

On the Geodetic Domination Sombor Index of Graphs: Structural Characterization and Applications to Resilient IoT Networks

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Abstract

In this paper a new invariant of the graph based on the Sombor index is proposed, which we call the Geodetic Domination Sombor Index (GDSO) that combines together geodetic domination degrees and the geometric structure which lies behind the Sombor index. “The geodetic domination degree determines how often a vertex appears in minimal geodetic dominating sets and is thus a genuine indicator of the importance of a vertex in the problem of shortest-path coverage and the above-mentioned network-control problem with domination in mind. The spectral properties of the geodetic domination Sombor matrix of fundamental graphs are analyzed as well as exact expressions for corresponding geodetic domination Sombor matrix are obtained, such as complete graphs, star graphs and complete bipartite graphs. Theoretical bounds are also laid out, and the extremal/or boundary cases are identified as well. To show the applicability of the proposed invariant, a resilient IoT-based environmental monitoring network application framework for achieving resilient GDSO is formulated, where the sensing positions and communication connections with environmental importance in the network can be identified through participation in optimal monitoring arrangements. This provides a graph-theoretical foundation to assess sensor placement for sensor-dense urban forests and other distributed ecological monitoring environments. As a result, the proposed index can be used as a global structural determinant and be used for structural descriptor binding when using shortest-path coverage and/or domination based control in critical infrastructure supervision, network reliability, environmental IoT systems, construction of molecular descriptors and complex system resilience, particularly when a shortest-path solution is desired in addition to the domination control solution.

Keywords: Geodetic domination, Sombor index, Geodetic domination degree, Graph energy, Network resilience, Topological descriptor, Spectral graph theory

1. Introduction

Perhaps most importantly, graph invariants are a fundamental but important mathematical method for quantifying and elucidating the structural features of networks and molecular structures. The topological index has been extensively studied and gained an increased popularity as a descriptor and has been shown to correlate well with the physicochemical properties of chemical compounds [1-3] due to its low level of computerization. Sombor index was introduced for the first time in 2021 by Gutman [1]. Since then, the Sombor index has received a great deal of attention in the fields of extremal graph theory, the spectral graph theory, and chemical applications [4–6]. Meanwhile, domination theory has turned into a basic region of graph theoretical research, utilized in correspondence frameworks, capacity positioning and resource dispersion [7]. Geodetic domination was introduced in [1] and later studied by Chartrand et al. [4] and Nuenay and Jamil [8] where a dominating set is needed to cover all the vertices by shortest-path intervals

simultaneously. In turn, this idea creates a more rigid structure than the cover by a geodesic or the simple generality of "domination".

Both regions have reached substantial maturity but in combination, comparatively few studies have focused on these two. Most of the Sombor-type descriptors are based on the conventional vertex degrees, eccentricities or similar information about the local structure and are thus incapable of incorporating the salience (occurrence) of vertices in minimal geodetic dominating configurations [9]. On the other hand, the geometric attributes of Sombor-type descriptors are seldom studied in the studies focused on domination theory, which center mainly on cardinality-based quantities such as domination numbers and geodetic domination numbers. This gap makes up the motivation for the present study.

This is the first work introducing a new invariant with a vertex called the Geodetic Domination Sombor Index (GDSO) which is based on the geodetic domination degree, defined as the number of times a vertex appears in a minimal geodetic dominating set. The main contributions of this study are: (i) closed-form expressions for GDSO of complete graphs, of star graphs and of complete bipartite graphs; (ii) determination of their spectral characteristics as well as their exact energy value; and (iii) sharp bounds between GDSO and the extreme geodetic domination degree, and between GDSO and the number of minimal geodetic dominating sets.

2. Preliminaries

In this paper, all graphs considered are finite, simple, undirected, and connected. Let $G=(V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$, where $|V(G)|=n$ and $|E(G)|=m$. For a vertex $v \in V(G)$, the degree of v , denoted by $d(v)$, is the number of vertices adjacent to v . The minimum and maximum degrees of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The distance between two vertices u and v , denoted by $d(u,v)$, is the length of a shortest ($u-v$) path in G .

