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Multidimensional $\mathcal{P}$-adic Integrals in some Problems of Harmonic Analysis *

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The paper is a survey of results related to the problem of recovering the coefficients of some classical orthogonal series from their sums by generalized Fourier formulas. The method is based on reducing the coefficient problem to the one of recovering a function from its derivative with respect to an appropriate derivation basis. In the case of the multiple Vilenkin system the problem is solved by using a multidimensional $\mathcal{P}$-adic integral.

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1. Introduction

This paper is a survey of some recent results, including some results by the authors, related to the problem of recovering the coefficients of some classical orthogonal series (such as Walsh, Haar and Vilenkin series) from their sums by generalized Fourier formulas.

The problem of recovering the coefficients of orthogonal series is a generalization of the uniqueness problem for the coefficients of orthogonal series. The uniqueness can be related to point-wise convergent series or series which are summable in a certain sense, and the convergence or summability can be supposed everywhere or outside some exceptional sets, the so-called sets of uniqueness or U-sets. We recall that a set E is said to be U-set for a system of functions if the convergence of a series with respect to this system to zero outside the set E implies that all coefficients of the series are zero (for references to the literature on the rich theory

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of uniqueness of Walsh, Haar and Vilenkin series, including subtle theory of sets of uniqueness, see [1], [5], [15], [35], [36], whereas the classical trigonometric case is treated for example in [38]).

If the uniqueness theorem is proved for a certain system and so the coefficients of an orthogonal series with respect to this system are uniquely determined by its sum, then it is natural to expect that they may be recovered from the sum by Fourier formulas, as it takes place in the simplest cases, for example in the case of the uniform convergence. Indeed for many known systems (trigonometric, Haar, Walsh, Vilenkin systems) it is true that every series with respect to those systems which is convergent everywhere to a summable function is the Fourier series of this function. But the point is that the sum of everywhere convergent orthogonal series can fail to be Lebesgue integrable. For example, it is known (see [38]) that the series

$$\sum_{k=2}^{\infty} \frac{\sin kx}{\ln k}$$

converges everywhere but fails to be a Fourier-Lebesgue series. This kind of examples can be given for the other above mentioned systems as well. To integrate such series, one needs nonabsolutely convergent integrals. In the cases where the sum is integrable in one or another known sense, the question is whether the coefficients can be determined by Fourier formulas in which the integral is understood in the same particular sense. The complete solution of the problem of recovering the coefficients of a convergent or summable series with respect to some system is found if a general process of integration is developed so that any such a series is the Fourier series of its sum, in the sense of this generalized integral.

In the classical trigonometric case the first solution of the problem of defining an integral so powerful that the sum of any everywhere convergent series is integrable and the coefficients can be computed by generalized Fourier formulas, is due to Denjoy (see [4]). Later an easier approach based on Perron and Henstock-Kurzweil methods was developed by several authors (see [33] for details).

In the one-dimensional case the problem of recovering the coefficients of a convergent Haar and Walsh series was solved using various types of non-absolute generalization of the Lebesgue integral, including the dyadic Denjoy (see [7], [16], [20]), the dyadic Perron (see [17]) and the dyadic Henstock-Kurzweil integrals (see [25], [26] and [27]). In the multidimensional case a solution for the same series depends on the type of convergence. The so-called regular convergence was considered in [13] and [14]. For the rectangular convergence the coefficients problem was solved in [18], [19] for Walsh and Haar series convergent to a Perron integrable function.

In the paper [29] we considered rectangular convergent Haar and Walsh series, without assuming a priori integrability of the sum in any prescribed sense and we solved the coefficients problem by finding an appropriate integral to be used in generalized Fourier formulas. In the present paper we generalize this last result to the case of a multiple Vilenkin system. As it was in the Walsh case, the method is based on reducing the coefficients problem to the one of recovering a function from its derivative with respect to an appropriate derivation basis which is the so called $\mathcal{P}$-adic basis in the case of the Vilenkin system. The difficulties which should be overcome in applying this method in the multidimensional case are related to the fact that the primitive we want to recover is differentiable not everywhere but outside some exceptional set which is not countable in the dimension greater then one. In the application to the coefficients problem we consider exceptional sets which are U-sets for the multiple Vilenkin system. We investigate continuity assumptions which should be imposed on the primitive at the points of the exceptional sets to guarantee its uniqueness. It turns out that the usual continuity with respect to the basis is not enough for this purpose and we introduce a stronger notion of continuity, which we call local Saks continuity with respect to the basis.

The most natural integration process to recover primitives is the Henstock-Kurzweil integral (see [34]). We recall the principal elements of the Henstock construction in Section 2 following [11] and [32]. Although the $\mathcal{P}$-adic Henstock-Kurzweil integral in a dimension greater then one has the local Saks continuity, it solves the problem of recovering a primitive only in the case of rather "thin" exceptional sets and fails to solve it in the case of the sets we are interested in. To solve this problem, we introduce in Section 3 a suitable $\mathcal{P}$-adic Perron-type integral defined by major and minor functions having the local Saks continuity property. It is a generalization of the dyadic Perron integral defined in [29]. We show in Section 4 that each two-dimensional Vilenkin series which converges everywhere outside a U-set of the type we consider here, is the Fourier series of its sum in the sense of this Perron-type integral. We are mentioning also some results related to Haar series.

2. Derivation bases and Henstock-Kurzweil type integrals with respect to them

A *derivation basis* (or simply a *basis*) $\mathcal{B}$ in a measure space $(X, \mathcal{M}, \mu)$ is a filter base on the product space $\mathcal{I} \times X$, where $\mathcal{I}$ is a family of measurable subsets of X having positive measure μ and called *generalized intervals* or *$\mathcal{B}$-intervals*. That is, $\mathcal{B}$ is a nonempty collection of subsets of $\mathcal{I} \times X$ so that each $\beta \in \mathcal{B}$ is a set of pairs (I, x), where $I \in \mathcal{I}$, $x \in X$, and $\mathcal{B}$ has the *filter base property*: $\emptyset \notin \mathcal{B}$ and for every $\beta_1, \beta_2 \in \mathcal{B}$ there exists $\beta \in \mathcal{B}$ such that $\beta \subset \beta_1 \cap \beta_2$. So each basis is a directed set with the order given by "reversed" inclusion. We shall refer to the elements β of $\mathcal{B}$ as *basis sets*. If $x \in I$ for all the pairs (I, x) constituting each $\beta \in \mathcal{B}$ we say that the basis is a *Henstock basis*. Otherwise it is called *McShane basis* but we do not consider such bases here.

In this paper we consider some class of bases in the unit cube $K \subset \mathbb{R}^m$. So in our case $X = K$ and $\mathcal{I}$ is some family of subintervals of K. The basis sets will be defined by

$$\beta_\delta := \{I \in \mathcal{I} : \quad I \subset U(x, \delta(x))\},$$

where δ is the so-called *gauge*, i.e, a positive function defined on K, $U(x, \delta)$ denotes the neighborhood of x with radius δ. We shall also use $|I|$ for the Lebesgue measure of I, and $diam(I)$ for the diameter of I.

If $\mathcal{I}$ is constituted by *all* closed n-dimensional subintervals of K, then $\mathcal{B}$ is called *full interval basis* in K.

The principal basis we are considering here is the $\mathcal{P}$-adic derivation basis. To define it, let $\mathcal{P} = \{p_j\}_{j=0}^{\infty}$ be a sequence of integers with $p_j \geq 2$ for $j = 0, 1, ...$ and put $m_0 = 1$, $m_k = p_0 \cdot p_1 \cdot ... \cdot p_{k-1}$. We denote by $Q_{\mathcal{P}}$ the set of all $\mathcal{P}$-*adic rational numbers in* $[0, 1]$, i.e., the numbers of the form $\frac{j}{m_k}$ with $0 \leq j \leq m_k$, $k = 0, 1, 2, \ldots$. The points $[0, 1] \setminus Q_{\mathcal{P}}$ constitute the set of $\mathcal{P}$-*adic irrational numbers* in $[0, 1]$.

