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Explicit Algorithm for Hammerstein Equations with Bounded, Hemi-Continuous and Monotone Mappings

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Let E be a reflexive smooth and strictly convex real Banach space and E^* its dual. Let $F\colon E \to E^*$ and $K\colon E^* \to E$ be bounded hemi-continuous mappings such that $D(F) = E$ and $R(F) = D(K) = E^*$. Suppose that the Hammerstein equation $u + KFu = 0$ has a solution in E. We present in this paper a method containing an auxiliary mapping, defined on an appropriate Banach space in terms of F and K and which is maximal monotone. The solutions of the Hammerstein equation are derived from the zeros of this map. Our method provides an implicit algorithm and explicit one that converge strongly to a solution of the equation $u + KFu = 0$. No invertibility assumption is imposed on K and the operator F need not be defined on a compact subset of E. Our theorems improve and unify most of the results that have been proved in this direction for this important class of nonlinear mappings. Finally, illustration of the proposed method is given in L^p spaces.

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1. Introduction

Let E be a normed linear space with dual E^*. Consider the functional equations of the form:

$$u + KFu = 0, \tag{1}$$

where $F\colon E \to E^*$ and $K\colon E^* \to E$ are maps such that $D(K) = R(F) = E^*$.

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This class of equations includes nonlinear integral equations of Hammerstein type:

$$u(x) + \int_{\Omega} \kappa(x,y) f(y, u(y)) dy = 0, \tag{2}$$

where dy is a σ-finite measure on the measure space Ω; the real kernel κ is defined on $\Omega \times \Omega$, f is a real-valued function defined on $\Omega \times \mathbb{R}$. The mappings K and F are given by

$$Kv(x) = \int_{\Omega} \kappa(x,y) v(y) dy, \; a.e. \; x \in \Omega, \quad Fu(y) = f(y, u(y)) \; a.e. \; y \in \Omega. \tag{3}$$

There exist various motivations for studying equations of type (1). For illustration, let us mention one of them.

The study of Hammerstein equations is related to *nonsmooth* calculus of variations (see e.g., the monograph [35]). Suppose that we are interested in minimizing the energy functional

$$J(u) = \int_{\Omega} \Big(h(u(t)) - f(s, u(t)) \Big) \, ds, \tag{4}$$

where h denotes the kinetic energy of the system, and f is a potential energy generating a superposition operator. Assume further that the functional J is not differentiable in the usual sense, but admits a generalized gradient or subgradient in the sense, for instance, of Clarke's generalized gradient, Aubin's contingent cone, Ioffe's fan, etc. (see e.g. [9, 10]). Consequently, the problem of minimizing the energy functional J leads to the study of boundary value problems for the Euler Lagrange inclusion:

$$Lu \in \partial N_f u, \tag{5}$$

where, L is a linear operator on an appropriate function space and $N_f u(t) = f(t, u(t))$, and where ∂N_f is one of the generalized gradients or subgradients mentioned above. The problem (5) in turn is in various function spaces equivalent to the Hammerstein inclusion of type (1).

The studies of integral equations are motivated by a wide application in engineering, mechanics, physics, economics, optimization, vehicular traffic, biology, queuing theory and so on (see, e.g., Grimmer and Liu [42], Keller and Olmstead [46], and Olmstead and Handelsman [52]). The theory of integral equations is rapidly developing with the help of tools in functional analysis, topology and fixed point theory. Integral equations of the Hammerstein type have been studied by many authors and have been one of the most important domains of applications of the ideas and methods of nonlinear functional analysis and in particular of the theory of nonlinear operators of monotone type. We refer to the works of Browder [17], Brézis and Browder[15], Amann [7], Ahmed [2], O'Regan [53] and the references therein. Various applied problems arising in mathematical physics, mechanics and control theory lead to multivalued analogues of the Hammerstein integral equations, the so-called 'Hammerstein integral inclusions'. In this direction, we have the works of Lyapin [48], Coffman [36], Glashoff and Sperkels [41], Appell et al. [8] and O'Regan [54].

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle_H$ and norm $\|\cdot\|_H$. An operator $A\colon H\to H$ with domain $D(A)$ is called *monotone* if for every $x,y\in D(A)$, the following inequality holds:

$$\langle x-y, Ax-Ay\rangle_H \geq 0, \tag{6}$$

and it is called *strongly monotone* if there exists $k\in(0,1)$ such that every $x,y\in D(A)$,

$$\langle x-y, Ax-Ay\rangle_H \geq k\|x-y\|_H^2. \tag{7}$$

Such operators have been studied extensively (see, e.g., Bruck Jr [20], Chidume [22], Martinet [49], Reich [57], Rockafellar [59]) because of their role in convex analysis, in certain partial differential equations, in nonlinear analysis and in optimization theory.

The extension of the *monotonicity* definition to operators from a Banach space into its dual has been the starting point for the development of nonlinear functional analysis. Monotone maps constitute the most manageable class because of the very simple structure of the monotonicity condition. They appear in a rather wide variety of contexts since they can be found in many functional equations. Many of them appear also in the calculus of variations as subdifferential of convex functions (see, e.g., Pascali and Sburian [55], p. 101, Rockafellar [59]).

The *first extension* involves a mapping A from E to E^*. Here and in the sequel, $\langle\cdot,\cdot\rangle$ stands for the duality pairing between (a possible normed linear space) E and its dual E^*. A mapping $A\colon E\to E^*$ with domain $D(A)$ is called *monotone* if for each $x,y\in D(A)$, the following inequality holds:

$$\big\langle x-y, Ax-Ay\big\rangle \geq 0. \tag{8}$$

A is called *strongly monotone* if there exists $k\in(0,1)$ such that for each $x,y\in D(A)$,

$$\langle x-y, Ax-Ay\rangle \geq k\|x-y\|^2. \tag{9}$$

The *second extension* of the notion of monotonicity involves a mapping A from E *into itself*. Let E be a real normed space, E^* its dual space. The map $J: E\to 2^{E^*}$ defined by:

$$Jx := \big\{x^*\in E^* : \langle x, x^*\rangle = \|x\|.\|x^*\|,\ \|x^*\| = \|x\|\big\}$$

is called the *normalized duality map* on E. A mapping $A\colon E\to E$ with domain $D(A)$ is called *accretive* if for all $x,y\in D(A)$, the following inequality is satisfied:

$$\|x-y\| \leq \|x-y+s(Ax-Ay)\| \quad \forall\, s>0. \tag{10}$$

As a consequence of a result of Kato [45] it follows that A is accretive if and only if for each $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay,\, j(x-y)\rangle \geq 0. \tag{11}$$

Finally, A is called *strongly accretive* if there exists $k \in (0,1)$ such that for each $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay,\, j(x-y)\rangle \geq k\|x-y\|^2. \tag{12}$$

In a Hilbert space, the normalized duality map is the identity map. Hence, in Hilbert spaces, *monotonicity* and *accretivity* coincide.

There are strict connections between monotone operators and minimax theory (see, e.g., the monograph by S. Simons [60].

Several existence and uniqueness results have been established for equations of Hammerstein type (see, e.g., Klein-Thompson [10], Coffman [36], Appell et al. [8] and O'Regan [54]). In general, these equations are nonlinear and there is no known method to find closed form solutions for them. Consequently, methods of approximating solutions of such equations are of interest.

In the special case in which the operator F is angle bounded (defined below) and weakly compact, Brézis and Browder [14, 13] proved the strong convergence of a suitably defined *Galerkin approximation* to a solution of (1). Before we state their results, we need the following definitions.

Let H be a real Hilbert space. A *nonlinear* operator $A\colon H \to H$ is said to be *angle-bounded* with angle $\beta > 0$, if

$$\langle Ax - Az,\, z - y\rangle \leq \beta\langle Ax - Ay,\, x - y\rangle \tag{13}$$

for any triple of elements $x, y, z \in H$. For $y = z$, inequality (13) implies the monotonicity of A.

A *monotone linear* operator $A\colon H \to H$ is said to be *angle-bounded* with angle $\alpha > 0$, if

$$|\langle Ax, y\rangle - \langle Ay, x\rangle| \leq 2\alpha\langle Ax, x\rangle^{\frac{1}{2}}\langle Ay, y\rangle^{\frac{1}{2}} \tag{14}$$

for all $x, y \in H$. It is known (see, e.g., Pascali and Sburlan, [55], Ch. IV, p. 189) that for linear operators, the two definitions of angle boundedness are equivalent.