A set $D \subseteq V(G)$ is called a dominating set of G if every vertex in $V(G) \setminus D$ is adjacent to at least one vertex in D . The domination number of a graph G , denoted by $\gamma(G)$, is the minimum cardinality among all dominating sets of G . Domination theory constitutes one of the central research areas in graph theory because of its numerous applications in network monitoring, facility location problems, communication systems, and resource allocation.

For vertices u and v of a graph G , the interval $I[u,v]$ consists of all vertices lying on some shortest path between u and v . A subset $S \subseteq V(G)$ is called a geodetic set if every vertex of G belongs to the interval of some pair of vertices in S , that is, $V(G) = \bigcup_{u,v \in S} I[u,v]$. The minimum cardinality of a geodetic set is called the geodetic number of G and is denoted by $g(G)$. A set $S \subseteq V(G)$ is called a geodetic dominating set if it is both a geodetic set and a dominating set of G . The minimum cardinality among all geodetic dominating sets of G is called the geodetic domination number and is denoted by $\gamma_g(G)$. The concept of geodetic domination combines the shortest-path coverage property of geodetic sets with the neighborhood coverage property of dominating sets and therefore provides a stronger structural characterization of graphs.

A geodetic dominating set S is said to be minimal if no proper subset of S is itself a geodetic dominating set. The collection of all minimal geodetic dominating sets of G is denoted by $\mathcal{GD}(G)$. The total number of minimal geodetic dominating sets of G is represented by $T_{\gamma_g}(G) = |\mathcal{GD}(G)|$. Since the proposed graph invariant introduced in this work depends on the participation of vertices in minimal geodetic dominating sets, the following definition is useful.

Definition 2.1. Let G be a connected graph and let $\mathcal{GD}(G)$ denote the family of all minimal geodetic dominating sets of G . The geodetic domination degree of a vertex $v \in V(G)$, denoted by $d_{\gamma_g}(v)$, is defined as $d_{\gamma_g}(v) = |\{S \in \mathcal{GD}(G) : v \in S\}|$.

Thus, $d_{\gamma_g}(v)$ represents the number of minimal geodetic dominating sets containing the vertex v . The sequence $(d_{\gamma_g}(v_1), d_{\gamma_g}(v_2), \dots, d_{\gamma_g}(v_n))$ is called the geodetic domination degree sequence of the graph G .

The Sombor index is a recently introduced degree-based topological index. For a graph G , the Sombor index is defined as $SO(G) = \sum_{uv \in E(G)} \sqrt{d^2(u) + d^2(v)}$, where $d(u)$ and $d(v)$ denote the vertex degrees of the end vertices of the edge uv . The Sombor index has attracted considerable attention due to its strong discriminating ability and its applications in mathematical chemistry and network analysis.

Several variants of the Sombor index have subsequently been introduced by replacing the vertex degree with alternative graph-theoretic parameters. Motivated by this development and the increasing importance of domination-related graph descriptors, the present work introduces a Sombor-type index based on geodetic domination degrees.

For a graph G , let $\delta_{\gamma_g} = \min_{v \in V(G)} d_{\gamma_g}(v)$ and $\Delta_{\gamma_g} = \max_{v \in V(G)} d_{\gamma_g}(v)$ denote the minimum and maximum geodetic domination degrees of vertices of G , respectively.

The adjacency matrix of a graph G is denoted by $A(G) = [a_{ij}]_{n \times n}$, where $a_{ij} = \begin{cases} 1, & \text{if } v_i v_j \in E(G) \\ 0, & \text{otherwise} \end{cases}$. The eigenvalues of a real symmetric matrix M are denoted by $\lambda_1(M) \geq \lambda_2(M) \geq \dots \geq \lambda_n(M)$. The spectral radius of M is defined as $\rho(M) = \max_i |\lambda_i(M)|$. The energy of a graph-related matrix M is given by $E(M) = \sum_{i=1}^n |\lambda_i(M)|$. These concepts play an important role in the spectral study of graph invariants and will be employed in the subsequent sections for defining the geodetic domination Sombor matrix and its energy. The following elementary inequality will be used repeatedly in deriving bounds for the proposed index.

Lemma 2.2. For any non-negative real numbers a and b , $\sqrt{a^2 + b^2} \leq a + b$.

