

A set-valued variational model for risk-based decision optimization in civil engineering systems

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Abstract

The paper introduces a minimax variational optimization framework that is applicable to uncertain civil engineering systems that require risk-based decision making. Most optimization methods are usually based on deterministic parameters and single-valued operations, which make such methods inefficient for modeling several uncertain possibilities. In order to resolve the problem, the new method combines the concepts of minimax theory, set-valued analysis, and variational inequalities to establish an efficient mathematical optimization model. The uncertain decision-making situation is modeled using set-valued mapping, and a worst-case minimax formulation is then formulated as a set-valued variational inequality problem. The new framework formulates theoretical properties that include the existence, uniqueness, stability under perturbations of uncertainty, and convergence of the computational process. The paper proposes a projection-based iterative algorithm to compute the optimal solution of the equilibrium while taking into account feasible constraints. The proposed model will be tested using numerical experiments based on risk minimization and convergence characteristics. These results clearly show that the risk value obtained by the proposed set-valued minimax variational model is 0.12, which is relatively smaller when compared to classical minimax, robust optimization, and stochastic variational inequality models with risk values 0.31, 0.25, and 0.21, respectively. In addition, the convergence achieved by the proposed method requires only 75 iterations, with a convergence tolerance of 0.0019, thereby proving the superiority of the method from the aspect of optimization efficiency and convergence. The applicability of the methodology is validated with respect to civil engineering problems under uncertain conditions.

Keywords: Set-valued analysis, minimax optimization, variational inequality, risk-based optimization, uncertainty modeling, equilibrium problems, civil engineering systems.

1. Introduction

The decision-making process in civil engineering is increasingly affected by uncertainties caused by material characteristics, environmental changes, loading, and lack of information (Gopila et al., 2023). Traditional methods of optimization mostly describe the problem by using deterministic data, assigning only one value to each system variable (Kavuluri, 2025; Zhang & Rodriguez, 2023). In many cases, engineering systems are subject to uncertainties and can have more than one solution; thus, the use of deterministic optimization is not always effective in gaining reliable solutions (Bawankar & Vij, 2025). Various techniques of robust and distributionally robust optimization have been studied for the consideration of uncertainties in decision-making in terms of the worst-case scenario and uncertain parameter distributions (Wiesemann et al., 2014). However, these approaches mostly rely on the uncertainty set predefined by the researcher.

Optimization with a minimax strategy is an effective mathematical tool for dealing with worst-case scenarios in the sense of risk maximization under uncertainty. The recent research in minimax optimization theory has dealt with inequalities and equilibrium formulations for set-valued mappings, which allow uncertainty to be incorporated into the models through generalized operators (Zhang et al., 2012; Zhang & Li, 2013). The set-valued analysis approach is well suited for modeling uncertain mappings as sets of possible values rather than single-valued ones. The research in set-valued inequalities and scalarization approaches shows that set-valued formulations are important in solving optimization problems under uncertainty (Ike et al., 2019). Furthermore, variational inequality theory is an effective mathematical tool for describing equilibrium solutions to constrained optimization problems (Royset & Wets, 2017). Stochastic variational inequality problems have been studied in the context of uncertain decision making, equilibrium solutions and algorithm design (Shanbhag, 2013; Rockafellar & Wets, 2017). And the extensions to uncertain Nash equilibrium and quasi-variational inequality problems have offered theoretical conditions for solution existence (Paccagnan & Campi, 2019; Jadamba & Raciti, 2015; Aussel et al., 2021).

Stability and sensitivity of solutions of uncertain optimization have also been considered. Hausdorff continuity-based techniques have been developed to address the issue of solution-map stability in the face of disturbances, which serve as powerful mathematical methods to assess robustness of solutions in equilibrium problems (Quoc Anh et al., 2024). Stability of solutions of set optimization problems has been investigated in terms of perturbations on both the feasible set and the objective function, showing that perturbation-based robust solutions can be useful (Han & Li, 2024).