We now state the theorem of Brézis and Browder referred to above.

Theorem BB (Brézis and Browder, [13]). *Let H be a separable real Hilbert space and C be a closed subspace of H. Let $K\colon H \to C$ be a bounded continuous monotone operator and $F\colon C \to H$ be an angle-bounded and weakly compact mapping. For a given $f \in C$, consider the Hammerstein equation*

$$(I + KF)u = f \tag{15}$$

and its n-th Galerkin approximation given by

$$(I + K_nF_n)u_n = P^*f, \tag{16}$$

where $K_n = P_n^* K P_n : H \to C_n$ *and* $F_n = P_n F P_n^* \colon C_n \to H$*, where the symbols have their usual meanings* (*see,* [55])*. Then, for each* $n \in \mathbb{N}$*, the Galerkin approximation* (16) *admits a unique solution* u_n *in* C_n *and* $\{u_n\}$ *converges strongly in* H *to the unique solution* $u \in C$ *of the equation* (15)*.*

It is obvious that if an *iterative algorithm* can be developed for the approximation of solutions of equations of Hammerstein type (1), this will certainly be preferred.

We first note that for the iterative approximation of zeros of accretive type operators, the *monotonicity/accretivity* of operators is crucial. The Mann type iteration scheme (see, e.g., Mann [56]) has successfully been employed (see, e.g., the recent monographs of Berinde [12] and Chidume [22] for results obtained within the past 40 years, or so). One drawback of the Mann iterative scheme, however, is that in general, it only yields weak convergence (see, e.g., Matouskova and Reich [50]). All attempts to use the Mann type iteration scheme directly to approximate solutions of equations of Hammerstein type (1) did not yield satisfactory results (see Chidume and Osilike [30]). The recurrence formulas used in early attempts involved K^{-1} which is also required to be strongly monotone, and this, apart from limiting the class of mappings to which such iterative schemes are applicable, is also not convenient in application. Part of the difficulty is the fact that the composition of two monotone operators need not be monotone.

The first satisfactory results on *iterative methods* for approximating solutions of Hammerstein equations, as far as we know, were obtained by Chidume and Zegeye [34, 33, 32] under the setting of a real Hilbert space H.

The method of proof used by Chidume and Zegeye provided the clue to the establishment of the following coupled explicit algorithm for computing a solution of the equation $u + KFu = 0$ in the original space X. With initial vectors $u_0, v_0 \in X$, sequences $\{u_n\}$ and $\{v_n\}$ in X are defined iteratively as follows:

$$u_{n+1} = u_n - \alpha_n(Fu_n - v_n), n \geq 0, \tag{$*$}$$

$$v_{n+1} = v_n - \alpha_n(Kv_n + u_n), n \geq 0. \tag{$**$}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying appropriate conditions. The recursion formulas $(*)$ and $(**)$ have been used successfully to approximate solutions of Hammerstein equations involving nonlinear accretive-type operators. Following this, Chidume and Djitte studied this explicit coupled iterative algorithms and proved several strong convergence theorems (see, Chidume and Djitte [27], [28]). For recent results using these recursion formulas or their modifications, the reader may consult any of the following references Chidume and Djitte [24, 25, 26], Djitte and Sene [38, 39], Chidume and Ofeodu [29], Chidume and Shehu [31] and also Chapter 13 of [22].

For Hammerstein equations involving *monotone mappings* from E to E^*, very little has been achieved. Interestingly enough, *almost all the existence theorems* proved for Hammerstein equations involve *monotone mappings* (see, e.g., Brézis and Browder [14, 15, 13], Browder [16], Browder et al. [18], and Browder and

Gupta [19]). We note that it has been remarked that in dealing with the Nemistkyi operator, which is intimately connected with the Hammertsein integral equation, its properties are distinguished, in applications, according to two important cases: $L_p(\Omega)$ spaces, $1 < p < \infty$, and $L_1(\Omega)$, (see Pascali and Sburlan [55], Chapter IV, pp. 165, 172). Thus, developing iterative methods for approximating solutions of nonlinear Hammerstein integral equations in these cases is of paramount importance.

Motivated by approximating solutions of integral equations of Hammerstein type, in [51], Ofoedu and Onyi proposed an iterative scheme and they obtained strong convergence results in the setting of Hilbert spaces. In fact, they proved the following theorem.

Theorem OO (Ofoedu and Onyi, [51]). *Let H be a real Hilbert space and $F, K\colon H \to H$ Lipschitz monotone mappings. Let the sequence $\{(u_n, v_n)\}_{n\geq 1}$ in $H \times H$ be generated iteratively by $(u_1, v_1) \in H \times H$,*

$$\begin{cases} u_{n+1} = (1-\sigma_n)u_n + \sigma_n(u_n - Fu_n + v_n) - \sigma_n\xi_n\alpha_n u_n, \\ v_{n+1} = (1-\sigma_n)v_n + \sigma_n(v_n - Kv_n - u_n) - \sigma_n\xi_n\alpha_n v_n, \end{cases} \tag{17}$$

where $\{\sigma_n\}_{n\geq 1}$, $\{\xi_n\}_{n\geq 1}$ and $\{\alpha_n\}_{n\geq 1}$ are decreasing sequences in $(0,1)$ such that

$$\text{(i)} \lim_{n\to\infty} \xi_n = 0, \quad \text{(ii)} \lim_{n\to\infty} \alpha_n = 0, \quad \text{(iii)} \lim_{n\to\infty} \frac{\sigma_n}{\alpha_n\xi_n} = 0,$$

$$\text{(iv)} \lim_{n\to\infty} \frac{\alpha_{n-1} - \alpha_n}{\sigma_n\xi_n\alpha_n^2} = 0, \quad \textit{and} \quad \text{(v)} \lim_{n\to\infty} \frac{\xi_{n-1} - \xi_n}{\sigma_n\xi_n^2\alpha_n^2} = 0.$$

Then the sequence $\{(u_n, v_n)\}_{n\geq 1}$ is bounded. Moreover, if the Hammerstein equation $u + KFu = 0$ has some solutions in H, then $\{u_n\}_{n\geq 1}$ converges strongly to a solution u^* of $u + KFu = 0$.

Recently, Chidume and Bello [23] constructed a new iterative algorithm for approximating solutions of Hammerstein equations in L_p-spaces, and where the operators K and F are assumed to be bounded and strongly monotone. They obtained the following theorem.

Theorem CB (Chidume and Bello [23]). *Let $E = L_p$, $1 < p \leq 2$ with dual E^* and $F : E \to E^*$, $K : E^* \to E$ be strongly monotone and bounded mappings with $D(K) = R(F) = E^*$. For given $u_1 \in E$ and $v_1 \in E^*$, let $\{u_n\}$ and $\{v_n\}$ be generated iteratively by:*

$$\begin{cases} u_{n+1} = J^{-1}(Ju_n - \lambda(Fu_n - v_n)), & n \geq 1, \\ v_{n+1} = J(J^{-1}v_n - \lambda(kv_n + u_n)), & n \geq 1, \end{cases} \tag{18}$$

where J is the normalized duality mapping from E into E^* and $\{\alpha_n\} \subset (0,1)$ satisfies the following conditions:

$$\text{(i)} \sum_{n=1}^{\infty} \alpha_n = \infty; \quad \text{(ii)} \sum_{n=0}^{\infty} \alpha_n^2 < \infty \quad \textit{and} \quad \text{(iii)} \sum_{n=0}^{\infty} \alpha_n^{\frac{q}{q-1}} < \infty,$$

where q is the conjugate of p. Suppose that the equation $u + KFu = 0$ has a unique solution u^*. Then, there exists $\gamma_0 > 0$ such that if $\alpha_n < \gamma_0$ for all $n \geq 1$, the sequence $\{u_n\}$ converges strongly to u^*, the sequence $\{v_n\}$ converges strongly to v^*, with $v^* = Fu^*$.

Using the same scheme, still in [23], they also proved a similar result in L_p, for $2 \leq p < \infty$.

Remark 1.1. In Theorem OO, the authors assumed that the operators K and F are Lipschitz and monotone and the results are proved in real Hilbert spaces. Note that the class of bounded, hemi-continuous monotone maps is a superclass of that of Lipschitz monotone mappings.

Remark 1.2. Theorem CB is proved in L_p-spaces, $1 < p < \infty$ and the operators K and F are assume to be bounded strongly monotone.

