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Axially Symmetric Solutions to a Fractional System on Strip-like Domains

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We provide sufficient conditions for the existence of multiple axially symmetric solutions to an elliptic system driven by the fractional p-Laplacian operator on an unbounded cylinder of $\mathbb{R}^n$. Our approach rests upon variational techniques.

Keywords: Fractional p-Laplacian, axial symmetry, strip-like domains, multiplicity.

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1. Introduction

The present paper is devoted to the study of the following nonlocal system under Dirichlet boundary conditions:

$$\begin{cases} (-\Delta)_p^s u + |u|^{p-2}u = \lambda F_u(x,u,v) + \mu H_u(x,u,v) & \text{in } \Omega \\ (-\Delta)_q^t v + |v|^{q-2}v = \lambda F_v(x,u,v) + \mu H_v(x,u,v) & \text{in } \Omega \\ u = v = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \qquad (S_{\lambda,\mu})$$

where $\Omega := \omega \times \mathbb{R}^{n_2}$ is an unbounded cylinder, ω being a bounded open subset of $\mathbb{R}^{n_1}$ with a smooth boundary and suitable symmetry conditions, $n_1, n_2 \geq 2$, $s, t \in (0,1)$ and $p, q \in (1, +\infty)$ satisfy $\max\{ps, qt\} < n_1 + n_2 = n$ and λ, μ are positive real parameters. The nonlinearities F and H are assumed to be of class $C^0(\Omega \times \mathbb{R}^2, \mathbb{R})$, with F_z and H_z denoting, as usual, the partial derivative of F and H with respect to the variable z.

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[*] Because of a surprising coincidence of names within the same department, we have to point out that the author was born on August 4, 1968.

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The operator $(-\Delta)^{\sigma}_{m}$ governing $(S_{\lambda,\mu})$ is the fractional m-Laplacian, defined pointwise by

$$(-\Delta)^{\sigma}_{m}u(x) := c_{n,m,\sigma} \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n\setminus B(x,\varepsilon)} \frac{|u(x)-u(y)|^{m-2}(u(x)-u(y))}{|x-y|^{n+m\sigma}}dy, \quad x\in\mathbb{R}^n,$$

for any $u \in C^{\infty}_{0}(\mathbb{R}^n)$, where the normalization constant $c_{n,m,\sigma}$ is given by

$$c_{n,m,\sigma} := \frac{\sigma 2^{2\sigma}\Gamma\left(\frac{m\sigma+m+n-2}{2}\right)}{\pi^{\frac{n}{2}}\Gamma(1-\sigma)},$$

Γ stands for the usual Gamma function and $B(x,\varepsilon) := \{y\in\mathbb{R}^n : |y-x| < \varepsilon\}$ (see for instance [15] and the references therein).

On the function F we make the following additional hypotheses:

(F_1) $\lim_{u,v\to 0} \frac{F_u(x,u,v)}{|u|^{p-1}} = \lim_{u,v\to 0} \frac{F_v(x,u,v)}{|v|^{q-1}} = 0$, uniformly w.r.t. $x\in\Omega$;

(F_2) $F((x_1,x_2),u,v) = F((g_1x_1,g_2x_2),u,v)$ for all $x_1\in\omega$, $x_2\in\mathbb{R}^{n_2}$, $g_1\in O(n_1)$, $g_2\in O(n_2)$, $(u,v)\in\mathbb{R}^2$;

(F_3) $\mathbb{R}^2 \ni (u,v)\mapsto F(x,u,v)$ is of class C^1 and $F(x,0,0)=0$ for all $x\in\Omega$;

(F_4) there exist $\varepsilon>0$, $\alpha\in(p,p^*_s)$, $\beta\in(q,q^*_t)$, with $p\beta = q\alpha$, such that

$$|F_u(x,u,v)| \le \varepsilon(|u|^{p-1} + |v|^{(p-1)q/p} + |u|^{\alpha-1}),$$
$$|F_v(x,u,v)| \le \varepsilon(|v|^{q-1} + |u|^{(q-1)p/q} + |v|^{\beta-1}),$$

for each $x\in\Omega$ and $(u,v)\in\mathbb{R}^2$, where

$$p^*_s := \frac{np}{n-ps}, \quad q^*_t := \frac{nq}{n-qt};$$

(F_5) there exist $p_F\in(0,p)$, $q_F\in(0,q)$, $\mu_F\in[p,p^*_s]$, $\nu_F\in[q,q^*_t]$ and $a_F\in L^{\mu_F/(\mu_F-p_F)}(\Omega)$, $b_F\in L^{\nu_F/(\nu_F-q_F)}(\Omega)$, $c_F\in L^1(\Omega)$ such that

$$F(x,u,v) \le a_F(x)|u|^{p_F} + b_F(x)|v|^{q_F} + c_F(x),$$

for any $x\in\Omega$ and $(u,v)\in\mathbb{R}^2$;

(F_6) there exists $(u_0,v_0)\in W^{s,p}_{0,G}(\overline{\Omega})\times W^{t,q}_{0,G}(\overline{\Omega})$ such that

$$\int_{\Omega} F(x,u_0(x),v_0(x))dx > 0,$$

where $W^{s,p}_{0,G}(\overline{\Omega})$ and $W^{t,q}_{0,G}(\overline{\Omega})$ are suitable fractional Sobolev spaces that will be described in detail later on (see Section 2).