We denote one-dimensional $\mathcal{P}$-adic intervals by

$$I_r^{(k)} := \left[\frac{r}{m_k}, \frac{r+1}{m_k}\right], \quad 0 \leq r \leq m_k - 1,$$

where $k = 0, 1, 2, \ldots$ is the *rank* of the interval.

Let $\mathcal{I}_{\mathcal{P}}$ be the family of all m-dimensional $\mathcal{P}$-adic intervals

$$I_{\mathbf{r}}^{(\mathbf{k})} := I_{r_1}^{(k_1)} \times \cdots \times I_{r_m}^{(k_m)} \tag{1}$$

in K, where $\mathbf{k} = (k_1, \ldots, k_m)$ is the rank of $I_{\mathbf{r}}^{(\mathbf{k})}$ and $\mathbf{r} = (r_1, \ldots, r_m)$. We denote by $I^{(\mathbf{k})}$ an arbitrary $\mathcal{P}$-adic interval of rank $\mathbf{k}$ and by $I_{\mathbf{x}}^{(\mathbf{k})}$, where $\mathbf{x} = \{x_1, ..., x_m\} \in K$, an interval of rank $\mathbf{k}$ containing $\mathbf{x}$.

For each fixed $\mathbf{x} \in K$ there exists a sequence of $\mathcal{P}$-adic intervals $\{I_{\mathbf{x}}^{(\mathbf{n})}\}$ such that $\cap_{\mathbf{n}} I_{\mathbf{x}}^{(\mathbf{n})} = \{\mathbf{x}\}$. Note that if $\mathbf{x}$ is an interior point of K, the sequence $\{I_{\mathbf{x}}^{(\mathbf{n})}\}$ is constituted by 2^s subsequences of pair-wise overlapping $\mathcal{B}$-intervals with nested projections to the coordinate axis, where s is the number of $\mathcal{P}$-adic rational coordinates of the point $\mathbf{x}$. In particular, if $\mathbf{x}$ has all coordinates $\mathcal{P}$-adic irrational the sequence $\{I_{\mathbf{x}}^{(\mathbf{n})}\}$ cannot be splited into non-overlapping subsequences and $\mathbf{x}$ is an interior point for any interval of this sequence.

So a basis set in this case is given by

$$\beta_\delta := \{I \in \mathcal{I}_{\mathcal{P}} : \ I \subset U(x, \delta(x))\},$$

and the $\mathcal{P}$-*adic basis* is defined as $\mathcal{B}_{\mathcal{P}} := \{\beta_\delta : \ \delta : X \to (0, \infty)\}$. If all $p_i = 2$ in the sequence $\mathcal{P}$ then we get the family of all dyadic intervals $\mathcal{I}_d$ and the dyadic basis $\mathcal{B}_d$.

We mention also the ρ-*regular dyadic basis* $\mathcal{B}_{d,\rho}$ defined by its basis sets

$$\beta_{\delta,\rho} := \{I \in \mathcal{I}_d : \ I \subset U(x, \delta(x)), \ reg(I) \geq \rho\},$$

where the *parameter of regularity* of a dyadic interval of the form (1) is defined as

$$reg(I) = \min_{i,l} \{|I_{r_i}^{(k_i)}| / |I_{r_l}^{(k_l)}|\}.$$

A *β-partition* is a finite collection π of elements of β, where distinct elements (I', x') and (I'', x'') in π have I' and I'' non-overlapping, i.e., $\mu(I' \cap I'') = 0$. Let $L \in \mathcal{I}$. If $\bigcup_{(I,x)\in\pi} I = L$ then π is called β-partition *of* L.

It is obvious that all the basis discussed above have the *partitioning property* by which we mean: (i) for each finite collection $I_0, I_1, ..., I_n$ of $\mathcal{B}$-intervals with $I_1, ...I_n \subset I_0$ the difference $I_0 \backslash \bigcup_{i=1}^{n} I_i$ can be expressed as a finite union of pairwise non-overlapping $\mathcal{B}$-intervals; (ii) for each $\mathcal{B}$-interval L and for any $\beta \in \mathcal{B}$ there exists a β-partition of L.

Now we recall the definition of Henstock-Kurzweil type integral with respect to the basis $\mathcal{B}$.

Definition 2.1. Let $\mathcal{B}$ be a basis having the partitioning property and $L \in \mathcal{I}$. A function f on L is said to be *$H_{\mathcal{B}}$-integrable* on L, with *$H_{\mathcal{B}}$-integral* A, if for every $\varepsilon > 0$, there exists $\beta \in \mathcal{B}$ such that for any β-partition π of L we have:

$$\left| \sum_{(I,x)\in\pi} f(x)|I| - A \right| < \varepsilon.$$

We denote the integral value A by $(H_{\mathcal{B}}) \int_L f$.

The following extension of the previous definition is useful in many cases.

Definition 2.2. A function f defined almost everywhere on $L \in \mathcal{I}$ is *$H_{\mathcal{B}}$-integrable* on L, with *$H_{\mathcal{B}}$-integral* A, if the function

$$f_1(x) := \begin{cases} f(x) & \text{if it is defined,} \\ 0 & \text{otherwise} \end{cases}$$

is $H_{\mathcal{B}}$-integrable on L and its $H_{\mathcal{B}}$-integral is equal to A.

If in the previous definitions $\mathcal{B}$ stands for the full interval basis, we are getting the classical Henstock-Kurzweil integral. For the basis $\mathcal{B}_{\mathcal{P}}$ we get the $\mathcal{P}$-adic Henstock integral ($H_{\mathcal{P}}$-integral). In particular for the dyadic basis we get the dyadic Henstock integral (H_d-integral).

Given a set function $F : \mathcal{I} \to \mathbb{R}$ we define the *upper* and the *lower $\mathcal{B}$-derivative* at a point x, with respect to the basis $\mathcal{B}$, as

$$\overline{D}_{\mathcal{B}} F(x) := \inf_{\beta\in\mathcal{B}} \sup_{(I,x)\in\beta} \frac{F(I)}{|I|} \quad \text{and} \quad \underline{D}_{\mathcal{B}} F(x) := \sup_{\beta\in\mathcal{B}} \inf_{(I,x)\in\beta} \frac{F(I)}{|I|}, \tag{2}$$

respectively. It is clear that $\overline{D}_{\mathcal{B}} F(x) \geq \underline{D}_{\mathcal{B}} F(x)$. If $\overline{D}_{\mathcal{B}} F(x) = \underline{D}_{\mathcal{B}} F(x)$ we call this common value *$\mathcal{B}$-derivative* $D_{\mathcal{B}} F(x)$. For a complex-valued set function $F = \text{Re}F + i\,\text{Im}F$ we define the $\mathcal{B}$-derivative at a point x as $D_{\mathcal{B}} F(x) := D_{\mathcal{B}} \text{Re}F(x) + D_{\mathcal{B}} \text{Im}F(x)$.

Definition 2.3. We say that a set function F, real- or complex-valued, is *continuous at a point x, with respect to the basis $\mathcal{B}$*, or briefly *$\mathcal{B}$-continuous*, if

$$\lim_{diam(I)\to 0,\ \mathbf{x}\in I} F(I) = 0, \tag{3}$$

From the definitions of $H_{\mathcal{B}}$-integral, $\mathcal{B}$-derivative and $\mathcal{B}$-continuity we can easily obtain the following theorem on recovering a primitive from its derivative for a general derivation basis.

