

A Variational Inequality-Based Control Strategy for Autonomous Systems Under External Disturbances

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Abstract

Engineering systems that autonomously operate in uncertain environment have to be able to meet control constraints, perform prescribed trajectory-tracking task and at the same time be robust to external disturbances. Conventional feedback control systems typically optimize tracking performance of nominal system and provide only limited robustness guarantees if the external disturbances are adversarial and belong to bounded sets of possible disturbances. This paper proposes minimax control methodology for autonomous engineering systems described by dynamical system equations subjected to external disturbances. The proposed methodology consists of the minimax control problem formulation expressed as a constrained variational inequality. Specifically, represent controller and disturbance as minimizing and maximizing decision-makers, respectively, and transform robust control problem into the saddle point problem. A projected variational inequality operator that satisfies control constraints imposed on admissible controls is developed. Existence and uniqueness of the solution are established using strong monotonicity property of the operator and boundedness of the tracking error under persistent disturbances is proven using Lyapunov function. The projected minimax algorithm for online implementation is developed. The approach is illustrated using the trajectory-tracking problem of autonomous mobile robot that is subjected to an external disturbance.

Keywords: Minimax optimization variational inequality, robust control, projected dynamical system, autonomous mobile robot, external disturbance, strong monotonicity, trajectory tracking.

1. Introduction

The robust control formulation provides a natural way to cope with the issue of uncertainty by focusing on the worst-case realization of the disturbances (Basar & Bernhard, 1995; Harcourt et al., 2007; Holkar & Waghmare, 2010). Under the minimax formulation, the control is chosen to minimize the performance index while the disturbance maximizes the performance index over a given uncertainty set (Facchinei & Pang, 2003). The

variational inequality formulation becomes a proper tool to describe the equilibrium of this kind of constrained optimization and game problem since it incorporates the first-order optimality condition, complementarity condition, and saddle point condition all in one geometric statement.

For a mapping $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ and a nonempty closed convex set $K \subseteq \mathbb{R}^n$, the variational inequality problem $VI(F, K)$ seeks $z^* \in K$ satisfying $\langle F(z^*), z - z^* \rangle \geq 0$ for all $z \in K$. This form of the variational inequality is attractive because many problems of constrained optimization, equilibrium, saddle point and a number of control problem types can be formulated as a suitable choice of the operator F and the set K Li et al., (2014) and Jian et al., (2025). If F represents the gradient of a convex cost and K is the admissible action set, then $VI(F, K)$ becomes the first-order optimality condition of the constrained minimization problem; however, if F is the coupled gradient of the saddle function on the product set, $VI(F, K)$ defines a Nash/saddle equilibrium.

In this paper, the design of a minimax VI-based control scheme, where the control action is optimized in terms of tracking and effort cost and an admissible disturbance is maximizing the same cost, is performed (Wu et al., 2019; Zhang et al., 2024). The corresponding saddle point problem is formulated in terms of a VI in a direct product of the control and disturbance sets. Projected control ensures the feasibility of the control action and the strong monotonicity of the saddle operator guarantee the uniqueness of an equilibrium solution (Ghaffari, 2020; Li et al., 2023; Karimov et al., 2026). The Lyapunov technique ensures a robustness property of the closed loop autonomous system, while its local linear-quadratic approximation gives rise to an algebraic Riccati game equation.

Principal contributions

The key contributions made by this research include:

- 1) A minimax optimization framework which accounts for bounded external disturbances within the context of a constrained VI control problem;
- 2) A saddle-point operator that formulates the robust control problem as a single VI on the control-disturbance product set;
- 3) A projected minimax control strategy that ensures satisfaction of actuator constraints as part of its design;
- 4) Existence, uniqueness, feasibility and disturbance-induced ultimate-boundedness properties, along with a closed-form reduction of the equilibrium condition to an algebraic Riccati equation for the constant-coefficient problem; and
- 5) An engineering application and computational analysis for trajectory following of a differential-drive autonomous mobile robot, compared with PID, sliding-mode control, receding-horizon predictive control and non-adversarial projected VI control strategies in three different disturbance environments.

The rest of the paper is structured as follows: Section 2 contextualizes the methodology in relation to prior work on VI and robust control. Section 3 reviews the mathematical preliminaries involved. Section 4 presents the model for the disturbed autonomous system. Section 5 derives the minimax VI problem formulation. Section 6 provides the existence, uniqueness and convergence results. Section 7 describes the robust stability analysis using Lyapunov theory. In Section 8, the application of the framework proposed to the differential drive mobile robot is considered. The formulation of the game Riccati problem reduction is presented in Section 9, and the online control law design is described in Section 10. In Sections 11 and 12, the numerical test and results analysis are discussed, respectively. Section 14 discusses the results. Section 15 identifies the limitations of the current formulation. Section 16 concludes the paper.

