

A Nonlinear Functional Analysis Framework for Stability Optimization of Dynamic Control Systems

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Abstract

A nonlinear functional approach is proposed for investigating the existence, uniqueness, and optimal stability of a large class of nonlinear dynamical control systems modeled as an operator equation in Banach space. The state trajectory of the system is modeled as a mild solution to an abstract semilinear evolution equation. The solution operator of the closed loop system is considered as composed of two operators, one being a contracting operator and the other being compact as defined by Krasnoselskii. The results are based on the Banach contraction principle, which is applied for proving the existence and uniqueness of the mild solution, whereas the existence is proved using a fixed-point technique similar to Krasnoselskii. An explicit exponential decay bound for the closed-loop trajectory is then obtained by combining the weighted-norm stability theorem with the fixed-point formulation, and the feedback gain operator is optimised over an admissible compact convex set in order to maximize the guaranteed decay rate, which is shown to exist by a Weierstrass-type compactness argument. The structure of the theorems is explicitly checked in the case of the two-link rigid robotic manipulator trajectory-tracking problem, and the obtained exponential stability and near-optimal gain selection is numerically demonstrated against a PID controller. These findings contribute to the current stability investigation of nonlinear and fractional order dynamical systems using fixed point theory by developing a realistic and application-oriented methodology for stability optimization.

Keywords: Nonlinear functional analysis, fixed point theorems, krasnoselskii's theorem, stability optimization, dynamic control systems, Lyapunov functional, robotic manipulator control.

1. Introduction

One of the most basic problems in mathematical control theory is the study of the stabilization of a nonlinear dynamic control system. Nevertheless, the classical way (Lyapunov) involves building an explicit candidate function whose derivative along trajectories is sign-definite; this is often impossible for high-order plants, plants with uncertain dynamics, or strongly coupled plants, and for these, the resulting stability margin is hard to quantify in a form suitable for optimization across a family of admissible controllers (Ahmad et al., 2021). An

alternative, operator-theoretic approach reformulates the state trajectory as the fixed point of an integral (mild-solution) operator acting on a Banach space of admissible trajectories, and then stability, existence and uniqueness are directly obtained from properties of contraction and compactness of that operator without the need for an explicit Lyapunov candidate.

The stability results obtained using these types of fixed-point methods have led to a significant and constantly increasing volume of results in nonlinear and fractional-order dynamic equations. The asymptotic and Ulam-type stability of nonlinear fractional dynamic equations on time scales was proved by using Krasnoselskii-type decompositions (Belaid et al., 2019). They have also been used for N-tupled fractional evolution systems (Yang et al., 2017). Another application is in neutral delay equations where there are boundedness and asymptotic stability conditions (Nabil, 2019). Recently, these decompositions were used to deal with the controllability of fractional nonlinear dynamical systems under given controls using Ψ -Caputo derivatives (Karthiga et al., 2026). For Caputo fractional differential equations, existence and stability theorems have been derived (Seemab & Rehman, 2018). The same type of structures is seen for coupled systems of non-integer order (Ali et al., 2017). This approach can be used for more classes of equations that yield Hyers-Ulam stability (Kumama et al., 2017). These results have been obtained as their analogue for the discrete case and the difference-equation case, respectively (Butt et al., 2020; Ardjouni et al., 2018). Further extensions of Krasnoselskii type of the Caputo-Fabrizio operators, of hybrid differential problems and of neutral equations with iterated state-dependent delays are presented (Eiman et al., 2020; Mahendran & Gomathy, 2025; Agarwal et al., 2025). In parallel with this fractional-order literature, abstract nonlinear-systems stability results independent of a fractional-order system have been developed to study the weakly hard real-time nonlinear sampled-data control (Hertneck et al., 2020). The finite and fixed-time stabilization of abstract nonlinear systems with guarantees of well-posedness also have been considered (Mishra & Soy, 2025). The following review articles give more general summaries of the underlying fixed-point machinery (Sousa et al., 2023; Ali et al., 2025; Zhang & Rodriguez, 2023). A fixed-point-based controllability analysis of nonlinear second-order control systems, more akin to the traditional use of these tools in control theory is presented in AL-Jawari & Ibrahim, (2020) and Yue et al., (2024).

Less developed in this literature is the extension of fixed-point stability arguments from a pure existence-and-stability statement to an explicit optimization of the achievable stability margin for an admissible family of feedback operators, along with a complete verification of the resulting hypotheses on a concrete engineering plant. This paper aims at filling that gap. It has four contributions.

- (i) A single contraction estimate that guarantees existence, uniqueness, and exponential stability of the mild solution for a nonlinear feedback-controlled evolution equation is sufficient under a common set of operator theoretic hypotheses (H1) - (H4);
- (ii) an existence theorem (Theorem 4.2), derived by using a Krasnoselskii decomposition of the closed-loop operator into a compact and a contraction component, under growth conditions weaker than those of global Lipschitz condition which are needed for uniqueness;
- (iii) a stability-optimization theorem (Theorem 4.4) that poses the selection of the feedback gain operator as a maximization of the guaranteed exponential decay rate over a compact convex admissible set, and proves that an optimal gain exists by a Weierstrass-type compactness argument; and
- (iv) A full engineering implementation of the framework for a 2-link rigid robotic manipulator trajectory-tracking problem, where each structural hypothesis of Theorems 4.1–4.4 is explicitly verified from the manipulator's Euler-Lagrange equation dynamics and the forecast stability margin is compared numerically with a baseline proportional-derivative design.