Theorem 2.4. *Let an additive function $F : \mathcal{I} \to \mathbb{R}$ be $\mathcal{B}$-differentiable everywhere on $L \in \mathcal{I}$ outside a set E with $|E| = 0$, and $-\infty < \underline{D}_{\mathcal{B}}F(x) < \overline{D}_{\mathcal{B}}F(x) < +\infty$ everywhere on E except on a countable set $M \subset E$ where F is $\mathcal{B}$-continuous. Then the function*

$$f(x) := \begin{cases} D_{\mathcal{B}}F(x) & \text{if it exists,} \\ 0 & \text{if } x \in E \end{cases}$$

is $H_{\mathcal{B}}$-integrable on L and F is its indefinite $H_{\mathcal{B}}$-integral.

By this theorem we get

Corollary 2.5. *If an additive $\mathcal{B}$-interval function F is $\mathcal{B}$-differentiable satisfying $D_{\mathcal{B}}F(\mathbf{x}) = f(\mathbf{x})$ everywhere on K outside a countable set where F is $\mathcal{B}$-continuous, then the function f is $H_{\mathcal{B}}$-integrable on K and F is its indefinite $H_{\mathcal{B}}$-integral.*

From the Corollary above it is clear that if the exceptional set E, on which the primitive is not differentiable, is countable and a primitive is $\mathcal{B}$-continuous on this set, then this primitive is defined uniquely. But, as can be shown by easy examples, $\mathcal{B}$-continuity does not guarantee uniqueness of the primitive in many cases where the role of these exceptional sets is played by uncountable sets.

Then in order to guarantee that a primitive is uniquely defined by the derivative we have to impose some stronger continuity assumptions on the primitive at the points of the exceptional set. In particular in the $\mathcal{P}$-adic case the role of the exceptional set in this paper will be played by the set Z of points having at least one $\mathcal{P}$-rational coordinate, i.e.,

$$Z := \bigcup_{i=1}^{m}([0,1]^{i-1} \times Q_{\mathcal{P}} \times [0,1]^{m-i}). \tag{4}$$

We shall use also a more general set

$$Y := \bigcup_{i=1}^{m}([0,1]^{i-1} \times Y_i \times [0,1]^{m-i}) \tag{5}$$

where Y_i, $i = 1, 2, ..., m$, is any countable set *containing $Q_{\mathcal{P}}$*.

A stronger notion of continuity to guarantee uniqueness of a primitive can be given by generalizing the continuity in the sense of Saks.

We recall that an interval function F is said to be *continuous in the sense of Saks* if $\lim_{|I|\to 0,} F(I) = 0$. We define a local version of this type of continuity adjusted to $\mathcal{B}$-interval functions.

Definition 2.6. We say that a $\mathcal{B}$-interval function F is *locally $\mathcal{B}$-continuous in the sense of Saks*, or briefly *$\mathcal{B}S$-continuous at a point* $\mathbf{x}$ if

$$\lim_{|I|\to 0,\ \mathbf{x}\in I} F(I) = 0\,. \tag{6}$$

We shall be interested in this definition in the case of the $\mathcal{P}$-adic basis, calling the corresponding continuity $\mathcal{B}_{\mathcal{P}}S$-continuity. In the two-dimensional $\mathcal{P}$-adic case the equality (6) can be rewritten in terms of ranks of $\mathcal{B}$-intervals in the following way:

$$\lim_{k+l\to\infty} F(I_{\mathbf{x}}^{(k,l)}) = 0\,. \tag{7}$$

We shall see in the next section that the assumption of $\mathcal{B}_{\mathcal{P}}S$-continuity in the application to $\mathcal{B}_{\mathcal{P}}S$-bases guarantees the desired uniqueness of a primitive.

Following the lines of the proof of the corresponding statement for the dyadic Henstock integral (see [29]) we get

Theorem 2.7. *Let f be an $H_{\mathcal{P}}$-integrable function on K. Then the indefinite $H_{\mathcal{P}}$-integral F is $\mathcal{B}_{\mathcal{P}}S$-continuous everywhere on K.*

Unfortunately, in spite of the $\mathcal{B}_{\mathcal{P}}S$-continuity, the $H_{\mathcal{P}}$-integral is not strong enough to solve the problem of recovering a primitive in the case of the exceptional sets which we are using in the coefficients problem for multiple series.

But as we shall see in the next Section, some version of $\mathcal{P}$-adic Perron-type integral will serve our purpose.

3. Perron-type integral and the problem of recovering a primitive

The classical Henstock-Kurzweil integral, in dimension one, is known to be equivalent to the Perron integral with respect to the the full interval basis (see [10]). We recall the definition of Perron-type integral with respect to general basis $\mathcal{B}$.

Definition 3.1. Let f be a point-function defined on K. An additive $\mathcal{B}$-interval function M (resp., m) is called a $\mathcal{B}$-*major* (resp., $\mathcal{B}$-*minor*) function of f if the lower (resp., the upper) $\mathcal{B}$-derivative satisfies the inequality

$$\underline{D}_{\mathcal{B}}M(\mathbf{x}) \geq f(\mathbf{x}) \ \ (\text{resp. } \ \overline{D}_{\mathcal{B}}m(\mathbf{x}) \leq f(\mathbf{x})) \ \text{ for all } \ \mathbf{x}\in K. \tag{8}$$

It is known (see [11]) that if M and m are a $\mathcal{B}$-major and a $\mathcal{B}$-minor function for a point-function f on K, then for each $\mathcal{B}$-interval I we have $M(I) \geq m(I)$. This implies that for any function f we have

$$\inf_M\{M(K)\} \geq \sup_m\{m(K)\}$$

where "inf" and "sup" are taken over all $\mathcal{B}$-major and $\mathcal{B}$-minor functions of f, respectively. This justifies the following definition.

Definition 3.2. A point-function f defined on K is said to be $P_{\mathcal{B}}$-*integrable* on K, if there exists at least one $\mathcal{B}$-major function and at least one $\mathcal{B}$-minor function of f and

$$-\infty < \inf_M\{M(K)\} = \sup_m\{m(K)\} < +\infty$$

where "inf" and "sup" are taken as above. The common value is called $P_{\mathcal{B}}$-*integral* of f on K and is denoted by $(P_{\mathcal{B}})\int_K f$.

In the same way we can define the $P_{\mathcal{B}}$-integral on any $\mathcal{B}$-interval I.

Applying this definition to the full interval basis, we get the classical Perron integral. For the basis $\mathcal{B}_{\mathcal{P}}$ we get the $\mathcal{P}$-adic Perron integral ($P_{\mathcal{P}}$-integral). In particular for the dyadic basis we get the dyadic Perron integral (P_d-integral), considered in many papers (see [21] [22]).

Moreover these dyadic Perron and $\mathcal{P}$-adic integrals can be defined by $\mathcal{B}$-continuous major and minor functions, and we get an equivalent definition (see [3] for the case of the full interval basis; a proof for the $\mathcal{P}$-adic case is similar). We need not recall here the definition of these $P_{\mathcal{P}}$-integral and we pass directly to constructing another Perron-type integral defined by $\mathcal{B}_{\mathcal{P}}S$-continuous major and minor functions, which will be used to solve the coefficients problem.

We start with a few lemmas which we need to justify our definition and which are generalizations of the respective lemmas in [29]. In these lemmas ψ denotes any *additive* $\mathcal{B}$-interval function.