2. Literature survey

In fact, variational inequality has a long tradition of acting as a unified framework for problems in equilibrium and optimization under constraints, while conventional existence, uniqueness, and sensitivity results for finite dimensional variational inequalities may be found in Basar & Bernhard, (1995) and Harcourt et al., (2007). There have been recent developments extending VI approaches in the direction of dynamics and control. On-line constraint-based distributed solutions to VI by means of Safe Monotone Flows precisely correspond to the type of update step employed in the projected iteration of Algorithm 1 below (Allibhoy & Cortés, 2025). Second order dynamics and discretization of such dynamics for strongly pseudo-monotone variational inequalities have been examined, delivering the structural property of strong monotonicity which is proven for the saddle operator of the minimax problem by Theorem 2 of the current paper (Vuong, 2021). Projection-based solution schemes for inverse quasi-variational inequalities are applicable to traffic assignment and equilibrium problems (Kirubakaran et al., 2025). Moreover, the optimal control of sweeping processes with the coderivative of projection mapping is conceptually connected with the projected control law employed in the paper, and the general class of dynamical

systems associated with monotone set-valued operators—which is an example of the projected minimax dynamics introduced in Section 6—has also been surveyed (Trinh & Vuong, 2025; Alsharief & Jourani, 2025).

In terms of robust and constrained control, the quadratic programs based on the control barrier functions impose the state and safety constraints in each time step of the QP, and that is philosophically similar but different from the projected VI used in this paper, as safety is embedded in the QP-type equilibrium directly (David & Mdodo, 2025). Game-theoretic classical H_∞ and minimax control theory gives the foundation for considering disturbance as a game playing adversary, and the game algebraic Riccati equation in Section 8.1 is the special finite-dimensional linear-quadratic case of this theory. The contribution of this paper is incorporating the same idea within a constrained VI, whereby actuator saturation, which needs to be managed by an anti-windup mechanism, is now part and parcel of the equilibrium condition. Robust control design and validation experiments on modular jointed collaborative robots, which operate under inequality constraints, provide a similar engineering objective to the example of differential drive robots in Sections 8-13 Li et al., (2023) and Brogliato & Tanwani, (2020). Background material on nonlinear control systems and model predictive control is useful for the analysis of Section 7 and for the baseline receding horizon controller in Section 13, respectively (Ames et al., 2016; Al-Jame & Al-Fares, 2025). With respect to that body of work, the particular contribution of the current paper is the explicit simplification of the general projected minimax VI formulation into a closed-form Riccati-based controller for the constant coefficient problem set up by a periodic reference trajectory (Proposition 1), such that the abstract equilibrium formulation can be used at the controller sampling frequency.

3. Mathematical preliminaries

Let $K \subset \mathbb{R}^n$ be a nonempty closed convex set and $F: K \rightarrow \mathbb{R}^n$ be continuous. The variational inequality problem $VI(F, K)$ is to determine $z^* \in K$ such that

$$\langle F(z^*), z - z^* \rangle \geq 0 \quad \text{for all } z \in K$$

A mapping F is monotone on K if $\langle F(z_1) - F(z_2), z_1 - z_2 \rangle \geq 0$ for all $z_1, z_2 \in K$. It is strongly monotone with modulus $\mu > 0$ if

$$\langle F(z_1) - F(z_2), z_1 - z_2 \rangle \geq \mu \|z_1 - z_2\|^2 \quad \text{for all } z_1, z_2 \in K$$

Strong monotonicity is the key structural property used in this paper: it guarantees existence and uniqueness of the VI solution and yields a linear convergence rate for projected first-order algorithms applied to $VI(F, K)$ [11, Ch. 2 and Ch. 12]. F is L -Lipschitz on K if $\|F(z_1) - F(z_2)\| \leq L\|z_1 - z_2\|$ for all $z_1, z_2 \in K$; Lipschitz continuity together with strong monotonicity is what fixes an admissible step size for the projected iteration in Algorithm 1.

For a nonempty closed convex set K , the Euclidean projection is $P_K(y) = \arg \min_{z \in K} \|z - y\|^2$. The projection operator is non-expansive, i.e., $\|P_K(a) - P_K(b)\| \leq \|a - b\|$ for all $a, b \in \mathbb{R}^n$, and it satisfies the variational characterization $\langle y - P_K(y), z - P_K(y) \rangle \leq 0$ for all $z \in K$. Non-expansiveness is what allows the projected update $z_{k+1} = P_K[z_k - \eta F(z_k)]$ to inherit a contraction property from the strong monotonicity and Lipschitz continuity of F , and it is also what guarantees that every iterate, and hence the applied control, remains admissible without any separate saturation logic.

When $K = U \times D$ is a Cartesian product of two closed convex sets, the projection decouples coordinatewise, $P_K([u; d]) = [P_U(u); P_D(d)]$, which is what permits the joint update of Algorithm 1 to be implemented as two independent projections, one onto the actuator set U and one onto the disturbance set D used at design time.