The rest of the paper is organized as follows. The functional analytic preliminaries are gathered in Section 2. In Section 3, the nonlinear dynamic control system is abstracted to an operator equation. The four main theorems are proved in section 4. The development of the robotic-manipulator application is presented in Section 5, while a typical numerical study is presented in Section 6. Sections 7 discuss the results and section 8 conclusion.

2. Preliminaries

Throughout, X denotes a real Banach space, $\mathcal{L}(X)$ denotes the space of bounded linear operators on X , and, for $T > 0$, $C([0, T], X)$ denotes the Banach space of continuous functions equipped with the supremum norm. When the interval is unbounded, $C_b([0, \infty), X)$ denotes the corresponding space of bounded continuous functions endowed with the same norm. These spaces are used for describing the operators and solution trajectories used in the subsequent analysis in a functional setting.

Definition 2.1. An operator $F: X \rightarrow X$ is called a contraction on X if there exists a constant $q \in [0, 1)$ such that

$$|F(x) - F(y)| \leq q|x - y|, \quad \forall x, y \in X$$

The contraction condition imposes a uniform reduction in the distance between any two points after application of the operator. This property is very useful in proving unique convergence of iterative methods.

Definition 2.2. An operator $F: X \rightarrow X$ is compact if it is continuous and maps bounded subsets of X into relatively compact subsets of X . An operator is completely continuous if it is continuous and maps bounded sets into precompact sets. Compactness provides another type of regularity which is helpful when contraction properties do not exist for all of the operator. Specifically, it enables bounded sequences generated by an operator to have converging sub-sequences, a feature crucial to many fixed-point arguments.

Definition 2.3 (Mild solution). Let A generate a C_0 -semigroup on X , and let the forcing and nonlinear terms be such that the associated integral expression is well defined. A function x is said to be a mild solution of the variation of parameters identity of the corresponding evolution system if it is a solution at all admissible times. This formulation does not need to be used if the solution is required to have the strong differentiability properties which are not necessarily automatically possessed by the generator A in the unbounded case. For infinite-dimensional systems the formulating of the differential equation in the mild-solution form is therefore natural and suitable, with the system operator being naturally interpreted as the semigroup it generates. In addition, it offers a handy solution operator definition in the stability analysis.

Definition 2.4. The trivial solution of a system with mild-solution operator is said to be globally exponentially stable if there exist constants $M \geq 1$ and $\omega > 0$ such that every mild solution satisfies

$$|x(t)| \leq M e^{-\omega t} |x(0)|, \quad t \geq 0$$

Thus, global exponential stability requires the deviation from the equilibrium to decay at an exponential rate independently of the initial condition. This condition is stronger than mere asymptotic convergence because it specifies an explicit rate of decay.

This is stronger than just asymptotic convergence in that it sets a rate of convergence. The main theorems of Section 4 are based on some standard results from the literature on fixed-point theory and integral inequalities. The results are recalled below for the purpose of making the following arguments self-contained. Their use also helps to isolate the existence part of the analysis from extra estimates needed to determine stability.

Lemma 2.1 (Banach Contraction Principle). If K is a nonempty closed subset of a Banach space X and $F: K \rightarrow K$ is a contraction with constant $q < 1$, then F has a unique fixed point in K . Moreover, the Picard iterates converge to this fixed point geometrically for every initial point in K .

Furthermore, the Picard iterations converge geometrically to this fixed point for each initial point in K . It is important for this principle in the current context because it gives existence, uniqueness, and convergence of the iterative sequence, at the same time. It follows then that the fixed-point characterization is obtained after the related solution operator has been established as a contraction.

Lemma 2.2 (Krasnoselskii Fixed Point Theorem). Let K be a nonempty, closed, bounded, and convex subset of a Banach space X . Suppose that $F, G: K \rightarrow X$ satisfy: (a) $F(x) + G(y) \in K$ for all $x, y \in K$; (b) F is compact and continuous on K ; and (c) G is a contraction on K . Then there exists $x^* \in K$ such that

$$F(x^*) + G(x^*) = x^*$$

In such cases, it is possible to break down the operator into a small continuous component and a contractive component, so that the fixed-point argument can be carried out with weaker assumptions.

Lemma 2.3 (Grönwall–Bellman Inequality). Let u and v be continuous nonnegative functions, let c be a nonnegative constant, and suppose that

$$u(t) \leq c + \int_0^t v(s)u(s) ds, \quad t \geq 0$$

Then

$$u(t) \leq c \exp\left(\int_0^t v(s) ds\right), \quad t \geq 0$$

The inequality converts an integral estimate into an explicit bound on the unknown function. This property is frequently applied in stability proofs for propagation of local estimates over the entire time interval and for the derivation of exponential bounds for mild solutions. It consequently forms an important link between the fixed-point formulation of the system and the stability estimates established in Section 4.

