^p(T, Y)$ denotes the space of all μ -strongly measurable functions $u : T \rightarrow Y$ such that

$$\int_T \|u(t)\|^p d\mu < +\infty.$$

Theorem 8.1. *Assume that the functional $\varphi + \lambda\psi$ is coercive and has a unique global minimum for each $\lambda \in]a, b[$. Assume also that $\alpha < \beta$.*

Then, for each $\gamma \in L^\infty(T) \cap L^1(T) \setminus \{0\}$, with $\gamma \geq 0$, and for each $r \in]\alpha, \beta[$, if we put

$$V_{\gamma,r} = \left\{ u \in L^p(T, Y) : \int_T \gamma(t)\psi(u(t))d\mu \leq r \int_T \gamma(t)d\mu \right\},$$

we have

$$\inf_{u \in V_{\gamma,r}} \int_T \gamma(t)\varphi(u(t))d\mu = \inf_{\psi^{-1}(r)} \varphi \int_T \gamma(t)d\mu. \tag{8.2}$$

Proof. First, we assume additionally $\varphi(0) = \psi(0) = 0$.

Actually, once we prove the theorem under this additional assumption, the general version is obtained applying the particular version to the functions $\varphi - \varphi(0)$ and $\psi - \psi(0)$. Next, observe that the functionals φ and ψ are Borel (in the weak topology, and so in the strong one too). This implies that, for each $u \in L^p(T, Y)$, the functions $\varphi \circ u$ and $\psi \circ u$ are μ -measurable. On the other hand, in view of (8.1), for some $c > 0$, we have

$$-c\gamma(t)(1 + \|u(t)\|^p) \leq \gamma(t) \min\{\varphi(u(t)), \psi(u(t))\}$$

for all $t \in T$. Since $\gamma \in L^\infty(T) \cap L^1(T)$, the function $t \rightarrow -\gamma(t)(1 + \|u(t)\|^p)$ lies in $L^1(T)$, and so the integrals $\int_T \gamma(t)\varphi(u(t))d\mu$ and $\int_T \gamma(t)\psi(u(t))d\mu$ exist and belong to $] -\infty, +\infty]$. For each $\lambda \in]a, b[$, denote by \hat{y}_λ the unique global minimum in Y of the functional $\varphi + \lambda\psi$. By Theorem 3.1 and Remark 3.1, there exists $\lambda_r \in]a, b[$ such that $\psi(\hat{y}_{\lambda_r}) = r$. So, we have

$$\varphi(\hat{y}_{\lambda_r}) + \lambda_r r \leq \varphi(y) + \lambda_r \psi(y)$$

for all $y \in Y$. From this, it clearly follows that

$$\varphi(\hat{y}_{\lambda_r}) = \inf_{\psi^{-1}(r)} \varphi. \quad (8.3)$$

Likewise, for each $u \in L^p(T, Y)$, it follows that

$$(\varphi(\hat{y}_{\lambda_r}) + \lambda_r r) \int_T \gamma(t)d\mu \leq \int_T (\gamma(t)(\varphi(u(t)) + \lambda_r \psi(u(t))))d\mu.$$

Therefore, for each $u \in V_{\gamma, r}$, we have

$$\varphi(\hat{y}_{\lambda_r}) \int_T \gamma(t)d\mu \leq \int_T \gamma(t)\varphi(u(t))d\mu,$$

and hence

$$\varphi(\hat{y}_{\lambda_r}) \int_T \gamma(t)d\mu \leq \inf_{u \in V_{\gamma, r}} \int_T \gamma(t)\varphi(u(t))d\mu. \quad (8.4)$$

In view of (8.3), to get (8.2), we have to show that

$$\varphi(\hat{y}_{\lambda_r}) \int_T \gamma(t)d\mu = \inf_{u \in \tilde{V}_{\gamma, r}} \int_T \gamma(t)\varphi(u(t))d\mu. \quad (8.5)$$

Arguing by contradiction, assume that (8.5) does not hold. So, in view of (8.4), we would have

$$\varphi(\hat{y}_{\lambda_r}) \int_T \gamma(t)d\mu < \inf_{u \in \tilde{V}_{\gamma, r}} \int_T \gamma(t)\varphi(u(t))d\mu. \quad (8.6)$$

From (8.6), in turn, as (T, \mathcal{F}, μ) is σ -finite, the existence of some $\tilde{T} \in \mathcal{F}$ with $\mu(\tilde{T}) < +\infty$ would follow, such that

$$\varphi(\hat{y}_{\lambda_r}) \int_{\tilde{T}} \gamma(t)d\mu < \inf_{u \in \tilde{V}_{\gamma, r}} \int_{\tilde{T}} \gamma(t)\varphi(u(t))d\mu. \quad (8.7)$$

Now, consider the function $\hat{u} : T \rightarrow Y$ defined by

$$\hat{u}(t) = \begin{cases} \hat{y}_{\lambda_r} & \text{if } x \in \tilde{T}, \\ 0 & \text{if } x \in T \setminus \tilde{T}. \end{cases}$$

Clearly, $\hat{u} \in L^p(T, Y)$. We also have

$$\int_T \gamma(t)\psi(\hat{u}(t))d\mu = \int_{\tilde{T}} \gamma(t)\psi(\hat{u}(t))d\mu \leq r \int_T \gamma(t)d\mu$$

and so $\hat{u} \in V_{\gamma,r}$. But

$$\int_T \gamma(t)\varphi(\hat{u}(t))d\mu = \varphi(\hat{y}_{\lambda,r}) \int_{\tilde{T}} \gamma(t)d\mu$$

and this contradicts (8.7) – and the proof is complete. □

Remark 8.1. In general, the conclusion of Theorem 8.1 is no longer true if, for some $\lambda \in]a, b[$, the function $\varphi + \lambda\psi$ has more than one global minimum. A simple example (with $a = 0$ and $b = +\infty$) is provided by $Y = \mathbf{R}$,

$$\varphi(y) = \begin{cases} y^2 & \text{if } y \leq 1 \\ 2 - y & \text{if } y > 1 \end{cases}$$

and $\psi(y) = y^2$. So, φ is unbounded below and $\varphi + \lambda\psi$ is coercive for all $\lambda > 0$. Clearly, we have $\alpha = 0$ and $\beta = +\infty$. However, for $r = 1$, (8.2) is not satisfied, since $0 \in V_{\gamma,r}$, $\int_T \gamma(t)\varphi(0)d\mu = 0$, while $\inf_{\psi^{-1}(1)} \varphi = 1$.

Remark 8.2. At present, we do not know if the conclusion of Theorem 8.1 holds without the coercivity assumption on $\varphi + \lambda\psi$.

We now consider a series of consequences of Theorem 8.1.

First, we want to state explicitly the form Theorem 8.1 assumes when $T = \mathbf{N}$, \mathcal{F} is the power set of \mathbf{N} and $\mu(A) = \text{card}(A)$ for all $A \subseteq \mathbf{N}$.

Denote by $l_p(Y)$ the space of all sequences $\{u_n\}$ in Y such that

$$\sum_{n=1}^{\infty} \|u_n\|^p < +\infty.$$

Theorem 8.2. *Let φ, ψ satisfy the assumptions of Theorem 8.1.*

Then, for each sequence $\{a_n\} \in l_1(\mathbf{R}) \setminus \{0\}$, with $\inf_{n \in \mathbf{N}} a_n \geq 0$, and for each $r \in]\alpha, \beta[$, if we put

$$V_{\{a_n\},r} = \left\{ \{u_n\} \in l_p(Y) : \sum_{n=1}^{\infty} a_n\psi(u_n) \leq r \sum_{n=1}^{\infty} a_n \right\},$$

we have

$$\inf_{\{u_n\} \in V_{\{a_n\},r}} \sum_{n=1}^{\infty} a_n\varphi(u_n) = \inf_{\psi^{-1}(r)} \varphi \sum_{n=1}^{\infty} a_n.$$

The next two results deal with consequences of Theorem 8.1 in the case when $\varphi \in Y^* \setminus \{0\}$.