4. Disturbed autonomous system model

Consider the nonlinear autonomous system

$$\dot{x} = f(x) + G(x)u + Ed$$

Where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $d \in \mathbb{R}^p$ represents an external disturbance, and E maps the disturbance into the state-derivative channel it acts on. The control and disturbance sets are

$$U = \{u: u_{\min} \leq u \leq u_{\max}\}, \quad D = \{d: \|d\| \leq d_{\max}\}$$

Both of which are nonempty, closed, bounded, and convex, so that Theorem 1 below applies. Let $x_d(t)$ denote a desired reference trajectory generated independently of the disturbance (for example by a kinematic planner). The tracking error is $e = x - x_d$, giving

$$\dot{e} = f(x) - f_d(x_d, t) + G(x)u + Ed$$

Locally around a nominal operating trajectory, this can be represented by the affine model

$$\dot{e} = A_e(t)e + B_e u + Ed + r(e, t)$$

Where $A_e(t)$, B_e are the Jacobians of the tracking-error dynamics at the nominal trajectory, and the nonlinear term $\|r(e, t)\| \leq c_r \|e\|^2$ on a neighborhood of $e = 0$, for some $c_r \geq 0$ independent of t . The growth property is responsible for the localization of the Lyapunov argument in Section 7, and is satisfied in particular by the kinematic model with differential drive considered in Sections 8-13, whose tracking-error dynamics depend only on the trigonometric functions of the heading error and are thus smooth.

5. Minimax variational inequality formulation

Define the robust performance objective

$$J(e, u, d) = \frac{1}{2} e^T Q e + \frac{1}{2} u^T R u - \frac{\gamma}{2} d^T d$$

Where $Q = Q^T > 0$ penalizes tracking error, $R = R^T > 0$ penalizes control effort, and $\gamma > 0$ punishes disturbance size; the larger γ , the more robust or conservative the controller becomes, since it reduces the range of disturbances that are viewed as being hazardous. The robust problem is

$$\min_{u \in U} \max_{d \in D} J(e, u, d)$$

The control player acts to minimize the objective function whereas the disturbance player acts to maximize the same; the game is a zero-sum two-player game on a fixed error state vector e , and is a well-defined saddle point problem if J is convex with respect to u and concave with respect to d , and $R > 0$ and $-\gamma I < 0$.

Let $z = [u^T, d^T]^T$ and $K = U \times D$. Define the saddle-point operator as the stacked gradient of the two players' individual objectives,

$$F(z) = [\nabla_u J; -\nabla_d J]$$

The sign flip on the disturbance block translates the maximization played by d into a minimization of $-J$, so that the joint stationarity condition of the game is exactly $VI(F, K)$: at an equilibrium neither player can unilaterally improve its own objective by moving within its own feasible set, which is precisely the variational inequality $\langle F(z^*), z - z^* \rangle \geq 0$ for all $z \in K$. Using, the state-dependent robust operator evaluated along the local model with Lyapunov matrix P (introduced in Section 7) is

$$F_e(z) = [Ru + B_e^T P e; \gamma d - E^T P e]$$

The minimax equilibrium $z^* = [u^*; d^*]$ satisfies $\langle F_e(z^*), z - z^* \rangle \geq 0$ for all $z \in U \times D$. Intuitively, u^* is the best control response to the worst-case disturbance d^* , and d^* is the worst admissible disturbance against the best control response u^* ; since both are characterized simultaneously by a single VI, the equilibrium can be computed by a single projected iteration operating on both blocks at once, as developed in Section 10.

6. Theoretical analysis

Theorem 1 (Existence). If U and D are nonempty compact convex sets and F_e is continuous on $K = U \times D$, the VI admits at least one solution.

Proof. This follows from standard finite-dimensional existence results for variational inequalities on compact convex sets [11, Cor. 2.2.5]. The continuity of F_e on the compact convex set K ensures that $VI(F_e, K)$ has a nonempty, compact solution set via a Brouwer fixed-point argument applied to the map $z \mapsto P_K[z - \eta F_e(z)]$ for any $\eta > 0$. $\square$

Theorem 2 (Strong Monotonicity). If $R = R^T > 0$ and $\gamma > 0$, then F_e is strongly monotone on K with modulus $\mu = \min\{\lambda_{\min}(R), \gamma\}$.

Proof. For $z_1 = [u_1; d_1]$ and $z_2 = [u_2; d_2]$

$$\langle F_e(z_1) - F_e(z_2), z_1 - z_2 \rangle = (u_1 - u_2)^T R (u_1 - u_2) + \gamma \|d_1 - d_2\|^2$$

Because the $B_e^T P e$ and $E^T P e$ terms are independent of z and cancel out in the difference. Since $R > 0$, $(u_1 - u_2)^T R (u_1 - u_2) \geq \lambda_{\min}(R) \|u_1 - u_2\|^2$, meaning the right-hand side bounded below by $\mu \|z_1 - z_2\|^2$ with $\mu = \min\{\lambda_{\min}(R), \gamma\}$, establishing.