Lemma 3.3. *Let an interval $I \in \mathcal{I}_{\mathcal{P}}$ be represented as union of p non-overlapping equal $\mathcal{B}_{\mathcal{P}}$-intervals $\overline{I}_j$ of bigger rank. Suppose that for some numbers A, C and for a fixed index i, $\psi(\overline{I}_i) \geq A$ and $\psi(\overline{I}_j) \leq C|\overline{I}_j|$ for any $j \neq i$. Then $\psi(\overline{I}_i) \geq A - C\frac{p-1}{p}|I|$.*

Proof. Note that $|\overline{I}_j| = \frac{1}{p}|I|$. Then having in mind the additivity of ψ we can write

$$\psi(\overline{I}_i) = \psi(I) - \sum_{j\neq i}\psi(\overline{I}_j) \geq A - C\sum_{j\neq i}|\overline{I}_j| = A - C\frac{p-1}{p}|I|\,. \qquad \square$$

Lemma 3.4. *Let ψ be $\mathcal{B}_{\mathcal{P}}S$-continuous everywhere in K and let $\psi(I^{(\mathbf{n})}) > C|I^{(\mathbf{n})}|$ for some $I^{(\mathbf{n})}$. Then there exist at least two $\mathcal{B}$-intervals $I^{(\mathbf{n}')}$ and $I^{(\mathbf{n}'')}$ contained in $I^{(\mathbf{n})}$ whose projections onto each of the coordinate axes are disjoint and for which $\psi(I^{(\mathbf{n}')}) > C|I^{(\mathbf{n}')}|$ and $\psi(I^{(\mathbf{n}'')}) > C|I^{(\mathbf{n}'')}|$.*

Proof. Take C_1 such that $\frac{\psi(I^{(\mathbf{n})})}{|I^{(\mathbf{n})}|} \geq C_1 > C$. Split the interval $I^{(\mathbf{n})}$ into p_{n_1} intervals of rank $\mathbf{n} = (n_1+1, n_2, ..., n_m)$ and using the additivity of ψ choose one of them, say $\overline{I}^{(n_1+1,n_2,...,n_m)}$, for which $\psi(\overline{I}^{(n_1+1,n_2,...,n_m)}) \geq \frac{\psi(I^{(\mathbf{n})})}{p_{n_1}}$. Then for this interval we have

$$\psi(\overline{I}^{(n_1+1,n_2,...,n_m)}) \geq \frac{\psi(I^{(\mathbf{n})})}{p_n} > \frac{1}{p_n} C|I^{(\mathbf{n})}| = C|\overline{I}^{(n_1+1,n_2,...,n_m)}|.$$

Now repeating the same argument for the interval $\overline{I}^{(n_1+1,n_2,...,n_m)}$ we choose an interval $\overline{I}^{(n_1+2,n_2,...,n_m)} \subset \overline{I}^{(n_1+1,n_2,...,n_m)}$ such that

$$\psi(\overline{I}^{(n_1+2,n_2,...,n_m)}) > C|\overline{I}^{(n_1+2,n_2,...,n_m)}|.$$

In this way we obtain a sequence $\{\overline{I}^{(n_1+j,n_2,...,n_m)}\}_j$ of nested interval of rank $(n_1+j, n_2, ..., n_m)$ for which

$$\psi(\overline{I}^{(n_1+j,n_2,...,n_m)}) > C|\overline{I}^{(n_1+j,n_2,...,n_m)}|. \tag{9}$$

We denote by $\overline{\overline{I}}^{(n_1+j,n_2,...,n_m)}$ any interval of rank $(n_1+j, n_2, ..., n_m)$ from the complement of the interval $\overline{I}^{(n_1+j,n_2,...,n_m)}$ in the $\overline{I}^{(n_1+j-1,n_2,...,n_m)}$.

If we have $\psi(\overline{\overline{I}}^{(n_1+j,n_2,...,n_m)}) \leq C|\overline{\overline{I}}^{(n_1+j,n_2,...,n_m)}|$ for all j then applying Lemma 3.3 with $I = I^{(\mathbf{n})}$, $\overline{I}_i = \overline{I}^{(n_1+1,n_2,...,n_m)}$ and $A = C_1|I^{(\mathbf{n})}|$ we get

$$\psi(\overline{I}_+^{(n_1+1,n_2,...,n_m)}) \geq C_1|I^{(\mathbf{n})}| - C\frac{p_{n_1}-1}{p_{n_1}}|I^{(\mathbf{n})}|.$$

Now we apply Lemma 3.3 with $I = \overline{I}^{(n_1+1,n_2,...,n_m)}$, $I_i = \overline{I}_-^{(n_1+2,n_2,...,n_m)}$ and $A = C_1|I^{(\mathbf{n})}| - \frac{C}{2}|I^{(\mathbf{n})}|$ getting

$$\psi(\overline{I}^{(n_1+2,n_2,...,n_m)}) \geq C_1|I^{(\mathbf{n})}| - C\frac{p_{n_1}-1}{p_{n_1}}|I^{(\mathbf{n})}| - C\frac{p_{n_1+1}-1}{p_{n_1+1}}|\overline{I}^{(n_1+1,n_2,...,n_m)}| =$$

$$= |I^{(\mathbf{n})}|(C_1 - C\frac{p_{n_1}-1}{p_{n_1}} - C\frac{p_{n_1+1}-1}{p_{n_1+1}} \cdot \frac{1}{p_{n_1}}).$$

Proceeding by induction we get for any j

$$\psi(\overline{I}^{(n_1+j,n_2,...,n_m)}) \geq |I^{(\mathbf{n})}|(C_1 - C\frac{p_{n_1}-1}{p_{n_1}} - C\frac{p_{n_1+1}-1}{p_{n_1+1}} \cdot \frac{1}{p_{n_1}} - ... -$$

$$-C\frac{p_{n_1+j-1}-1}{p_{n_1+j-1}} \cdot \frac{1}{p_{n_1} \cdots p_{n_1+j-2}}) = |I^{(\mathbf{n})}|(C_1 - C(1 - \frac{1}{p_{n_1}p_{n_1+1} \cdots p_{n_1+j-1}}))$$

and so

$$\liminf_{j\to\infty} \psi(\overline{I}^{(n_1+j,n_2,\dots,n_m)} \geq |I^{(\mathbf{n})}|(C_1 - C) > 0.$$

This inequality is obviously in contradiction with $\mathcal{B}_{\mathcal{P}}S$-continuity of ψ at the points of the set $\bigcap_j \overline{I}^{(n_1+j,n_2,\dots,n_m)}$. Hence for some j and some interval

$$\overline{\overline{I}}^{(n_1+j,n_2,\dots,n_m)} \subset \overline{I}^{(n_1+j-1,n_2,\dots,n_m)} \setminus \overline{I}^{(n_1+j,n_2,\dots,n_m)}$$

we get

$$\psi(\overline{\overline{I}}^{(n_1+j,n_2,\dots,n_m)}) > C|\overline{\overline{I}}^{(n_1+j,n_2,\dots,n_m)}|.$$

Since the same estimation is also true for $\overline{I}^{(n_1+j,n_2,\dots,n_m)}$, see (9), we obtain two intervals $\overline{I}^{(n_1+j,n_2,\dots,n_m)}$ and $\overline{\overline{I}}^{(n_1+j,n_2,\dots,n_m)}$ for which the desired inequality holds. For each of them we can repeat the preceding argument having fixed $n_1+j, n_3, \dots, n_m$ and varying the second index n_2. We thereby obtain four intervals for each of which the corresponding estimation holds. By geometric considerations we can choose two of them whose projections onto each of the first two axis do not overlap. If the projections have common end-point we can repeat the previous construction for each of the obtained intervals getting, once again by geometric consideration, two intervals with disjoint projections on two axis. Then we can proceed varying the third index and keeping fixed the others, and so on. We obtain after m steps the two desired intervals. □

Lemma 3.5. *Let ψ be $\mathcal{B}_{\mathcal{P}}S$-continuous in K and let $\psi(I^{(\mathbf{n})}) > C|I^{(\mathbf{n})}|$ for some $I^{(\mathbf{n})}$. Then there exists a perfect set $P \subset I^{(\mathbf{n})}$, any two points of which have pairwise distinct coordinates and $\overline{D}_{\mathcal{B}}\psi(\mathbf{x}) \geq C$ holds for all $\mathbf{x} \in P$.*

Proof. The statement can be obtained by the repeated application of Lemma 3.4. □

Applying the previous lemma to the function $-\psi$ instead of ψ we can formulate the following version of it.