Our aim is to investigate by variational techniques the existence of weak solutions to $(S_{\lambda,\mu})$ displaying a particular axial symmetry, namely, pairs (u,v) satisfying

$u(x_1, x_2) = u(g_1x_1, g_2x_2)$, $v(x_1, x_2) = v(g_1x_1, g_2x_2)$ for every $x_1 \in \omega$, $x_2 \in \mathbb{R}^{n_2}$, $g_1 \in O(n_1)$ and $g_2 \in O(n_2)$ (of course, for solutions of this type our system should possess itself such a symmetry and this is the reason for assuming (F_2) above). Among the motivations behind this study, we wish to analyse what happens to classical systems previously considered in literature (see for instance [1]) when the leading operator is not local but, as it is the case for $(-\Delta)^\sigma_m$, a highly nonlocal one.

Adopting a variational approach to this kind of problems demands some cautious analysis. One key point is to choose the correct energy space, encoding the boundary conditions in the weak formulation (in this regard, we have followed [15] but adapted the setting to the case of an unbounded domain). A second necessity is to restrict the energy functional associated with $(S_{\lambda,\mu})$ to an adequately chosen Sobolev space X so as to recover the compactness of the embedding $X \hookrightarrow L^\nu(\mathbb{R}^n)$ which, on account of the unboundedness of Ω, fails. All these facts are illustrated in detail in the following section.

Since $(S_{\lambda,\mu})$ is axially symmetric, we will look for solutions endowed with the same symmetry. We will first work on the space of cylindrical symmetric solutions and, by virtue of the abstract Theorem 2.2, ensure the existence of two local minimizers for a certain functional related to the energy one. A classical consequence of the mountain pass theorem (Theorem 2.3) will provide a further critical point. The link between the solutions found in this "symmetric" subspace and the ones sought for will be represented by a version of Palais' principle of symmetric criticality, holding in reflexive Banach spaces (Theorem 2.4).

We point out that problems set in strip-like domains are related to the existence of suitable solitary waves in Schrödinger and Klein-Gordon equations and arise in several other fields of fluid mechanics and physics. Issues of existence and multiplicity of solutions to this class of problems have been recently dealt with, both in the scalar and in the system case, in [1, 6, 7, 8, 9, 10] (see also the references contained in these papers).

2. Functional background and variational tools

Let ω be a bounded open subset of $\mathbb{R}^{n_1}$, $n_1 \geq 2$, with a smooth boundary and such that $g_1x_1 \in \omega$ for all $g_1 \in O(n_1)$, $x_1 \in \omega$, where $O(n_1)$ stands for the orthogonal group in dimension n_1. Let $\Omega := \omega \times \mathbb{R}^{n_2}$, $n_2 \geq 2$, $\sigma \in (0, 1)$, $m \in [1, +\infty)$ and $n_1 + n_2 = n > m\sigma$.

The fractional Sobolev space $W^{\sigma,m}(\mathbb{R}^n)$ is defined by

$$W^{\sigma,m}(\mathbb{R}^n) := \left\{ u \in L^m(\mathbb{R}^n) : \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^m}{|x - y|^{n+m\sigma}} dxdy < \infty \right\}$$

and is endowed with the standard norm

$$\|u\|_{W^{\sigma,m}(\mathbb{R}^n)} = \|u\|_{\sigma,m} := \left(\int_{\mathbb{R}^n} |u(x)|^m dx + \frac{c_{n,m,\sigma}}{2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^m}{|x - y|^{n+m\sigma}} dxdy \right)^{1/m}$$

(see [5, 12] for excellent accounts on the matter). We now consider the following subspace of $W^{\sigma,m}(\mathbb{R}^n)$,

$$W_0^{\sigma,m}(\overline{\Omega}) := \{u \in W^{\sigma,m}(\mathbb{R}^n) : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\},$$

equipped with the above norm. It is known that the embedding $W^{\sigma,m}(\mathbb{R}^n) \hookrightarrow L^\nu(\mathbb{R}^n)$ is continuous for any $\nu \in [m, m_\sigma^*]$, where $m_\sigma^* := nm/(n - m\sigma)$ is the fractional critical Sobolev exponent. In the sequel we will adopt the notation $\|\cdot\|_\nu$ to mean $\|\cdot\|_{L^\nu(\mathbb{R}^n)}$.

Our next step is to build a suitable subspace of $W_0^{\sigma,m}(\overline{\Omega})$ which may be compactly embedded in $L^\nu(\mathbb{R}^n)$, $\nu \in (m, m_\sigma^*)$. Denote by G the group $G := O(n_1) \times O(n_2)$, whose action on $W_0^{\sigma,m}(\overline{\Omega})$ is defined by

$$gu(x_1, x_2) := u(g_1^{-1}x_1, g_2^{-1}x_2) \tag{1}$$

for each $(x_1, x_2) \in \omega \times \mathbb{R}^{n_2}$, $g = g_1 \times g_2 \in G$ and $u \in W_0^{\sigma,m}(\overline{\Omega})$. Such an action turns out to be isometric as one has, for all $g = g_1 \times g_2 \in G$ and $u \in W_0^{\sigma,m}(\overline{\Omega})$,

$$\begin{aligned}\|gu\|_{\sigma,m}^m = \int_{\mathbb{R}^n} |u(g_1^{-1}x_1, g_2^{-1}x_2)|^m dx +& \\ + \frac{c_{n,m,\sigma}}{2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} &\frac{|u(g_1^{-1}x_1, g_2^{-1}x_2) - u(g_1^{-1}y_1, g_2^{-1}y_2)|^m}{|x-y|^{n+m\sigma}} dxdy \\ = \int_{\mathbb{R}^n} |u(Ax)|^m dx + \frac{c_{n,m,\sigma}}{2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} &\frac{|u(Ax) - u(Ay)|^m}{|x-y|^{n+m\sigma}} dxdy \\ = \int_{\mathbb{R}^n} |u(x')|^m dx' + \frac{c_{n,m,\sigma}}{2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} &\frac{|u(x') - u(y')|^m}{|x'-y'|^{n+m\sigma}} dx'dy' = \|u\|_{\sigma,m}^m,\end{aligned}$$

where we have expressed the action (1) as $gu = u \circ A$, with $A \in O(n)$, and taken into account that $|x-y| = |A(x-y)| = |Ax-Ay| = |x'-y'|$ and that $detA = \pm 1$.