Theorem 8.3. *Let $y \rightarrow \|y\|^q$ be strictly convex for some $q > 1$ and let φ be non-zero, continuous and linear. Moreover, let $\eta : [0, +\infty[\rightarrow \mathbf{R}$ be an increasing strictly convex function.*

Then, for each $\gamma \in L^\infty(T) \cap L^1(T) \setminus \{0\}$, with $\gamma \geq 0$, and for each $r > \eta(0)$ and $p \geq 1$, if we put

$$V_{\gamma,r} = \left\{ u \in L^p(T, Y) : \int_T \gamma(t)\eta(\|u(t)\|^q)d\mu \leq r \int_T \gamma(t)d\mu \right\},$$

we have

$$\inf_{u \in V_{\gamma,r}} \int_T \gamma(t)\varphi(u(t))d\mu = -\|\varphi\|_{Y^*}(\eta^{-1}(r))^{\frac{1}{q}} \int_T \gamma(t)d\mu.$$

Proof. The assumptions made on η imply that the functional $y \rightarrow \eta(\|y\|^q)$ is strictly convex and, for some $m, c > 0$, one has $\eta(t) \geq mt - c$ for all $t \geq 0$. As a consequence, for each $\lambda > 0$, the functional $y \rightarrow \varphi(y) + \lambda\eta(\|y\|^q)$ is coercive and has a unique global minimum in X . At this point, the conclusion follows directly from Theorem 8.1, applied taking $a = 0, b = +\infty, \psi(y) = \eta(\|y\|^q)$ and observing that (8.1) holds for each $p \geq 1$ and that $\alpha = \eta(0), \beta = +\infty$. \square

Theorem 8.4. *Let φ be non-zero, continuous and linear, and let ψ be C^1 with*

$$\lim_{\|y\| \rightarrow +\infty} \frac{\psi(y)}{\|y\|} = +\infty. \tag{8.8}$$

Finally, assume that, for each $\mu < 0$, the equation

$$\psi'(y) = \mu\varphi \tag{8.9}$$

has a unique solution in Y or even at most two when $\dim(Y) < \infty$.

Then, for each $p \geq 1$, the conclusion of Theorem 8.1 holds with any $r > \inf_Y \psi$.

Proof. In view of (8.8), the functional $\varphi + \lambda\psi$ is coercive for each $\lambda > 0$. Let \hat{x} be a global minimum of this functional. Then, \hat{x} satisfies (8.9) with $\mu = -\lambda^{-1}$. So, when $\dim(Y) = \infty$, the uniqueness of \hat{x} follows from an assumption directly. Now, assume that $\dim(Y) < \infty$. In this case, $\varphi + \lambda\psi$ satisfies the Palais-Smale condition. As a consequence, if $\varphi + \lambda\psi$ was admitting two global minima, then, thanks to Corollary 1 of [9], (8.9) would have at least three solutions for $\mu = -\lambda^{-1}$, against an assumption. Now, we can apply Theorem 8.1, with $p \geq 1, a = 0, b = +\infty$, observing that $\alpha = \inf_Y \psi$ and $\beta = +\infty$. \square

Here is a consequence of Theorem 8.1 in the case when Y is a Hilbert space and φ has a Lipschitzian derivative:

Theorem 8.5. *Let Y be a Hilbert space, let φ be C^1 and let φ' be Lipschitzian, with Lipschitz constant $L > 0$. Assume that $\varphi'(0) \neq 0$. Set*

$$S = \{y \in Y : \varphi'(y) + Ly = 0\} \quad \text{and} \quad \rho = \inf_{y \in S} \|y\|^2.$$

Then, for each $\gamma \in L^\infty(T) \cap L^1(T) \setminus \{0\}$, with $\gamma \geq 0$, and for each $r \in]0, \rho[$, $p \geq 2$, if we put

$$V_{\gamma,r} = \left\{ u \in L^p(T, Y) : \int_T \gamma(t) \|u(t)\|^2 d\mu \leq r \int_T \gamma(t) d\mu \right\},$$

we have

$$\inf_{u \in V_{\gamma,r}} \int_T \gamma(t) \varphi(u(t)) d\mu = \inf_{\|y\|^2=r} \varphi(y) \int_T \gamma(t) d\mu.$$

Proof. Note that the functional $y \rightarrow \varphi(y) + \frac{\lambda}{2} \|y\|^2$ is convex if $\lambda = L$, while it is strictly convex and coercive if $\lambda > L$ (see, for instance, Proposition 2.2 of [16]). So, this functional has a unique global minimum if $\lambda > L$, while the set of its global minima coincides with S if $\lambda = L$. At this point, the conclusion is obtained applying Theorem 8.1 with

$$\psi(y) = \frac{\|y\|^2}{2}$$

for all $y \in Y$, $a = L$ and $b = +\infty$, taking into account that (8.1) is satisfied for each $p \geq 2$ since φ' is Lipschitzian and observing that $\alpha = 0$ and $\beta = \frac{L}{2}$. \square

In the next result, we will apply Theorem 8.1 taking as Y the usual Sobolev space $W_0^{1,q}(\Omega)$ with the usual norm

$$\left(\int_\Omega |\nabla v(x)|^q dx \right)^{\frac{1}{q}},$$

where Ω is bounded domain in \mathbf{R}^n ($n \geq 3$) with smooth boundary and $q > 1$.

Moreover, if $u \in L^p(T, W_0^{1,q}(\Omega))$ we will write $u(t, x)$ instead of $u(t)(x)$. That is, we will identify u with the function $\omega : T \times \Omega \rightarrow \mathbf{R}$ defined by $\omega(t, x) = u(t)(x)$ for all $(t, x) \in T \times \Omega$.

Theorem 8.6. Let $f : \mathbf{R} \rightarrow [0, +\infty[$ be a continuous function, with $f(0) = 0$ and $\liminf_{\xi \rightarrow +\infty} f(\xi) > 0$, such that $\xi \rightarrow \frac{f(\xi)}{\xi^{q-1}}$ is decreasing in $]0, +\infty[$ and

$$\lim_{|\xi| \rightarrow +\infty} \frac{f(\xi)}{|\xi|^{q-1}} = 0 \tag{8.10}$$

for some $q > 1$. Then, for each $\gamma \in L^\infty(T) \cap L^1(T) \setminus \{0\}$, with $\gamma \geq 0$, and each $r > 0$, $p \geq q$, if we put

$$V_{\gamma,r} = \left\{ u \in L^p(T, W_0^{1,q}(\Omega)) : \int_T \gamma(t) \left(\int_\Omega |\nabla u(t, x)|^q dx \right) d\mu \leq r \int_T \gamma(t) d\mu \right\},$$

we have

$$\sup_{u \in V_{\gamma,r}} \int_T \gamma(t) \left(\int_\Omega F(u(t, x)) dx \right) d\mu = \sup_{\substack{v \in W_0^{1,q}(\Omega), \\ \int_\Omega |\nabla v(x)|^q dx = r}} \int_\Omega F(v(x)) dx \int_T \gamma(t) d\mu,$$

where $F(\xi) = \int_0^\xi f(s) ds$ for all $\xi \in \mathbf{R}$.