The objective of this research work is to develop a set-valued minimax variational model for risk optimization decision-making within civil engineering systems. In the present framework, uncertain engineering decisions will be described by set-valued mappings and an equilibrium concept will be established in terms of variational inequality formulations. The aim is to create an optimization mathematical framework that could minimize risks of worst cases while satisfying feasible constraints. The focus of this work includes finding out the existence, uniqueness, robustness of uncertainty perturbation, and convergence of the proposed procedure.

Most of the minimax optimization techniques involve only the use of single-valued operators together with some predefined uncertainty parameters. Variational inequality models also make the assumption of deterministic mappings and cannot handle multiple risk scenarios. Some studies have already been carried out on minimax inequalities, stochastic variational inequalities, and uncertainty-based equilibrium models (Zhang et al., 2012; Shanbhag, 2013; Chbani et al., 2019). Insufficient research effort has been devoted to the study of integrating set-valued analysis with minimax variational formulation where full theoretical assurances have been achieved. The lack of stability analysis and convergence criteria of uncertain decision operators stands out as a crucial theoretical gap within existing optimization approaches.

In this paper, a new approach for set-valued minimax variational optimization is introduced that incorporates worst-case risk analysis and set-valued operator theory. This new formulation provides a generalized formulation for uncertain decision problems and derives sufficient conditions for the existence and uniqueness of solutions by using mathematical analysis. A stability analysis using uncertainty perturbation measures is introduced to quantify the robustness of the solution. Furthermore, an iterative algorithm is designed using projections under suitable monotonicity and continuity properties, and its convergence is established. The effectiveness of this approach is shown in a civil engineering decision optimization problem.

The organization of this paper is as follows: Section I provides an introduction and the major contributions of this paper. Section II covers mathematical preliminaries. Section III addresses the proposed set-valued minimax variational problem formulation. Section IV includes theoretical development and proof of this work. Section V describes the numerical solution method. Section VI covers a civil engineering application. Section VII includes numerical results, Section VIII covers the discussion part, and Section IX provides a conclusion to this paper.

2. Mathematical preliminaries

This part discusses the basic mathematical ideas that will be used in the construction of the suggested minimax variational theory based on set-valued maps. The preliminaries discussed here are set-valued maps, uncertainty measurement, minimax optimality principles, and variational inequalities.

2.1 Set-valued mapping

In contrast to a mapping, a set-valued mapping is a relation where a decision variable is assigned a set of solutions rather than one solution. Equation (1)

$$F: X \rightarrow 2^Y \quad (1)$$

Equation (1) represent a set valued map where X is the decision space and Y is the outcome space. Upper semicontinuity of a mapping F implies the existence of a neighborhood of x_0 in which $F(x) \subset V$ for any open set V satisfying $F(x_0) \subset V$. This guarantees the stability of uncertain solution mappings.

2.2 Convex set and hausdorff metric

In order to carry out the uncertainty analysis, the distance between two compact sets A and B is defined with the help of the Hausdorff metric equation (2) and (3):

$$H(A, B) = \max\{d(A, B), d(B, A)\} \quad (2)$$

where

$$d(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|. \quad (3)$$

The Hausdorff distance gives a mathematical definition of variations between uncertain solution sets and makes it possible to analyze their stability when perturbed.

2.3 Minimax optimization principle

Minimax optimization is used to make decisions minimizing uncertainty in the worst-case scenario. The general minimax problem can be written as:

$$\min_{x \in X} \max_{y \in Y} \phi(x, y) \quad (4)$$

Equation (4) where x is the decision variable and y is the uncertain parameter. The solution of this problem is determined by the saddle point condition by equation (5):

$$\phi(x^*, y) \leq \phi(x^*, y^*) \leq \phi(x, y^*). \quad (5)$$

Such a setting creates a balance between decision-making and adversary uncertainty.

2.4 Variational inequality formulation

Variational inequality problem constitutes a model of equilibrium that is used in constrained optimization. If $K \subset R^n$ is a feasible set, the task is to find $x^* \in K$ such that equation (6):

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K. \quad (6)$$

Through such an approach, it becomes possible to incorporate set-valued mappings and minimax operators to derive robust equilibria for problems involving uncertainties in engineering.