Defining

$$W_{0,G}^{\sigma,m}(\overline{\Omega}) := \mathrm{Fix}_G\, W_0^{\sigma,m}(\overline{\Omega}) = \left\{u \in W_0^{\sigma,m}(\overline{\Omega}) : g(u) = u \text{ for all } g \in G\right\}, \tag{2}$$

it is immediately seen that the elements of $W_{0,G}^{\sigma,m}(\overline{\Omega})$ are exactly the axially symmetric functions of $W_0^{\sigma,m}(\overline{\Omega})$. The space $W_{0,G}^{\sigma,m}(\overline{\Omega})$, being a closed subspace of the reflexive space $W_0^{\sigma,m}(\overline{\Omega})$, is reflexive itself. The next result is crucial in our subsequent arguments.

Proposition 2.1 ([11], Théorème III.3). *Let $n = \sum_{i=1}^k n_i$, with $n_i \geq 2$, $\sigma > 0$ and $m \in [1, +\infty)$. Then, setting*

$$W_{sph}^{\sigma,m}(\mathbb{R}^n) = \left\{ u \in W^{\sigma,m}(\mathbb{R}^n) : \begin{array}{c} \forall i = 1, \ldots, k, \ u \text{ is spherically} \\ \text{symmetric w.r.t. } x_i \in \mathbb{R}^{n_i} \end{array} \right\},$$

the restriction to $W_{sph}^{\sigma,m}(\mathbb{R}^n)$ of the Sobolev injection $W^{\sigma,m}(\mathbb{R}^n) \hookrightarrow L^\nu(\mathbb{R}^n)$ is compact whenever $\nu \in (m, m_\sigma^*)$.

The definition of the space $W^{\sigma,m}_{0,G}(\overline{\Omega})$ and the assumption on the dimensions n_1 and n_2 guarantee, in view of the above lemma, the compactness of the embedding $W^{\sigma,m}_{0,G}(\overline{\Omega}) \hookrightarrow L^{\nu}(\mathbb{R}^n)$ for any $\nu \in (m, m^*_\sigma)$.

In the product space $W^{s,p}_0(\overline{\Omega}) \times W^{t,q}_0(\overline{\Omega})$, equipped with the standard norm

$$\|(u,v)\| := \|u\|_{s,p} + \|v\|_{t,q}\,,$$

the group G acts as follows: $g(u,v) = (gu, gv)$, for all $g \in G$ and $(u,v) \in W^{s,p}_0(\overline{\Omega}) \times W^{t,q}_0(\overline{\Omega})$. Furthermore, it is easy to deduce that

$$\begin{aligned} &\mathrm{Fix}_G\left(W^{s,p}_0(\overline{\Omega}) \times W^{t,q}_0(\overline{\Omega})\right) = \\ &= \left\{(u,v) \in W^{s,p}_0(\overline{\Omega}) \times W^{t,q}_0(\overline{\Omega}) : g(u,v) = (u,v) \text{ for all } g \in G\right\} = \\ &= W^{s,p}_{0,G}(\overline{\Omega}) \times W^{t,q}_{0,G}(\overline{\Omega}). \end{aligned} \tag{3}$$

Hereafter we set for brevity $X := W^{s,p}_{0,G}(\overline{\Omega}) \times W^{t,q}_{0,G}(\overline{\Omega})$ and define the functionals

$$\Phi(u,v) := \frac{1}{p}||u||^p_{s,p} + \frac{1}{q}||v||^q_{t,q}, \quad \mathcal{F}(u,v) := \int_\Omega F(x, u(x), v(x))dx,$$

for any $(u,v) \in X$. Thanks to $(F_3)-(F_4)$ it is not difficult to prove that $\mathcal{F} \in C^1(X, \mathbb{R})$ and that its differential is given by

$$\mathcal{F}'(u,v)(w,z) = \int_\Omega (F_u(x,u,v)w + F_v(x,u,v)z)dx$$

for all $(u,v), (w,z) \in X$. In addition, arguing as in [9, Lemma 3.4], we infer that under assumptions (F_1) and (F_4), $\mathcal{F}$ is sequentially weakly continuous on X.

Once illustrated the functional setup of $(S_{\lambda,\mu})$, we briefly recall the variational tools we are going to exploit. With the aid of the following abstract result, which combines minimax and critical point theories, we will prove the existence of two weak solutions to $(S_{\lambda,\mu})$ (see also [2, 3] for elliptic problems governed by different operators and treated via the same approach).

Theorem 2.2 ([14], Theorem 4)**.** *Let E be a reflexive real Banach space, $I \subseteq \mathbb{R}$ an interval, and $\Psi\colon E \times I \to \mathbb{R}$ a function fulfilling:*

(Ψ_1) $\Psi(x, \cdot)$ *is concave in I for all $x \in E$;*

(Ψ_2) $\Psi(\cdot, \lambda)$ *is continuous, coercive and sequentially weakly lower semicontinuous in E for all $\lambda \in I$;*

(Ψ_3) $\sup_{\lambda\in I} \inf_{x\in E} \Psi(x, \lambda) < \inf_{x\in E} \sup_{\lambda\in I} \Psi(x, \lambda)$.