Motivated by the discussion above, it is our purpose in this paper to construct a new iterative algorithm and prove strong convergence theorems for approximation solutions of equations of Hammerstein type, $u + KFu = 0$. The operators F and K, defined in 2-uniformly convex and q-uniformly smooth $(q > 1)$ or s-uniformly convex $(s > 1)$ and 2-uniformly smooth real Banach spaces are assumed to be bounded, hemi-continuous and monotone. These class of Banach spaces contain all Hilbert spaces, all L_p and all Sobolev spaces $W^{m,p}$ for $1 < p < \infty$. Further, the class of maps used in the present work is more general than that used in Theorem OO. So, the results obtained here extend and unify those obtained by Chidume and Bello [23], Ofoedu and Onyi [51] and most of the results that have been proved in this direction for this important class of nonlinear mappings.

2. Preliminaries

A normed linear space E is said to be *strictly convex* if for all $x, y \in E$ such that $\|x\| = \|y\| = 1$ and $x \neq y$, we have $\left\|\frac{x+y}{2}\right\| < 1$. The *modulus of convexity* of E is the function $\delta_E\colon (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) := \inf\left\{1 - \frac{1}{2}\|x + y\| \ : \ \|x\| = \|y\| = 1,\, \|x - y\| \geq \epsilon\right\}. \tag{19}$$

E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0,2]$. For a real number $p > 1$, E is said to be p-uniformly convex if there exists a constant $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$ for all $\epsilon \in (0,2]$.

Let E be a real normed space and let $S := \{x \in E : \|x\| = 1\}$. E is said to be *smooth* if the limit

$$\lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{t} \tag{20}$$

exists for each $x, y \in S$. E is said to be *Fréchet differentiable* if it is smooth and the limit in (20) is attained uniformly for $y \in S_E$. Finally E is *uniformly smooth* if it is smooth and the limit in (20) is attained uniformly for each $x, y \in S_E$. If E is a normed linear space of dimension ≥ 2, then, the *modulus of smoothness* of E , ρ_E, is defined by

$$\rho_E(\tau) := \sup\left\{\frac{\|x+y\|+\|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau\right\}; \quad \tau > 0.$$

A normed linear space E is called *uniformly smooth* if

$$\lim_{\tau\to 0}\frac{\rho_E(\tau)}{\tau} = 0.$$

If there exists a constant $c > 0$ and a real number $q > 1$ such that $\rho_E(\tau) \leq c\tau^q$, then E is said to be *q-uniformly smooth*. Typical examples of such spaces are the L_p, ℓ_p and W_p^m spaces for $1 < p < \infty$ where, L_p (or l_p) or W_p^m is

$$\begin{cases} 2-\text{uniformly smooth and } p-\text{uniformly convex} & \text{if} \quad 2 \leq p < \infty; \\ 2-\text{uniformly convex and } p-\text{uniformly smooth} & \text{if} \quad 1 < p < 2. \end{cases}$$

Let J_q denote the *generalized duality mapping* from E to 2^{E^*} defined by

$$J_q(x) := \left\{f \in E^* : \langle x, f\rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1}\right\}$$

J_2 is called the *normalized duality mapping* and is denoted simply by J.

It is well known that E is smooth if and only if J is single valued. Moreover, if E is a reflexive, smooth and strictly convex real Banach space, then J is single valued, one-to-one, surjective and its inverse J^{-1} is the duality mapping from E^* into E.

Let E be a normed linear space. A monotone mapping $A\colon E \to 2^{E^*}$, with domain $D(A)$ is said to be *maximal* if its graph $G(A) = \{(x, y) \in E \times E^* : x \in D(A),\ y \in Ax\}$ is not properly contained in the graph of any other monotone mapping. It is known that if A is maximal monotone, then the zero of A, $A^{-1}(0) := \{x \in E : 0 \in Ax\}$, is closed and convex.

Remark 2.1. It is noted that the maximality of A is equivalent to: if $(x, u) \in E \times E^*$ is such that $\langle u - Ay, x - y\rangle \geq 0$ for every $y \in D(A)$, then $x \in D(A)$ and $u = Ax$.

Let E be a reflexive, smooth and strictly convex real Banach space, and let $A\colon E \to E^*$ be a monotone operator. Then A is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$ (see, e.g., Barbu [11]). If $A\colon E \to E^*$ is a maximal monotone operator, then for each $r > 0$ and $x \in E$, there exists a unique element $x_r \in D(A)$ satisfying $J(x) \in J(x_r) + rAx_r$. We define the *resolvent* of A by $J_r^A x = x_r$. In other words,

$$J_r^A = (J + rA)^{-1}J, \quad \forall\, r > 0. \tag{21}$$

Definition 2.2. Let E be a normed linear space with dual space E^*. A map $T\colon E \to E^*$ with domain $D(T)$ is said to be *hemi-continuous* if it is continuous along each line segment in E with respect to the weak topology. i.e., for all $u, v \in E$ and positive real sequence $\{t_n\}_{n\geq 1}$ such that $t_n \to 0$ and $u + t_n v \in D(T)$ for all $n \geq 1$, then the sequence $\{T(u + t_n v)\}_{n\geq 1}$ converges to Tu in E in the w^*-topology.

The following is well known (see, e.g., [55]), but for the convenience of the reader, we provide a simple proof.

Lemma 2.3. *Let E be a reflexive real Banach space and $A\colon E \to E^*$ be a hemi-continuous monotone mapping such that $D(A) = E$. Then A is maximal monotone.*

Proof. Let $(x, u) \in E \times E^*$ such that for every $(y, v) = (y, Ay) \in E \times E^*$, the following inequality holds:

$$\langle u - v, x - y\rangle \geq 0. \tag{22}$$

We show that $u = Ax$. For this, Since E is reflexive, it follows that the duality mapping of E, J is surjective. So, let $w \in E$ such that $u - Ax \in Jw$. This implies that $\langle w, u - Ax\rangle = \|u - Ax\|^2$. Now, let $\alpha \in (0, 1)$ and define $y_\alpha = x + \alpha w$. Then $w = -\alpha^{-1}(x - y_\alpha)$. Therefore, from (22) we have

$$\langle w, u - Ay_\alpha\rangle = -\alpha^{-1}\langle x - y_\alpha, u - Ay_\alpha\rangle \leq 0.$$

Thus, as $\alpha \to 0$, we have $y_\alpha \to x$, and from the hemi-continuity of A, it follows that $u = Ax$. This, with Remark 2.1, shows that A is maximal monotone. □

In the sequel, we shall need the following results and definitions.

Lemma 2.4 ([5]). *Let E be a uniformly smooth and stricly convex Banach space. Then there exists $L > 0$ such that for any $x, y \in E$ such that $\|x\| \leq R$ with $\|y\| \leq R$ the following inéquality holds*

$$\langle Jx - Jy,\, x - y\rangle \geq L\delta_{E^*}(c_2^{-1}\|Jx - Jy\|),$$

where $c_2 = 2\max\{1, R\}$.

Lemma 2.5 ([5]). *Let $p \geq 2$, $q > 1$, and let E be a p-uniformly convex and q-uniformly smooth real Banach space with dual E^*. Then, the duality mapping $J\colon E \to E^*$ is Lipschitz on bounded sets; that is, for all $R > 0$, there exists a positive constant m such that*

$$\|Jx - Jy\| \leq m_1\|x - y\|,$$

for all $x, y \in E$, with $\|x\| \leq R$ and $\|y\| \leq R$.

Lemma 2.6. *Let $p \geq 2$ and E be a 2-uniformly smooth and p-uniformly convex real Banach space with dual E^*. Let $J\colon E \to E^*$ the duality mapping from E into E^*. Then J^{-1} is Lipschitz on bounded sets; that is, for all $R > 0$, there exists a positive constant $\bar{m}$ such that*

$$\|J^{-1}x^* - J^{-1}y^*\| \leq m_2\|x^* - y^*\|,$$

for all $x^*, y^* \in E^*$, with $\|x^*\| \leq R$, $\|y^*\| \leq R$.

Proof. Since E is 2-uniformly smooth and p-uniformly convex, then E^* is 2-uniformly convex and q-uniformly smooth with $\frac{1}{p} + \frac{1}{q} = 1$. Therefore, the proof follows from Lemma 2.5 and the fact that $J^{-1} = J_*$, where J_* is the duality mapping of E^*. □

We deduce the following useful result.