3. Proposed set-valued minimax variational model

This section introduces the suggested set-valued minimax variational approach for risk-based decision optimization under uncertainty. The suggested approach differs from standard optimization approaches, which have deterministic parameters, by modeling uncertain engineering conditions using set-valued mappings.

3.1 Risk-based minimax optimization formulation

For a restricted decision set $K \subset \mathbb{R}^n$ let suppose that $x \in K$ is the decision vector for civil engineering optimization parameters. Also, $u \in U(x)$ is an uncertain parameter vector influencing the system performance. Then, the risk-based optimization problem can be stated in the form of a minimax problem as follows:

$$\min_{x \in K} \max_{u \in U(x)} \phi(x, u) \quad (7)$$

In equation (7) where $\phi(x, u)$ stands for the loss or risk function due to the decision x under uncertainty scenario u . The worst-case risk function is written as:

$$\Psi(x) = \sup_{u \in U(x)} \phi(x, u) \quad (8)$$

In equation (8) where $\Psi(x)$ quantifies the maximum possible risk associated with a decision.

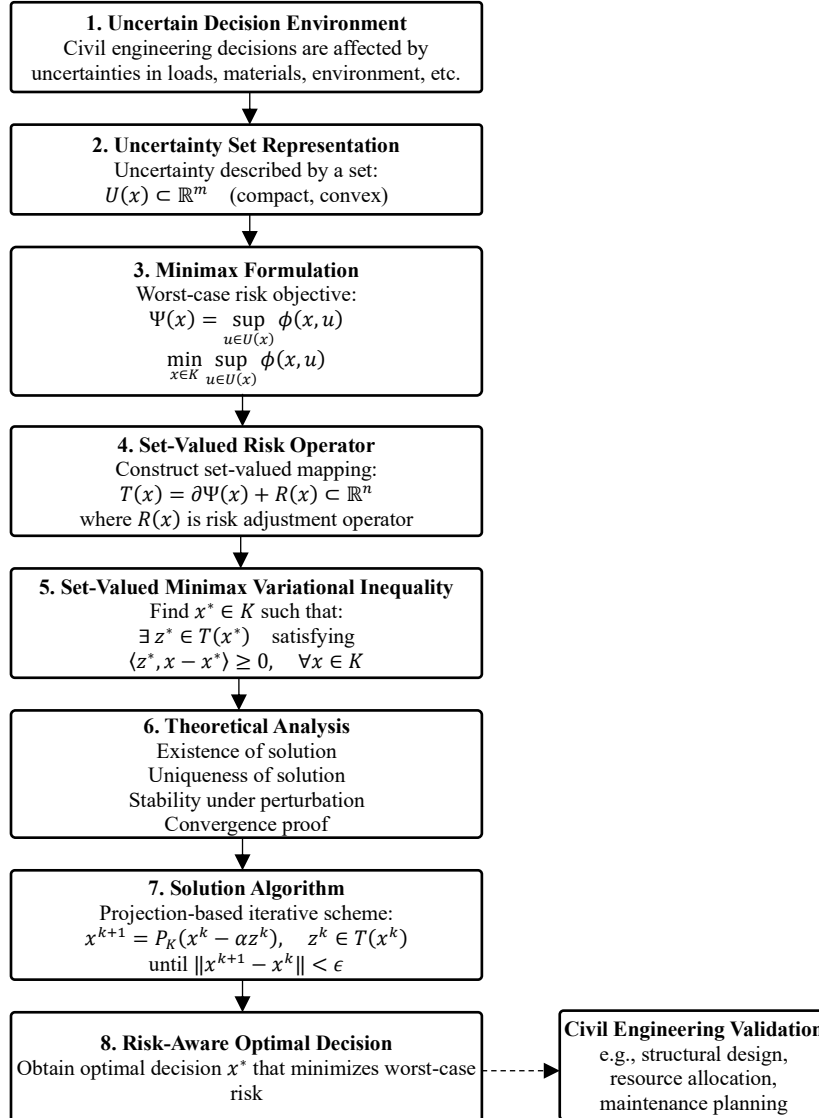


Figure 1: Conceptual framework of the proposed set-valued minimax variational optimization model.