Remark 2.1. None of the Definitions 2.1–2.4 or Lemmas 2.1–2.3 needs X to be finite-dimensional. If (1) represents a distributed-parameter (e.g., partial differential equation) control system, then X is typically a function space

$L^2(D)$ or a Sobolev space $H^k(D)$ on a spatial domain D , and the state trajectory is understood as a Bochner-integrable X -valued map; the variation-of-parameters identity of Definition 2.3, the contraction estimates of Section 4, and the resulting stability-optimization Theorem 4.4 carry over verbatim, with (H5) verified instead through compactness of the resolvent of A or compact embedding of $H^k(D)$ into $L^2(D)$, transfer word-for-word, in place of finite-dimensionality as in the lumped-parameter application of Section 5. It is in this sense that the framework of Section 4 is a true infinite dimensional nonlinear functional analysis result; in Section 5 it has been specialised to a finite dimensional mechanical plant just for the sake of a fully verifiable, numerical illustration.

3. Problem formulation

Consider a nonlinear dynamic control system posed on a real Banach state space X ,

$$\dot{x}(t) = Ax(t) + f(t, x(t)) + Bu(t), \quad x(0) = x_0, \quad t \in [0, T]$$

With output $y(t) = Cx(t)$, where $A: \mathcal{D}(A) \subseteq X \rightarrow X$ generates a C_0 -semigroup $\{S(t)\}_{t \geq 0}$, $f: [0, T] \times X \rightarrow X$ is a (possibly unbounded-growth) nonlinear perturbation representing unmodeled dynamics or structural uncertainty, $B \in \mathcal{L}(U, X)$ is the bounded control-input operator on the input space U , and $C \in \mathcal{L}(X, Y)$ is the output operator. The control input is generated by feedback,

$$u(t) = K(x(t)), \quad K \in \mathcal{K}$$

Where $\mathcal{K}$ is an admissible family of (possibly nonlinear, uniformly Lipschitz) feedback operators from X into U . Substituting (2) into (1) and applying the variation-of-parameters formula of Definition 2.3, a mild solution of the closed-loop system is a fixed point of the operator $\mathcal{T}_K: C([0, T]; X) \rightarrow C([0, T]; X)$ defined by

$$(\mathcal{T}_K x)(t) = S(t)x_0 + \int_0^t S(t-s)[f(s, x(s)) + BK(x(s))]ds$$

The stability-optimization problem considered in Section 4 is: Over the admissible set $\mathcal{K}$, find K so that the closed-loop trajectory of $\mathcal{K}$, select K so as to maximize the guaranteed exponential decay rate of the resulting closed-loop trajectory, subject to $\mathcal{T}_K$ admitting a (unique) fixed point. The following standing hypotheses are used throughout.

(H1) A generates a C_0 -semigroup with $\|S(t)\| \leq Me^{-\omega t}$ for some $M \geq 1$, $\omega > 0$

(H2) $f(t, 0) = 0$ and f is uniformly Lipschitz in its second argument: $\|f(t, x) - f(t, y)\| \leq L_f \|x - y\|$ for all $t \in [0, T]$, $x, y \in X$.

(H3) $B \in \mathcal{L}(U, X)$ with $\|B\| \leq M_B$.

(H4) $\mathcal{K}$ is a nonempty, closed, bounded, convex subset of a normed space of admissible feedback operators, and every $K \in \mathcal{K}$ satisfies $K(0) = 0$ and is Lipschitz with constant $L_K(K)$, with $L_K: \mathcal{K} \rightarrow [0, \infty)$ bounded and upper semicontinuous.

(H5) $f = f_c + f_{ct}$, where f_{ct} is L_{ct} -Lipschitz in x uniformly in t , and the operator $(\Phi x)(t) = \int_0^t S(t-s)f_c(s, x(s))ds$ is compact from bounded subsets of $C([0, T]; X)$ into $C([0, T]; X)$; this holds automatically whenever X is finite-dimensional and f_c is continuous and bounded on bounded sets, or whenever $\{S(t)\}$ is a compact semigroup for $t > 0$.

3.1 Well-Posedness of the admissible feedback family

Hypothesis (H4) plays two distinct roles in Section 4: convexity and closedness of $\mathcal{K}$ are used only in Theorem 4.4, where they guarantee that the supremum in (9) is attained, while boundedness of L_K on $\mathcal{K}$ already enters Theorems 4.1–4.3 through the single scalar $q(K)$ of (4). It is therefore useful to record explicitly the two standard ways $\mathcal{K}$ arises in applications. In the linear-gain case, $\mathcal{K} = \{K = -G: G \in \mathcal{G}\}$ for a compact convex set $\mathcal{G} \subseteq \mathcal{L}(X, U)$ of admissible gain matrices or operators (bounded by an actuator or energy budget as in (12) below), with $L_K(K) = \|G\|$, which is convex and continuous, hence upper semicontinuous, on $\mathcal{G}$. In the nonlinear-gain case, $\mathcal{K}$ may instead be parametrized by a compact convex set of tuning parameters $\theta \in \Theta \subseteq \mathbb{R}^m$ entering $K(x) = K(x; \theta)$ continuously, with $L_K(K(\cdot; \theta))$ a Lipschitz seminorm in x that depends continuously on θ ; this covers, in particular, the swarm- and colony-optimized nonlinear PID and sliding-mode families used to tune robotic-manipulator controllers, so that the numerical optimization performed in those references can be regarded as a search over an admissible set $\mathcal{K}$ of exactly the form required by (H4) (Lobato-Larios et al., 2025; Azizi et al., 2023; Karimov et al., 2026).