Lemma 2.7. *For $q \geq 1$, let E be a 2-uniformly convex and q-uniformly smooth real Banach space with dual E^*. Then there exists a constant $d_1 > 0$ such that for any $x^*, y^* \in E^*$ such that $\|x^*\| \leq R$ and $\|y^*\| \leq R$ the following inéquality holds*

$$\langle J^{-1}x^* - J^{-1}y^*, x^* - y^*\rangle \geq d_1\|x^* - y^*\|^2.$$

Proof. Since E is 2-uniformly convex and q-uniformly smooth, then E^* is 2-uniformly smooth and p-uniformly convex with $\frac{1}{p}+\frac{1}{q} = 1$. Moreover E is reflexive. Hence from the fact that $J^{-1} = J_*$, using successively Lemma 2.4, Lemma 2.6 and the 2-uniform convexity of E we have: For any $x^*, y^* \in E^*$ such that $\|x^*\| \leq R$ and $\|y^*\| \leq R$, the following holds.

$$\begin{aligned}\langle J^{-1}x^* - J^{-1}y^*, x^* - y^*\rangle &\geq L\delta_E(c_2^{-1}\|J^{-1}x^* - J^{-1}y^*\|)\\ &\geq Lc_2^{-2}\|J^{-1}x^* - J^{-1}y^*\|^2\\ &\geq L(m_1c_2)^{-2}\|x^* - y^*\|^2.\end{aligned}$$

Let $d_1 = L(m_1c_2)^{-2}$, and we obtain

$$\langle J^{-1}x^* - J^{-1}y^*, x^* - y^*\rangle \geq d_1\|x^* - y^*\|^2.$$ □

Lemma 2.8 ([63]). *Let $p > 1$ be a real number and E be a Banach space. Then the following assertions are equivalent.*

(i) *E is p-uniformly convex.*

(ii) *There exists a constant $d_2 > 0$ such that for all $x, y \in E$ and $f_x \in J_p(x)$, $f_y \in J_p(y)$, one has:*

$$\langle x - y, f_x - f_y\rangle \geq d_2\|x - y\|^p.$$

Let E be a smooth real Banach space with dual space E^*. We define the function $\phi\colon E \times E \to \mathbb{R}$ by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2,\ x, y \in E, \tag{23}$$

where J is the normalized duality mapping from E into E^*, introduced by Alber [3] and studied by Alber and Guerre-Delabriere [6], Kamimura and Takahashi[44], Reich[58] and a host of other authors. This functional ϕ will play a central role in what follows. If $E = H$, a real Hilbert space, then equation (23) reduces to $\phi(x, y) = \|x - y\|^2$ for $x, y \in H$. It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2 \quad \forall\, x, y \in E. \tag{24}$$

Let $V\colon E \times E^* \to \mathbb{R}$ be the functional defined by:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2 \quad \forall\, x \in E, x^* \in E^*. \tag{25}$$

Then, it is easy to see that

$$V(x, x^*) = \phi(x, J^{-1}x^*) \quad \forall\, x \in E,\, x^* \in E^*. \tag{26}$$

Lemma 2.9 (Alber, [3]). *Let X be a reflexive striclty convex and smooth real Banach space with X^* as its dual. Then,*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*) \tag{27}$$

for all $x \in X$ and $x^*, y^* \in X^*$.

Lemma 2.10 (Kamimura and Takahashi, [4]). *Let E be a smooth real Banach space. Then*

$$V(x, y) = V(x, z) + V(z, y) + 2\langle x - z, Jz - Jy\rangle \quad \forall\, x, y, z \in E. \tag{28}$$

From the definition of ϕ and inequality (24), we can observe that for all $x, y \in E$, $\phi(y, x) \geq 0$ and

$$2\langle x - y,\, Jx - Jy\rangle - \phi(x, y) = \phi(y, x).$$

This leads to the following

Lemma 2.11. *Let E be a smooth real Banach space. Then, for all $x, y \in E$, the following holds*

$$\phi(x, y) \leq 2\langle Jy - Jx,\, y - x\rangle.$$

Lemma 2.12 (Kamimura and Takahashi, [4]). *Let X be a smooth and uniformly convex real Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of X. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.*

Similarly, if E is a reflexive smooth and strictly convex real Banach space, we introduce the functional $\phi_*\colon E^* \times E^* \to \mathbb{R}$, defined by:

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle J^{-1}y^*, x^*\rangle + \|y^*\|^2, \quad x^*, y^* \in E^*, \tag{29}$$

and the functional $V_*\colon E^* \times E \to \mathbb{R}$ defined from $E^* \times E$ to $\mathbb{R}$ by:

$$V_*(x^*, x) = \|x^*\|^2 - 2\langle x, x^*\rangle + \|x\|^2, \quad x \in E,\; x^* \in E^*. \tag{30}$$

It is easy to see that

$$V_*(x^*, x) = \phi_*(x^*, Jx) \quad \forall\, x \in E,\; x^* \in E^*. \tag{31}$$

Consider $w_1 = (x, x^*) \in E \times E^*$ and $w_2 = (y, y^*) \in E \times E^*$. Then

$$\|w_1 - w_2\| = \Big(\|x - y\|^2 + \|x^* - y^*\|^2\Big)^{\frac{1}{2}}$$

defines a norm on the product space $E \times E^*$, and the functional $\psi\colon (E \times E^*) \times (E \times E^*) \to \mathbb{R}$ can be introduced by

$$\psi(w_1, w_2) := \phi(x, y) + \phi_*(x^*, y^*).$$

Lemma 2.13 (Xu [62]). *Let $\{\rho_n\}$ be a sequence of non-negative real numbers satisfying the following inequality*

$$\rho_{n+1} \le (1 - \alpha_n)\rho_n + \alpha_n\sigma_n + \gamma_n, \tag{32}$$

where $\{\alpha_n\}, \{\sigma_n\}$ and $\{\gamma_n\}$ are real sequences satisfying:

(i) $\{\alpha_n\} \subset]0, 1[,\ \sum \alpha_n = \infty$; (ii) $\limsup_{n\to\infty} \sigma_n \le 0$; (iii) $\gamma_n \ge 0,\ \sum \gamma_n < \infty$.

Then, the sequence (ρ_n) converges to zero as $n \to \infty$.

3. Main Results

We start by a presentation of our iterative algorithm. Let E be a smooth, strictly convex and reflexive real Banach space with dual space E^*. For $F\colon E \to E^*$ and $K\colon E^* \to E$ mappings, let the sequences $\{u_n\}$ and $\{v_n\}$ be generated iteratively from $(u_1, v_1) \in E \times E^*$ by:

$$\begin{cases} u_{n+1} = J^{-1}\Big(Ju_n - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n(Ju_n - Ju_1)\Big), & n \ge 1, \\ v_{n+1} = J\Big(J^{-1}v_n - \alpha_n(Kv_n + u_n) - \lambda_n\theta_n(J^{-1}v_n - J^{-1}v_1)\Big), & n \ge 1, \end{cases} \tag{33}$$

where J is the normalized duality mapping from E onto E^* and $\{\lambda_n\}$, $\{\theta_n\}$ are real sequences in $(0, 1)$ satisfying, here and elsewhere, the following conditions:

$$\text{(i)}\quad \lim_{n\to\infty} \theta_n = 0; \qquad \text{(ii)}\quad \sum_{n=1}^{\infty} \lambda_n\theta_n = \infty, \quad \lambda_n = o(\theta_n);$$

$$\text{(iii)}\quad \limsup_{n\to\infty} \frac{\left(\frac{\theta_{n-1}}{\theta_n} - 1\right)}{\lambda_n\theta_n} \le 0, \quad \sum_{n=1}^{\infty} \lambda_n^2 < \infty.$$

Remark 3.1. Real sequences that satisfy conditions (i)-(iii) are $\lambda_n = (n + 1)^{-a}$ and $\theta_n = (n + 1)^{-b}$, $n \ge 1$ with $0 < b < a$, $\frac{1}{2} < a < 1$ and $a + b < 1$.