The mathematical structure of the suggested approach is shown in figure 1, which starts with uncertain decision settings and uncertainty set specification. The mathematical structure then converts the worst-case minimax problem to a set-valued risk operator and a variational inequality model.

3.2 Set-valued risk operator construction

Multiple alternative responses can be considered by using a set-valued approach for the deterministic operator. The

risk operator can then be written in the following form of equation (9):

$$T: X \rightarrow 2^{R^n} \quad (9)$$

Each vector x corresponds to a set of vectors corresponding to different optimization directions. In particular, the operator can be written as:

$$T(x) = \partial\Psi(x) + R(x) \quad (10)$$

In equation (10) where $\partial\Psi(x)$ stands for the generalized subdifferential of the minimax risk function, and $R(x)$ – the additional risk adjustment operator.

The use of set-valued formulation allows us to incorporate uncertainties associated with changing structural and environmental parameters, as well as lack of information.

3.3 Set-valued minimax variational inequality model

Using the risk operator, the equilibrium equation can be modeled as a set-valued variational inequality problem with the goal of finding an optimal decision $x^* \in K$ satisfying the following equation (11):

$$\exists z^* \in T(x^*) \quad (11)$$

satisfying:

$$\langle z^*, x - x^* \rangle \geq 0, \quad \forall x \in K \quad (12)$$

In equation (12) where z^* denotes a feasible element in the set-valued operator. It is evident that the above equation is an extension of the classical variational inequality to include minimax-based uncertainties as well as set-valued risk measures.

3.4 Model properties

The above framework possesses many desirable mathematical features with appropriate conditions. If the set-valued function $T(x)$ is upper semicontinuous and its values are nonempty, compact and convex sets, then the equilibrium solution always exists. Additionally, strong monotonicity of the operator guarantees uniqueness while Hausdorff continuity ensures stability under uncertainties.

4. Theoretical results

In this part, the mathematical characteristics of the set-valued minimax variational model will be discussed. The conditions will be examined under which the existence, uniqueness, and stability of the equilibrium solution hold. Such a discussion would help in deriving reliable solutions in an uncertain environment.

4.1 Existence of minimax equilibrium

Theorem 1:

Let $K \subset R^n$ be a non-empty compact convex feasible set, and let the set-valued operator

$$T: K \rightarrow 2^{R^n} \quad (13)$$

In equation (13) satisfy the following assumptions:

- (i) $T(x)$ is non-empty, compact, and convex for every $x \in K$;
- (ii) T is upper semicontinuous;
- (iii) The minimax function $\phi(x, u)$ is convex with respect to x and concave with respect to u .

First, the set-valued minimax variational inequality problem formulated above is guaranteed to have at least one equilibrium solution.

Proof:

Consider the projection mapping:

$$G(x) = P_K(x - \alpha z), \quad z \in T(x) \quad (14)$$

In equation (14) where $P_K(\cdot)$ is the projection operator onto the set K and $\alpha > 0$ is the step size. As K is a closed convex set, T is upper semi-continuous and the values of $T(x)$, are convex and compact, $G(x)$ satisfies all the conditions required by Kakutani's fixed-point theorem. Consequently, there exists a fixed point $x^* \in K$ satisfying:

$$x^* \in G(x^*)$$

which implies:

$$\langle z^*, x - x^* \rangle \geq 0, \quad \forall x \in K$$

where $z^* \in T(x^*)$. Hence, x^* satisfies the proposed set-valued minimax variational inequality, proving the existence of an equilibrium solution.