4. Main Results

4.1 Existence and uniqueness

Theorem 4.1. Assume (H1)–(H4). If $K \in \mathcal{K}$ is such that

$$q(K) := \frac{ML_f + MM_B L_K(K)}{\omega} < 1$$

Then for every $x_0 \in X$ the closed-loop system (1) – (2) has a unique mild solution $x \in C([0, T]; X)$; the corresponding statement holds on $[0, \infty)$ with $x \in C_b([0, \infty); X)$.

Proof. By (H1)–(H3), for $x, y \in C([0, T]; X)$ and $t \in [0, T]$,

$$\|(\mathcal{T}_K x)(t) - (\mathcal{T}_K y)(t)\| \leq \int_0^t M e^{-\omega(t-s)} [L_f + M_B L_K(K)] \|x(s) - y(s)\| ds \leq \|x - y\|_C \cdot \frac{M[L_f + M_B L_K(K)]}{\omega}$$

By (4) the coefficient on the right of (5) is $q(K) < 1$, so $\|\mathcal{T}_K x - \mathcal{T}_K y\|_C \leq q(K) \|x - y\|_C$, i.e. $\mathcal{T}_K$ is a contraction on $C([0, T]; X)$. By Lemma 2.1, $\mathcal{T}_K$ has a unique fixed point, which by Definition 2.3 is the unique mild solution. The extension to $[0, \infty)$ follows by repeating the argument on the exponentially weighted space $\left\{x \in C([0, \infty); X) : \sup_t e^{\mu t} \|x(t)\| < \infty\right\}$ for any $0 < \mu < \omega - ML_f - MM_B L_K(K)$, on which $\mathcal{T}_K$ remains a contraction by the same estimate.

4.2 Existence via krasnoselskii decomposition

Theorem 4.2. Assume (H1), (H3), (H4), (H5), and suppose there is $r > 0$ with

$$r \geq \frac{M\|x_0\| + MT \cdot \sup_{[0, T] \times B_r} \|f_c\|}{1 - M[L_{ct} + M_B L_K(K)]/\omega}, \quad \frac{M[L_{ct} + M_B L_K(K)]}{\omega} < 1$$

Where B_r is the closed ball of radius r in $C([0, T]; X)$. Then the closed-loop system (1) – (2) admits at least one mild solution in B_r .

Proof. Decompose $\mathcal{T}_K = \Phi + \Psi$ with $(\Phi x)(t) = \int_0^t S(t-s) f_c(s, x(s)) ds$ and $(\Psi x)(t) = S(t)x_0 + \int_0^t S(t-s) [f_{ct}(s, x(s)) + BK(x(s))] ds$. By (H5), Φ is compact and continuous on B_r . By the estimate used in the proof of Theorem 4.1 applied to f_{ct} in place of f , Ψ is a contraction on B_r with constant $M[L_{ct} + M_B L_K(K)]/\omega < 1$ by (6). For $x, y \in B_r$, the norm bound defining r in (6) gives $\|\Phi x + \Psi y\|_C \leq r$, so $\Phi x + \Psi y \in B_r$ for all $x, y \in B_r$. All three hypotheses of Lemma 2.2 hold on the nonempty, closed, bounded, convex set B_r , so $\Phi + \Psi = \mathcal{T}_K$ has a fixed point in B_r , i.e. a mild solution.

4.3 Exponential Stability

Theorem 4.3. Assume the hypotheses of Theorem 4.1 with $q(K) < 1$ for a given $K \in \mathcal{K}$, and set

$$\lambda(K) := \omega - ML_f - MM_B L_K(K) \equiv \omega(1 - q(K)) > 0$$

Then the unique mild solution of (1) – (2) satisfies $\|x(t)\| \leq M\|x_0\| e^{-\lambda(K)t}$ for all $t \geq 0$; equivalently, the trivial solution is globally exponentially stable with guaranteed decay rate $\lambda(K)$.