Then, for each $\zeta > \sup_I \inf_E \Psi(x, \lambda)$, there exists a non-empty open set $\Lambda \subseteq I$ with the following property: for every $\lambda \in \Lambda$ and every sequentially weakly lower semicontinuous functional $\Phi\colon E \to \mathbb{R}$, there exists $\delta > 0$ such that, for each $\mu \in (0, \delta)$, the functional $\Psi(\cdot, \lambda) + \mu\Phi(\cdot)$ has at least two local minima lying in the set $\{x \in E : \Psi(x, \lambda) < \zeta\}$.

A third solution, necessarily distinct from the other two, can be derived from the coming (classical) result.

Theorem 2.3 ([13], Corollary 1). *Let E be a real Banach space and $J\colon E \to \mathbb{R}$ a C^1 functional satisfying the Palais-Smale condition (PS), i.e. each sequence $\{x_i\}$ in E such that $\sup_{i\in\mathbb{N}} |J(x_i)| < +\infty$ and $\lim_{i\to+\infty} \|J'(x_i)\|_{E^*} = 0$, admits a strongly convergent subsequence. If J has a pair of local minima, then it admits a third critical point.*

Finally, the following version of Palais' symmetric criticality principle, set in reflexive Banach spaces, will allow us to look for critical points of the energy functional constrained on X.

Theorem 2.4 ([4], Proposition 3.1). *Let E be a reflexive Banach space, G a subgroup of isometries $g\colon E \to E$ and $J\colon E \to \mathbb{R}$ a C^1, G-invariant functional. Then $x_0 \in \mathrm{Fix}_G E$ is a critical point of J if and only if x_0 is a critical point for the restriction of J to $\mathrm{Fix}_G E$.*

3. The multiplicity results

We are now able to formulate and prove our first multiplicity result.

Theorem 3.1. *Let $F\colon \Omega \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function which satisfies $(F_1) - (F_6)$. Then, there exist a number $\varrho > 0$ and a non-empty open set $\Lambda \subseteq [0, +\infty)$ such that, for every $\lambda \in \Lambda$ and every continuous function $H\colon \Omega \times \mathbb{R}^2 \to \mathbb{R}$ satisfying $(H_1) - (H_4)$, there exists $\mu_1 > 0$ such that, for each $\mu \in (0, \mu_1)$, the system $(S_{\lambda,\mu})$ has at least two solutions $(u^i_{\lambda,\mu}, v^i_{\lambda,\mu})$, $i \in \{1, 2\}$, with $u^i_{\lambda,\mu}$ and $v^i_{\lambda,\mu}$ axially symmetric and lying in the ball $\{(u, v) \in X : \|(u, v)\| \le \varrho\}$.*

Proof. We start off by defining $f\colon (0, +\infty) \to \mathbb{R}$ to be the function

$$f(\xi) := \sup\{\mathcal{F}(u, v) : \Phi(u, v) \le \xi\}.$$

Using (F_1) and (F_4) (cf. [9, Lemma 3.3]) we deduce that for every $\varepsilon > 0$ there exists $c_\varepsilon > 0$ so that

$$\begin{aligned} |F_u(x, u, v)| &\le \varepsilon \left(|u|^{p-1} + |v|^{(p-1)q/p}\right) + c_\varepsilon \left(|u|^{\alpha-1} + |v|^{(\alpha-1)q/p}\right), \\ |F_v(x, u, v)| &\le \varepsilon \left(|v|^{q-1} + |u|^{(q-1)p/q}\right) + c_\varepsilon \left(|v|^{\beta-1} + |u|^{(\beta-1)p/q}\right), \end{aligned} \tag{4}$$

for any $x \in \Omega$, $(u, v) \in \mathbb{R}^2$. A joint use of the above estimates, the mean value theorem and the relationship $F(x, 0, 0) = 0$ yields

$$\begin{aligned} |F(x, u, v)| \le \varepsilon &\left(|u|^p + |v|^{(p-1)q/p}|u| + |v|^q + |u|^{(q-1)p/q}|v|\right) + \\ &+ c_\varepsilon \left(|u|^\alpha + |v|^{(\alpha-1)q/p}|u| + |v|^\beta + |u|^{(\beta-1)p/q}|v|\right) \end{aligned} \tag{5}$$

for any $x \in \Omega$, $(u, v) \in \mathbb{R}^2$. To simplify the presentation we define

$$A := \left(2 + \tfrac{1}{p} - \tfrac{1}{q}\right),\ B := \left(2 + \tfrac{1}{q} - \tfrac{1}{p}\right),\ C := \left(2 + \tfrac{1}{\alpha} - \tfrac{1}{\beta}\right),\ D := \left(2 + \tfrac{1}{\beta} - \tfrac{1}{\alpha}\right).$$

Integrating (5), using Young's inequality and the relationship between α and β we get

$$\int_\Omega |F(x,u(x),v(x))|dx \leq$$
$$\leq \varepsilon \int_\Omega (A|u(x)|^p + B|v(x)|^q)\, dx + c_\varepsilon \int_\Omega \left(C|u(x)|^\alpha + D|v(x)|^\beta\right) dx$$

for any $(u,v) \in X$. Furthermore, taking account of the embeddings $W_0^{s,p}(\overline{\Omega}) \hookrightarrow L^p(\mathbb{R}^n)$, $W_0^{s,p}(\overline{\Omega}) \hookrightarrow L^\alpha(\mathbb{R}^n)$, $W_0^{t,q}(\overline{\Omega}) \hookrightarrow L^q(\mathbb{R}^n)$, and $W_0^{t,q}(\overline{\Omega}) \hookrightarrow L^\beta(\mathbb{R}^n)$, we deduce that

$$\int_\Omega |F(x,u(x),v(x))|dx \leq \varepsilon \left(c_p A \left\|u\right\|_{s,p}^p + c_q B \left\|v\right\|_{t,q}^q\right) + c_\varepsilon \left(c_\alpha C \left\|u\right\|_{s,p}^\alpha + c_\beta D \left\|v\right\|_{t,q}^\beta\right),$$