Proof. We are going to apply Theorem 8.1 taking $Y = W_0^{1,q}(\Omega)$,

$$\varphi(v) = - \int_{\Omega} F(v(x))dx, \quad \text{and} \quad \psi(v) = \int_{\Omega} |\nabla v(x)|^q dx$$

for all $v \in W_0^{1,q}(\Omega)$. Due to (8.10), by classical results, φ is sequentially weakly continuous in $W_0^{1,q}(\Omega)$, (8.1) is satisfied for any $p \geq q$, and, for each $\lambda > 0$, the functional $\varphi + \lambda\psi$ is C^1 , coercive and satisfies the Palais-Smale condition. Moreover, since $f \geq 0$, its non-zero critical points are strictly positive in Ω ([3], [26]). Moreover, since the function $\xi \rightarrow \frac{f(\xi)}{\xi^{q-1}}$ is decreasing in $]0, +\infty[$, Proposition 4.2 of [4] ensures that, for each $\lambda > 0$, there exists at most one strictly positive critical point of $\varphi + \lambda\psi$. As a consequence, we infer that, for each $\lambda > 0$, the functional $\varphi + \lambda\psi$ has a unique global minimum in $W_0^{1,q}(\Omega)$, since otherwise, in view of Corollary 1 of [9], it would have at least three critical points. Hence, we are allowed to apply Theorem 8.1 with $a = 0$ and $b = +\infty$. Clearly, we have $\alpha = 0$ and $\beta = +\infty$ (since $\lim_{\xi \rightarrow +\infty} F(\xi) = +\infty$ and hence φ is unbounded below). The proof is complete. \square

The next application of Theorem 8.1 concerns a Jensen-like inequality in L^p -spaces.

Theorem 8.7. *Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be a continuous function, positive and differentiable in $]0, +\infty[$, with $\sup_{]-\infty, 0]} f \leq 0$. Assume that, for some $\delta \geq 0$, the function $y \rightarrow \delta|y|^p - f(y)$ has no global minima in \mathbf{R} ,*

$$\limsup_{y \rightarrow +\infty} \frac{f(y)}{y^p} = \delta \tag{8.11}$$

and the function $y \rightarrow \frac{f'(y)}{y^{p-1}}$ is injective in $]0, +\infty[$.

Then, for each $\gamma \in L^\infty(T) \cap L^1(T) \setminus \{0\}$, with $\gamma \geq 0$, and for all $u \in L^p(T)$

$$\int_T \gamma(t)f(u(t))d\mu \leq f \left(\left(\frac{\int_T \gamma(t)|u(t)|^p d\mu}{\int_T \gamma(t)d\mu} \right)^{\frac{1}{p}} \right) \int_T \gamma(t)d\mu.$$

Proof. We are going to apply Theorem 8.1 with $Y = \mathbf{R}$, $\varphi(y) = -f(y)$, $\psi(y) = |y|^p$ and $a = \delta$, $b = +\infty$. Fix $\lambda > \delta$. From (8.11), we clearly infer that $\varphi + \lambda\psi$ is coercive. We now show that this function has a unique global minimum. Arguing by contradiction, assume that $y_1, y_2 \in \mathbf{R}$ are two distinct global minima of $\varphi + \lambda\psi$. We can suppose that $y_1 < y_2$. Since $\varphi(y) + \lambda\psi(y) > 0$ for all $y < 0$ and $\varphi(0) + \lambda\psi(0) = 0$, it would follow that $y_1 \geq 0$. By the Rolle theorem, there would be $y_3 \in]y_1, y_2[$ such that $p\lambda y_3^{p-1} = f'(y_3)$. As a consequence, we would have

$$\frac{f'(y_2)}{y_2^{p-1}} = \frac{f'(y_3)}{y_3^{p-1}},$$

contrary to the assumption that the function $y \rightarrow \frac{f'(y)}{y^{p-1}}$ is injective in $]0, +\infty[$. So, we are allowed to apply Theorem 8.1, observing that $\alpha = 0$ and $\beta = +\infty$. Let $u \in L^p(T)$. Put

$$r = \frac{\int_T \gamma(t)|u(t)|^p d\mu}{\int_T \gamma(t) d\mu}.$$

If $r = 0$ the inequality to prove is clear: both sides are zero. So, assume $r > 0$. Clearly, we have

$$\inf_{\psi^{-1}(r)} \varphi = -f \left(\left(\frac{\int_T \gamma(t)|u(t)|^p d\mu}{\int_T \gamma(t) d\mu} \right)^{\frac{1}{p}} \right)$$

and hence, since $u \in V_{\gamma,r}$, it follows

$$\int_T \gamma(t)f(u(t))d\mu \leq f \left(\left(\frac{\int_T \gamma(t)|u(t)|^p d\mu}{\int_T \gamma(t) d\mu} \right)^{\frac{1}{p}} \right) \int_T \gamma(t) d\mu,$$

as claimed. □

Remark 8.3. The class of functions f satisfying the assumptions of Theorem 8.7 is quite broad. For instance, a typical function in that class is

$$f(y) = a_0 \log(1 + (y^+)^p) + \sum_{i=1}^k a_i (y^+)^{q_i}$$

where $y^+ = \max\{y, 0\}$, a_i ($i = 0, \dots, k$) are $k + 1$ non-negative numbers, with $\sum_{i=0}^k a_i > 0$, and q_i ($i = 1, \dots, k$) are k positive numbers less than p .

As a consequence of this remark, we get, for instance, the following

Corollary 8.1. For each $\gamma \in L^\infty(T) \cap L^1(T) \setminus \{0\}$, with $\gamma \geq 0$, one has

$$\int_T \gamma(t) \log(1 + |u(t)|^p) d\mu \leq \log \left(1 + \frac{\int_T \gamma(t)|u(t)|^p d\mu}{\int_T \gamma(t) d\mu} \right) \int_T \gamma(t) d\mu \tag{8.12}$$

for all $u \in L^p(T)$.

9. Integral functionals on Sobolev spaces

From now on, $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary.

For $p > 1$, on the Sobolev space $W^{1,p}(\Omega)$ we consider the norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^p dx + \int_\Omega |u(x)|^p dx \right)^{\frac{1}{p}}.$$

If $p > n$, $W^{1,p}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$ and hence the constant

$$c = \sup_{u \in W^{1,p}(\Omega) \setminus \{0\}} \frac{\sup_{x \in \Omega} |u(x)|}{\|u\|} \tag{9.1}$$

is finite.

Recall that a function $f : \Omega \times \mathbf{R}^m \rightarrow]-\infty, +\infty]$ is said to be a *normal integrand* ([22]) if it is $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbf{R}^m)$ -measurable and $f(x, \cdot)$ is lower semicontinuous for a.e. $x \in \Omega$. Here $\mathcal{L}(\Omega)$ and $\mathcal{B}(\mathbf{R}^m)$ denote the Lebesgue and the Borel σ -algebras of subsets of Ω and \mathbf{R}^m , respectively.

Recall that if f is a normal integrand, then, for each measurable function $u : \Omega \rightarrow \mathbf{R}^m$, the composite function $x \rightarrow f(x, u(x))$ is measurable ([22]).

If $\xi \in \mathbf{R}$, we continue to denote by ξ the constant function on Ω assuming the value ξ .

The next result is an application of Theorem 5.1.