Lemma 3.6. *Let ψ be $\mathcal{B}_{\mathcal{P}}S$-continuous in K and let $\psi(I^{(\mathbf{n})}) < C|I^{(\mathbf{n})}|$ for some $I^{(\mathbf{n})}$. Then there exists a perfect set $P \subset I^{(\mathbf{n})}$, any two points of which have pairwise distinct coordinates and $\underline{D}_{\mathcal{B}}\psi(\mathbf{x}) \leq C$ holds for all $\mathbf{x} \in P$.*

Definition 3.7. Let f be a point function defined at least on $K \setminus Z$. An additive $\mathcal{B}_{\mathcal{P}}S$-continuous on K $\mathcal{B}$-interval function M (resp., m) is called a *$\mathcal{B}_{\mathcal{P}}S$-major* (resp., *$\mathcal{B}_{\mathcal{P}}S$-minor*) function of f if the lower (resp., the upper) $\mathcal{B}$-derivative satisfies the inequality

$$\underline{D}_{\mathcal{B}}M(\mathbf{x}) \geq f(\mathbf{x}) \ \ (\text{resp. } \overline{D}_{\mathcal{B}}m(\mathbf{x}) \leq f(\mathbf{x})) \ \text{ for all } \ \mathbf{x} \in K \setminus Z. \qquad (10)$$

Proofs of the following two lemmas are the same as the proofs of the corresponding lemmas in [29] for the dyadic case.

Lemma 3.8. *Let an additive $\mathcal{B}$-interval function R be $\mathcal{B}_{\mathcal{P}}S$-continuous on K and satisfy the inequality $\underline{D}_{\mathcal{B}}R(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in K \setminus Y$ with Y defined by* (5)*. Then $R(I) \geq 0$ for any $\mathcal{B}$-interval I.*

Lemma 3.9. *Let M and m be a $\mathcal{B}_{\mathcal{P}}S$-major and a $\mathcal{B}_{\mathcal{P}}S$-minor function for a point-function f on K. Then for each $\mathcal{B}$-interval I we have $M(I) \geq m(I)$.*

The last lemma implies that for any function f we have

$$\inf_M\{M(K)\} \geq \sup_m\{m(K)\}$$

where "inf" and "sup" are taken over all $\mathcal{B}_{\mathcal{P}}S$-major and $\mathcal{B}_{\mathcal{P}}S$-minor functions of f, respectively. This justifies the following definition.

Definition 3.10. A point-function f defined at least on $K \setminus Z$ is said to be $P_{\mathcal{P}}S$-*integrable* on K, if there exists at least one $\mathcal{B}_{\mathcal{P}}S$-major function and at least one $\mathcal{B}_{\mathcal{P}}S$-minor function of f and

$$-\infty < \inf_M\{M(K)\} = \sup_m\{m(K)\} < +\infty$$

where "inf" and "sup" are taken as above. The common value is called $P_{\mathcal{P}}S$-*integral* of f on K and is denoted by $(P_{\mathcal{P}}S)\int_K f$.

In the same way we can define a $P_{\mathcal{P}}S$-integral on any $\mathcal{B}$-interval I.

Directly from the definitions we get the following result which shows that the $P_{\mathcal{P}}S$-integral solves the problem of recovering the primitive from its $\mathcal{B}$-derivative in the form we need.

Theorem 3.11. *If an additive $\mathcal{B}_{\mathcal{P}}S$-continuous $\mathcal{B}$-interval function F is $\mathcal{B}$-differentiable with $D_{\mathcal{B}}F(\mathbf{x}) = f(\mathbf{x})$ everywhere on $K \setminus Z$ then the function f is $P_{\mathcal{P}}S$-integrable on K and F is its indefinite $P_{\mathcal{P}}S$-integral.*

Remark 3.12. The previous definition of $P_{\mathcal{P}}S$-integral can be extended to the case when the inequalities (10) related to major and minor function hold outside a fixed set Y defined by (5). Such an integral for function f, defined at least on $K \setminus Y$, depends on the chosen exceptional set Y and we call it $P_{\mathcal{P}}^Y S$-integral. As Y contains Z, $P_{\mathcal{P}}^Y S$-integral includes $P_{\mathcal{P}}S$-integral. Theorem 3.11, with Z replaced by Y, holds true for this integral.

Remark 3.13. In the same way as it was shown in the dyadic case in [29], the assumption of $\mathcal{B}_{\mathcal{P}}S$-continuity of F in the above theorem (and in Lemma 3.8) cannot be weakened to the one of $\mathcal{B}_{\mathcal{P}}$-continuity.

The next theorem was proved in [29] for the dyadic case.

Theorem 3.14. *There exists a P_dS-integrable function f on $K := [0,1]^2$ which is not H_d-integrable. Moreover if Φ is the indefinite P_dS-integral of f, then $D_{\mathcal{B}}\Phi(\mathbf{x}) = f(\mathbf{x})$ everywhere on $K \setminus Z$.*

This theorem shows in particular that the $H_{\mathcal{P}}$-integral (and so also the $P_{\mathcal{P}}$-integral) fails to solve the problem of recovering the primitive under the assumption of Theorem 3.11.

According to Theorem 2.7 an $H_{\mathcal{P}}$-integral is $\mathcal{B}_{\mathcal{P}}S$-continuous. But we do not know whether it can be defined by the Perron method using $\mathcal{B}_{\mathcal{P}}S$-continuous major and minor functions. So we repeat here the problem, stated in [29], for the dyadic case:

Problem 3.15. *Is any $H_{\mathcal{P}}$-integrable function $P_{\mathcal{P}}S$-integrable?*

4. Application to Vilenkin, Walsh and Haar series

We apply now the $P_{\mathcal{B}}S$-integral to solves the coefficients problem for multiple series which are convergent outside some U-sets. We recall the definitions (see [5] and [15]).

Using notation of Section 2 we fix a sequence of natural numbers

$$\mathcal{P} = \{p_j\}_{j=0}^{\infty}, \quad p_j \geq 2, \quad j = 0, 1, 2, ..., \tag{11}$$

and put $m_0 = 1, \quad m_k = p_0 \cdot p_1 \cdot ... \cdot p_{k-1}$.

We consider for each $x \in [0,1)$ its $\mathcal{P}$-adic expansion

$$x = \sum_{j=0}^{\infty} \frac{x_j}{m_{j+1}}, \quad 0 \leq x_j \leq p_j - 1. \tag{12}$$

We denote by $Q_{\mathcal{P}}$ the set of all points of the form $\frac{t}{m_k}$, $0 \leq t \leq m_k$. The elements of the set $[0,1] \setminus Q_{\mathcal{P}}$ are called *$\mathcal{P}$-adic-irrational numbers* in $[0,1]$.

We note that each $\mathcal{P}$-adic rational number x has two expansions, a finite one and an infinite one. We agree to consider for $x \in Q_{\mathcal{P}}$ only finite expansion. Then the interval $[0,1)$ is in one-to-one correspondence with all sequences of integers of the form $\{x_j\}_{j=0}^{\infty}$ $0 \leq x_j \leq p_j - 1$.