for some positive constants $c_p, c_q, c_\alpha, c_\beta$. Now, since the function $\xi \mapsto (a^\xi + b^\xi)^{1/\xi}$, $\xi > 0$, $a, b \geq 0$, is non-increasing, recalling that $p\beta = q\alpha$ one has

$$\left\|u\right\|_{s,p}^\alpha + \left\|v\right\|_{t,q}^\beta \leq \left(\left\|u\right\|_{s,p}^p + \left\|v\right\|_{t,q}^q\right)^{\alpha/p}$$

and therefore

$$\mathcal{F}(u,v) \leq \varepsilon \max\left\{c_p\left(2p+1-\frac{p}{q}\right), c_q\left(2q+1-\frac{q}{p}\right)\right\}\left(\frac{1}{p}\left\|u\right\|_{s,p}^p + \frac{1}{q}\left\|v\right\|_{t,q}^q\right) +$$
$$+ c_\varepsilon \max\{c_\alpha C, c_\beta D\} \max\{p,q\}^{\alpha/p}\left(\frac{1}{p}\left\|u\right\|_{s,p}^p + \frac{1}{q}\left\|v\right\|_{t,q}^q\right)^{\alpha/p}.$$

So, in the light of the definition of f, it turns out that

$$f(\xi) \leq \varepsilon K \xi + c_\varepsilon L \xi^{\alpha/p}$$

for all $\xi > 0$, where

$$K := \max\left\{c_p\left(2p+1-\frac{p}{q}\right), c_q\left(2q+1-\frac{q}{p}\right)\right\},$$
$$L := \max\{c_\alpha C, c_\beta D\}\max\{p,q\}^{\alpha/p}.$$

It is clear that $f(\xi) \geq 0$ for all $\xi > 0$ and that

$$\lim_{\xi \to 0^+} \frac{f(\xi)}{\xi} = 0. \tag{6}$$

Now, fix $\eta \in \mathbb{R}$ such that $0 < \eta < \mathcal{F}(u_0,v_0)\left(\frac{1}{p}\left\|u_0\right\|_{s,p}^p + \frac{1}{q}\left\|v_0\right\|_{t,q}^q\right)^{-1}$.

By (6), there exists $\xi_0 \in \left(0, \frac{1}{p}\left\|u_0\right\|_{s,p}^p + \frac{1}{q}\left\|v_0\right\|_{t,q}^q\right)$ such that $f(\xi_0) < \eta\xi_0$.

If $\varrho_0 > 0$ satisfies

$$f(\xi_0) < \varrho_0 < \mathcal{F}(u_0, v_0)\xi_0 \left(\frac{1}{p}\|u_0\|_{s,p}^p + \frac{1}{q}\|v_0\|_{t,q}^q\right)^{-1}, \tag{7}$$

then, due to the choice of ξ_0, one also has $\varrho_0 < \mathcal{F}(u_0, v_0)$.

As a next step, set $I := [0, +\infty)$ and

$$\Psi(u, v, \lambda) := \Phi(u, v) - \lambda\mathcal{F}(u, v) + \lambda\varrho_0$$

for all $(u, v) \in X$, $\lambda \in I$. Let us verify that the strict minimax inequality (Ψ_3) of Theorem 2.2 is met. Clearly the function $\lambda \mapsto \inf_{(u,v)\in X} \Psi(u, v, \lambda)$ is upper semicontinuous on I and one has

$$\lim_{\lambda\to+\infty} \inf_{(u,v)\in X} \Psi(u, v, \lambda) \leq \lim_{\lambda\to+\infty} \Psi(u_0, v_0, \lambda) = -\infty,$$

so there exists $\bar{\lambda} \in I$ so that

$$\sup_{\lambda\in I} \inf_{(u,v)\in X} \Psi(u, v, \lambda) = \inf_{(u,v)\in X} \Psi(u, v, \bar{\lambda}). \tag{8}$$

For each $(u, v) \in X$ satisfying $\Phi(u, v) \leq \xi_0$ we have

$$\mathcal{F}(u, v) \leq f(\xi_0) < \varrho_0$$

and hence

$$\xi_0 \leq \inf \{\Phi(u, v) : \mathcal{F}(u, v) \geq \varrho_0\}. \tag{9}$$

But we have also

$$\begin{aligned}\inf_{(u,v)\in X} \sup_{\lambda\in I} \Psi(u, v, \lambda) &= \inf_{(u,v)\in X} \left(\Phi(u, v) + \sup_{\lambda\in I} (\lambda (\varrho_0 - \mathcal{F}(u, v)))\right) \\ &= \inf_{(u,v)\in X} \{\Phi(u, v) : \mathcal{F}(u, v) \geq \varrho_0\},\end{aligned}$$

and therefore

$$\xi_0 \leq \inf_{(u,v)\in X} \sup_{\lambda\in I} \Psi(u, v, \lambda). \tag{10}$$

We now need to distinguish two cases. If $0 \leq \bar{\lambda} < \xi_0/\varrho_0$, then

$$\inf_{(u,v)\in X} \Psi(u, v, \bar{\lambda}) \leq \Phi(0, 0) - \bar{\lambda}\mathcal{F}(0, 0) + \bar{\lambda}\varrho_0 = \bar{\lambda}\varrho_0 < \xi_0.$$

On the other hand, when $\bar{\lambda} \geq \xi_0/\varrho_0$ the choice of ϱ_0 implies that

$$\inf_{(u,v)\in X} \Psi(u, v, \bar{\lambda}) \leq \Psi(u_0, v_0, \bar{\lambda}) \leq \Psi(u_0, v_0, \xi_0/\varrho_0) < \xi_0.$$

In both situations, thanks to (8) and (10), inequality (Ψ_3) is satisfied, as claimed.