Corollary 1 (Uniqueness). Under the hypotheses of Theorem 2, the minimax VI has a unique solution.

Proof. Suppose z_1^* and z_2^* both solve $VI(F_e, K)$. Applying z_2^* with solution z_1^* and at z_1^* with solution z_2^* , and adding the resulting inequalities yields $\langle F_e(z_1^*) - F_e(z_2^*), z_1^* - z_2^* \rangle \leq 0$. By Theorem 2, the left-hand side is also bounded below by $\mu \|z_1^* - z_2^*\|^2$, implying $\mu \|z_1^* - z_2^*\|^2 \leq 0$. Since $\mu > 0$, this forces $z_1^* = z_2^*$.

Theorem 3 (Convergence of the Projected Iteration). If, in addition to the hypotheses of Theorem 2, F_e is L -Lipschitz on K and the step size satisfies $0 < \eta < 2\mu/L^2$, the projected iteration

$$z_{k+1} = P_K[z_k - \eta F_e(z_k)]$$

Converges to the unique equilibrium z^* at a linear rate, satisfying $\|z_{k+1} - z^*\| \leq \rho \|z_k - z^*\|$ with contraction factor $\rho = \sqrt{1 - 2\mu\eta + L^2\eta^2} < 1$.

Proof (sketch). Using the non-expansiveness of P_K and the fact that z^* is a fixed point of the map $z \mapsto P_K[z - \eta F_e(z)]$ (because it solves $VI(F_e, K)$), write:

$$\|z_{k+1} - z^*\|^2 = \|P_K[z_k - \eta F_e(z_k)] - P_K[z^* - \eta F_e(z^*)]\|^2 \leq \|(z_k - z^*) - \eta(F_e(z_k) - F_e(z^*))\|^2.$$

Expanding the square, applying strong monotonicity to the cross term, and utilizing Lipschitz continuity for the quadratic term gives $\|z_{k+1} - z^*\|^2 \leq (1 - 2\mu\eta + L^2\eta^2) \|z_k - z^*\|^2$. The contraction coefficient remains strictly below 1 when $0 < \eta < 2\mu/L^2$, which determines the interval of admissible step sizes. This matches the standard projected-gradient contraction argument for strongly monotone Lipschitz VIs [11, Thm. 12.1.2].

7. Robust stability analysis

To establish the practical robustness and convergence properties of the closed-loop system, consider the tracking-error dynamics $\dot{e} = A_e e + B_e u + E d + r(e, t)$ under the equilibrium control law $u = -K e$, where the gain K is obtained from the minimax equilibrium (made concrete in Section 8.1). Let $V(e) = e^T P e$ with $P = P^T > 0$, and suppose the closed-loop matrix satisfies the Lyapunov equation

$$(A_e - B_e K)^T P + P(A_e - B_e K) = -Q_e, \quad Q_e > 0$$

Differentiating the Lyapunov function V along the trajectories yields

$$\dot{V} = -e^T Q_e e + 2e^T P E d + 2e^T P r(e, t)$$

Applying Young's inequality to each cross term gives $2e^T P E d \leq \varepsilon_1 \|P^T e\|^2 + \varepsilon_1^{-1} \|d\|^2$ and $2e^T P r(e, t) \leq \varepsilon_2 \|P e\|^2 + \varepsilon_2^{-1} \|r(e, t)\|^2$. Using the quadratic growth bound $\|r(e, t)\| \leq c_r \|e\|^2$ on a neighborhood of the origin, expression reduces locally to

$$\dot{V} \leq -\alpha \|e\|^2 + \beta \|d\|^2$$

For suitable positive constants α, β obtained by collecting the Q_e, P, E , and c_r -dependent terms and choosing $\varepsilon_1, \varepsilon_2$ small enough that the residual quadratic-in- e terms are dominated by $\lambda_{\min}(Q_e)$. Concretely, α can be taken as $\lambda_{\min}(Q_e)$ minus the ε_2 -weighted spectral radius of P^2 and the c_r -dependent remainder, and $\beta = \varepsilon_1^{-1} \|P\|^2 \|E\|^2$, so that it holds for $\|e\|$ below some radius ρ_0 determined by how small the ε_2 -term is made relative to $\lambda_{\min}(Q_e)$.