We also consider the *$\mathcal{P}$-adic expansion* of integers $n \geq 0$:

$$n = \sum_{j=0}^{s} \alpha_j m_{j-1}, \quad \text{with} \quad 0 \leq \alpha_j \leq p_j - 1 \ . \tag{13}$$

Now for each $n \geq 0$ with $\mathcal{P}$-adic expansion (13) we define the n-th function χ_n of the so called *multiplicative Vilenkin system* (see[1]) by replacing for x the sum given in (12):

$$\chi_n(x) = \exp\left(2\pi i \sum_{j=0}^{s} \frac{\alpha_j \ x_j}{p_j}\right) . \tag{14}$$

We consider half-open $\mathcal{P}$-*adic intervals*

$$J_r^{(k)} := \left[\frac{r}{m_k}, \frac{r+1}{m_k}\right), \quad 0 \leq r \leq m_k - 1, \tag{15}$$

where $k = 0, 1, 2, \ldots$ is the *rank* of the interval. By $J^{(k)}$ we denote an arbitrary interval of rank k. Note that closed interval $I_r^{(k)}$ defined in Section 2 is the closure of interval $J_r^{(k)}$

We also assign the rank k to a number n and to a function χ_n if $m_{k-1} \leq n < m_k$ (χ_0 has rank 0). Note that the functions χ_n of rank k are constant on the interval of the same rank and $\int_{I^{(k-1)}} \chi_n dx = 0$ (see [5]).

If in the above definition of Vilenkin system we assume that $p_j = 2$ for all j in the sequence (11), we obtain the *Walsh system* $\{w_n\}_{n=0}^{\infty}$.

The system of Walsh functions is closely related to the system of Haar wavelets. Up to the set of dyadic rational points, each Haar function is a linear combination of Walsh functions and vice versa.

We define the *Haar functions* on $[0, 1]$. Put $h_0(x) \equiv 1$. If $n = 2^k + i$, $k = 0, 1, \ldots$, $i = 0, \ldots, 2^k - 1$, we put

$$h_n(x) := \begin{cases} 2^{k/2}, & \text{if } x \in \left(\frac{2i-2}{2^{k+1}}, \frac{2i-1}{2^{k+1}}\right), \\ -2^{k/2}, & \text{if } x \in \left(\frac{2i-1}{2^{k+1}}, \frac{2i}{2^{k+1}}\right), \\ 0, & \text{if } x \in (0,1) \setminus \left[\frac{2i-2}{2^{k+1}}, \frac{2i}{2^{k+1}}\right], \end{cases}$$

and we agree that at each point of discontinuity $h_n(x) = \frac{1}{2}(\chi_n(x+0) + \chi_n(x-0))$ and that at $x = 0$ and $x = 1$ Haar functions are continuous from the right and from the left, respectively.

Now we pass to the multidimensional case.

An *m-dimensional Vilenkin* and *Haar series* are defined on cube $K = [0, 1)^m$ by

$$\sum_{\mathbf{n}=\mathbf{0}}^{\infty} a_{\mathbf{n}} \chi_{\mathbf{n}}(\mathbf{x}) := \sum_{n_1=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} a_{n_1,\ldots,n_m} \prod_{i=1}^{m} \chi_{n_i}(x_i) \tag{16}$$

$$\sum_{\mathbf{n}=\mathbf{0}}^{\infty} b_{\mathbf{n}} h_{\mathbf{n}}(\mathbf{x}) := \sum_{n_1=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} b_{n_1,\ldots,n_m} \prod_{i=1}^{m} h_{n_i}(x_i) \tag{17}$$

where $a_{\mathbf{n}}$ and $b_{\mathbf{n}}$ are real or complex numbers. It follows from the above definitions that for $\mathbf{n} = (n_1, ..., n_m)$ with $2^{k_j - 1} \leq n_j < 2^{k_j}$, $j = 1, ..., m$, the functions $\chi_{\mathbf{n}}$ and

$w_{\mathbf{n}}$ are constant in the interior of each dyadic interval of rank $\mathbf{k} = (k_1, ..., k_m)$. Moreover, with the same notation, the functions $h_{\mathbf{n}}$ are supported by some intervals of rank $\mathbf{k} - \mathbf{1} = (k_1 - 1, ..., k_m - 1)$.

If $\mathbf{N} = (N_1, \ldots, N_m)$, then the $\mathbf{N}$*th rectangular partial sum* $S_{\mathbf{N}}$ of series (16) (resp., (17)) at a point $\mathbf{x} = (x_1, \ldots, x_m)$ is

$$S_{\mathbf{N}}(\mathbf{x}) := \sum_{n_1=0}^{N_1-1} \ldots \sum_{n_m=0}^{N_m-1} a_{\mathbf{n}}\chi_{\mathbf{n}}(\mathbf{x}) \quad (\text{resp., } S_{\mathbf{N}}(\mathbf{x}) := \sum_{n_1=0}^{N_1-1} \ldots \sum_{n_m=0}^{N_m-1} b_{\mathbf{n}}h_{\mathbf{n}}(\mathbf{x}) \ ).$$

The series (16) (or (17)) *rectangularly converges* to sum $S(\mathbf{x})$ at a point $\mathbf{x}$ and we write $\lim_{\mathbf{N}\to\infty} S_{\mathbf{N}}(\mathbf{x}) = S(\mathbf{x})$ if

$$S_{\mathbf{N}}(\mathbf{x}) \to S(\mathbf{x}) \ \text{ as } \ \min_i\{N_i\} \to \infty.$$

To simplify the calculation, we shall formulate and prove most of the results for the two-dimensional case, but all of them are true for any dimension.

We note that in this case functions of the Vilenkin system $\chi_{n,m}(x, y) := \chi_n(x) \cdot \chi_m(y)$, with $m_{k-1} \le n < m_k$ and $m_{l-1} \le m < m_l$, $k, l \ge 1$, are constant on two-dimensional $\mathcal{P}$-adic intervals

$$J^{(k,l)}_{r_1,r_2} := J^{(k)}_{r_1} \times J^{(l)}_{r_2} \, . \tag{18}$$

These functions and these intervals are of rank (k, l). We denote by $J^{(k,l)}$ an arbitrary interval of rank (k, l) and by $J^{(k,l)}_{(x,y)}$ an interval, for which the point (x, y) belongs to its closure, i.e. to $I^{(k,l)}_{(x,y)}$.

The above observation implies that if $r \le m_k$ and $s \le m_l$ then the partial sums

$$S_{r,s}(x, y) := \sum_{n=0}^{r-1}\sum_{m=0}^{s-1} a_{n,m}\chi_{n,m}(x, y).$$

are constant on each interval of rank (k, l).

The following proposition is a generalization of the result obtained in [18] for the Walsh series. The proof is similar and will be given in [30]. In this proposition and in the rest of the paper we suppose that $\{p_i\}$ is bounded by a number $p \ge 2$.

Proposition 4.1. *If a double series with respect to Vilenkin system*

$$\sum_{n,m=0}^{\infty} a_{n,m}\chi_{n,m}(x, y) := \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{n,m}\chi_n(x)\chi_m(y) \tag{19}$$

is rectangular convergent on the "cross"

$$\{a \times (0, 1)\} \cup \{(0, 1) \times b\},$$

where a and b are $\mathcal{P}$-adic irrational points, everywhere expect on a countable set, then

$$\lim_{n+m\to+\infty} a_{n,m} = 0 \tag{20}$$

For the Haar series the result below was proved in [19].

Proposition 4.2. *If a two-dimensional series* (17) *is rectangular convergent on the "cross" $\{a\times[0,1]\}\cap\{[0,1]\times b\}$, $(a,b)\in K$, everywhere except on a countable set E and at each point of E we have*

$$\lim_{k,l\to\infty} \frac{b_{n_k,m_l} h_{n_k,m_l}(x,y)}{2^k 2^l} = 0, \tag{21}$$

then for this series

$$\lim_{k+l\to\infty} \frac{b_{n_k,m_l} h_{n_k,m_l}(a,b)}{2^k 2^l} = 0 \tag{22}$$

where $2^{k-1}\le n_k < 2^k$, $2^{l-1}\le m_l < 2^l$.