Proof. Since $(a+b)^2 = a^2 + b^2 + 2ab \geq a^2 + b^2$, taking square roots on both sides yields $\sqrt{a^2 + b^2} \leq a + b$. Hence the result follows.

Lemma 2.3. Cauchy Schwarz inequality: For any real sequences $a = (a_1, a_2, \dots, a_n)$ and $b = (b_1, b_2, \dots, b_n)$, then $(\sum_{i=1}^n a_i b_i)^2 \leq (\sum_{i=1}^n a_i^2)(\sum_{i=1}^n b_i^2)$.

The above definitions and results from the theoretical foundation is used for the Geodetic Domination Sombor Index and its associated spectral invariants introduced in the next section.

3. Geodetic Domination Sombor index of graphs

Definition 3.1 For a simple connected graph $G = (V, E)$ the geodetic domination Sombor index is denoted as $\text{GDSO}(G)$ and is defined as

$$\text{GDSO}(G) = \sum_{uv \in E(G)} |d_{\gamma_g}^2(u) + d_{\gamma_g}^2(v)|,$$

where $d_{\gamma_g}(u)$ is the number of minimal geodetic dominating sets containing u called the geodetic domination degree of the vertex u .

The geodetic domination degree $d_{\gamma_g}(u)$ counts how many alternative minimum-cost configurations include vertex u when the network must be monitored by a set that both dominates all nodes and covers all shortest paths between monitors. In power grid PMU placement, for instance, a substation with $d_{\gamma_g}(u) = 0$ never appears in any optimal configuration and can be deprioritized for backup investment, whereas a substation with high $d_{\gamma_g}(u)$ is structurally indispensable and requires hardened security. The GDSO aggregates these criticalities across transmission lines, yielding a single scalar that grid operators use to compare regional topologies for vulnerability assessment.

Example 3.2 Consider the graph G with vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ and edges as shown in figure 1.

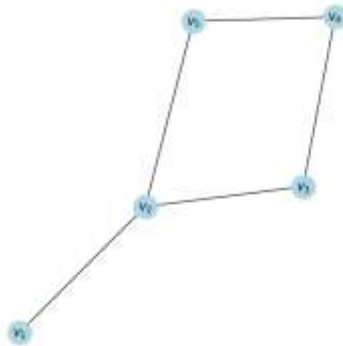


Figure 1

The minimal geodetic dominating sets are $S_1 = \{v_1, v_4\}$ and $S_2 = \{v_1, v_3, v_5\}$. Thus $d_{\gamma_g}(v_1) = 2$, $d_{\gamma_g}(v_2) = 0$, $d_{\gamma_g}(v_3) = 1$, $d_{\gamma_g}(v_4) = 1$, $d_{\gamma_g}(v_5) = 1$. Then the geodetic domination Sombor index is $\text{GDSO}(G) = 4 + 2\sqrt{2} \approx 6.83$.

Remark 3.3 The Sombor index as well as GDSO employ the Euclidean norm to add contributions from the edges; however, the quantities used in the calculations of these two indices are quite

different regarding the vertices involved in the calculation. Handout methods refer to the degree $d(v)$ that is commonly used, which is the number of neighbors of a vertex and being a local connectivity property, is used in the Sombor index. In contrast, GDSO uses the geodetic domination degree $d_{vg}(v)$ which is the number of minimal geodetic dominating sets where v belongs. Thereby this substitution makes the invariant to become a structural local adjacency measure, considering the shortest path geodesic influence and the domination coverage of the network. The Sombor index gives the number of neighbors a node has, while GDSO gives the frequency that the node has to be checked, under optimal network monitoring configurations.

3.1 Exact Values for fundamental Graph families

Theorem 3.4 The geodetic domination Sombor index for the star graph $K_{1, n-1}$ is $n-1$.

Proof: Let $G = K_{1, n-1}$ with center v and leaves $v_1, v_2, \dots, v_{n-1}$. The unique minimal geodetic dominating set is $S = \{v_1, v_2, \dots, v_{n-1}\}$. Since the leaves dominates the center and all leaf-to-leaf geodesics pass through the center. Thus $d_{vg}(v_i) = 1$ for all leaves and $d_{vg}(v) = 0$. Each of the $n-1$ edges contributes $\sqrt{0^2 + 1^2} = 1$ giving $\text{GDSO}(K_{1, n-1}) = n-1$.