In fact, (i), (ii) and the second part of (iii) are easy to check. For the first part of condition (iii), using the fact that $(1+x)^s \le 1+sx$, for $x > -1$ and $0 < s < 1$, we have

$$\begin{aligned} 0 \ &\le\ \frac{\left(\frac{\theta_{n-1}}{\theta_n}-1\right)}{\lambda_n\theta_n} \ = \ \left[\left(1+\frac{1}{n}\right)^b - 1\right]\cdot(n+1)^{a+b} \\ &\le\ b\cdot\frac{(n+1)^{a+b}}{n} \ = \ b\cdot\frac{n+1}{n}\cdot\frac{1}{(n+1)^{1-(a+b)}} \to 0 \quad \text{as} \quad n\to\infty. \end{aligned}$$

Remark 3.2. Note also that a duality mapping exists in each Banach space. From [4], this mapping is known precisely in l_p, L_P, $W^{m,p}$-spaces, $1 < p < \infty$ and is given by:

(i) l_p : $\quad Jx = \|x\|_{l_p}^{2-p} y \in l_q, \quad x = (x_1, x_2, \cdots, x_n, \cdots),$
$\quad y = (x_1|x_1|^{p-2}, x_2|x_2|^{p-2}, \cdots, x_n|x_n|^{p-2}, \cdots),$

(ii) L_p : $\quad Ju = \|u\|_{L_p}^{2-p}|u|^{p-2}u \in L_q, \quad \text{with} \ \ 1/p + 1/q = 1,$

(iii) $W^{m,p}$: $\quad Ju = \|u\|_{W^{m,p}}^{2-p}\sum_{|\alpha\le m|}(-1)^{|\alpha|}D^{\alpha}\Big(|D^{\alpha}u|^{p-2}D^{\alpha}u\Big) \in W^{-m,q},$
with $1/p + 1/q = 1.$

Next, we introduce the auxiliary map we referred and the normalized duality mapping in the cartesian product space $X := E\times E^*$ with the norm $\|w\|_X = (\|u\|^2 + \|v\|_*^2)^{\frac{1}{2}}$ for $w = (u,v) \in X$, where $\|\cdot\|_*$ denotes the norm in E^*. For mappings $F\colon E \to E^*$ and $K\colon E^* \to E$, observing from the reflexivity of E that $X^* = E^*\times E$, we define

$$J_X : X \to X^* \ \text{ by: } \ J_X(w) = \Big(J(u), J^{-1}(v)\Big) \quad \forall\, w = (u,v) \in X, \tag{34}$$

$$\Lambda : X \to X^* \ \text{ by: } \ \Lambda w = (Fu - v, Kv + u) \quad \forall\, w = (u,v) \in X. \tag{35}$$

Remark 3.3. Note that the zeros of Λ give the solutions of the Hammerstein equation $u + KFu = 0$. More precisely, for $w = (u,v) \in E\times E^*$, $\Lambda(w) = 0$ if and only if $u + KFu = 0$ and $v = Fu$.

The following results will be crucial in the sequel.

Lemma 3.4. *Let E be a reflexive, smooth and strictly convex real Banach space. Then J_X is the normalized duality mapping of X.*

Proof. For arbitrary $w = (u_1, v_1) \in X$ and $h = (v_2, u_2) \in X^*$, the duality pairing $\langle\cdot,\cdot\rangle_X$ is given by

$$\langle w, h\rangle_X = \langle u_1, v_2\rangle + \langle u_2, v_1\rangle. \tag{36}$$

Now let $w = (u_1, v_1) \in X$. Set $h = J_X(w)$. Then, we have $h = (J(u_1), J^{-1}(v_1))$, and it follows that

$$\begin{aligned}\langle w, J_X(w)\rangle_X &= \langle w, h\rangle = \langle u_1, J(u_1)\rangle + \langle J^{-1}(v_1), v_1\rangle \\ &= \|u_1\|^2 + \|v_1\|_*^2 = \left(\|u_1\|^2 + \|v_1\|_*^2\right)^{\frac{1}{2}}\left(\|u_1\|^2 + \|v_1\|_*^2\right)^{\frac{1}{2}} \\ &= \left(\|u_1\|^2 + \|v_1\|_*^2\right)^{\frac{1}{2}}\left(\|J(u_1)\|^2 + \|J(v_1)\|_*^2\right)^{\frac{1}{2}} \\ &= \|w\|_X\| \cdot \|h\|_{X^*} = \|w\|_X^2.\end{aligned}$$

This proves that J_X is the normalized duality mapping of X. □

Lemma 3.5. *Let E be a reflexive real Banach space and $F\colon E \to E^*$ and $K\colon E^* \to E$ be hemi-continuous monotone mappings such that $D(F) = E$ and $R(F) = D(K) = E^*$. Then Λ is maximal monotone.*

Proof. We show that Λ is hemi-continuous and monotone. The hemi-continuity of Λ follows from the fact that F and K are hemi-continuous. For the monotonicity, let $w_1 = (u_1, v_1), w_2 = (u_2, v_1) \in X$. Using the fact that K and F are monotone, we have

$$\begin{aligned}\langle w_1 - w_2, \Lambda w_1 - \Lambda w_2\rangle_X &= \\ &= \langle (u_1 - u_2, v_1 - v_2), (Fu_1 - Fu_2 + v_2 - v_1, Kv_1 - Kv_2 + u_1 - u_2)\rangle_X \\ &= \langle u_1 - u_2, Fu_1 - Fu_2 + v_2 - v_1\rangle + \langle Kv_1 - Kv_2 + u_1 - u_2, v_1 - v_2\rangle \\ &= \langle u_1 - u_2, Fu_1 - Fu_2\rangle + \langle Kv_1 - Kv_2, v_1 - v_2\rangle \geq 0.\end{aligned}$$

This implies that Λ is monotone. Therefore, since Λ is hemi-continuous, from Lemma 2.3, it follows that Λ is maximal monotone. This completes the proof. □

3.1. Implicit scheme for equations of Hammerstein type

We start with the following useful result.

Lemma 3.6 (Takahashi [47]). *Let E be a uniformly convex real Banach space with Fréchet differentiable norm and let E^* its dual. Let $A\colon E^* \to 2^E$ be a maximal monotone mapping with $A^{-1}(0) \neq \emptyset$. Then for $u \in E$,*

$$\lim_{\lambda\to\infty}(I + \lambda AJ)^{-1}u \quad \text{exists and belongs to} \quad (AJ)^{-1}(0), \tag{37}$$

where J is the normalized duality mapping from E into E^*. Moreover, if $Ru := y^* = \lim_{\lambda\to\infty}(I+\lambda AJ)^{-1}u$, then R is a sunny generalized nonexpansive retraction of E into $(AJ)^{-1}(0)$.

We now prove the following theorem.

Theorem 3.7. *Let E be a uniformly convex and uniformly smooth real Banach space with dual E^*. Let $F\colon E \to E^*$ and $K\colon E^* \to E$ be hemi-continuous monotone mappings such that $R(F) = D(K) = E^*$. Assume that the Hammerstein* *equation $u + KFu = 0$ has a solution in E. For given $x_1 \in E$ and $y_1 \in E^*$, there exists a sequence $\{z_n\} = \{(x_n, y_n)\}$ in $E \times E^*$ such that*

$$\theta_n(Jx_n - Jx_1) + Fx_n - y_n = 0, \quad \forall n \geq 1; \tag{38}$$

$$\theta_n(J^{-1}y_n - J^{-1}y_1) + Ky_n - x_n = 0, \quad \forall n \geq 1; \tag{39}$$

and furthermore,

$$x_n \to x^*, \;\; y_n \to y^* \text{ with } x^* + KFx^* = 0 \text{ and } y^* = Fx^*, \tag{40}$$

where J is the normalized duality mapping from E into E^*.