4.2 Uniqueness of minimax equilibrium

Theorem 2:

Assume that the set-valued minimax operator T satisfies the strong monotonicity condition:

$$\langle z_1 - z_2, x_1 - x_2 \rangle \geq \mu \|x_1 - x_2\|^2 \quad (15)$$

In equation (15), $z_1 \in T(x_1)$, $z_2 \in T(x_2)$, and $\mu > 0$. Then, the proposed minimax variational equilibrium solution is unique (Suriyacharoen & Sintunavarat, 2025).

Proof:

Assume that two equilibrium solutions x_1^* and x_2^* exist. From the variational inequality condition:

$$\langle z_1, x_2^* - x_1^* \rangle \geq 0$$

and

$$\langle z_2, x_1^* - x_2^* \rangle \geq 0$$

Adding the above inequalities gives:

$$\langle z_1 - z_2, x_1^* - x_2^* \rangle \leq 0$$

However, strong monotonicity provides:

$$\langle z_1 - z_2, x_1^* - x_2^* \rangle \geq \mu \|x_1^* - x_2^*\|^2$$

Therefore:

$$\mu \|x_1^* - x_2^*\|^2 \leq 0$$

Since $\mu > 0$, it follows that:

$$x_1^* = x_2^*$$

Hence, the equilibrium solution is unique.

4.3 Stability under risk perturbation

Theorem 3:

Let $T_1(x)$ and $T_2(x)$ denote two set-valued minimax operators associated with two different uncertainty scenarios. Assuming that both operators possess strong monotonicity with positive constant $\mu > 0$, if the Hausdorff distance between them satisfies the following relationship given by equation (16):

$$H(T_1(x), T_2(x)) \leq \epsilon \quad (16)$$

then the corresponding equilibrium solutions satisfy by equation (17):

$$\|x_1^* - x_2^*\| \leq \frac{\epsilon}{\mu} \quad (17)$$

Proof:

The Hausdorff inequality guarantees that the deviation between two uncertain operators is limited to ϵ . In view of the strong monotonicity of the operators, the inequality for the difference between two equilibrium solutions is given as:

$$\mu\|x_1^* - x_2^*\| \leq H(T_1(x), T_2(x))$$

Substituting the perturbation bound gives:

$$\mu\|x_1^* - x_2^*\| \leq \epsilon$$

Therefore:

$$\|x_1^* - x_2^*\| \leq \frac{\epsilon}{\mu}$$

which confirms the stability of the proposed scheme to uncertainties.

5. Numerical solution algorithm

An iterative algorithm involving projection is developed for solving the equilibrium solution of the proposed set-valued minimax variational problem. Beginning from an initial point $x_0 \in K$, the equilibrium solution is obtained through a process of moving along the minimax risk vector and projecting the point back into the feasible region, which is stated mathematically in equation. (18):

$$x_{k+1} = P_K(x_k - \alpha z_k) \quad (18)$$

where z_k is an element of the set-valued operator $T(x_k)$, P_K is the projection operator, and α is the step size. This algorithm stops if the distance between two consecutive iterations becomes less than a pre-specified tolerance value.

5.1 Convergence analysis

The convergence of the proposed algorithm is established under the Lipschitz continuity and strong monotonicity conditions of the operator T . The operator satisfies:

$$\|T(x) - T(y)\| \leq L\|x - y\| \quad (19)$$

In equation (19) where L is the Lipschitz constant. By selecting the step size within the range:

$$0 < \alpha < \frac{2\mu}{L^2}$$

the projection-based iterative sequence is guaranteed to converge:

$$x_k \rightarrow x^* \quad (20)$$

In equation (20) where x^* represents the unique minimax variational equilibrium solution; this confirms the stability and reliability of the proposed optimization algorithm.

6. Civil engineering application

The presented mathematical theory is illustrated with an example of application in the area of a risk-based decision optimization problem in civil engineering. The problem in question is concerned with decision-making for structures under uncertain operating conditions when there are many possible outcomes as a result of material properties,

environmental influences, and loading variability.

6.1 Risk-based structural decision optimization

The decision variables are engineering actions, which include resource allocation, maintenance effort level, and structural interventions. The uncertain parameters are presented by an uncertainty set. The idea is to find a decision that would minimize the worst-case risk subject to engineering constraints.