Remark 3.5 Star graphs: used to describe an airline hub and spoke network in which there are one central station that is connected to $n-1$ stations. Together, the above theorem shows that the hub does not belong to the minimal geodetic dominating set, as all spoke-to-spoke shortest paths have to go through that hub, whereas all the spokes are enough to be a geodetic dominating set by themselves.

This observation gives an insight on one of the Weak Points: Though the hub is not required for coverage, if it fails, the network will not be operational. This means that it is important for airlines to have the ability to have multiple hub locations not because they must have hubs to cover, but because they have one possible location for failure of the hub. In addition, the diminishing resilience brought by more additional destinations in a single hub is reflected in the linear trend of GDSO with n , in line with the industry trend towards multi-hub network configurations.

Theorem 3.6 The geodetic domination Sombor index for the complete graph K_n is $\frac{n(n-1)}{\sqrt{2}}$.

Proof: In K_n the unique minimal geodetic dominating set is $V(K_n)$ itself, so $d_{vg}(v) = 1$ for all v . With $\binom{n}{2}$ edges each contributing $\sqrt{1^2 + 1^2} = \sqrt{2}$, we obtain $\text{GDSO}(K_n) = \frac{n(n-1)}{\sqrt{2}}$.

Theorem 3.7 The geodetic domination Sombor index for the complete bipartite graph $K_{r,s}$ is $\sqrt{2} rs$.

Proof: The minimal geodetic dominating sets are precisely the two partite sets X and Y . Thus $d_{vg}(v) = 1$ for all $v \in V(K_{r,s})$. With rs edges each contributing $\sqrt{2}$, we have $\text{GDSO}(G) = \sqrt{2} rs$.

Remark 3.8 Complete bipartite graph models coupled with infrastructures, such as power stations and water treatment plants, where each facility depends on all facilities in the other sector. The two minimal geodetic dominating sets represent two alternative complete coverage strategies: monitoring all power stations or monitoring all water plants. $\text{GDSO}(G) = \sqrt{2} rs$ reveals that balanced systems ($r \approx s$) maximize the index for fixed total resources $n = r + s$, indicating the richest set of redundant monitoring configurations. Conversely, highly asymmetric systems ($r \ll s$) yield a lower GDSO, signaling fewer alternative dominating configurations and higher systemic risk. This guides infrastructure policy: national protection plans should prioritize balanced diversification of interdependent sectors rather than concentrating investment in one sector, even if that sector appears more critical by traditional metrics.

Definition 3.9 If $d_{vg}(v) = k$, for all $v \in V(G)$, then G is a k -geodetic domination regular graph.

Proposition 3.10 If G is k -geodetic domination regular graph with m edges then $\text{GDSO}(G) = \sqrt{2} k m$.

3.2 The geodetic domination Sombor matrix and spectral invariants

Definition 3.11 The geodetic domination Sombor matrix is defined as

$$SO_{vg}(G) = \begin{cases} \sqrt{d_{vg}(u) + d_{vg}(v)} & \text{when } (u, v) \in E(G) \\ 0, \text{ otherwise} \end{cases}$$

Let the eigenvalues be $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$, with spectral radius $\mu_1(G)$. The geodetic domination Sombor energy is $\text{GDE}_{so}(G) = \sum_{i=1}^n |\mu_i(G)|$.

The spectral properties of $SO_{vg}(G)$ enable single-number robustness certification for infrastructure networks. Municipal planners can compare candidate road network designs by their GDE_{so} values: higher energy indicates more spectrally active domination configurations, suggesting greater resilience to simultaneous node and edge failures.

3.3 Spectral Energy results

Theorem 3.12 The geodetic domination Sombor energy of K_n is $2\sqrt{2} (n-1)$.

Proof. The geodetic domination Sombor matrix of K_n is

$$SO_{\gamma_g}(K_n) = \begin{pmatrix} 0 & \sqrt{2} & & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 0 & \dots & \sqrt{2} & \sqrt{2} \\ \dots & \dots & \dots & \dots & \dots \\ \sqrt{2} & \sqrt{2} & \dots & 0 & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & \dots & \sqrt{2} & 0 \end{pmatrix}$$

The matrix $SO_{\gamma_g}(K_n)$ has eigen values $(n-1)\sqrt{2}$ (multiplicity 1) and $-\sqrt{2}$ (multiplicity $n-1$).