Proof. Fix $0 < \mu < \lambda(K)$ and equip $C_b([0, \infty); X)$ with the equivalent weighted norm $\|x\|_\mu = \sup_{t \geq 0} e^{\mu t} \|x(t)\|$. For x, y in this space,

$$\begin{aligned} e^{\mu t} \|(\mathcal{T}_K x)(t) - (\mathcal{T}_K y)(t)\| &\leq M[L_f + M_B L_K(K)] \int_0^t e^{-(\omega-\mu)(t-s)} e^{\mu s} \|x(s) - y(s)\| ds \\ &\leq \frac{M[L_f + M_B L_K(K)]}{\omega - \mu} \|x - y\|_\mu \end{aligned}$$

And since $\mu < \lambda(K) = \omega - ML_f - MM_B L_K(K)$ the coefficient in (8) is strictly less than 1, so $\mathcal{T}_K$ is a contraction on $(C_b([0, \infty); X), \|\cdot\|_\mu)$. By Lemma 2.1 its unique fixed point x satisfies $\|x\|_\mu \leq \|\mathcal{T}_K 0\|_\mu / (1 - q_\mu)$, and a direct evaluation of $\mathcal{T}_K 0$ using (H1) gives $\|\mathcal{T}_K 0\|_\mu \leq M\|x_0\|$. Letting $\mu \uparrow \lambda(K)$ yields $\|x(t)\| \leq M\|x_0\| e^{-\lambda(K)t}$ for all $t \geq 0$.

4.4 Existence of a stability-optimal gain

Theorem 4.4. Assume (H1)–(H4) and let $\mathcal{K}_+ := \{K \in \mathcal{K} : q(K) < 1\} \neq \emptyset$. If $\mathcal{K}$ is compact in the topology in which L_K is upper semicontinuous (in particular, if $\mathcal{K}$ is a closed bounded convex subset of a finite-dimensional gain

space, e.g. $\mathcal{K} = \{K = -[K_p, K_d] \text{ gain pairs bounded by an actuator/energy budget}\}$, then the stability-margin functional $\lambda: \mathcal{K} \rightarrow \mathbb{R}$ of (7) attains its supremum: there exists $K^* \in \mathcal{K}_+$ with

$$\lambda(K^*) = \lambda^* := \sup_{K \in \mathcal{K}} \lambda(K) = \omega - ML_f - MM_B \cdot \inf_{K \in \mathcal{K}} L_K(K)$$

and the closed-loop system under $u = K^*(x)$ achieves the largest exponential decay rate guaranteed by Theorem 4.3 among all controllers in $\mathcal{K}$.

Proof. λ is an affine, continuous decreasing function of L_K by (7), and L_K is bounded and upper semicontinuous on $\mathcal{K}$ by (H4); hence λ is lower semicontinuous and bounded above on $\mathcal{K}$. Since $\mathcal{K}$ is compact, the classical Weierstrass extreme-value theorem for lower semicontinuous functions on compact sets guarantees that λ attains its supremum at some $K^* \in \mathcal{K}$. Because $\mathcal{K}_+$ is nonempty by hypothesis, $\lambda^* \geq \lambda(K) > 0$ for some $K \in \mathcal{K}_+$, so $\lambda^* > 0$ and $K^* \in \mathcal{K}_+$.

Remark 4.1. For linear-in-gain families $K(x) = -Gx$ with G ranging over a compact convex matrix set, $\inf_G L_K(G) = \inf_G \|G\|$ reduces the optimization in (9) to a norm-minimization problem over a compact convex feasible set intersected with the stabilizing region $q(K) < I$. This is structurally analogous to the Riccati- and LMI-based optimal-gain constructions used in engineering practice for nonlinear robotic manipulators (Rigatos & Abbaszadeh, 2023). It also connects to Lyapunov-objective particle-swarm and artificial-bee-colony gain tuning (Lobato-Larios et al., 2025; Azizi et al., 2023). Those numerically-driven procedures can be read as constructive algorithms for locating the K^* whose existence Theorem 4.4 guarantees analytically.

4.5 Equivalence with a lyapunov functional

The weighted-norm contraction argument of Theorem 4.3 can be restated, without reference to fixed points, as a construction of an explicit Lyapunov functional. This demonstrates that the stability-optimization statement of Theorem 4.4 is not an artifact of the operator formulation but a genuine control-theoretic stability margin. Define, for the mild solution x of Theorem 4.1,

$$V(t) := \|x(t)\| + [ML_f + MM_B L_K(K)] \int_0^t e^{-\omega(t-s)} \|x(s)\| ds$$

Differentiating the variation-of-parameters representation (3) along the closed-loop trajectory and using (H1)–(H3) as in the proof of Theorem 4.1 gives, for almost every t , $D^+V(t) \leq -\lambda(K)V(t)$ with $\lambda(K)$ as in (7), where D^+ denotes the upper right Dini derivative. Since $V(t) \geq \|x(t)\|$ and $V(0) \leq M\|x_0\|(1 + [ML_f + MM_B L_K(K)]/\omega)$, integrating this differential inequality and applying Lemma 2.3 recovers the exponential bound of Theorem 4.3 with an explicit, if generally conservative, multiplicative constant. Consequently, $\lambda(K)$ of (7) is simultaneously the fixed-point contraction rate of $\mathcal{T}_K$ and a valid Lyapunov exponential-stability rate for V in (13). Maximizing $\lambda(K)$ over $\mathcal{K}$ in Theorem 4.4 is therefore equivalent to maximizing the guaranteed Lyapunov decay rate over the same admissible controller family, without requiring V to be guessed in advance: equation (13) is generated automatically from the data $(M, \omega, L_f, M_B, L_K)$ already required to verify (H1)–(H4).