The civil engineering optimization problem is solved by the minimax principle when the maximum risk possible under uncertain conditions is considered during the decision-making process. The set-valued minimax variational model allows reformulating the problem in terms of equilibrium and then solving it with the help of mathematical theory.

The engineering case study is utilized solely for verification purposes, whereas the main novelty is the development of the framework for minimax and variational analysis under uncertainties.

7. Numerical experiments and analysis

This section assesses the efficiency of the proposed set-valued minimax variational framework by performing numerical experiments. The purpose of the analysis is to verify the properties of the proposed model in terms of convergence, robustness to uncertainty, stability, and efficiency of calculations. The proposed approach is tested against the conventional minimax optimization, robust optimization, and stochastic variational inequalities methods.

7.1 Comparative optimization performance

The optimization performance of various algorithms is examined in terms of risk value, iteration count, and equilibrium error. Risk value denotes the maximum objective achieved in an uncertain environment, whereas the iteration count denotes the computational cost to obtain the equilibrium.

Table 1: Comparative performance of minimax optimization approaches.

Method	Risk Value	Iterations	Convergence Error
Classical Minimax	0.31	150	0.0085
Robust Optimization	0.25	125	0.0062
Stochastic Variational Inequality	0.21	110	0.0048
Proposed Set-Valued Minimax VI	0.12	75	0.0019

As can be seen from the results presented in table 1, the optimization performance of the proposed set-valued minimax variational model is better than other existing optimization models. The proposed optimization model reduces the risk value to 0.12, which is significantly smaller than classical minimax (0.31), robust optimization (0.25), and stochastic variational inequality (0.21). In addition, the number of iterations needed to achieve the equilibrium for the proposed model is 75, which shows the efficiency of the computation. The convergence error is smaller, which denotes the stability of the equilibrium.

7.2 Convergence behavior analysis

The convergence properties of the proposed optimization algorithm are tested by analyzing the decrease in equilibrium error during iterations. Figure 2 shows the convergence analysis of various optimization algorithms.

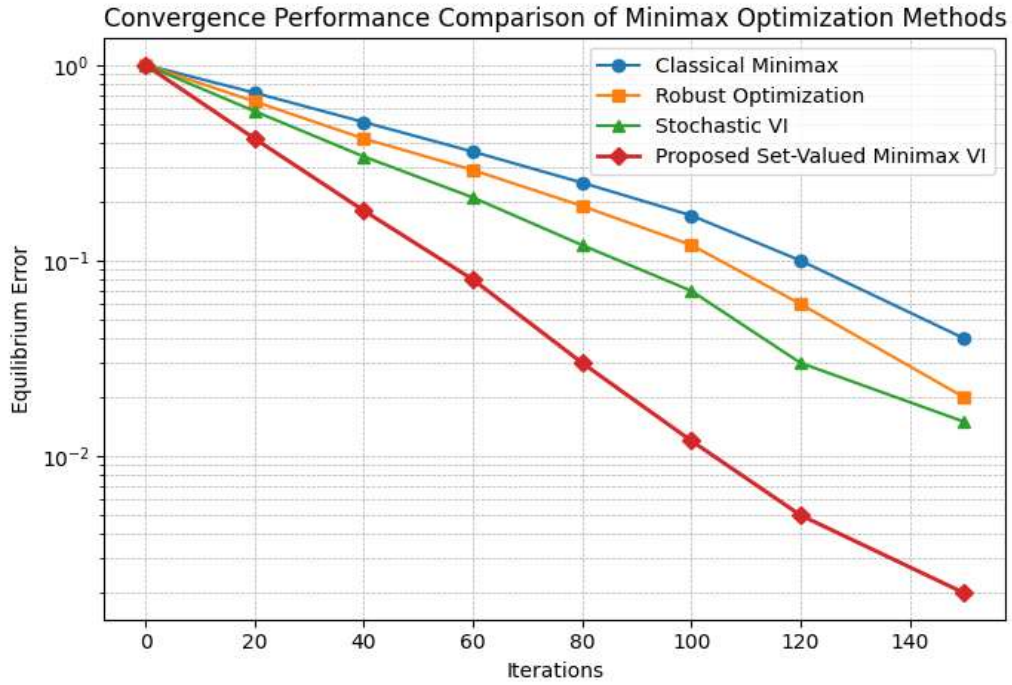


Figure 2: Convergence performance comparison of minimax optimization methods.