Thus $GDE_{SO}(K_n) = \sqrt{2}\sqrt{2}|n-1| + |-\sqrt{2}| + \dots + |-\sqrt{2}| + (n-1)\sqrt{2} = 2\sqrt{2}(n-1)$.

Theorem 3.13 The geodetic domination Sombor energy of $K_{1,n-1}$ is $2\sqrt{n-1}$.

Proof.

$$\text{The geodetic domination Sombor matrix is } \begin{pmatrix} 0 & 0 & & 0 & 1 \\ 0 & 0 & \dots & 0 & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & & 1 & 0 \end{pmatrix}$$

The matrix has eigen values $\sqrt{n-1}$, $-\sqrt{n-1}$ and 0 (multiplicity $n-2$).

Thus $GDE_{SO}(K_{1,n-1}) = |\sqrt{n-1}| + |-\sqrt{n-1}| + 0 + \dots + 0 = 2\sqrt{n-1}$

Theorem 3.14 For $K_{r,s}$ with $r, s \geq 2$, the geodetic domination Sombor energy is $2\sqrt{2rs}$.

Proof.

$$\text{The geodetic domination sombor matrix is } \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \sqrt{2} & \dots & \sqrt{2} \\ 0 & 0 & 0 & \dots & 0 & \sqrt{2} & \dots & \sqrt{2} \\ 0 & 0 & 0 & \dots & 0 & \sqrt{2} & \dots & \sqrt{2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & \sqrt{2} & \dots & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & \sqrt{2} & \dots & \sqrt{2} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \sqrt{2} & \sqrt{2} & \sqrt{2} & \dots & \sqrt{2} & 0 & \dots & 0 \end{pmatrix}$$

The characteristic polynomial of the geodetic domination sombor matrix is $(-1)^{r+s}(\lambda^{r+s} - 2rs\lambda^{r+s-2})$. The eigen values are 0, $0, \dots, 0$ ($r+s-2$ times), $\pm\sqrt{2rs}$. Thus $GDE_{SO}(K_{r,s}) = 2\sqrt{2rs}$.

Theorem 3.15 For the characteristic polynomial $c_0\mu^n + c_1\mu^{n-1} + c_2\mu^{n-2} + \dots + c_n$ of $SO_{\gamma_g}(G)$,

(i) $c_0 = 1$

(ii) $c_1 = \sum_{i=1}^n \mu_i = \text{tr}(SO_{\gamma_g}) = 0$

(iii) $c_2 = \sum_{1 \leq i \leq j \leq n} S_{ij} = -\sum_{1 \leq i \leq j \leq n} S_{ij}^2 = -\sum_{uv \in E(G)} (d_{\gamma_g}^2(u) + d_{\gamma_g}^2(v))$

(iv) If $Q_{GD}(G) = \sum_{uv \in E(G)} (d_{\gamma_g}^2(u) + d_{\gamma_g}^2(v))$ then

$\text{tr}(SO_{\gamma_g}(G))^2 = \sum_{i=1}^n \mu_i^2 = 2 \sum_{uv \in E(G)} d_{\gamma_g}^2(u) + d_{\gamma_g}^2(v) = 2Q_{GD}$.

3.4 General Bounds

Theorem 3.16 For a graph G with m edges and $T_{\gamma_g}(G)$ is the total number of minimal geodetic dominating sets, $GDSO(G) \leq \sqrt{2mT_{\gamma_g}(G)}$.

Proof. Since $d_{\gamma_g}(u) \leq T_{\gamma_g}(G)$ for all u , each edge contributes at most $\sqrt{T_{\gamma_g}^2(G) + T_{\gamma_g}^2(G)} = \sqrt{2}mT_{\gamma_g}(G)$. Summing over m edges yields the bound,

$GDSO(G) \leq \sqrt{2mT_{\gamma_g}(G)}$.

Corollary 3.17 If $G = K_n$, equality holds.