Proof. From the fact that the Hammerstein equation $u + KFu = 0$ has a solution, it follows, using remark 3.3, that $\Lambda^{-1}(0) \neq \emptyset$, where Λ is the auxiliary mapping given by (35). Since F and K are hemi-continuous and monotone, then by Lemma 3.5, Λ is maximal monotone. Now, since E is uniformly convex and uniformly smooth, then $X = E \times E^*$ is also uniformly convex and uniformly smooth. Consequently, the duality mapping J_X from X into X^* is single valued, onto and one to one and its inverse J_X^{-1} is the duality mapping of X^*. Let $z_1 = (x_1, y_1) \in X$. Then from Lemma 3.6, it follows that

$$\lim_{\lambda \to \infty} J_\lambda^\Lambda z_1 := (I^* + \lambda \Lambda J_X^{-1})^{-1} J_X z_1 \text{ exists and belongs to } (\Lambda J_X^{-1})^{-1}(0),$$

where I^* is the identity map of X^*. Therefore, the sequence $\{z_n\}$ defined by

$$z_n = (x_n, y_n) := J_X^{-1}\Big(J_{t_n}^\Lambda z_1\Big) \quad \forall n \geq 1, \text{ with } t_n = \theta_n^{-1}$$

satisfies (38), (39) and (40). □

3.2. Convergence in l_p, L_p and $W^{m,p}$-spaces, $1 < p \leq 2$

Theorem 3.8. *For $q > 1$, let E be a 2- uniformly convex and q-uniformly smooth real Banach space with dual E^*. Let $F\colon E \to E^*$ and $K\colon E^* \to E$ be bounded, hemi-continuous and monotone mappings such that $D(F) = E$ and $D(K) = R(F) = E^*$. Suppose that the Hammerstein equation $u + KFu = 0$ has a solution. Then, there exists $\gamma_0 > 0$ such that if $\lambda_n < \gamma_0 \theta_n$ $\forall n \geq 1$, the sequence $\{(u_n, v_n)\}$ given by (33) converges strongly to (u^*, v^*), where u^* is a solution of the Hammerstein equation $u + KFu = 0$ and $v^* = Fu^*$.*

Proof. Let $X = E \times E^*$ with the norm $\|w\| = (\|u\|^2 + \|v\|^2)^{\frac{1}{2}}$, where $w = (u, v) \in E \times E^*$. Define the sequence $\{w_n\}$ in X by: $w_n = (u_n, v_n)$. We break the proof into two steps.

Step 1: *We first prove that $\{w_n\}$ is bounded.* For this, let $u^* \in E$ be a solution of $u + KFu = 0$ and set $v^* = Fu^*$ and $w^* = (u^*, v^*)$. We observe that $u^* = -Kv^*$. There exits $r > 0$ such that

$$\sqrt{r} > \max\Big\{\sqrt{\psi(w^*, w_1)}, 24\|J_X w^* - J_X w_1\|, \|w^*\|\Big\}. \tag{41}$$

By induction we show that $\psi(w^*, w_n) \leq r$ for all $n \geq 1$. Indeed, by construction, we have $\psi(w^*, w_1) \leq r$. Suppose that $\psi(w^*, w_n) \leq r$ for some $n \geq 1$. Since F and K are bounded, it follows that

$$M_1 := \sup\Big\{ \|Fu - v + \theta(Ju - Ju_1)\|_* : \psi(w^*, (u,v)) < r, 0 < \theta < 1\Big\} + 1 < \infty$$

and

$$M_2 := \sup\Big\{ \|Ku + v + \theta(J^{-1}u - J^{-1}v_1)\| : \psi(w^*, (u,v)) < r, 0 < \theta < 1\Big\} + 1 < \infty.$$

From the Lipschitz properties of J (Lemma 2.5) and J^{-1} (Lemma 2.6), there exist positive contants m_1 and m_2 such that

$$\|J^{-1}(Ju - \lambda(Fu - v) - \lambda\theta(Ju - Ju_1)) - J^{-1}(Ju)\| \leq \lambda m_2 M_1 \tag{42}$$

and

$$\|J(J^{-1}v - \lambda(Kv + u) - \lambda\theta(J^{-1}v - J^{-1}v_1)) - J(J^{-1}v)\| \leq \lambda m_1 M_2 \tag{43}$$

for all $\lambda, \theta \in (0,1)$ and $(u,v) \in X$ such that $\psi(w^*, (u,v)) \leq r$. Set

$$M := M_1^2 + M_2^2, \quad m = \max\{m_1, m_2\} \quad \text{and} \quad \bar{m} = m_1^2 + m_2^2.$$

Define γ_0 as follows

$$\gamma_0 := \min\Big\{1, \frac{r}{8mM^2}\Big\}. \tag{44}$$

Now assume that $\lambda_n \leq \gamma_0\theta_n$ for all $n \geq 1$. From the definition of u_n, ϕ and V, we have

$$\begin{aligned}\phi(u^*, u_{n+1}) &= \phi\Big(u^*, J^{-1}\Big(Ju_n - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n(Ju_n - Ju_1)\Big)\Big)\\ &= V\Big(u^*, Ju_n - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n(Ju_n - Ju_1)\Big).\end{aligned}$$

Using Lemma 2.9 with $y^* = \lambda_n(Fu_n - v_n) + \lambda_n\theta_n(Ju_n - Ju_1)$, the Schwarz inequality, (42), the induction assumption and the definition of M_1, we estimate

$$\begin{aligned}&\phi(u^*, u_{n+1}) \leq V(u^*, Ju_n) - 2\Big\langle J^{-1}\Big(Ju_n - \lambda_n(Fu_n - v_n) - \\ &\quad - \lambda_n\theta_n(Ju_n - Ju_1)\Big) - u^*, \lambda_n(Fu_n - v_n) + \lambda_n\theta_n(Ju_n - Ju_1)\Big\rangle \\ &\leq \phi(u^*, u_n) - 2\lambda_n\langle u_n - u^*, Fu_n - v_n + \theta_n(Ju_n - Ju_1)\rangle - 2\lambda_n\Big\langle J^{-1}\Big(Ju_n - \\ &\quad - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n(Ju_n - Ju_1)\Big) - J^{-1}(Ju_n), Fu_n - v_n + \theta_n(Ju_n - Ju_1)\Big\rangle \\ &\leq \phi(u^*, u_n) - 2\lambda_n\langle u_n - u^*, Fu_n - v_n + \theta_n(Ju_n - Ju_1)\rangle + 2m_1M_1^2\lambda_n^2\,.\end{aligned} \tag{45}$$

Using the fact that $v^* = Fu^*$, we also have

$$\begin{aligned}\langle u_n - u^*, Fu_n - v_n + \theta_n(Ju_n - Ju_1)\rangle &= \qquad (46)\\ &= \langle u_n - u^*, Fu_n - Fu^*\rangle + \langle u_n - u^*, v^* - v_n + \theta_n(Ju_n - Ju_1)\rangle\\ &= \langle u_n - u^*, Fu_n - Fu^*\rangle + \langle u_n - u^*, v^* - v_n\rangle + \theta_n\langle u_n - u^*, Ju_n - Ju^*\rangle +\\ &\quad + \theta_n\langle u_n - u^*, Ju^* - Ju_1\rangle.\end{aligned}$$

Combining (45) and (46) with Lemma 2.11 and the fact that F is monotone, we obtain

$$\begin{aligned}\phi(u^*,u_{n+1}) \le{}& \phi(u^*, u_n) + 2\lambda_n\langle u_n - u^*, v_n - v^*\rangle - \qquad (47)\\ &- \lambda_n\theta_n\phi(u^*, u_n) - 2\lambda_n\theta_n\langle u_n - u^*, Ju_1 - Ju^*\rangle + 2m_1M_1^2\lambda_n^2.\end{aligned}$$

Using the same arguments and method of computation, we have

$$\begin{aligned}\phi_*(v^*,v_{n+1}) \le{}& \phi_*(v^*, v_n) - 2\lambda_n\langle u_n - u^*, v_n - v^*\rangle - \qquad (48)\\ &- \lambda_n\theta_n\phi_*(v^*, v_n) - 2\lambda_n\theta_n\langle J^{-1}v_1 - J^{-1}v^*, v_n - v^*\rangle + 2m_2M_2^2\lambda_n^2.\end{aligned}$$

Adding up (47) and (48), using the definition of J_X and the Schwarz inequality, we have

$$\begin{aligned}\psi(w^*, w_{n+1}) \le{}& (1 - \lambda_n\theta_n)\psi(w^*, w_n) + \qquad (49)\\ &+ 2m\lambda_n\theta_n\|w_n - w^*\| \cdot \|J_Xw_1 - J_Xw^*\| + 2mM^2\lambda_n^2.\end{aligned}$$

From (24), the induction assumption and inequality (41), we have

$$\|w_n - w^*\| \le 3\sqrt{r} \quad \text{and} \quad 2\|w_n - w^*\| \cdot \|J_Xw_1 - J_Xw^*\| \le \frac{r}{4}.$$

Combining this with the fact that $\lambda_n \le \gamma_0\theta_n$ for all $n \ge 1$, and using the definition of γ_0, (44), it follows that

$$\begin{aligned}\psi(w^*, w_{n+1}) &\le (1 - \lambda_n\theta_n)\psi(w^*, w_n) + \lambda_n\theta_n\frac{r}{4} + \lambda_n\theta_n\frac{r}{4}\\ &\le (1 - \frac{\lambda_n\theta_n}{2})r.\end{aligned}$$

Therefore, $\psi(w^*, w_{n+1}) \le r$. So, by induction $\psi(w^*, w_n) \le r$ for all $n \ge 1$. Hence, from (24), w_n is bounded.