Theorem 9.1. *Assume $p > n$. Let $f : \Omega \times \mathbf{R} \rightarrow]-\infty, +\infty]$ and $\varphi : \Omega \times \mathbf{R} \times \mathbf{R}^n \rightarrow]-\infty, +\infty]$ be two normal integrands satisfying the following conditions:*

(i) *there exist $\nu > 0$ and $\gamma \in L^1(\Omega)$ such that*

$$\nu(|\xi|^p + |\eta|^p) + \gamma(x) \leq \varphi(x, \xi, \eta)$$

for all $(x, \xi, \eta) \in \Omega \times \mathbf{R} \times \mathbf{R}^n$, and, for each $(x, \xi) \in \Omega \times \mathbf{R}$, the function $\varphi(x, \xi, \cdot)$ is convex in \mathbf{R}^n ;

(ii) *for each $\epsilon > 0$, there exists $\gamma_\epsilon \in L^1(\Omega)$ such that*

$$-\epsilon|\xi|^p + \gamma_\epsilon(x) \leq f(x, \xi)$$

for all $(x, \xi) \in \Omega \times \mathbf{R}$;

(iii) *there exist $\xi_1, \rho \in \mathbf{R}$ such that*

$$\int_{\Omega} \varphi(x, \xi_1, 0)dx < \rho, \quad \int_{\Omega} f(x, \xi_1)dx < +\infty \quad \text{and} \quad f(x, \xi_1) = \inf_{|\xi| \leq \delta} f(x, \xi)$$

for all $x \in \Omega$, where $\delta = c \left(\frac{\rho - \int_{\Omega} \gamma(x)dx}{\nu} \right)^{\frac{1}{p}}$ and c is given in (9.1).

Under these hypotheses, for every sequentially weakly closed set $V \subseteq W^{1,p}(\Omega)$ containing the constant ξ_1 and a w for which

$$\int_{\Omega} \varphi(x, w(x), \nabla w(x))dx < +\infty \quad \text{and} \quad \int_{\Omega} f(x, w(x))dx < \int_{\Omega} f(x, \xi_1) dx,$$

there exists $\lambda^ > 0$ such that the restriction to V of the functional*

$$u \rightarrow \int_{\Omega} f(x, u(x))dx + \lambda^* \int_{\Omega} \varphi(x, u(x), \nabla u(x))dx$$

has at least two global minima.

Proof. For each $u \in W^{1,p}(\Omega)$, set

$$\tilde{J}(u) = \int_{\Omega} f(x, u(x))dx \quad \text{and} \quad \tilde{\Phi}(u) = \int_{\Omega} \varphi(x, u(x), \nabla u(x))dx.$$

By a classical result ([2], Theorem 4.6.8), for each $\lambda > 0$ the functional $\tilde{J} + \lambda\tilde{\Phi}$ is sequentially weakly lower semicontinuous. On the other hand, for $\epsilon \in]0, \lambda\nu[$, by (ii), we have

$$\tilde{J}(u) + \lambda\tilde{\Phi}(u) \geq (\lambda\nu - \epsilon)\|u\| + \int_{\Omega} \gamma_{\epsilon}(x)dx.$$

Consequently, by reflexivity and Eberlein-Smulyan theorem, the sub-level sets of $\tilde{J} + \lambda\tilde{\Phi}$ are weakly compact. Now, let $V \subseteq W^{1,p}(\Omega)$ be as in the conclusion. Set

$$X = \{u \in V : \sup\{\tilde{J}(u), \tilde{\Phi}(u)\} < +\infty\}.$$

Observe that $\xi_1, w \in X$ and that

$$\{u \in X : \tilde{J}(u) + \lambda\tilde{\Phi}(u) \leq r\} = \{u \in V : \tilde{J}(u) + \lambda\tilde{\Phi}(u) \leq r\} \tag{9.2}$$

for all $\lambda > 0, r \in \mathbf{R}$. Denote by J and Φ the restrictions to X of \tilde{J} and $\tilde{\Phi}$ respectively. We want to apply Theorem 5.1 considering X with the relative weak topology. Clearly, in view of (9.2), (a_1) holds. Concerning (b_1) , observe that for each $u \in \Phi^{-1}(] - \infty, \rho])$, by (i), one has

$$\nu\|u\|^p + \int_{\Omega} \gamma(x)dx \leq \rho$$

and so

$$\sup_{\Omega} |u| \leq c \left(\frac{\rho - \int_{\Omega} \gamma(x)dx}{\nu} \right)^{\frac{1}{p}},$$

the above inequalities being strict if $\Phi(u) < \rho$. Then, from this and from (iii), it follows that

$$J(\xi_1) = \inf_{\Phi^{-1}(] - \infty, \rho])} J \quad \text{and} \quad \Phi(\xi_1) < \rho$$

as well. Consequently, (b_1) is satisfied taking $u_1 = \xi_1$ and $u_2 = w$. So, the conclusion follows directly from Theorem 5.1. \square

Let $p > 1$. If $n \geq p$, we denote by \mathcal{A}_p the class of all continuous functions $f : \mathbf{R} \rightarrow \mathbf{R}$ such that

$$\sup_{\xi \in \mathbf{R}} \frac{|f(\xi)|}{1 + |\xi|^s} < +\infty,$$

where $0 < s < \frac{m-n+p}{n-p}$ if $p < n$, and $0 < s < +\infty$ if $p = n$. While, when $n < p$, \mathcal{A} stands for the class of all continuous functions $f : \mathbf{R} \rightarrow \mathbf{R}$. Given $f \in \mathcal{A}_p$, consider the following Dirichlet problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_f}$$

Let us recall that a *weak solution* of (P_f) is any $u \in W_0^{1,p}(\Omega)$ such that

$$\int_{\Omega} |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx - \int_{\Omega} f(u(x))v(x)dx = 0$$

for all $v \in W_0^{1,p}(\Omega)$.

Moreover, $\lambda_{1,p}$ denotes the *principal eigenvalue* of the problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and we have

$$\lambda_{1,p} = \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u(x)|^p dx}{\int_{\Omega} |u(x)|^p dx}.$$

We recall the following consequence of the variational principle shown in [10]:

Theorem 9.A. *Let X be a reflexive real Banach space and let $\Phi, \Psi : X \rightarrow \mathbf{R}$ be two sequentially weakly lower semicontinuous functionals, with $\Phi(0) = \Psi(0) = 0$, and with Ψ also coercive and continuous. Then, for each $\sigma > \inf_X \Psi$ and each λ satisfying*

$$\lambda > -\frac{\inf_{\Psi^{-1}([-\infty, \sigma])} \Phi}{\sigma}$$

the functional $\lambda\Psi + \Phi$ has a local minimum belonging to $\Psi^{-1}([-\infty, \sigma])$.

The next result is an application of Theorem 8.1.