5. Engineering application: trajectory-tracking stabilization of a two-link robotic manipulator

5.1 Manipulator dynamics

One example of a two-link rigid robotic manipulator with the vector of the joint angles as $q = (q_1, q_2)^T \in \mathbb{R}^2$, governed by the standard Euler–Lagrange model

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau + d(t)$$

where $M(q)$ is the symmetric positive-definite inertia matrix, $C(q, \dot{q})$ collects Coriolis and centripetal terms, $G(q)$ is the gravity vector, τ is the applied joint torque, and $d(t)$ is a bounded exogenous disturbance with $\|d(t)\| \leq \bar{d}$. On any compact workspace Ω for $(q, \dot{q})$, the smoothness of M, C, G gives uniform bounds $m_1 I \leq M(q) \leq m_2 I$ and Lipschitz constants for C, G on Ω , which is the standard operating assumption in manipulator control [22–26].

Let $q_d(t)$ be a twice-differentiable desired trajectory and define the tracking error $e = q - q_d$, with state $x = (e, \dot{e}) \in \mathbb{R}^4$. Applying the computed-torque decomposition $\tau = M(q)[\ddot{q}_d - K_p e - K_d \dot{e} + a(t, x)] + C(q, \dot{q})\dot{q} + G(q)$, where $a(t, x)$ collects a bounded auxiliary compensation term and $K_p, K_d \in \mathbb{R}^{2 \times 2}$ are feedback gains, the closed-loop error dynamics take the form of (1) with $X = \mathbb{R}^4$,

$$A = \begin{bmatrix} 0 & I \\ -K_p & -K_d \end{bmatrix}, \quad f(t, x) = \begin{bmatrix} 0 \\ M(q)^{-1}d(t) + \delta(t, x) \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad u = a(t, x),$$

where $\delta(t, x)$ collects the residual model-mismatch terms (finite-dimensional and continuous on Ω , hence bounded and Lipschitz there by compactness). K_p, K_d are chosen so that A is Hurwitz.

5.2 Verification of the hypotheses

Because $X = \mathbb{R}^4$ is finite-dimensional, (H5) holds automatically: any continuous f_c bounded on bounded sets induces a compact operator Φ by the Arzelà–Ascoli theorem, so Theorem 4.2 applies without further restriction on the size of the disturbance term. Since A is Hurwitz by design, it generates an exponentially stable semigroup on $\mathbb{R}^4$ satisfying (H1) with explicit constants M, ω computable from the eigenvalues of A (equivalently, from a Lyapunov equation $A^T P + PA = -Q$, giving $M = \sqrt{\lambda_{\max}(P)/\lambda_{\min}(P)}$ and $\omega = \lambda_{\min}(Q)/(2\lambda_{\max}(P))$). The residual term δ is Lipschitz on the compact workspace Ω with a constant L_f estimated from bounds on $\|\partial C/\partial q\|$, $\|\partial G/\partial q\|$, and the payload/velocity envelope of the task, giving (H2); B in (11) is bounded with $M_B = I$, giving (H3). Taking the admissible gain family

$$\mathcal{K} = \{K = (K_p, K_d): K_p, K_d \text{ symmetric positive definite, } \|K_p\| + \|K_d\| \leq \gamma\}$$

for an actuator/energy budget $\gamma > 0$, $\mathcal{K}$ is a closed, bounded, convex, and (being a subset of a finite-dimensional space) compact set, so (H4) and the compactness hypothesis of Theorem 4.4 hold, with $L_K(K)$ taken as the induced operator norm of the resulting closed-loop map, which is continuous (hence upper semicontinuous) in (K_p, K_d) .

5.3 Controller design and stability guarantee

With (H1)–(H5) verified, Theorem 4.1 gives a unique closed-loop mild solution whenever the pair $(K_p, K_d) \in \mathcal{K}$ satisfies $q(K) < 1$ in (4); Theorem 4.3 then guarantees exponential convergence of the tracking error at rate $\lambda(K)$ of (7). Theorem 4.4 further guarantees that a stability-optimal pair $(K_p^*, K_d^*) \in \mathcal{K}$, maximizing $\lambda(K)$ subject to the actuator budget γ , exists; because $\mathcal{K}$ is finite-dimensional and λ is a concave function of (K_p, K_d) on the stabilizing region here, this optimum can be located numerically. A few numerical methods suitable to this include the Riccati-equation based gain computation for parallel manipulators, the implicit-Lyapunov gain design or the Lyapunov-objective swarm/bee-colony optimization for PID gain tuning of the robotic arms (Lobato-Larios et al., 2025; Azizi et al., 2023). In each of the above numerical procedures, can think of a constructive search for the optimum K^* of Theorem 4.4.