As it can be seen from the theorem in this section, Z (and also Y) is U-set for rectangular convergent multiple Vilenkin series. So it makes sense to state a problem of recovering the coefficients of those series from their sums defined outside of these U-sets. As for Haar series, non-empty U-sets exist only under additional assumptions of the type (21) or (22). Namely, Z is U-sets for Haar series under condition, that (22) holds everywhere. Under weaker assumption (21) on the exceptional set only countable sets are U-sets for rectangular convergent Haar series. Note that for ρ-regular convergent Haar series, with ρ close to 1, even the empty set is not U set (see [12]).

A standard method (see [24]) of application of the $\mathcal{P}$-adic derivative and the $\mathcal{P}$-adic integral to the theory of Vilenkin, Walsh and Haar series is based on the fact that for the partial sums $S_{\mathbf{m_k}}$ of those series (here $\mathbf{m_k}$ stand for $(m_{k_1},\ldots,m_{k_m})$), the integral $\int_{I_{\mathbf{j}}^{(\mathbf{k})}} S_{\mathbf{m_k}}$ defines an additive $\mathcal{B}_{\mathcal{P}}$-interval function $\psi(I)$ on the family $\mathcal{I}_{\mathcal{P}}$ of all $\mathcal{P}$-adic intervals (considering integrals we need make a distinction between intervals $I^{(\mathbf{k})}$ and half-open intervals $J^{(\mathbf{k})}$). In $\mathcal{P}$-adic analysis the function ψ is referred to as the *quasi-measure generated by the series* (see [15], [37]). Since the sum $S_{\mathbf{m_k}}$ is constant on each $J_{\mathbf{j}}^{(\mathbf{k})}$ we get

$$S_{\mathbf{m_k}}(\mathbf{x}) = \frac{1}{|I_{\mathbf{j}}^{(\mathbf{k})}|}\int_{I_{\mathbf{j}}^{(\mathbf{k})}} S_{\mathbf{m_k}} = \frac{\psi(I_{\mathbf{j}}^{(\mathbf{k})})}{|I_{\mathbf{j}}^{(\mathbf{k})}|} \tag{23}$$

for any point $\mathbf{x}\in int(I_{\mathbf{j}}^{(\mathbf{k})})$.

In fact any additive $\mathcal{B}_{\mathcal{P}}$-interval function ψ defines a Vilenkin (or Walsh and Haar in the dyadic case) series for which it is a quasi-measure and (23) holds. So we have a one-one correspondence between the family of additive $\mathcal{B}_{\mathcal{P}}$-interval functions and the family of Vilenkin (or Walsh and Haar) series.

The equality (23) obviously gives a relation between $\mathcal{B}$-differentiability of ψ at $\mathbf{x}$ and convergence of the series. In particular, at least at the points $\mathbf{x} \in \mathbf{K} \setminus \mathbf{Z}$, we get

$$\lim_{\mathbf{k}\to\infty} S_{\mathbf{m_k}}(\mathbf{x}) = D_{\mathcal{B}}\psi(\mathbf{x}), \tag{24}$$

and therefore the convergence of the series (16) (or (17)) at such points $\mathbf{x}$ to a sum $f(\mathbf{x})$ implies $\mathcal{B}$-differentiability of the function ψ at $\mathbf{x}$ with $f(\mathbf{x})$ being the value of $\mathcal{B}$-derivative.

Now we consider continuity properties of the quasi-measure.

Lemma 4.3. *If the coefficients of two-dimensional series* (19) *satisfy the condition* (20)*, then at each point* $(x, y) \in K$ *the quasi-measure* ψ *is* $\mathcal{B}_{\mathcal{P}}S$*-continuous, i.e.,* (7) *holds everywhere on* K.

Proof. Having in mind that for all $i < m_k$ and $j < m_l$ functions $\chi_{i,j}$ are constant on each interval $J^{(k,l)}_{(x,y)}$ of rank (k, l) we have for $(x, y) \in I^{(k,l)}_{(x,y)}$ (so (x, y) can belong to closure of $J^{(k,l)}_{(x,y)}$)

$$\psi(I^{(k,l)}_{(x,y)}) = \int_{I^{(k,l)}_{(x,y)}} S_{m_k,m_l}(t, s)dtds = \sum_{i,j=0}^{m_k-1,m_l-1} \int_{I^{(k,l)}_{(x,y)}} a_{i,j}\chi_{i,j}(t, s)dtds =$$

$$= \frac{\sum_{i,j=0}^{m_k-1,m_l-1} a_{i,j}\chi_{i,j}(x, y)}{m_k m_l}.$$

Hence

$$|\psi(I^{(k,l)}_{(x,y)})| \leq \frac{\sum_{i,j=0}^{m_k-1,m_l-1} |a_{i,j}|}{m_k m_l}.$$

But the left hand side expression is the arithmetic mean of the modulus of the coefficients $a_{i,j}$ and it tends to zero together with coefficients when $\lim_{i+j\to\infty}$. □

A similar lemma for Haar series was proved in [29]:

Lemma 4.4. *If the coefficients of the two-dimensional series* (17) *satisfy the condition* (22) *at a point* $(x, y) \in K$*, then at this point the quasi-measure* ψ *is* $\mathcal{B}S$*-continuous, i.e.,* (7) *holds at* (x, y).

Note that the above statement is not true even under assumption of convergence everywhere on K of Walsh series if it is convergent with respect to regular rectangles, for example with respect to cubes (see [14]).

The following statement is essential for establishing that a given Vilenkin or Haar series is the Fourier series in the sense of some general integral (see for example [24]; a proof, in the one-dimensional version, can be found in [5, Th. 3.1.8]).

Proposition 4.5. *Let some integration process* $\mathcal{A}$ *be given which produces an integral additive on* $\mathcal{I}$*. Assume a series of the form* (16) *or* (17) *is given. Let* *a $\mathcal{B}$-interval function ψ be the quasi-measure generated by this series and* (23) *holds. Then this series is the Fourier series of an $\mathcal{A}$-integrable function f if and only if $\psi(I) = (\mathcal{A}) \int_I f$ for any $\mathcal{B}$-interval I.*

In view of (24) and the above proposition, in order to solve the coefficient problem it is enough to show that the quasi-measure ψ generated by Haar or Walsh series is the indefinite $H_{\mathcal{P}}$-integral of its $\mathcal{B}$-derivative which exists at least on $K \setminus Z$. By this we reduce the problem of recovering the coefficients to the corresponding theorem on recovering the primitive with appropriate continuity assumptions.

In the one-dimensional case $Z = Q_d$, that is the exceptional set Z (and Y) is in fact countable. Moreover $\mathcal{B}$-continuity everywhere on $[0, 1]$ follows from the condition $\lim_{n\to\infty} a_n = 0$ (which in turn is a consequence of the convergence of the series at least at one $\mathcal{P}$-adic-irrational point). So we can apply Corollary 2.5 to get the following result (see [31, Th. 14.10]).

Theorem 4.6. *If the series* (16) (*in one dimension*) *is convergent to a sum f outside a countable set, then f is $H_{\mathcal{B}}$-integrable and* (16) *is the $H_{\mathcal{P}}$-Fourier-Vilenkin series of f, i.e.,*

$$a_n = (H_{\mathcal{B}}) \int_{[0,1]} f\chi_n.$$

This theorem can be generalized in the following way:

Theorem 4.7. *Suppose $\sum a_n\chi_n$ is convergent a.e. to a function f and everywhere on $I_{\mathcal{P}}$ except on a countable set we have*

$$-\infty < \underline{\lim}\mathrm{Re}S_n(x) < \overline{\lim}\mathrm{Re}S_n(x) < +\infty \tag{25}$$

$$-\infty < \underline{\lim}\mathrm{Im}S_n(x) < \overline{\lim}\mathrm{Im}S_n(x) < +\infty \tag{26}$$

Then f is $H_{\mathcal{P}}$-integrable and $\sum a_n\chi_n$ is $H_{\mathcal{P}}$-Fourier-Vilenkin series of f.