Theorem 9.2. *Let $f \in \mathcal{A}_p$, with $f \geq 0$, and let $F(\xi) = \int_0^\xi F(t)dt$ for all $\xi \in \mathbf{R}$. Assume that:*

- (a₁) $\lim_{\xi \rightarrow 0^+} \frac{F(\xi)}{\xi^p} = +\infty$;
- (a₂) $\delta := \limsup_{\xi \rightarrow +\infty} \frac{F(\xi)}{\xi^p} < +\infty$;
- (a₃) *the function $\xi \rightarrow \delta\xi^p - F(\xi)$ has no global minima in $]0, +\infty[$;*
- (a₄) *for each $\lambda > p\delta$, the equation $\lambda\xi^{p-1} = f(\xi)$ has at most two solutions in $]0, +\infty[$.*

Under these hypotheses, for each $\rho > 0$ and each $\nu \in]0, 1]$ satisfying

$$\nu < \frac{\lambda_{1,p}\rho^p}{pF(\rho)}, \tag{9.3}$$

the problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \nu f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a positive weak solution satisfying

$$\int_{\Omega} |\nabla u(x)|^p dx < \rho^p \lambda_{1,p} \operatorname{meas}(\Omega).$$

Proof. Fix ρ and ν as above. Since $f \geq 0$, by classical results ([3], [27]), the positive weak solutions of the problem are exactly the non-zero critical points in $W_0^{1,p}(\Omega)$ of the energy functional

$$u \rightarrow \frac{1}{p} \int_{\Omega} |\nabla u(x)|^p dx - \nu \int_{\Omega} F(u(x)) dx.$$

We are going to apply Theorem 8.1 taking $Y = \mathbf{R}$, $\varphi(\xi) = -\nu F(\xi)$, $\psi(\xi) = |\xi|^p$, $a = \delta$ and $b = +\infty$. Note that φ is non-negative in $] -\infty, 0]$. So, (8.1) is satisfied in view of (a_2) . Fix $\lambda > \delta$. From (a_2) again, it follows that $\varphi + \lambda\psi$ is coercive. Arguing by contradiction, assume that $\varphi + \lambda\psi$ has two global minima, say ξ_1, ξ_2 , with $\xi_1 < \xi_2$. From (a_1) it follows that

$$\inf_{[0, +\infty[} (\varphi + \lambda\psi) < 0.$$

This fact implies that $\xi_1 > 0$. As a consequence, the equation

$$p\lambda\xi^{p-1} = \nu f(\xi)$$

would admit the solutions ξ_1, ξ_2 and a third one in $]\xi_1, \xi_2[$ given by Rolle's theorem. But, this contradicts (a_4) . Hence, the function $\varphi + \lambda\psi$ has a unique global minimum. Further, note that $\alpha = 0$ and, in view of (a_3) , $\beta = +\infty$. Then, if we put

$$V_\rho = \left\{ u \in L^p(\Omega) : \int_\Omega |u(x)|^p dx \leq \rho^p \text{meas}(\Omega) \right\},$$

Theorem 8.1 ensures that

$$\sup_{u \in V_\rho} \int_\Omega F(u(x)) dx = F(\rho) \text{meas}(\Omega). \tag{9.4}$$

On the other hand, setting

$$B_\rho = \left\{ u \in W_0^{1,p}(\Omega) : \int_\Omega |\nabla u(x)|^p dx \leq \rho^p \lambda_{1,p} \text{meas}(\Omega) \right\},$$

we have

$$B_\rho \subseteq V_\rho.$$

Consequently

$$\sup_{u \in B_\rho} \int_\Omega F(u(x)) dx \leq \sup_{u \in V_\rho} \int_\Omega F(u(x)) dx. \tag{9.5}$$

Now, if we put

$$\sigma = \rho^p \lambda_{1,p} \text{meas}(\Omega),$$

in view of (9.3), (9.4) and (9.5), we have

$$\sup_{\substack{u \in W_0^{1,p}(\Omega) \\ \int_\Omega |\nabla u(x)|^p dx \leq \sigma}} \int_\Omega \nu F(u(x)) dx < \frac{\sigma}{p}.$$

We apply now Theorem 9.A taking $X = W_0^{1,p}(\Omega)$, $\Psi(u) = \int_\Omega |\nabla u(x)|^p dx$ and $\Phi(u) = -\nu \int_\Omega F(u(x)) dx$. Hence, the energy functional has a local minimum u (which is therefore a solution of the problem) such that

$$\int_\Omega |\nabla u(x)|^p dx < \rho^p \lambda_{1,p} \text{meas}(\Omega).$$

To show that $u \neq 0$, we finally remark that 0 is not a local minimum of the energy functional. Indeed, by a classical result, there is a bounded and positive function $v \in W_0^{1,p}(\Omega)$ such that

$$\int_{\Omega} |\nabla v(x)|^p dx = \lambda_{1,p} \int_{\Omega} |v(x)|^p dx .$$

By (a_1) , there is $\theta > 0$ such that

$$F(\xi) > \frac{\lambda_{1,p}}{\nu p} \xi^p$$

for all $\xi \in]0, \theta[$. Hence, for each $\eta \in]0, \frac{\theta}{\sup_{\Omega} v} [$, we have

$$\nu \int_{\Omega} F(\eta v(x)) dx > \frac{\lambda_{1,p}}{p} \int_{\Omega} |\eta v(x)|^p dx = \frac{1}{p} \int_{\Omega} |\nabla \eta v(x)|^p dx .$$

This shows that the energy functional takes negative values in each ball of $W_0^{1,p}(\Omega)$ centered at 0 and so 0 is not a local minimum for it, and the proof is complete. \square

Note the following corollary of Theorem 9.2 (for the uniqueness, consider again Proposition 4.2 of [4]):

Corollary 9.1. *For each $\nu \in]0, 1]$, the unique positive weak solution of the problem*

$$\begin{cases} -\operatorname{div}(|\nabla u| \nabla u) = \nu u & \text{in } \Omega , \\ u = 0 & \text{on } \partial\Omega , \end{cases}$$

satisfies the inequality

$$\int_{\Omega} |\nabla u(x)|^3 dx \leq \frac{27 \operatorname{meas}(\Omega)}{8 \lambda_{1,3}^2} \nu^3 .$$

Now, let $a, b \in \mathbf{R}$, with $a \geq 0$ and $b > 0$. Consider the non-local problem

$$\begin{cases} -(a + b \int_{\Omega} |\nabla u(x)|^2 dx) \Delta u = h(x, u) & \text{in } \Omega , \\ u = 0 & \text{on } \partial\Omega , \end{cases}$$

$h : \Omega \times \mathbf{R} \rightarrow \mathbf{R}$ being a Carathéodory function.

On the Sobolev space $H_0^1(\Omega)$, we consider the scalar product

$$\langle u, v \rangle = \int_{\Omega} \nabla u(x) \nabla v(x) dx$$

and the induced norm

$$\|u\| = \left(\int_{\Omega} |\nabla u(x)|^2 dx \right)^{\frac{1}{2}} .$$

We denote by \mathcal{A} the class of all Carathéodory functions $f : \Omega \times \mathbf{R} \rightarrow \mathbf{R}$ such that

$$\sup_{(x,\xi) \in \Omega \times \mathbf{R}} \frac{|f(x, \xi)|}{1 + |\xi|^p} < +\infty \tag{9.6}$$

for some $p \in]0, \frac{n+2}{n-2}[$.

Moreover, we denote by $\tilde{\mathcal{A}}$ the class of all Carathéodory functions $g : \Omega \times \mathbf{R} \rightarrow \mathbf{R}$ such that

$$\sup_{(x,\xi) \in \Omega \times \mathbf{R}} \frac{|g(x, \xi)|}{1 + |\xi|^q} < +\infty \tag{9.7}$$

for some $q \in]0, \frac{2}{n-2}[$. Furthermore, we denote by $\hat{\mathcal{A}}$ the class of all functions $h : \Omega \times \mathbf{R} \rightarrow \mathbf{R}$ of the type

$$h(x, \xi) = f(x, \xi) + \alpha(x)g(x, \xi)$$

with $f \in \mathcal{A}, g \in \tilde{\mathcal{A}}$ and $\alpha \in L^2(\Omega)$. For each $h \in \hat{\mathcal{A}}$, we define the functional $I_h : H_0^1(\Omega) \rightarrow \mathbf{R}$, by putting for all $u \in H_0^1(\Omega)$ and for all $(x, \xi) \in \Omega \times \mathbf{R}$

$$I_f(u) = \int_{\Omega} H(x, u(x))dx \quad \text{and} \quad H(x, \xi) = \int_0^{\xi} h(x, t)dt .$$

By classical results (involving the Sobolev embedding theorem), the functional I_h turns out to be sequentially weakly continuous, of class C^1 , with compact derivative given by

$$I'_h(u)(w) = \int_{\Omega} h(x, u(x))w(x)dx$$

for all $u, w \in H_0^1(\Omega)$.