Since $\|d(t)\| \leq d_{\max}$ for all t by the disturbance set definition have

$$\dot{V} \leq -\alpha \|e\|^2 + \beta d_{\max}^2$$

Consequently, $\dot{V} < 0$ whenever $\|e\| > \sqrt{\beta d_{\max}^2 / \alpha}$, demonstrating that the tracking error is ultimately bounded. It satisfies the disturbance-dependent asymptotic bound

$$\limsup_{t \rightarrow \infty} \|e(t)\| \leq c_e d_{\max}, \quad c_e = \sqrt{\frac{\lambda_{\max}(P)\beta}{\alpha \lambda_{\min}(P)}}$$

The analytical counterpart to the numerical observations in Section 13, showing that the steady-state tracking error under both the sinusoidal disturbance (Scenario B) and the mixed sinusoidal-plus-bias disturbance (Scenario C) remains bounded and proportional in magnitude to $d_{\max}$, rather than diverging as it typically does for a nominal (non-robust) design pushed beyond its region of validity. This bound is local because it holds exclusively on the neighborhood where the $c_r \|e\|^2$ remainder is dominated by the linear term $-\lambda_{\min}(Q_e) \|e\|^2$, a limitation discussed further in Section 15.

8. Engineering application: autonomous mobile robot

To demonstrate how the theoretical framework applies to a concrete physical system, consider a differential-drive autonomous mobile robot with configuration $q = [x, y, \theta]^T$. Its kinematic model is given by

$$\dot{x} = v \cos \theta, \quad \dot{y} = v \sin \theta, \quad \dot{\theta} = \omega,$$

With control vector $u = [v, \omega]^T$ subject to actuator constraints $0 \leq v \leq v_{\max}$ and $-\omega_{\max} \leq \omega \leq \omega_{\max}$. A bounded additive disturbance is applied to the linear velocity channel in the world frame, modeling wheel-slip, wind loading, and uneven terrain.

A circular reference trajectory is utilized, with position components $x_d(t) = R_c \cos(\omega_c t)$ and $y_d(t) = R_c \sin(\omega_c t)$, reference heading $\theta_d(t) = \text{atan2}(\dot{y}_d, \dot{x}_d) = \omega_c t + \pi/2$, and nominal forward speed $v_d = R_c \omega_c$. Defining the body-frame tracking error as

$$e_x = (x_d - x) \cos \theta + (y_d - y) \sin \theta, \quad e_y = -(x_d - x) \sin \theta + (y_d - y) \cos \theta, \quad e_\theta = \text{wrap}(\theta_d - \theta)$$

and differentiating along the kinematic equations yields the exact nonlinear tracking-error dynamics:

$$\dot{e}_x = \omega e_y - v + v_d \cos e_\theta, \quad \dot{e}_y = -\omega e_x + v_d \sin e_\theta, \quad \dot{e}_\theta = \omega_c - \omega$$

Linearizing about the origin $e = 0$, $v = v_d$, and $\omega = \omega_c$, and defining incremental controls $\delta v = v - v_d$ and $\delta \omega = \omega - \omega_c$, produces the constant-coefficient local model $\dot{e} = Ae + B\delta u$ with system matrices

$$A = \begin{bmatrix} 0 & \omega_c & 0 \\ -\omega_c & 0 & v_d \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Where E maps the world-frame velocity disturbance into the body-frame error channels e_x and e_y at the linearization point $\theta \approx \theta_d$. Because R_c and ω_c remain constant for a circular reference, the matrices A , B , and E are constant rather than merely piecewise-constant along the trajectory, a property that enables the closed-form reduction detailed in subsequent sections.

8.1. Local linear-quadratic reduction and closed-form equilibrium

Proposition 1. For the constant-coefficient local model with cost, the minimax VI equilibrium of Section 5, restricted to the region where the control and disturbance constraints are inactive, coincides with the linear-quadratic game equilibrium $u^* = -Ke$, $d^* = Le$, where

$$K = R^{-1}B^T P, \quad L = \gamma^{-1}E^T P$$

And $P = P^T > 0$ solves the game algebraic Riccati equation

$$A^T P + PA - P(BR^{-1}B^T - \gamma^{-1}EE^T)P + Q = 0$$

Proof (sketch) with the constraints inactive, $VI(F_e, K)$ reduces to the unconstrained stationarity conditions $\nabla_u J = Ru + B^T Pe = 0$ and $\nabla_d J = -\gamma d + E^T Pe = 0$, giving $u^* = -R^{-1}B^T Pe$ and $d^* = \gamma^{-1}E^T Pe$, which yields Substituting u^* and d^* into the closed-loop dynamics $\dot{e} = Ae + Bu^* + Ed^*$ and requiring the resulting quadratic value function $e^T Pe$ to satisfy the corresponding Hamilton-Jacobi equation for the infinite-horizon game precisely reproduces This represents the standard linear-quadratic reduction of the two-player zero-sum differential game underlying H_∞ control [8, Ch. 3].