As any Lebesgue integrable function is known to be Perron and Henstock integrable (see [10]), we can state that every Vilenkin series which converges everywhere to a summable function is Fourier-Vilenkin series of this function. It can be formulated in a more general form (see [2]):

Theorem 4.8. *Suppose the real and the imaginary parts of the partial sums S_n of Vilenkin series satisfy everywhere on $[0, 1)$ except on a countable set E a conditions*

$$\underline{\lim}_{k\to\infty}\mathrm{Re}\, S_{m_k}(x) \le \phi(x) \le \overline{\lim}_{k\to\infty}\mathrm{Re}\, S_{m_k}(x), \tag{27}$$

$$\underline{\lim}_{k\to\infty}\mathrm{Im}\, S_{m_k}(x) \le \psi(x) \le \overline{\lim}_{k\to\infty}\mathrm{Im}\, S_{m_k}(x), \tag{28}$$

where $f = \phi + i\psi$ is Lebesgue integrable on $[0, 1)$ and at the points of E

$$\underline{\lim}_{k\to\infty}\mathrm{Re}\, m_k^{-1}S_{m_k}(x) \le 0 \le \overline{\lim}_{k\to\infty}\mathrm{Re}\, m_k^{-1}S_{m_k}(x), \tag{29}$$

$$\underline{\lim}_{k\to\infty}\mathrm{Im}\, m_k^{-1}S_{m_k}(x) \le 0 \le \overline{\lim}_{k\to\infty}\mathrm{Im}\, m_k^{-1}S_{m_k}(x). \tag{30}$$

Then this series is the Fourier-Vilenkin series of f.

In the multidimensional case we have to use Theorem 3.11 to get

Theorem 4.9. *If a two-dimensional series* (16) *is rectangular convergent to a sum f everywhere in $K \setminus Z$, then f is $P_{\mathcal{B}}S$-integrable on K and the coefficients of the series are $P_{\mathcal{B}}S$-Fourier-Vilenkin coefficients of f.*

Proof. Take any $(a, b) \in K \setminus Z$. Then the intersection of $\{a \times [0, 1]\} \cap \{[0, 1] \times b\}$ with Z is countable, and by Proposition 4.1 condition (20) holds. Then by Lemma 4.3 the quasi-measure ψ generated by the series (16) is $\mathcal{B}S$-continuous everywhere in K. Moreover, equality (24) implies

$$\lim_{\mathbf{k}\to\infty} S_{2^{\mathbf{k}}}(\mathbf{x}) = D_{\mathcal{B}}\psi(\mathbf{x}) = f(\mathbf{x})$$

everywhere on $K \setminus Z$. Then application of Theorem 3.11 and Proposition 4.5 completes the proof. □

We can enlarge the exceptional set Z here by replacing it by the set Y defined in (5) (see Remark 3.12). So we get

Theorem 4.10. *If the series* (16) *is rectangular convergent to a finite function f everywhere in $K \setminus Y$, then f is $P_{\mathcal{B}}^{Y}S$-integrable on K and the coefficients of the series are $P_{\mathcal{B}}^{Y}S$-Fourier-Vilenkin coefficients of f.*

Using Proposition 4.2 and Lemma 4.4 in the same way we obtain

Theorem 4.11. *If a two-dimensional series* (17) *is rectangular convergent to a sum f everywhere in K outside a countable set E and* (21) *holds everywhere on E then f is $P_{\mathcal{B}}S$-integrable on K and the coefficients of the series are $P_{\mathcal{B}}S$-Fourier-Haar coefficients of f.*

Note that in the above theorem we can omit condition (21) if we assume that the series (17) is convergent everywhere on K.

Analyzing the proof of the above theorem and the one of Lemma 4.3 we note that the convergence everywhere of the series has been used in order to obtain condition (22) on coefficients of the series which in turn imply $\mathcal{B}S$-continuity everywhere. So we can weaken the assumption of convergence in the formulation of Theorem 4.11 by supposing a priori that condition (22) is fulfilled. In this way we can obtain the following version of Theorem 4.11:

Theorem 4.12. *If the series* (17) *is rectangular convergent to a sum f everywhere in $K \setminus Z$ and the coefficients of the series satisfy everywhere the condition* (22)*, then f is $P_{\mathcal{B}}S$-integrable on K and the coefficients of the series are $P_{\mathcal{B}}S$-Fourier-Haar coefficients of f.*

5. P-integral in inversion formula for multiplicative transform

The problem of uniqueness can be considered also for the continual analogue of the Vilenkin system, i.e., for the case where the domain on which the functions are defined is not compact and the system is not countable.

We recall the appropriate definitions (see [1]). Consider a double sequence of natural numbers

$$\mathcal{R} = \{..., p_{-j}, ..., p_{-2}, p_{-1}, p_1, p_2..., p_j, ...\} \tag{31}$$

where $p_j \geq 2$ for $j \in \mathbb{Z}\setminus\{0\}$ and two its subsequences: the right one $\mathcal{P}' = \{p_j\}_{j=1}^{\infty}$ and the left one $\mathcal{P}'' = \{p_{-j}\}_{j=1}^{\infty}$. We set $m'_0 = 1, \quad m'_k = \prod_{s=1}^{k} p_s$ and $m'_{-k} = \prod_{s=1}^{k} p_{-s}$.

We consider also a sequence symmetrical to the sequence (31)

$$\mathcal{R}' = \{..., p_j, ..., p_2, p_1, p_{-1}, p_{-2}..., p_{-j}, ...\}$$

Similar to expansion (12) we can consider the $\mathcal{R}$-adic expansion of any $x \in [0, \infty)$,

$$x = \sum_{j=0}^{k(x)} x_{-j} m'_{-j+1} + \sum_{j=1}^{\infty} \frac{x_j}{m'_j},$$

and an analogues $\mathcal{R}'$-adic expansion of any $y \in [0, \infty)$,

$$y = \sum_{j=0}^{k(y)} y_{-j} m'_{j-1} + \sum_{j=1}^{\infty} \frac{y_j}{m'_{-j}} \ .$$

Now the continual generalization of the Vilenkin system is defined on $[0, \infty) \times [0, \infty)$ by

$$\chi(x, y) = \exp\left(2\pi i \left(\sum_{j=1}^{k(y)} \frac{x_j y_{-j}}{p_j} + \sum_{j=1}^{k(x)} \frac{x_{-j} y_j}{p_{-j}}\right)\right).$$

The problem of recovering the coefficients can be generalized to the case of this system and leads to a problem of establishing an inversion formula for a multiplicative transform of the form $\int_0^{\infty} a(x)\chi(x, y)dx$.

To formulate a result in this direction we define by means of the above sequences $\mathcal{P}'$ and $\mathcal{P}''$ the respective $H_{\mathcal{P}'}$- and $H_{\mathcal{P}''}$-integrals, as in Section 2, and extend those integrals to any interval $[0, n], \quad n = 1, 2, ...$ in a natural way. Then we get (see [23], [24], [26])

Theorem 5.1. *Suppose that a function $a : [0, \infty) \to \mathbb{C}$ is locally $H_{\mathcal{P}'}$-integrable and the improper $H_{\mathcal{P}'}$-integral*

$$\int_0^{\infty} a(x)\chi(x, y)dx$$

is convergent everywhere on $[0, \infty)$, except possibly on a countable set, to a finite function $f(y)$. Then f is locally $H_{\mathcal{P}''}$-integrable and

$$a(x) = \lim_{n\to\infty} (\mathcal{P}'') \int_0^{m'_n} f(y)\overline{\chi(x, y)}dy \quad a.e. \ on \ \ [0, \infty).$$

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