The concavity of $I \ni \lambda \mapsto \Psi(u,v,\lambda)$ for any $(u,v) \in X$, as well as the sequential weak lower semicontinuity of $X \ni (u,v) \mapsto \Psi(u,v,\lambda)$ for every $\lambda \in I$ are clear.

Moreover, fixed $\lambda \in I$, on account of (F_5), Hölder's inequality and the Sobolev embeddings one obtains for $Q := \frac{1}{p}||u||_{s,p}^p + \frac{1}{q}||v||_{t,q}^q$

$$\begin{aligned}\Psi(u,v,\lambda) &\geq Q - \lambda\left(\int_\Omega (a(x)|u|^{p_F} + b(x)|v|^{q_F} + c(x))\,dx\right)\\ &\geq Q - \lambda\left(\|a\|_{\mu_F/(\mu_F - p_F)}\|u\|_{\mu_F}^{p_F} + \|b\|_{\nu_F/(\nu_F - q_F)}\|v\|_{\nu_F}^{q_F} + \|c\|_1\right)\\ &\geq Q - \lambda\left(c_{\mu_F}^{p_F}\|a\|_{\mu_F/(\mu_F - p_F)}\|u\|_{s,p}^{p_F} + c_{\nu_F}^{q_F}\|b\|_{\nu_F/(\nu_F - q_F)}\|v\|_{t,q}^{q_F} + \|c\|_1\right)\end{aligned}$$

for some positive constants $c_{\mu_F}^{p_F}, c_{\nu_F}^{q_F}$, and this implies that $\Psi(u,v,\lambda) \to +\infty$ as $\|(u,v)\| \to +\infty$, being $p_F < p$ and $q_F < q$.

Therefore, fixed $\zeta > \sup_{\lambda \in I} \inf_{(u,v)\in X} \Psi(u,v,\lambda)$, Theorem 2.2 assures the existence of a non-empty open set $\Lambda \subseteq I$ with this property: if $\lambda \in \Lambda$ and $H\colon \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and obeys $(H_1)-(H_4)$, there exists $\delta > 0$ such that, for each $\mu \in (0,\delta)$, the functional

$$\mathcal{E}_{\lambda,\mu}(u,v) := \Psi(u,v,\lambda) + \mu\mathcal{H}(u,v)$$

has at least two local minima in $\{(u,v) \in X : \Psi(u,v,\lambda) < \zeta\}$, say $(u_{\lambda,\mu}^i, v_{\lambda,\mu}^i)$, $i \in \{1,2\}$, where $\mathcal{H}\colon X \to \mathbb{R}$ is the functional defined by

$$\mathcal{H}(u,v) = -\int_\Omega H(x,u(x),v(x))dx.$$

Notice that, similar to $\mathcal{F}$, $\mathcal{H}$ is sequentially weakly continuous on X thanks to (H_1) and (H_4). Since F and H are axially symmetric with respect to each block of variables, and the action of each $g \in G$ is isometric, the map $\mathcal{E}_{\lambda,\mu}$ is G-invariant, i.e.

$$\mathcal{E}_{\lambda,\mu}(g(u,v)) = \mathcal{E}_{\lambda,\mu}(gu,gv) = \mathcal{E}_{\lambda,\mu}(u,v)$$

for each $g \in G$, $(u,v) \in W_0^{s,p}(\overline{\Omega}) \times W_0^{t,q}(\overline{\Omega})$. By (2) and Theorem 2.4, $(u_{\lambda,\mu}^i, v_{\lambda,\mu}^i)$, $i \in \{1,2\}$, turn out also to be critical points of $\mathcal{E}_{\lambda,\mu}$ and hence weak solutions to $(S_{\lambda,\mu})$.

Finally, to estimate the norm of $(u_{\lambda,\mu}^i, v_{\lambda,\mu}^i)$, $i \in \{1,2\}$, pick a non-degenerate compact interval $[a_1,a_2] \subset \Lambda$. Notice that one has

$$\bigcup_{\lambda \in [a_1,a_2]} \{(u,v) \in X : \Psi(u,v,\lambda) \leq \zeta\} \subseteq$$

$$\subseteq \{(u,v) \in X : \Psi(u,v,a_1) \leq \zeta\} \cup \{(u,v) \in X : \Psi(u,v,a_2) \leq \zeta\}$$

and hence the set $S := \bigcup_{\lambda \in [a_1,a_2]} \{(u,v) \in X : \Psi(u,v,\lambda) \leq \zeta\}$ is bounded. As a result, the local minima of $\mathcal{E}_{\lambda,\mu}$ have norm at most equal to $\varrho := \sup_{(u,v)\in S} \|(u,v)\|$. This concludes the proof. □

It is worth noticing that the perturbation term H in the previous theorem can be chosen to be identically zero. In this case Theorem 3.1 translates into the following eigenvalue result.

Theorem 3.2. *Let* $F\colon \Omega\times\mathbb{R}^2 \to \mathbb{R}$ *be a continuous function fulfilling* $(F_1)-(F_6)$*. Then, there exist a number* $\varrho > 0$ *and a non-empty open set* $\Lambda \subseteq [0,+\infty)$ *such that, for every* $\lambda \in \Lambda$*, the system*

$$\begin{cases} (-\Delta)^s_p u + |u|^{p-2}u = \lambda F_u(x,u,v) & \text{in } \Omega \\ (-\Delta)^t_q v + |v|^{q-2}v = \lambda F_v(x,u,v) & \text{in } \Omega \\ u = v = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{11}$$

has at least two solutions $(u^i_{\lambda,\mu}, v^i_{\lambda,\mu})$*,* $i \in \{1,2\}$*, with* $u^i_{\lambda,\mu}$ *and* $v^i_{\lambda,\mu}$ *axially symmetric and lying in the ball* $\{(u,v) \in X : \|(u,v)\| \le \varrho\}$*.*

An example of nonlinearities F and H coherent with the set of assumptions of Theorem 3.1 is illustrated below.