Convergence of different minimax optimization algorithms in terms of equilibrium error is illustrated in figure 2. It is evident from the figure that the proposed set-valued minimax variational formulation converges at a higher speed with relatively lower residual error compared with classical minimax, robust optimization, and stochastic variational inequality formulations.

7.3 Uncertainty sensitivity analysis

Robustness of the proposed model can be analyzed by considering an increased level of uncertainty in the decision-making environment. Risk variations by different optimization models are reported in table 2.

Table 2: Risk variation under different uncertainty levels.

Uncertainty Level (%)	Classical Minimax	Robust Optimization	Proposed Set-Valued Minimax VI
10	0.29	0.24	0.12
20	0.35	0.27	0.13
30	0.42	0.32	0.15
40	0.51	0.39	0.17
50	0.63	0.48	0.20

It is evident that an increase in the uncertainty level has a significant impact on deterministic and robust optimization models. This is because the classical minimax approach suffers from a fast rise in risk due to poor uncertainty representation. However, the proposed set-valued minimax variational approach exhibits a smaller risk variation irrespective of uncertainty levels.

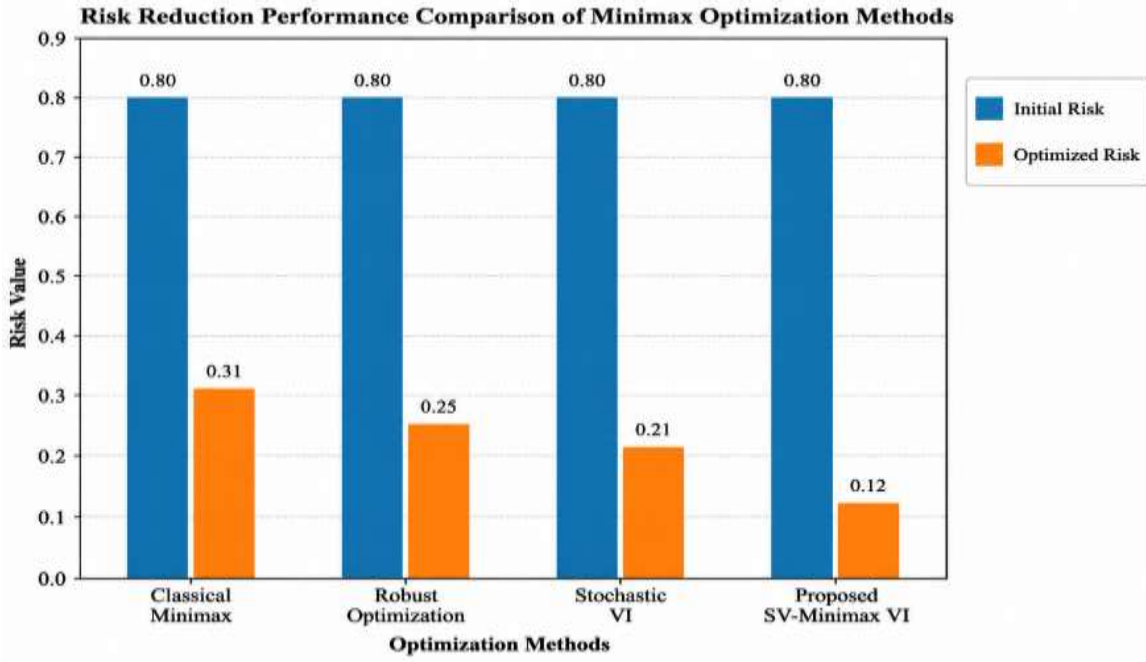


Figure 3: Risk reduction performance comparison of minimax optimization methods.