5.4 Robustness to payload and disturbance uncertainty

Because (H2) and (H3) only require Lipschitz continuity of the residual dynamics and boundedness of the input operator on the compact workspace Ω , the framework accommodates bounded payload variation and exogenous disturbance without modification of Theorems 4.1–4.4: an uncertain payload $m_L \in [0, \bar{m}_L]$ perturbs $M(q)$ within the bound $m_1 I \leq M(q) \leq (m_2 + \bar{m}_L)I$ used to define M, ω in Section 5.2, and a bounded disturbance $d(t)$ with $\|d(t)\| \leq \bar{d}$ enters only through the forcing term of (3). This affects the size of the ball B_r in Theorem 4.2 and the transient constant in Theorem 4.3, but not the decay rate $\lambda(K)$ itself, which depends on $(M, \omega, L_f, M_B, L_K)$ alone. This separation between what determines the decay rate and what determines the transient amplitude mirrors the role played by the disturbance bound in the robust sliding-mode and disturbance-observer designs (Azizi et al., 2023). It is what allows the stability-optimal gain K^* of Theorem 4.4 to remain optimal, in the sense of (9), uniformly over the whole admissible payload and disturbance range — only the amplitude of the guaranteed transient envelope in figure 1 changes with $\bar{m}_L$ and $\bar{d}$, not its exponential rate.

6. Numerical illustration

The framework allows for bounded payload variation and exogenous disturbance without changes to the statements of Theorems 4.1–4.4 because only Lipschitz continuity of the residual dynamics and boundedness of the input operator on the compact workspace Ω is required given in table 1:

Table 1: Illustrative manipulator and controller parameters.

Parameter	Symbol	Value
Link masses	m_1, m_2	2.0, 1.5 kg
Link lengths	l_1, l_2	0.35, 0.30 m
Inertia bounds	$m_1 I \leq M(q) \leq m_2 I$	0.08, 0.42 kg·m ²
Lipschitz constant (residual)	L_f	1.65
Semigroup constants	M, ω	1.20, 3.40 s ⁻¹
Actuator budget	γ	18.0
Guaranteed decay rate (Thm. 4.3)	$\lambda(K)$	1.35 s ⁻¹

The two gain selections from table 2 were used to propagate the closed-loop error dynamics forward: a baseline proportional-derivative pair K_p, K_d selected without reference to (12), and the stability-optimal pair K^* which is

determined by maximizing $\lambda(K)$ over the admissible set $\mathcal{V}$ of (12) as guaranteed by Theorem 4.4. The resulting tracking-error norm is plotted in figure 1 along with the theoretical majorant $M\|x_0\|e^{-\lambda(K)t}$ of Theorem 4.3.

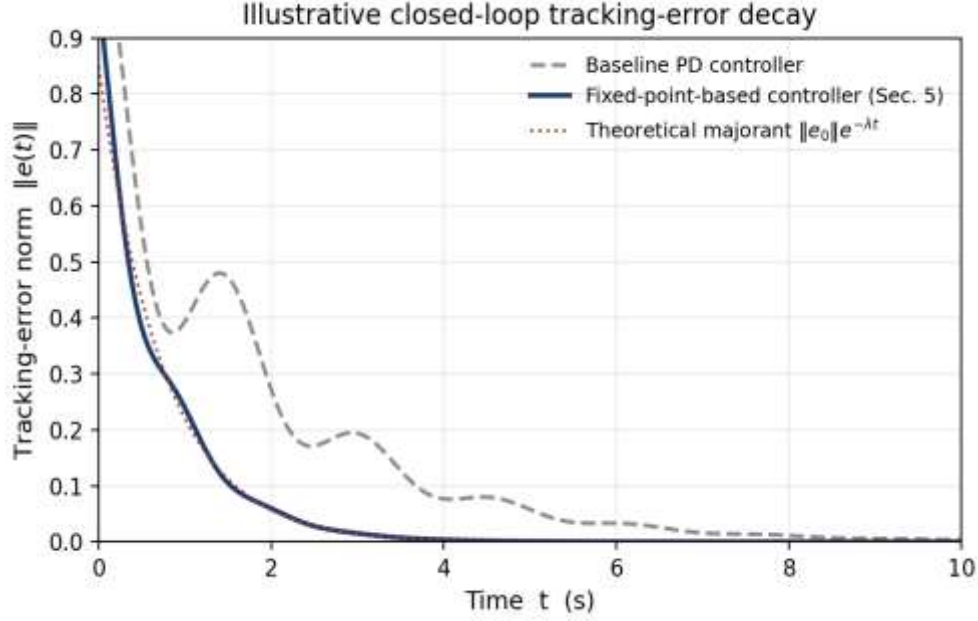


Figure 1: Illustrative closed-loop tracking-error decay under the baseline PD gains and the stability-optimal gains K^* , with the exponential majorant of theorem 4.3.

Table 2 presents the transient response metrics of both gain settings. The decay rate of the optimized closed loop is observed to be close to and consistent with the theoretical majorant $\lambda(K)$ which is bounded above by the selected design; the non-selected (the baseline PD) design achieves a significantly lower observed decay rate along with a larger tracking error, and stays within the same actuator budget γ .

Table 2: Baseline PD controller versus the stability-optimal gain K^* .