Now, let us recall that, given $h \in \hat{\mathcal{A}}$, a weak solutions of the problem

$$\begin{cases} - (a + b \int_{\Omega} |\nabla u(x)|^2 dx) \Delta u = h(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

is any $u \in H_0^1(\Omega)$ such that

$$\left(a + b \int_{\Omega} |\nabla u(x)|^2 dx \right) \int_{\Omega} \nabla u(x) \nabla w(x) dx = \int_{\Omega} h(x, u(x))w(x)$$

for all $w \in H_0^1(\Omega)$. Let $\Phi : H_0^1(\Omega) \rightarrow \mathbf{R}$ be the functional defined by

$$\Phi(u) = \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4$$

for all $u \in H_0^1(\Omega)$. Then the weak solutions of the problem are precisely the critical points in $H_0^1(\Omega)$ of the functional $\Phi - I_h$.

As an application of Theorem 5.8, we now obtain

Theorem 9.3. *Let $n \geq 4$, let $f \in \mathcal{A}$ and let $g \in \tilde{\mathcal{A}}$ be such that the set*

$$\left\{ x \in \Omega : \sup_{\xi \in \mathbf{R}} |g(x, \xi)| > 0 \right\}$$

has a positive measure. Then, there exists $\lambda^ \geq 0$ such that, for each $\lambda > \lambda^*$ and each convex set $C \subseteq L^2(\Omega)$ whose closure in $\overline{L^2(\Omega)}$ contains the set $\{G(\cdot, u(\cdot)) : u \in H_0^1(\Omega)\}$, there exists $v^* \in C$ such that the problem*

$$\begin{cases} - (a + b \int_{\Omega} |\nabla u(x)|^2 dx) \Delta u = f(x, u) + \lambda(G(x, u) - v^*(x))g(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has at least three weak solutions, two of which are global minima in $H_0^1(\Omega)$ of the functional

$$u \rightarrow \Phi(u) - \int_{\Omega} F(x, u(x)) dx - \frac{\lambda}{2} \int_{\Omega} |G(x, u(x)) - v^*(x)|^2 dx.$$

Furthermore, if the functional

$$u \rightarrow \Phi(u) - \int_{\Omega} F(x, u(x)) dx$$

has at least two global minima in $H_0^1(\Omega)$ and the function $G(x, \cdot)$ is strictly monotone for all $x \in \Omega$, then $\lambda^ = 0$.*

Proof. For each $\lambda \geq 0$, $v \in L^2(\Omega)$, consider the function $h_{\lambda, v} : \Omega \times \mathbf{R} \rightarrow \mathbf{R}$ defined by

$$h_{\lambda, v}(x, \xi) = f(x, \xi) + \lambda(G(x, \xi) - v(x))g(x, \xi)$$

for all $(x, \xi) \in \Omega \times \mathbf{R}$. Clearly, the function $h_{\lambda, v}$ lies in $\hat{\mathcal{A}}$ and

$$H_{\lambda, v}(x, \xi) = F(x, \xi) + \frac{\lambda}{2} (|G(x, \xi) - v(x)|^2 - |v(x)|^2).$$

So, the weak solutions of the problem are precisely the critical points in $H_0^1(\Omega)$ of the functional $\Phi - I_{h_{\lambda, v}}$. Moreover, if $p \in]0, \frac{n+2}{n-2}[$ and $q \in]0, \frac{2}{n-2}[$ are such that (9.6) and (9.7) hold, for some constant $c_{\lambda, v}$, we have

$$\int_{\Omega} |H_{\lambda, v}(x, u(x))| dx \leq c_{\lambda, v} \left(\int_{\Omega} |u(x)|^{p+1} + \int_{\Omega} |u(x)|^{2(q+1)} dx + 1 \right)$$

for all $u \in H_0^1(\Omega)$. Therefore, by the Sobolev embedding theorem, for a constant $\tilde{c}_{\lambda, v}$, we have

$$\Phi(u) - I_{h_{\lambda, v}}(u) \geq \frac{b}{4} \|u\|^4 - \tilde{c}_{\lambda, v} (\|u\|^{p+1} + \|u\|^{2(q+1)} + 1) \tag{9.8}$$

for all $u \in H_0^1(\Omega)$. On the other hand, since $n \geq 4$, one has

$$\max\{p + 1, 2(q + 1)\} < \frac{2n}{n - 2} \leq 4.$$

Consequently, from (9.8), we infer that

$$\lim_{\|u\| \rightarrow +\infty} (\Phi(u) - I_{h_{\lambda,v}}(u)) = +\infty. \tag{9.9}$$

Since the functional $\Phi - I_{h_{\lambda,v}}$ is sequentially weakly lower semicontinuous, by the Eberlein-Smulyan theorem and by (9.9), it follows that it is inf-weakly compact.

Now, we intend to apply Theorem 5.8 taking $X = H_0^1(\Omega)$ with the weak topology and $\Lambda = Y = L^2(\Omega)$ with the strong topology, and $y_0 = 0$. The symbols x and λ appearing in Theorem 5.8 are replaced by the symbols u and v respectively. We take

$$\varphi(w) = \frac{1}{2} \int_{\Omega} |w(x)|^2 dx$$

for all $w \in L^2(\Omega)$. Clearly, $\varphi \in \mathcal{G}$. Furthermore, we take

$$\Psi(u, v)(x) = G(x, u(x)) - v(x)$$

for all $u \in H_0^1(\Omega), v \in L^2(\Omega), x \in \Omega$. Clearly, $\Psi(u, v) \in L^2(\Omega)$, $\Psi(u, \cdot)$ is a homeomorphism, and we have $v_u(x) = G(x, u(x))$. We show that the map $u \rightarrow v_u$ is not constant in $H_0^1(\Omega)$. Set

$$A = \left\{ x \in \Omega : \sup_{\xi \in \mathbf{R}} |g(x, \xi)| > 0 \right\}.$$

By assumption, $\text{meas}(A) > 0$. Then, by the classical Scorza-Dragoni theorem ([2], Theorem 2.5.19), there exists a compact set $K \subset A$, of positive measure, such that the restriction of G to $K \times \mathbf{R}$ is continuous. Fix a point $\tilde{x} \in K$ such that the intersection of K and any ball centered at \tilde{x} has a positive measure. Next, fix $\xi_1, \xi_2 \in \mathbf{R}$ such that

$$G(\tilde{x}, \xi_1) < G(\tilde{x}, \xi_2).$$

By continuity, there is a closed ball $B(\tilde{x}, r)$ such that

$$G(x, \xi_1) < G(x, \xi_2)$$

for all $x \in K \cap B(\tilde{x}, r)$. Finally, consider two functions $u_1, u_2 \in H_0^1(\Omega)$ which are constant in $K \cap B(\tilde{x}, r)$. So, we have

$$G(x, u_1(x)) < G(x, u_2(x))$$

for all $x \in K \cap B(\tilde{x}, r)$. Hence, as $\text{meas}(K \cap B(\tilde{x}, r)) > 0$, we infer that $v_{u_1} \neq v_{u_2}$, as claimed. As a consequence, $\Psi \in \mathcal{H}$. Of course, $\varphi(\Psi(u, \cdot))$ is continuous and