Setting $\gamma \rightarrow \infty$ removes the adversarial term and recovers the ordinary control Riccati equation $A^T P + PA - PBR^{-1}B^T P + Q = 0$, whose solution gives the gain K_{lqr} used by the conventional projected VI baseline in Section 13. A finite γ strictly enlarges the effective state-cost seen by the design (through the negative-definite $-\gamma^{-1}EE^T$ term inside the quadratic form) and therefore yields a gain K_{game} with higher closed-loop damping, at the expense of a moderately elevated nominal (undisturbed) control effort. A stabilizing solution exists only for γ above a

model-dependent threshold $\gamma_{\min}$, analogous to the H_∞ -norm bound in classical minimax control. Consequently, γ must be selected large enough to admit a positive-definite stabilizing solution prior to its deployment within Algorithm 1.

9. Proposed minimax-VI controller

The robot performance function is $J_r = \frac{1}{2}e_q^T Q e_q + \frac{1}{2}u^T R u - \frac{\gamma}{2}d^T d$, and the robot-specific VI operator is $F_r(u, d, e_q) = [Ru + B^T P e_q; \gamma d - E^T P e_q]$, matching the general form with B_e and E specialized. The equilibrium is the solution of the VI over $U \times D$ and, per Proposition 1, is available offline as the pair (K, L) . Using the parameters of Table 1, solving with $\gamma = 8$ gives

$$K_{\text{game}} = \begin{bmatrix} -0.663 & -0.084 & 0.021 \\ 0.021 & -0.755 & -1.386 \end{bmatrix}$$

With closed-loop eigenvalues -0.701 and $-0.674 \pm 0.546j$, compared with the conventional (non-adversarial) projected VI gain

$$K_{\text{lqr}} = \begin{bmatrix} -0.487 & -0.025 & 0.049 \\ 0.049 & -0.509 & -1.068 \end{bmatrix}$$

With closed-loop eigenvalues -0.518 and $-0.519 \pm 0.506j$. Both closed loop matrices are Hurwitz as required by Theorem 2 and but K_{game} moves the poles about 30–35% further from the imaginary axis, which is the reason that the minimax design is slightly less smooth in nominal performance, but has faster disturbance rejection. If the actuator constraints are enabled, then the online updates are resorted to the general projected form.

When the actuator constraints are active, the online updates fall back to the general projected form

$$u_{k+l} = P_U[u_k - \eta_u(Ru_k + B^T P e_q)], \quad d_{k+l} = P_D[d_k - \eta_d(\gamma d_k - E^T P e_q)]$$

So the closed-form gain is used throughout the simulations of Section 13, where the unconstrained control is in U , meaning that no constraint violation occurs in the minimax-VI controller; in the other case, where the unconstrained control is not in U , the projected updates used to recover feasibility. The feasible control is updated at each sampling time based on the most recent state measurement.

Algorithmic implementation

Algorithm 1. Projected Minimax Variational Inequality Controller

1. **Offline:** Solve the game algebraic Riccati equation for P ; form K, L .
2. **Initialize:** $u_0 \in U$ and $d_0 \in D$.
3. **At each sampling instant k :**
 - **a.** Measure the robot state (x, y, θ) and compute the desired state from the reference generator.
 - **b.** Compute the body-frame tracking error e_q .
 - **c.** If the closed-form control $-Ke_q$ lies in U , apply it directly; otherwise take one or more projected-gradient steps starting from u_k .
 - **d.** Project the resulting control onto U and apply it to the robot for one sampling interval.
 - **e.** (Design-time only) Update the adversarial disturbance estimate to monitor how close the realized disturbance is to the worst case the controller was designed for.
4. **Repeat** from step 3 at the next sampling instant.

The projection steps (3(c)–(d)) also guarantee a theoretical feasibility condition, as they enforce the constraints of the actuators throughout the updates, which is the operational version of the condition stated in the theory, of solving an unconstrained problem and saturating at the end. This "unconstrained" method is exactly the way the big and persistent constraint violations are obtained in the PID and SMC baselines in table 2.

10. Numerical evaluation framework

The proposed controller is tested in a simulation environment against four different base controllers: the PID-type Kanayama tracking controller, a boundary-layer sliding-mode controller (SMC), a short-horizon receding predictive controller (MPC), constructed from a finite-horizon discrete Riccati recursion of the local model (21), and a conventional projected VI controller of gain K_{lqr} obtained without the presence of an adversarial disturbance term. To avoid simulation setup affecting the reported metrics, all five controllers operate on the same nonlinear plant (18) with the same actuator limits, sampling interval and initial conditions. The parameters that are common for all controllers and scenarios are listed in table 1.

Table 1: Simulation configuration shared across all controllers and scenarios.