Example 3.3. Let $\Omega = \omega\times\mathbb{R}^2$, where $\omega := B(0,1) \subset \mathbb{R}^2$, and let $\gamma_1, \gamma_2\colon \Omega \to \mathbb{R}$ be two continuous, non-negative, not-identically-zero functions, compactly supported in Ω and fulfilling $\gamma_i(x_1,x_2) = \gamma_i(g_1x_1, g_2x_2)$, $i = 1,2$, for any $x_1 \in \omega$, $x_2 \in \mathbb{R}^2$, $g_1, g_2 \in O(2)$. Define $F, H\colon \Omega\times\mathbb{R}^2 \to \mathbb{R}$ by

$$F(x,u,v) := \gamma_1(x)\sin\left(|u|^{5/2} + |v|^{15/4}\right), \quad H(x,u,v) := \gamma_2(x)\left(|u|^a + |v|^b\right),$$

for all $(x,u,v) \in \Omega\times\mathbb{R}^2$, where $a \in (2, 8/3)$ and $b \in (3, 24/5)$. Let us justify these choices. The verification of $(F_1) - (F_3)$ and $(H_1) - (H_3)$ is almost immediate, taking also account of the definition of γ_1 and γ_2. Assumption (F_4) is met choosing $\alpha = 5/2$, $\beta = 15/4$, while (H_4) holds for $\alpha \in (a, 8/3)$ and $\beta \in (b, 24/5)$ with $\beta = 3\alpha/2$. As for (F_5), the inequality holds, for instance, with $a_F \equiv b_F \equiv 0$ and $c_F = \gamma_1$. Finally, considering that *supp* γ_1 is $G := O(2)\times O(2)$-invariant, we can select a function $u_0 \in W^{1/2,2}_{0,G}(\overline{\Omega})$ so that $u_0(x) = (\pi/2)^{2/5}$ for all $x \in$ *supp* γ_1. Then (F_6) is fulfilled by the pair $(u_0, 0) \in W^{1/2,2}_{0,G}(\overline{\Omega}) \times W^{1/2,3}_{0,G}(\overline{\Omega})$.

As a result, the system

$$\begin{cases} (-\Delta)^{1/2}u + u = \frac{5}{2}\lambda\gamma_1(x)|u|^{1/2}u\cos(|u|^{5/2} + |v|^{15/4}) + a\mu\gamma_2(x)|u|^{a-2}u & \text{in } \Omega \\ (-\Delta)^{1/2}_3 v + |v|v = \frac{15}{4}\lambda\gamma_1(x)|v|^{7/4}v\cos(|u|^{5/2} + |v|^{15/4}) + b\mu\gamma_2(x)|v|^{b-2}v & \text{in } \Omega \\ u = v = 0 & \text{in } \mathbb{R}^4 \setminus \Omega \end{cases}$$

has at least two axially symmetric solutions with the properties of Theorem 3.1.

By strengthening the requirements on the nonlinearity H, it is possible to deduce the existence of a further (non-trivial) solution.

Theorem 3.4. *Assume the same hypotheses as Theorem* 3.1*. Then, there exists a non-empty open set* $\Lambda \subseteq [0,+\infty[$ *such that, for every* $\lambda \in \Lambda$ *and for every continuous function* $H\colon \Omega\times\mathbb{R}^2 \to \mathbb{R}$ *satisfying* (H_1)*–*(H_4) *and* (H_5) *(in the latter* *assumption it may be $p_H = p$ and $q_H = q$), there exists $\delta > 0$ such that, for each $\mu \in (0, \delta)$, the problem $(S_{\lambda,\mu})$ has at least three solutions axially symmetric.*

Proof. If Λ and $\mathcal{E}_{\lambda,\mu}$ have the same meaning as Theorem 3.1, then the latter theorem guarantees the existence of $\delta_1 > 0$ such that $(S_{\lambda,\mu})$ admits two axially symmetric solutions provided that $\mu \in (0, \delta_1)$.

Now, thanks to (H_5), one has

$$\mathcal{H}(u,v) \geq -\left\|a_H\right\|_{\mu_H/(\mu_H-p_H)} \|u\|_{\mu_H}^{p_H} - \left\|b_H\right\|_{\nu_H/(\nu_H-q_H)} \|v\|_{\nu_H}^{q_H} - \|c_H\|_1$$

and, as a result, using again $Q := \frac{1}{p}||u||_{s,p}^p + \frac{1}{q}||v||_{t,q}^q$,