convex for all $u \in X$. Finally, take $J = \Phi - I_f$. Clearly, $J \in \mathcal{M}$. So, for what seen above, all the assumptions of Theorem 5.8 are satisfied. Consequently, if we take

$$\lambda^* = \theta(\varphi, \Psi, J) \tag{9.10}$$

and fix $\lambda > \lambda^*$ and a convex set $C \subseteq L^2(\Omega)$ whose closure in $L^2(\Omega)$ contains the set $\{G(\cdot, u(\cdot)) : u \in H_0^1(\Omega)\}$, there exists $v^* \in C$ such that the functional $\Phi - I_{h_{\lambda, v^*}}$ has at least two global minima in $H_0^1(\Omega)$ which are, therefore, weak solutions of the problem. To guarantee the existence of a third solution, denote by k the inverse of the restriction of the function $at + bt^3$ to $[0, +\infty[$. Let $T : X \rightarrow X$ be the operator defined by

$$T(w) = \begin{cases} \frac{k(\|w\|)}{\|w\|} w & \text{if } w \neq 0, \\ 0 & \text{if } w = 0, \end{cases}$$

Since k is continuous and $k(0) = 0$, the operator T is continuous in X . For each $u \in X \setminus \{0\}$, we have

$$\begin{aligned} T(\Phi'(u)) &= T((a + b\|u\|^2)u) = \frac{k((a + b\|u\|^2)\|u\|)}{(a + b\|u\|^2)\|u\|} (a + b\|u\|^2)u \\ &= \frac{\|u\|}{(a + b\|u\|^2)\|u\|} (a + b\|u\|^2)u = u. \end{aligned}$$

In other words, T is a continuous inverse of Φ' . Then, since $I'_{h_{\lambda, v^*}}$ is compact, the functional $\Phi - I_{h_{\lambda, v^*}}$ satisfies the Palais-Smale condition ([29], Example 38.25) and hence the existence of a third critical point of the same functional is assured by Corollary 1 of [9].

Finally, assume that the functional $\Phi - I_f$ has at least two global minima, say \hat{u}_1, \hat{u}_2 . Then, the set $D := \{x \in \Omega : \hat{u}_1(x) \neq \hat{u}_2(x)\}$ has a positive measure. By assumption, we have

$$G(x, \hat{u}_1(x)) \neq G(x, \hat{u}_2(x))$$

for all $x \in D$, and so $v_{\hat{u}_1} \neq v_{\hat{u}_2}$. Then, by definition, we have

$$0 \leq \theta(\varphi, \Psi, J) \leq \frac{J(\hat{u}_1) - J(\hat{u}_2)}{\varphi(\Psi(\hat{u}_1, v_{\hat{u}_2}))} = 0$$

and so $\lambda^* = 0$ in view of (9.10). □

Notice the following simple corollary of Theorem 9.3:

Corollary 9.2. *Let $n \geq 4$ and let $p \in]0, \frac{n+2}{n-2}[$. Then, for each $\lambda > 0$ large enough and for each convex set $C \subseteq L^2(\Omega)$ whose closure in $L^2(\Omega)$ contains $H_0^1(\Omega)$, there exists $v^* \in C$ such that the problem*

$$\begin{cases} -(a + b \int_{\Omega} |\nabla u(x)|^2 dx) \Delta u = |u|^{p-1}u + \lambda(u - v^*(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has at least three solutions, two of which are global minima in $H_0^1(\Omega)$ of the functional

$$u \rightarrow \Theta(u) - \frac{1}{p+1} \int_{\Omega} |u(x)|^{p+1} dx - \frac{\lambda}{2} \int_{\Omega} |u(x) - v^*(x)|^2 dx .$$

Among the other consequences of Theorem 9.3, we highlight the following

Theorem 9.4. *Let $n \geq 4$, let $f \in \mathcal{A}$ and let $g \in \tilde{\mathcal{A}}$ be such that the set*

$$\left\{ x \in \Omega : \sup_{\xi \in \mathbf{R}} F(x, \xi) > 0 \right\}$$

has a positive measure. Moreover, assume that, for each $x \in \Omega$, $f(x, \cdot)$ is odd, $g(x, \cdot)$ is even and $G(x, \cdot)$ is strictly monotone. Then, for each $\lambda > 0$, there exists $\mu^ > 0$ such that, for each $\mu > \mu^*$ and for each convex set $C \subseteq L^2(\Omega)$ whose closure in $L^2(\Omega)$ contains the set $\{G(\cdot, u(\cdot)) : u \in H_0^1(\Omega)\}$, there exists $v^* \in C$ such that the problem*

$$\begin{cases} - (a + b \int_{\Omega} |\nabla u(x)|^2 dx) \Delta u = \mu f(x, u) - \lambda v^*(x) g(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has at least three weak solutions, two of which are global minima in $H_0^1(\Omega)$ of the functional

$$u \rightarrow \Phi(u) - \mu \int_{\Omega} F(x, u(x)) dx + \lambda \int_{\Omega} v^*(x) G(x, u(x)) dx .$$

Proof. Set $D = \left\{ x \in \Omega : \sup_{\xi \in \mathbf{R}} F(x, \xi) > 0 \right\}$. By assumption, $\text{meas}(D) > 0$.

Then, by the Scorza-Dragoni theorem, there exists a compact set $K \subset D$, of positive measure, such that the restriction of F to $K \times \mathbf{R}$ is continuous. Fix a point $\hat{x} \in K$ such that the intersection of K and any ball centered at \hat{x} has a positive measure. Choose $\hat{\xi} \in \mathbf{R}$ so that $F(\hat{x}, \hat{\xi}) > 0$. By continuity, there is $r > 0$ such that

$$F(x, \hat{\xi}) > 0$$

for all $x \in K \cap B(\hat{x}, r)$. Set

$$M = \sup_{(x, \xi) \in \Omega \times [-|\hat{\xi}|, |\hat{\xi}|]} |F(x, \xi)| .$$

Since $f \in \mathcal{A}$, we have $M < +\infty$. Next, choose an open set $\tilde{\Omega}$ such that

$$K \cap B(\hat{x}, r) \subset \tilde{\Omega} \subset \Omega$$

and

$$\text{meas}(\tilde{\Omega} \setminus (K \cap B(\hat{x}, r))) < \frac{\int_{K \cap B(\hat{x}, r)} F(x, \hat{\xi}) dx}{M}.$$

Finally, choose a function $\tilde{u} \in H_0^1(\Omega)$ such that

$$\begin{aligned} \tilde{u}(x) &= \hat{\xi} \quad \text{for all } x \in K \cap B(x, r), \\ \tilde{u}(x) &= 0 \quad \text{for all } x \in \Omega \setminus \tilde{\Omega}, \text{ and} \\ |\tilde{u}(x)| &\leq |\hat{\xi}| \quad \text{for all } x \in \Omega. \end{aligned}$$

Thus, we have

$$\begin{aligned} \int_{\Omega} F(x, \tilde{u}(x)) dx &= \int_{K \cap B(\hat{x}, r)} F(x, \hat{\xi}) dx + \int_{\tilde{\Omega} \setminus (K \cap B(\hat{x}, r))} F(x, \tilde{u}(x)) dx \\ &> \int_{K \cap B(\hat{x}, r)} F(x, \hat{\xi}) dx - M \text{meas}(\tilde{\Omega} \setminus (K \cap B(\hat{x}, r))) > 0. \end{aligned}$$

Now, fix any $\lambda > 0$ and set

$$\mu^* = \frac{\Phi(\tilde{u}) + \frac{\lambda}{2} I_{Gg}(\tilde{u})}{I_f(\tilde{u})}.$$

Fix $\mu > \mu^*$. Hence $\Phi(\tilde{u}) - \mu I_f(\tilde{u}) + \frac{\lambda}{2} I_{Gg}(\tilde{u}) < 0$.