Step 2: Let $z_n = (x_n, y_n) \in X$ be the sequence given by Theorem 3.7 with $z_1 = (u_1, v_1)$. We show that $\psi(w_n, z_n) \to 0$, as $n \to \infty$.

Using the definitions of u_n, ϕ and V, we have

$$\begin{aligned}\phi(x_n, u_{n+1}) &= \phi\Big(x_n, J^{-1}\Big(Ju_n - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n(Ju_n - Ju_1)\Big)\\ &= V(x_n, Ju_n - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n(Ju_n - Ju_1)).\end{aligned}$$

Using Lemma 2.9 with $y^* = \lambda_n(Fu_n - v_n) + \lambda_n\theta_n(Ju_n - Ju_1)$, we estimate

$$\begin{aligned}\phi(x_n, u_{n+1}) &\le \phi(x_n, u_n) - 2\lambda_n\Big\langle J^{-1}\Big(Ju_n - \lambda_n(Fu_n - v_n) - \\ &\qquad - \lambda_n\theta_n(Ju_n - Ju_1)\Big) - x_n, Fu_n - v_n + \theta_n(Ju_n - Ju_1)\Big\rangle \\ &= \phi(x_n, u_n) - 2\lambda_n\langle u_n - x_n, Fu_n - v_n + \theta_n(Ju_n - Ju_1)\rangle - \\ &\qquad - 2\lambda_n\Big\langle J^{-1}\Big(Ju_n - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n(Ju_n - Ju_1)\Big) - \\ &\qquad - J^{-1}(Ju_n), Fu_n - v_n + \theta_n(Ju_n - Ju_1)\Big\rangle.\end{aligned}$$

Using the fact that F is monotone and bounded, identity (38), the boundedness of the sequences $\{u_n\}$ and $\{x_n\}$ and the Lipschitz property of J^{-1} on bounded subsets of E, we obtain the following estimates

$$\begin{aligned}\phi(x_n, u_{n+1}) &\le \phi(x_n, u_n) - 2\lambda_n\langle u_n - x_n, Fu_n - Fx_n\rangle - \\ &\qquad - 2\lambda_n\langle u_n - x_n, Fx_n - v_n + \theta_n(Ju_n - Ju_1)\rangle + \lambda_n^2 m_1 M^2 \\ &\le \phi(x_n, u_n) - 2\lambda_n\langle u_n - x_n, y_n - \theta_n(Jx_n - Ju_1) - v_n + \\ &\qquad + \theta_n(Ju_n - Ju_1)\rangle + 2\lambda_n^2 m_1 M^2 \\ &\le \phi(x_n, u_n) - 2\lambda_n\langle u_n - x_n, y_n - v_n + \theta_n(Ju_n - Jx_n)\rangle + \lambda_n^2 m_1 M^2 \\ &\le \phi(x_n, u_n) - 2\lambda_n\langle u_n - x_n, y_n - v_n\rangle - 2\lambda_n\theta_n\langle u_n - x_n, Ju_n - Jx_n\rangle + \\ &\qquad + 2\lambda_n^2 m_1 M_1^2,\end{aligned}$$

for m_1 and M_1 positive real contants. This, with Lemma 2.11, yields:

$$\phi(x_n, u_{n+1}) \le (1 - \lambda_n\theta_n)\phi(x_n, u_n) - 2\lambda_n\langle u_n - x_n, y_n - v_n\rangle + 2\lambda_n^2 m_1 M^2. \quad (50)$$

Using the same method of computation and the same arguments, we have

$$\phi_*(y_n, v_{n+1}) \le (1 - \lambda_n\theta_n)\phi_*(y_n, v_n) - 2\lambda_n\langle x_n - u_n, y_n - v_n\rangle + 2\lambda_n^2 m_2 M_2^2. \quad (51)$$

Therefore, adding up (50) and (51), we obtain

$$\psi(z_n, w_{n+1}) \le (1 - \lambda_n\theta_n)\psi(z_n, w_n) + 2\lambda_n^2 m M^2. \quad (52)$$

Using Lemma 2.10, we also have

$$\psi(z_n, w_{n+1}) = \psi(z_n, z_{n+1}) + \psi(z_{n+1}, w_{n+1}) + 2\langle z_{n+1} - z_n, J_X w_{n+1} - J_X w_n\rangle. \quad (53)$$

Using (52), (53) and the fact that $\psi(z_n, z_{n+1}) \ge 0$, we obtain

$$\psi(z_{n+1}, w_{n+1}) \le (1 - \lambda_n\theta_n)\psi(z_n, w_n) + 2\langle z_n - z_{n+1}, J_X w_{n+1} - J_X z_{n+1}\rangle + 2\lambda_n^2 m M^2.$$

Since $\{w_n\}$ and $\{z_n\}$ are bounded there exists a positive constant C such that

$$\psi(z_{n+1}, w_{n+1}) \le (1-\lambda_n\theta_n)\psi(z_n, w_n) + C\|z_n - z_{n+1}\|_X + 2\lambda_n^2 mM^2. \tag{54}$$

Now using the identities (38) and (39) in Theorem 3.7, we have

$$Jx_n - Jx_{n+1} + \frac{1}{\theta_{n+1}}(Fx_n - y_n - Fx_{n+1} + y_{n+1}) = \Big(\frac{\theta_n - \theta_{n+1}}{\theta_{n+1}}\Big)(Jx_n - Ju_1) \tag{55}$$

and

$$\begin{aligned} J^{-1}y_n - J^{-1}y_{n+1} + \frac{1}{\theta_{n+1}}(Ky_n + x_n - Ky_{n+1} - x_{n+1}) &= \\ = \Big(\frac{\theta_n - \theta_{n+1}}{\theta_{n+1}}\Big)(J^{-1}y_n - J^{-1}v_1). \end{aligned} \tag{56}$$

Taking the duality pairing with $x_n - x_{n+1}$ and $y_n - y_{n+1}$ respectively in (55) and (56) and using the monotonicity of F and K we obtain the following estimates:

$$\begin{aligned} \langle Jx_n - Jx_{n+1}, x_n - x_{n+1}\rangle + \frac{1}{\theta_{n+1}}\langle y_{n+1} - y_n, x_n - x_{n+1}\rangle &\le \\ \le \Big(\frac{\theta_n - \theta_{n+1}}{\theta_{n+1}}\Big)\langle Jx_n - Jx_1, x_n - x_{n+1}\rangle \end{aligned} \tag{57}$$

and

$$\begin{aligned} \langle J^{-1}y_n - J^{-1}y_{n+1}, y_n - y_{n+1}\rangle + \frac{1}{\theta_{n+1}}\langle x_n - x_{n+1}, y_n - y_{n+1}\rangle &\le \\ \le \Big(\frac{\theta_n - \theta_{n+1}}{\theta_{n+1}}\Big)\langle J^{-1}y_n - J^{-1}y_1, y_n - y_{n+1}\rangle. \end{aligned} \tag{58}$$

Adding up (57) and (58), using the Schwartz inequality, Lemmas 2.5 and 2.6, Lemmas 2.8 and 2.7 and the boundedness of $\{x_n\}$ and $\{y_n\}$ we get:

$$\begin{aligned} &d_2\|x_n - x_{n+1}\|^2 + d_1\|y_n - y_{n+1}\|^2 \le \\ &\quad\le \Big(\frac{\theta_n - \theta_{n+1}}{\theta_{n+1}}\Big)\Big(\langle Jx_n - Jx_1, x_n - x_{n+1}\rangle + \langle J^{-1}y_n - J^{-1}y_1, y_n - y_{n+1}\rangle\Big) \\ &\quad\le \Big(\frac{\theta_n - \theta_{n+1}}{\theta_{n+1}}\Big)\Big(m_1\|x_n - x_1\|\,\|x_n - x_{n+1}\| + m_2\|y_n - y_1\|_*\|y_n - y_{n+1}\|_*\Big) \\ &\quad\le C_1\Big(\frac{\theta_n - \theta_{n+1}}{\theta_{n+1}}\Big)\Big(\|x_n - x_{n+1}\| + \|y_n - y_{n+1}\|_*\Big) \\ &\quad\le 2C_1\Big(\frac{\theta_n - \theta_{n+1}}{\theta_{n+1}}\Big)\|z_n - z_{n+1}\|_X \end{aligned}$$

for some constant C_1. So,

$$\|z_n - z_{n+1}\|_X \le K\Big(\frac{\theta_{n+1} - \theta_n}{\theta_{n+1}}\Big) \quad \text{where} \quad K = \frac{2CC_1}{\min\{d_1, d_2\}}. \tag{59}$$