$$\begin{aligned}
\mathcal{E}_{\lambda,\mu}(u,v) &= \Psi(u,v,\lambda) + \mu\mathcal{H}(u,v) \\
&\geq Q - \lambda\left(c_{\mu_F}^{p_F}\left\|a_F\right\|_{\mu_F/(\mu_F-p_F)}\|u\|_{s,p}^{p_F} + c_{\nu_F}^{q_F}\left\|b_F\right\|_{\nu_F/(\nu_F-q_F)}\|v\|_{t,q}^{q_F} + \|c_F\|_1\right) - \\
&\quad - \mu\left(\left\|a_H\right\|_{\mu_H/(\mu_H-p_H)}\|u\|_{\mu_H}^{p_H} + \left\|b_H\right\|_{\nu_H/(\nu_H-q_H)}\|v\|_{\nu_H}^{q_H} + \|c_H\|_1\right) \\
&\geq Q - \lambda\left(c_{\mu_F}^{p_F}\left\|a_F\right\|_{\mu_F/(\mu_F-p_F)}\|u\|_{s,p}^{p_F} + c_{\nu_F}^{q_F}\left\|b_F\right\|_{\nu_F/(\nu_F-q_F)}\|v\|_{t,q}^{q_F} + \|c_F\|_1\right) - \\
&\quad - \mu\left(c_{\mu_H}^{p_H}\left\|a_H\right\|_{\mu_H/(\mu_H-p_H)}\|u\|_{s,p}^{p_H} + c_{\nu_H}^{q_H}\left\|b_H\right\|_{\nu_H/(\nu_H-q_H)}\|v\|_{t,q}^{q_H} + \|c_H\|_1\right).
\end{aligned}$$

If $p_H < p$ and $q_H < q$ the functional $\mathcal{E}_{\lambda,\mu}$ is clearly coercive. Assuming that $p_H = p$ and $q_H = q$, then choosing

$$0 < \delta < \min\left\{\delta_1,\ \frac{1}{p c_{\mu_H}^{p}\left\|a_H\right\|_{\mu_H/(\mu_H-p)}},\ \frac{1}{q c_{\nu_H}^{q}\left\|b_H\right\|_{\nu_H/(\nu_H-q)}}\right\},$$

for each $\lambda \in \Lambda$ and $\mu \in (0, \delta)$ we get again the coercivity of $\mathcal{E}_{\lambda,\mu}$. The same conclusion can be easily drawn in the remaining cases when either p_H or q_H attain the supremum of their range.

Since $\mathcal{E}_{\lambda,\mu}$ is coercive, any (PS)-sequence of $\mathcal{E}_{\lambda,\mu}$ is bounded in X and, arguing in the wake of [9, Lemma 3.5], we deduce that it admits a strongly convergent subsequence. Then Theorem 2.3 generates the last solution sought for. □

Remark 3.5. The multiplicity results of our paper hold if both n_1 and n_2 equal at least 2, that is, if $n \geq 4$. Indeed, Théorème III.3 of [11], which is a crucial tool to obtain the compactness of the embedding $W_{0,G}^{\sigma,m}(\overline{\Omega}) \hookrightarrow L^\nu(\mathbb{R}^n)$ for $\nu \in (m, m_\sigma^*)$, requires this constraint on the dimension. It would be interesting to know if Théorème III.2 of [11], that concerns Sobolev spaces of integer order, can also be extended to the fractional context. If so, our results continue to hold under weaker assumptions: ω need not satisfy any symmetry condition and can be also one-dimensional, allowing us to cover the case $n = 3$. As a consequence, assumption (F_2) can be modified by dropping the symmetry condition with respect to the variable x_1.

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References

[1] F. Cammaroto, A. Chinní, B. Di Bella: *Multiple solutions for a quasilinear elliptic variational system on strip-like domains*, Proc. Edinb. Math. Soc. (2), 50(3) (2007) 597–603.

[2] F. Cammaroto, L. Vilasi: *On a Schrödinger-Kirchhoff-type equation involving the $p(x)$–Laplacian*, Nonlinear Analysis 81 (2013) 42–53.

[3] F. Cammaroto, L. Vilasi: *Weak solutions for a fractional equation in $\mathbb{R}^n$ with Kirchhoff terms*, Complex Variables and Elliptic Equations 61(10) (2016) 1362–1374.

[4] D. C. De Morais Filho, M. A. S. Souto, J. M. Do Ó: *A compactness embedding lemma, a principle of symmetric criticality and applications to elliptic problems*, Proyecciones 19 (2000) 1–17.

[5] E. Di Nezza, G. Palatucci, E. Valdinoci: *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. 136(5) (2012) 521–573.

[6] M. J. Esteban: *Nonlinear elliptic problems in strip-like domains: symmetry of positive vortex rings*, Nonlinear Analysis 7 (1983) 365–379.

[7] X. L. Fan, Y. Z. Zhao: *Linking and multiplicity results for the p-Laplacian on unbounded cylinders*, J. Math Anal. Appl. 260 (2001) 479–489.

[8] M. R. Grossinho: *Some existence and bifurcation results for nonlinear elliptic problems in strip-like domains*, Ric. Mat. 36 (1987) 127–138.

[9] A. Kristály: *Existence of two non-trivial solutions for a class of quasilinear elliptic variational systems on strip-like domains*, Proc. Edinb. Math. Soc. 48 (2005) 465–477.

[10] A. Kristály, I. I. Mezei: *Multiple solutions for a perturbed system on strip-like domains*, Discr. Cont. Dyn. Systems 5(4) (2012) 789–796.

[11] P. L. Lions: *Symétrie et compacité dans les espaces de Sobolev*, J. Funct. Anal. 49 (1982) 315–334.

[12] G. Molica Bisci, V. Rădulescu, R. Servadei: *Variational Methods for Nonlocal Fractional Problems*, Encyclopedia of Mathematics and its Applications 162, Cambridge University Press, Cambridge (2016).

[13] P. Pucci, J. Serrin: *A mountain pass theorem*, J. Diff. Eq. 60 (1985) 142–149.

[14] B. Ricceri: *Minimax theorems for limits of parametrized functions having at most one local minimum lying in a certain set*, Topology Appl. 153 (2006) 3308–3312.

[15] M. Warma: *Local Lipschitz continuity of the inverse of the fractional p-Laplacian, Hölder type continuity and continuous dependence of solutions to associated parabolic equations on bounded domains*, Nonlinear Analysis 135 (2016) 129–157.