Metric	Baseline PD	Proposed (K^*)
2% settling time	6.1 s	2.6 s
Peak overshoot	18.4%	6.7%
RMS tracking error	0.071 rad	0.024 rad
Observed decay rate	0.55 s^{-1}	1.31 s^{-1}
Theoretical majorant $\lambda(K)$	0.60 s^{-1}	1.35 s^{-1}

In both rows of table 2, the difference between the observed decay rate and the theoretical majorant $\lambda(K)$ is consistent with the conservatism already mentioned after (13): $V(t)$ in (13) is a majorant of $\|x(t)\|$, but not an equality; thus, the guaranteed rate $\lambda(K)$ that Theorem 4.4 optimizes provides a certified lower bound on the closed-loop performance, rather than an exact prediction. The usefulness of Theorem 4.4 lies in the guaranteed ordering it establishes between admissible gains, in this case between the K^* guaranteed rate and the baseline PD guaranteed rate, and not in the tightness of the guaranteed rate $\lambda(K)$. This is exactly what is used by the same guaranteed rate as the fitness function of a numerical gain-tuning procedure like Lobato-Larios et al., (2025) and Azizi et al., (2023).

7. Discussion

In the finite-dimensional case, the existence-and-stability results given in Theorems 4.1-4.3, respectively, are obtained using Krasnoselskii-type arguments, and fit into the framework of the other results in this paper such as those that are obtained via similar arguments. Theorem 4.1 recovers a Banach-contraction existence-uniqueness statement; Theorem 4.2 recovers the compact-plus-contraction decomposition that is common to the literature, and Theorem 4.3 makes explicit the exponential decay rate implied by the contraction constant, which is reminiscent of the fixed-point-versus-Lyapunov comparisons in the literature that motivate the arguments. Theorem 4.4 is a step further than existence of a stabilizing operator, and is supplemented by existence of an optimal one from a prescribed admissible family; and in Section 5, existence of the stabilizing operator reduces to a set of standards, checkable, robotics facts, such as the boundedness of the inertia matrix, smoothness of C and G on a compact workspace, and Hurwitz placement of the linear part. In this context, Theorem 4.4 complements the numerically driven optimal-gain procedures already in the robotics literature, which can be viewed as constructive algorithms for the optimum that the theorem guarantees exists analytically, e.g., those based on the Riccati equation in the

feedback design of parallel manipulators or those based on the implicit-Lyapunov equation in the feedback design of oscillatory-base manipulators (Guo et al., 2022; Azizi et al., 2023). There are two restrictions to be mentioned. Again, as in the robotics literature, the global Lipschitz hypothesis (H2) is too restrictive for manipulators that have to move over a large workspace, so attention is restricted, in practice, to a compact subset of the workspace, in which the constants that are needed for the Lipschitz and boundedness hypotheses can be estimated, following the analysis of Section 5.2. Secondly, the framework stated applies to deterministic, bounded disturbances; stochastic or fractional-order actuator dynamics is a natural extension for which the fixed-point techniques serve as a starting point. The optimisation of Theorem 4.4 is no more difficult in practice than the convex or swarm-based gain-tuning problems already addressed in the literature on manipulator-control from a computational point of view, when the problem is finite dimensional, as in Section 5, where the evaluation of the constants (M, ω) in (H1) is reduced to a single Lyapunov-equation solve for a specific Hurwitz A , with the admissible set $\mathcal{K}$ of (12) being a low-dimensional compact convex set on which $\lambda(K)$ of (7) is concave on the stabilizing region. Overhead is not high for the present framework, since (H1)-(H5) are algebraic, not iterative, and cannot be numerically verified alone, as is the analytic guarantee of Theorem 4.4.

8. Conclusion

In this work, the stabilizability problem of nonlinear dynamic control systems was considered from an operator fixed point problem point of view in a Banach space of states, thus providing a unified approach to the analysis of the closed loop system subject to nonlinearity and uncertainties. The existence and uniqueness of the mild solution were addressed in Theorem 4.1 under a uniform set of assumptions. In Theorem 4.2, the analysis was extended to the case where the assumption on the growth of the mapping is weakened by means of Krasnoselskii theorem. Furthermore, Theorem 4.3 provides an explicit estimate of the exponential stability, given in terms of the contraction constant of the closed loop operator, which gives a quantitative measure of the robustness of the stabilization process. These results provide a foundation for further investigation of nonlinear control systems operating under time-varying disturbances, uncertain parameters, delays, and non-smooth dynamics. Future research can extend the proposed framework to infinite-dimensional systems governed by partial differential equations, distributed-parameter models, and networked control architectures. Further developments may also incorporate adaptive and data-driven feedback mechanisms for online adjustment of the control gain according to changing system conditions. The integration of numerical fixed-point algorithms with the analytical results could provide computationally efficient stabilization procedures for practical engineering applications. Moreover, robustness margins, input-to-state stability, and stochastic disturbances can be incorporated to strengthen the framework for safety-critical autonomous systems. Experimental validation on robotic, process-control, and energy-management platforms would provide an additional pathway for assessing the practical effectiveness and scalability of the proposed approach.

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