Theorem 3.18 Let G be a connected graph with n vertices and m edges. If δ_{γ_g} and Δ_{γ_g} denote the minimum and maximum geodetic domination degree of vertices of G then $\sqrt{2}m\delta_{\gamma_g} \leq GDSO(G) \leq \sqrt{2}m\Delta_{\gamma_g}$.

Proof. For any edge $uv \in E(G)$, the monotonicity of geodetic domination degrees yields

$$\sqrt{\delta_{\gamma_g}^2 + \delta_{\gamma_g}^2} \leq \sqrt{d_{\gamma_g}^2(u) + d_{\gamma_g}^2(v)} \leq \sqrt{\Delta_{\gamma_g}^2 + \Delta_{\gamma_g}^2},$$

which simplifies $\sqrt{2}\delta_{\gamma_g} \leq \sqrt{d_{\gamma_g}^2(u) + d_{\gamma_g}^2(v)} \leq \sqrt{2}\Delta_{\gamma_g}$. Summing over all m edges establishes the desired inequality.

Theorem 3.19 For any connected graph G , $GDSO(G) \leq \sum_{v \in V(G)} d(v)d_{\gamma_g}(v)$, where $d(v)$ is the ordinary degree of vertex v in G .

Proof. By Lemma 2.2,

$GDSO(G) = \sum_{uv \in E(G)} \sqrt{d_{\gamma_g}^2(u) + d_{\gamma_g}^2(v)} \leq \sum_{uv \in E(G)} d_{\gamma_g}(u) + d_{\gamma_g}(v)$. Each term $d_{\gamma_g}(v)$ appears in the sum exactly $d(v)$ times, once for each edge incident with v . Therefore, $\sum_{uv \in E(G)} (d_{\gamma_g}(u) + d_{\gamma_g}(v)) = \sum_{v \in V(G)} d(v) d_{\gamma_g}(v)$. Hence $GDSO(G) \leq \sum_{v \in V(G)} d(v) d_{\gamma_g}(v)$.

Proposition 3.20 If μ_1 is the spectral radius of $SO_{\gamma_g}(G)$ then $\mu_1 \leq \sqrt{2Q_{GD}(G)}$.

Theorem 3.21 (McClelland type bound) For any graph G , $GDE_{so}(G) \leq \sqrt{2nQ_{GD}}$.

Proof. Applying Cauchy Schwarz inequality to the sequence $(|\mu_1|, |\mu_2|, \dots, |\mu_n|)$ and $(1, 1, 1, \dots, 1)$,

$$(\sum_{i=1}^n |\mu_i|)^2 \leq (\sum_{i=1}^n \mu_i^2) (\sum_{i=1}^n 1^2) = n \sum_{i=1}^n \mu_i^2.$$

$$\text{Now, } GDE_{so}(G) = \sum_{i=1}^n |\mu_i| \leq \sqrt{n \sum_{i=1}^n \mu_i^2} = \sqrt{2nQ_{GD}(G)}.$$

Theorem 3.22 For any graph G , $GDE_{so}(G) \geq \sqrt{2Q_{GD}(G)}$.

Proof. $(GDE_{so}(G))^2 = (\sum_{i=1}^n |\mu_i|)^2 = \sum_{i=1}^n |\mu_i|^2 + 2(\sum_{i < j} |\mu_i| |\mu_j|)$. Since $|\mu_i| |\mu_j| \geq 0$, $(GDE_{so}(G))^2 \geq \sum_{i=1}^n |\mu_i|^2 = 2Q_{GD}$. Hence, $GDE_{so}(G) \geq \sqrt{2Q_{GD}(G)}$.

Theorem 3.23 For any graph G , $GDE_{so}(G) \leq 2^{3/4} \sqrt{n \Delta_{\gamma_g} GDSO(G)}$.