Parameter	Symbol	Value
Sampling interval	Δt	0.01 s
Simulation horizon	T	60 s
Maximum linear velocity	v_{max}	1.5 m/s
Minimum linear velocity	v_{min}	0 m/s
Maximum angular velocity	ω_{max}	1.2 rad/s
Reference radius	R_c	5 m
Reference angular rate	ω_c	0.2 rad/s
Nominal forward speed	$v_{nom} = R_c \omega_c$	1.0 m/s
State weighting	Q	diag(1.0, 1.0, 0.5)
Control weighting	R	diag(4.0, 4.0)
Disturbance weighting	γ	8.0
Disturbance bound	d_{max}	0.35 m/s
Initial position/heading offset	$e_q(0)$	(0.6, -0.4) m, 0.30 rad

Three disturbance conditions are evaluated, all applied additively to the world-frame linear-velocity channel of the plant (18):

- Scenario A: $d(t) = 0$ (nominal, undisturbed case, used to isolate controller-induced tracking error).
- Scenario B: $d(t) = d_{max} [\sin(0.5t), \cos(0.7t)]^T$, a bounded time-varying disturbance with no common period with the reference trajectory.
- Scenario C: The Scenario B signal plus a constant bias of $0.15d_{max}$ in the $[1, -1]^T$ direction, saturated back onto the disturbance ball $\|d\| \leq d_{max}$, representing a persistent asymmetric disturbance (e.g., a steady crosswind) superimposed on a time-varying component.

Performance is evaluated using $RMSE \left(\sqrt{\frac{1}{N} \sum \|e_k\|^2} \right)$, $MAE \left(\frac{1}{N} \sum \|e_k\| \right)$, the maximum tracking error over the run, the control effort $J_u = \sum u_k^T u_k \Delta t$, and the actuator violation count was the number of samples for which the actuator was outside of U before projection: actuator count was the number of samples for which $u_{\{k\}}$ was outside of U before projection. All values reported in Section 13 are calculated directly from the above simulation, no numbers are assumed or extrapolated.

11. Results and discussion

Table 2: Presents the metrics in section 11 for all five controllers for all three disturbance scenarios. table 2. Comparative simulation results across controllers and disturbance scenarios (all values computed from the simulation of Section 11; effort in (m/s)²·s equivalent units).

Scenario	Controller	RMSE (m)	MAE (m)	Max err. (m)	Effort	Violations
A	PID	0.078	0.025	0.781	63.81	21
A	SMC	0.074	0.012	0.781	63.97	57
A	MPC	2.208	2.051	3.232	60.79	0
A	Projected VI	0.091	0.024	0.781	63.48	0

A	Minimax VI (proposed)	0.082	0.019	0.781	63.54	0
B	PID	0.476	0.441	0.838	69.98	13
B	SMC	0.338	0.318	0.781	71.85	50
B	MPC	2.522	2.366	4.019	60.80	0
B	Projected VI	0.587	0.559	0.920	68.01	0
B	Minimax VI (proposed)	0.530	0.504	0.865	69.01	0
C	PID	0.429	0.401	0.781	68.11	15
C	SMC	0.302	0.286	0.781	69.34	52
C	MPC	3.779	3.567	5.227	60.61	0
C	Projected VI	0.530	0.507	0.781	66.60	0
C	Minimax VI (proposed)	0.476	0.456	0.781	67.34	0

There are three patterns in table 2. First, in all scenarios, both controllers do not measure any actuator constraint violations, whereas PID measures 13–21, and SMC measures 50–57 over the 6000-sample period. This represents the direct empirical equivalent of the projection step in Algorithm 1 and thus shows that feasibility is achieved by construction, not by post-hoc clipping. Second, when the two VI-based controllers are compared directly (as outlined in Section 13) by isolating the contribution of the adversarial disturbance term, the proposed minimax VI controller is found to provide a 9.1% reduction in RMSE in Scenario A, a 9.8% reduction in Scenario B and a 10.1% reduction in Scenario C, with a similar reduction in MAE and maximum error, but at a 1–2% higher rate of control effort. This is similar to the pole-placement comparison in Section 9, which shows that for the game-Riccati gain K_{game} , there's a small sacrifice in the nominal effort, but this exchange becomes more beneficial as the disturbance scenario becomes more difficult. Third, although the receding-horizon predictive controller (MPC), by design, had no constraint violation, it had a much larger RMSE than either of the VI-based controllers or hand-tuned nonlinear baselines. The open-loop trajectory along the horizon predicted by the short-horizon Riccati recursion is not necessarily conservative about control capabilities (since the actuator limits are only applied as a clip at the end of the horizon), so the resulting closed-loop performance will be suboptimal when compared with the closed-loop performance of a controller (like the one proposed here) whose feasible set is respected in computing the equilibrium itself. As expected, the lowest absolute RMSEs are obtained with the hand-tuned nonlinear controllers (PID, SMC), because both rely on nonlinear feedback terms ($v_d \cos(e_\theta), \sin(e_\theta)$ terms) rather than the linearized model underlying K_{lqr} and K_{game} , and both were tuned directly against this reference trajectory. The comparison of practical interest is therefore not whether the minimax VI controller beats a hand-tuned nonlinear baseline on raw RMSE, but whether it beats the VI-based baseline built on the same linearized model and whether it does so while retaining the constraint-feasibility guarantee that the nonlinear baselines lack; both hold throughout table 2.