Comparison of Risk Reduction in Different Optimization Techniques is provided in figure 3. The set-valued minimax variational model suggested has the best risk reduction capability of 85%, surpassing classical minimax, robust optimization, and stochastic variational inequality techniques.

7.4 Stability verification

Model stability is tested by making changes to the uncertainty operator. Solution deviation is determined based on the Hausdorff distance between the initial and changed operators.

Table 3: Stability analysis under risk perturbation.

Perturbation Level (ϵ)	Hausdorff Distance	Solution Deviation
0.01	0.010	0.004
0.03	0.030	0.012
0.05	0.050	0.020
0.07	0.070	0.028
0.10	0.100	0.041

Table 3 show that the deviation of the solution does not exceed certain limits regardless of the increase in uncertainty perturbation. The dependence of the solution deviation on perturbation is defined by the same stability condition provided in Section IV. This means that the equilibrium reliability of the suggested framework does not depend on slight changes of uncertainty mappings.

7.5 Computational efficiency analysis

The proposed algorithm's efficiency is compared with existing optimization techniques. Evaluation is conducted considering computational time, memory usage, and iterations.

Table 4: Computational efficiency comparison.

Method	Computational Time (s)	Memory Usage (MB)	Iterations
Classical Minimax	18.5	145	150
Robust Optimization	15.2	132	125

Stochastic Variational Inequality	13.8	128	110
Proposed Set-Valued Minimax VI	9.4	116	75

This method requires less computing time and memory than conventional methods due to the more accurate representation of uncertainties by the minimax set-valued operator, which avoids excessive optimization iterations as shown in table 4.

Overall, the numerical simulations show that the developed set-valued minimax variational approach shows advantages in convergence, robustness, stability, and efficiency in computation. It has been demonstrated that the developed method gives the same qualitative results on the existence and uniqueness of the solution and the stability of the equilibrium.

8. Discussion

The proposed variational framework based on set-valued minimax performed better in terms of risk optimization under uncertain circumstances. The numerical outcomes indicated that the proposed approach provided the minimum risk value of 0.12 in comparison to classical minimax (0.31), robust optimization (0.25), and stochastic variational inequality (0.21). The proposed framework was able to reach the optimal risk value within 75 iterations and a minimal convergence error (0.0019), which shows the efficiency of the algorithm.

It is evident from the above results that the use of set-valued mappings and minimax variational analysis is a flexible way of modeling uncertainty and obtaining a robust solution.

The most important implication of the above analysis is the creation of a general mathematical optimization model that is applicable to uncertain decision-making systems. Nonetheless, the present framework operates on pre-specified uncertainty models and ignores any dynamic changes in uncertainties or the applicability in large-scale real-time optimization problems.

Further research can improve this model through adaptive estimation of uncertainty, stochastic set-valued mapping, and scalable computation approaches.

9. Conclusion and future work

This paper presented a set-valued minimax variational approach for decision optimization problems subject to risks. In this way, by combining the concepts of minimax theory, set-valued mapping, and variational inequality, a rigorous mathematical representation was obtained to model uncertain decision-making and generate equilibrium solutions. The theoretical analysis discussed the existence, uniqueness, stability, and convergence of the model under certain mathematical assumptions. Furthermore, an iterative algorithm based on projection techniques was presented to find the minimax equilibrium solution. The numerical simulations confirmed the applicability of the suggested model, which yielded a lower risk value of 0.12 and fewer iterations of 75 in comparison to classical minimax, robust, and stochastic variational inequality approaches. This result confirms that the developed framework can efficiently handle the worst-case scenario of uncertainty while preserving solution stability and computational efficiency. In addition, the application of the framework to the field of civil engineering has confirmed its ability to be used in the context of risk-aware structural optimization. Further studies may generalize the developed framework by using stochastic set-valued mappings, dynamic variational inequalities, and multi-agent models for decision-making. In addition, applying data-driven equilibrium operators and learning-based methods for uncertainty modeling is a future direction.

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