Proof. By theorem 3.21,

$$GDE_{so}(G) \leq \sqrt{2nQ_{GD}} \text{ where } Q_{GD} = \sum_{uv \in E(G)} (d_{\gamma_g}^2(u) + d_{\gamma_g}^2(v)).$$

For any $(u, v) \in E(G)$, $(d_{\gamma_g}^2(u) + d_{\gamma_g}^2(v)) \leq \Delta_{\gamma_g}(d_{\gamma_g}(u) + d_{\gamma_g}(v))$.

$$\text{Also using the inequality } \sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2}, \frac{d_{\gamma_g}(u) + d_{\gamma_g}(v)}{2} \leq \sqrt{\frac{(d_{\gamma_g}^2(u) + d_{\gamma_g}^2(v))}{2}}$$

$$\text{That is } (d_{\gamma_g}(u) + d_{\gamma_g}(v)) \leq \sqrt{2} \sqrt{(d_{\gamma_g}^2(u) + d_{\gamma_g}^2(v))}$$

Combining these two inequalities we obtain,

$$(d_{\gamma_g}^2(u) + d_{\gamma_g}^2(v)) \leq \Delta_{\gamma_g} \sqrt{2} \sqrt{(d_{\gamma_g}^2(u) + d_{\gamma_g}^2(v))}.$$

Taking the sum over all m edges,

$$\sum_{(u,v) \in E(G)} (d_{\gamma_g}^2(u) + d_{\gamma_g}^2(v)) \leq \sqrt{2} \Delta_{\gamma_g} \sum_{(u,v) \in E(G)} \sqrt{(d_{\gamma_g}^2(u) + d_{\gamma_g}^2(v))}$$

That is, $Q_{GD} \leq \sqrt{2} \Delta_{\gamma_g} GDSO(G)$

Applying theorem 3.21,

$$GDE_{so}(G) \leq \sqrt{2nQ_{GD}} \leq \sqrt{2n\sqrt{2}\Delta_{\gamma_g}GDSO(G)} = 2^{3/4} \sqrt{n\Delta_{\gamma_g}GDSO(G)}.$$

3.5 Application of GDSO in Resilient IoT-Based Environmental Monitoring Networks

The proposed Geodetic Domination Sombor Index can also be naturally applied in the design and resilience assessment of IoT based environmental monitoring networks. Imagine a section of an urban forest, Miyawaki Micro-forest, wetland or other ecological feature that has multiple monitoring sites. It can be modeled by communicating graph of GTset, which is indicated by a graph $G=(V, E)$, where vertices represent possible sensing locations and edges represent possible wireless communication links.

A dominating set is a collection of sensing locations that can provide neighborhood-based coverage of the environment; the geodetic condition also includes the minimum communication channels that exist among the sensing locations. Then, a minimal geodetic dominating set is the smallest set of sensors such that each sensor can monitor its neighbors and the shortest path structure of the sensor set can cover each sensor.

In this setting, there is an operational interpretation for geodetic domination degree $d_{vg}(v)$. It counts the number of minimum sensor-deployment configurations where the location v is in. Large magnitude of $d_{vg}(v)$ thus reveals that there are few structural alternatives available at the corresponding sensing location and the location is steadily being picked up by optimal deployment configurations. These sites should always be identified as "critical sensing nodes" and, as such, could be accorded privileges for redundant sensing equipment, enhanced communication, more energy resources, and preventive maintenance. On the contrary, the small geodetic domination degree shows a high chance of its being replaced with alternative monitoring points.

The GDSO consequently provides a network-level measure of the structural criticality associated with ~~environmental~~ sensing and communication links. For an edge uv , the contribution $\sqrt{(d_{vg}^2(u) + d_{vg}^2(v))}$ becomes larger when both endpoints participate frequently in minimal geodetic dominating configurations. Such links may therefore represent communication corridors whose failure could substantially affect the availability of optimal monitoring configurations. The GDSO can consequently be employed to compare alternative sensor-network topologies and to identify locations and communication links requiring enhanced redundancy.

When deployment of environmental-monitoring sensors is constrained due to installation costs, battery size, distance between the sensors, and physical access, this interpretation becomes crucial in the context of distributed environmental-monitoring systems. GDSO differs from traditional degree-based descriptors, which do not take into account participation in globally valid minimum monitoring configurations, since it takes customer participation into account as well." It thus offers an interpretable graph-theoretical test to distinguish between highly connected locations and monitoring locations that are truly and structurally important. Furthermore, geodetic domination Sombor matrix and its energy would be able to give global description of structural organization of critical sensing locations. Therefore, GDSO establishes a mathematical link between the geodetic domination theory and Sombor-type graph invariants and resilient IoT sensor-network design.

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