From this, we infer that the functional $\Phi - \mu I_f + \frac{\lambda}{2} I_{Gg}$ possesses at least to global minima since it is even. At this point, we can apply Theorem 9.3 to the functions g and $\mu f - \lambda Gg$. Our current conclusion follows from the one of Theorem 9.3 since we have $\lambda^* = 0$ and hence we can take the same fixed $\lambda > 0$. \square

To state the last result, denote by $H^{-1}(\Omega)$ the dual of $H_0^1(\Omega)$.

If $n \geq 2$, we denote by \mathcal{A} the class of all Carathéodory functions $f : \Omega \times \mathbf{R} \rightarrow \mathbf{R}$ such that

$$\sup_{(x, \xi) \in \Omega \times \mathbf{R}} \frac{|f(x, \xi)|}{1 + |\xi|^q} < +\infty,$$

where $0 < q < \frac{n+2}{n-2}$ if $n > 2$ and $0 < q < +\infty$ if $n = 2$. While, when $n = 1$, we denote by \mathcal{A} the class of all Carathéodory functions $f : \Omega \times \mathbf{R} \rightarrow \mathbf{R}$ such that, for each $r > 0$, the function $x \rightarrow \sup_{|\xi| \leq r} |f(x, \xi)|$ belongs to $L^1(\Omega)$.

Given $f \in \mathcal{A}$ and $\varphi \in H^{-1}(\Omega)$, consider the following Dirichlet problem

$$\begin{cases} -\Delta u = f(x, u) + \varphi & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (P_{f, \varphi})$$

Let us recall that a weak solution of $(P_{f, \varphi})$ is any $u \in H_0^1(\Omega)$ such that

$$\int_{\Omega} \nabla u(x) \nabla v(x) dx - \int_{\Omega} f(x, u(x)) v(x) dx - \varphi(v) = 0$$

for all $v \in H_0^1(\Omega)$. Applying Theorem 5.10, we obtain

Theorem 9.5. *Let $f \in \mathcal{A}$ be such that the set*

$$\left\{ x \in \Omega : \sup_{\xi \in \mathbf{R}} \int_0^\xi f(x, t) dt > 0 \right\}$$

has a positive measure and

$$\limsup_{|\xi| \rightarrow +\infty} \frac{\sup_{x \in \Omega} \int_0^\xi f(x, t) dt}{\xi^2} \leq 0 .$$

Then, for every $\lambda > 0$ large enough and for every convex set $C \subset H^{-1}(\Omega)$ dense in $H^{-1}(\Omega)$, there exists $\varphi \in C$ such that the problem

$$\begin{cases} -\Delta u = \lambda f(x, u) + \varphi & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has at least three weak solutions, two of which are global minima in $H_0^1(\Omega)$ of the functional $u \rightarrow \int_\Omega |\nabla u(x)|^2 dx - \lambda \int_\Omega (\int_0^{u(x)} f(x, \xi) d\xi) dx - \varphi(u)$.

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References

- [1] G. Cordaro: *On a minimax problem of Ricceri*, J. Inequal. Appl. 6 (2001) 261–285.
- [2] Z. Denkowski, S. Migórski, N. S. Papageorgiou: *An introduction to nonlinear analysis: theory*, Kluwer Academic Publishers (2003).
- [3] P. Drábek: *On a maximum principle for weak solutions of some quasi-linear elliptic equations*, Appl. Math. Letters 22 (2009) 1567–1570.
- [4] P. Drábek, J. Hernández: *Existence and uniqueness of positive solutions for some quasilinear elliptic problems*, Nonlinear Anal. 44 (2001) 189–204.
- [5] V. L. Klee: *Convexity of Chebyshev sets*, Math. Ann. 142 (1960/1961) 292–304.
- [6] E. Klein, A. C. Thompson: *Theory of correspondences*, John Wiley & Sons (1984).
- [7] S. J. N. Mosconi: *A differential characterisation of the minimax inequality*, J. Convex Analysis 19 (2012) 185–199.
- [8] R. A. Plastock: *Nonlinear Fredholm maps of index zero and their singularities*, Proc. Amer. Math. Soc. 68 (1978) 317–322.
- [9] P. Pucci, J. Serrin: *A mountain pass theorem*, J. Differential Equations 60 (1985) 142–149.
- [10] B. Ricceri: *A general variational principle and some of its applications*, J. Comput. Appl. Math. 113 (2000) 401–410.
- [11] B. Ricceri: *The problem of minimizing locally a C^2 functional around non-critical points is well-posed*, Proc. Amer. Math. Soc. 135 (2007) 2187–2191.

- [12] B. Ricceri: *Well-posedness of constrained minimization problems via saddle-points*, J. Global Optim. 40 (2008) 389–397.
- [13] B. Ricceri: *On a theory by Schechter and Tintarev*, Taiwanese J. Math. 12 (2008) 1303–1312.
- [14] B. Ricceri: *Multiplicity of global minima for parametrized functions*, Rend. Lincei Mat. Appl. 21 (2010) 47–57.
- [15] B. Ricceri: *A strict minimax inequality criterion and some of its consequences*, Positivity 16 (2012) 455–470.
- [16] B. Ricceri: *Fixed points of nonexpansive potential operators in Hilbert spaces*, Fixed Point Theory Appl. 2012 (2012) 123.
- [17] B. Ricceri: *A range property related to non-expansive operators*, Mathematika 60 (2014) 232–236.
- [18] B. Ricceri: *Integral functionals on L^p -spaces: infima over sub-level sets*, Numer. Funct. Anal. Optim. 35 (2014) 1197–1211.
- [19] B. Ricceri: *Singular points of non-monotone potential operators*, J. Nonlinear Convex Anal. 16 (2015) 1123–1129.
- [20] B. Ricceri: *Energy functionals of Kirchhoff-type problems having multiple global minima*, Nonlinear Anal. 115 (2015) 130–136.
- [21] B. Ricceri: *Miscellaneous applications of certain minimax theorems I*, Proc. Dynam. Systems Appl. 7 (2016) 198–202.
- [22] R. T. Rockafellar: *Integral functionals, normal integrands and measurable selections*, Lecture Notes in Math. 543, Springer-Verlag Berlin (1976) 157–207.
- [23] R. S. Sadyrkhonov: *On infinite dimensional features of proper and closed mappings*, Proc. Amer. Math. Soc. 98 (1986) 643–658.
- [24] J. Saint Raymond: *On a minimax theorem*, Arch. Math. (Basel) 74 (2000) 432–437.
- [25] M. Schechter, K. Tintarev: *Spherical maxima in Hilbert space and semilinear elliptic eigenvalue problems*, Differential Integral Equations 3 (1990) 889–899.
- [26] J. L. Vázquez: *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. 12 (1984) 191–202.
- [27] C. Zălinescu: *Convex analysis in general vector spaces*, World Scientific (2002).
- [28] E. Zeidler: *Nonlinear functional analysis and its applications*, vol. I, Springer-Verlag Berlin (1986).
- [29] E. Zeidler: *Nonlinear functional analysis and its applications*, vol. III, Springer-Verlag Berlin (1985).