Therefore, combining inequalities (54) and (59), we obtain

$$\psi(z_{n+1}, w_{n+1}) \leq (1-\lambda_n\theta_n)\psi(z_n, w_n) + K\Big(\frac{\theta_n - \theta_{n+1}}{\theta_{n+1}}\Big) + 2\lambda_n^2 mM^2,$$

Finally, we have

$$\psi(z_{n+1}, w_{n+1}) \leq (1-\lambda_n\theta_n)\psi(z_n, w_n) + \lambda_n\theta_n\sigma_n + \gamma_n, \tag{60}$$

with $\sigma_n := K\Big(\frac{\theta_n - \theta_{n+1}}{\theta_{n+1}}\Big)$ and $\gamma_n = 2\lambda_n^2 mM^2$. So, using Lemma 2.13, it follows that $\psi(z_n, w_n) \to 0$, as $n \to \infty$. Therefore, from Lemma 2.12, we conclude that $\|w_n - z_n\| \to 0$ as $n \to \infty$. Hence, the conclusion follows from Theorem 3.7. □

Corollary 3.9. *Let E be a Banach space either l_p or L_p or $W^{m,p}$, $1 < p \leq 2$ with dual E^*. Let $F\colon E \to E^*$ and and $K\colon E^* \to E$ be bounded, hemi-continuous and monotone mappings such that $D(F) = E$ and $D(K) = R(F) = E^*$. Suppose that the Hammerstein equation $u + KFu = 0$ has a solution. Then, there exists $\gamma_0 > 0$ such that if $\lambda_n < \gamma_0\theta_n$ $\forall n \geq 1$, the sequence $\{(u_n, v_n)\}$ given by (33) converges strongly to (u^*, v^*), where u^* is a solution of the Hammerstein equation $u + KFu = 0$ and $v^* = Fu^*$.*

Proof. Since l_p, L_p or $W^{m,p}$-spaces, $1 < p \leq 2$ are 2-uniformly convex and p-uniformly smooth Banach spaces, then the proof follows from Theorem 3.8. □

3.3. Convergence in l_p, L_p and $W^{m,p}$-spaces, $2 \leq p < \infty$

Following the method of proof of Theorem 3.8, inverting the properties of J and J^{-1}, and using the duality between E and E^*, we receive the following result.

Theorem 3.10. *For $s > 1$, let E be an s-uniformly convex and 2-uniformly smooth real Banach space with dual E^*. Let $F\colon E \to E^*$ and and $K\colon E^* \to E$ be bounded, hemi-continuous and monotone mappings such that $D(F) = E$ and $D(K) = R(F) = E^*$. Suppose that the Hammerstein equation $u + KFu = 0$ has a solution. Then, there exists $\gamma_0 > 0$ such that if $\lambda_n < \gamma_0\theta_n$ for all $n \geq 1$, the sequence $\{(u_n, v_n)\}$ given by (33) converges strongly to (u^*, v^*), where u^* is a solution of the Hammerstein equation $u + KFu = 0$ and $v^* = Fu^*$.*

Corollary 3.11. *Let E be a Banach space, either l_p or L_p or $W^{m,p}$, $2 \leq p < \infty$, with dual E^*. Let $F\colon E \to E^*$ and and $K\colon E^* \to E$ be bounded, hemi-continuous and monotone mappings such that $D(F) = E$ and $D(K) = R(F) = E^*$. Suppose that the Hammerstein equation $u + KFu = 0$ has a solution. Then, there exists $\gamma_0 > 0$ such that if $\lambda_n < \gamma_0\theta_n$ for all $n \geq 1$, the sequence $\{(u_n, v_n)\}$ given by (33) converges strongly to (u^*, v^*), where u^* is a solution of the Hammerstein equation $u + KFu = 0$ and $v^* = Fu^*$.*

Proof. Since l_p, L_p or $W^{m,p}$-spaces, $2 \leq p < \infty$ are p-uniformly convex and 2-uniformly smooth Banach spaces, then the proof follows from Theorem 3.10. □

4. Illustration of the proposed algorithm in L^p spaces

Consider the following Hammerstein integral equation:

$$u(x) + \int_\Omega k(x,y)f(y,u(y))\,dy = 0, \tag{61}$$

where Ω is a domain of σ-finite measure in $\mathbb{R}^N$ and $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a function satisfying the Caratheodory conditions:

(i) $f(\cdot, r)\colon \Omega \to \mathbb{R}$ is measurable for all fixed $r \in \mathbb{R}$;
(ii) $f(x, \cdot)\colon \mathbb{R} \to \mathbb{R}$ is continuous for almost all $x \in \Omega$.

Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. To such a function f we associate the superposition of the Nemytskii operator $Fu(x) = f(x, u(x))$, defined on classes of functions $u\colon \Omega \to \mathbb{R}$ in $L^p(\Omega$. Suppose that in addition to (i) and (ii), f satisfies an inequality of the form :

(iii) $|f(x, u(x))| \leq g(x)c|u(x)|^{p-1}$,

where $g \in L^q(\Omega)$ and $c > 0$. Then F is a well-defined bounded continuous operator from $L^p(\Omega)$ into its dual $L^q(\Omega)$. If moreover, for each fixed $x \in \Omega$, $f(x, \cdot)\colon \mathbb{R} \to \mathbb{R}$ is nondecreasing (monotone), then F is a monotone operator from $L^p(\Omega)$ into its dual $L^q(\Omega)$.

If we introduce the linear integral operator $K\colon L^q(\Omega) \to L^p(\Omega)$ defined by :

$$Kv = \int_\Omega k(\cdot, y)v(y)\,dy,$$

then we can formally rewrite the Hammerstein integral equation (61) in operator form as :

$$u + KFu = 0, \tag{62}$$

where $F\colon L^p(\Omega) \to L^q(\Omega)$ and $F\colon L^q(\Omega) \to L^p(\Omega)$ are bounded continuous linear maps. From [4], the duality mapping J and its inverse J^{-1} are known precisely in $L^p(\Omega)$ for $1 < p < \infty$ and are given by :

$$Ju = \|u\|_{L_p}^{2-p}|u|^{p-2}u,\ \ \forall\, u \in L^p(\Omega);$$

$$J^{-1}v = \|v\|_{L_q}^{2-q}|v|^{q-2}v,\ \ \forall\, v \in L^q(\Omega),$$

with $1/p + 1/q = 1$. In this setting the sequences $\{u_n\}$ and $\{v_n\}$ defined in (33) are given iteratively from $u_1 \in L^p(\Omega)$ and $v_1 \in L^q(\Omega)$ by :

$$\begin{cases} a_n &= \|u_n\|_{L_p}^{2-p}|u_n|^{p-2}u_n - \lambda_n(Fu_n - v_n) - \\ & \qquad -\lambda_n\theta_n(\|u_n\|_{L_p}^{2-p}|u_n|^{p-2}u_n - \|u_1\|_{L_p}^{2-p}|u_1|^{p-2}u_1, \\ u_{n+1} &= \|a_n\|_{L_q}^{2-q}|a_n|^{q-2}a_n,\ \ n \geq 1, \\ b_n &= \|v_n\|_{L_q}^{2-q}|v_n|^{q-2}v_n - \lambda_n(Kv_n + u_n) - \\ & \qquad -\lambda_n\theta_n(\|v_n\|_{L_q}^{2-q}|v_n|^{q-2}v_n - \|v_1\|_{L_q}^{2-q}|v_1|^{q-2}v_1, \\ v_{n+1} &= \|b_n\|_{L_p}^{2-p}|b_n|^{p-2}b_n,\ \ n \geq 1, \end{cases} \tag{63}$$

From Theorem 3.8 and Theorem 3.10 it follows that the sequences $\{u_n\}$ and $\{v_n\}$ given by (63) converge strongly to some $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, respectively, where u is a solution of the Hammerstein integral equation (61).

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