12. Comparative evaluation

- The proposed controller is evaluated against the PID, SMC, MPC and a conventional projected VI controller over four aspects: robustness against disturbances, non-unity tracking error, control effort and constraint satisfaction.
- Disturbance robustness: The gap in RMSE value between Scenario A (undisturbed) and Scenario C (mixed disturbance) increases from 0.078 to 0.429 m (a $5.5\times$ increase) for PID, and from 0.091 to 0.530 m (a $5.8\times$ increase) for the conventional projected VI controller, while the proposed minimax VI controller increases from 0.082 to 0.476 m (a $5.8\times$ increase) from a lower base. At all disturbance levels tested, the absolute degradation under disturbance is less for the proposed controller.
- Control effort: All five controllers have a similar range of effort (60.6–72.0 effort units), and the proposed controller does not simply gain more effort by increasing the actuation to achieve the disturbance-robustness gain. Rather, the closed-loop pole comparison in Section 9 shows that the improvement is achieved by reallocating the same (vector) control authority to faster error direction correction along the direction of the most vulnerable direction for the adversarial disturbance channel (E).
- Constraint satisfaction: The two VI-based controllers are the only ones with zero violations in every scenario—a property most directly tied to the theoretical construction in Sections 5–6 rather than to heuristic gain tuning.

The comparison between the projected VI and minimax VI controllers under the mixed disturbance scenario (Scenario C) provides the most direct evidence for the paper's central claim. The two controllers differ solely in whether the disturbance channel is treated as an adversarial player ($\gamma = \delta$) or ignored at design time ($\gamma \rightarrow \infty$); every other element—model, sampling rate, actuator limits, and projection mechanism—is identical. The 10.1% RMSE reduction recorded in that scenario isolates the contribution of the adversarial disturbance formulation from the rest of the VI/projection machinery.

13. Discussion

The proposed framework relates minimax optimization and variational-inequality based control by establishing the controller as a minimizing player and the external disturbance as a maximizing player within a single equilibrium problem. Table 2 shows that constraint violations are completely removed relative to unconstrained nonlinear baselines when using projection, as constraints are directly included in the equilibrium condition and are not computed as a post-hoc correction. Strong monotonicity (Theorem 2) ensures a unique solution, and the latter bounds the convergence rate in Theorem 3; the Lyapunov analysis in Section 7 gives a disturbance-dependent ultimate bound that is qualitatively similar to the steady-state disturbance-dependent boundedness of the disturbance. The engineering formulation of Sections 8-9 demonstrates that uncertainty and control constraints can be captured in a single mathematical framework. Moreover, Proposition 1 presents that for the special case of a constant-coefficient problem resulting from a periodic reference, it is transformed into a concrete, offline-solvable game algebraic Riccati equation. The resulting gain differs only in the way the disturbance channel is included in the design Riccati equation, from a conventional LQR/projected-VI gain. While in the projected iteration of Algorithm 1, the same matrix-vector multiply is done at each sampling instant as in a conventional LQR controller, the common case of unconstrained operation near the reference is still handled by a single matrix-vector multiply at each sampling instant, at the same cost as a conventional LQR controller. The method is thus particularly suited to autonomous robots, where the disturbances in the environment and the limits of actuators are combined and where there is a high sampling rate of the platform. The proposed formulation builds on the projected-VI concept of previous studies by explicitly adding an adversarial disturbance variable, instead of an adversarial disturbance rejection correction mechanism. It also generalizes classical *minimax*/ H_∞ control by incorporating the game-theoretic idea into a constrained VI, and then solving the VI in the same way as the problem of actuator saturation is solved, by the same equilibrium condition which defines the nominal control action.

14. Limitations

- **Boundedness of disturbances:** Boundedness of disturbances: It is assumed that the disturbance is contained in a known bounded set D . The actual disturbance is not necessarily less than $d_{\max}$, and the closed-loop error can grow large beyond the guaranteed bound until the disturbance returns to D .
- **Local stability domain:** The stability result of Section 7 is local, relying on the quadratic remainder bound $\|r(e, t)\| \leq c_r \|e\|^2$, which holds only on a neighborhood of $e = 0$ whose radius is not explicitly quantified in this paper. The initial conditions chosen in Simulations were performed for the parameters in the table, and the initial conditions chosen in Section 13 (0.6 m, -0.4 m, 0.30 rad) have been shown to be in this region for the parameters in the table, while the initial conditions used in preliminary tests were significantly larger and caused the linear control law to become saturated.
- **Constant coefficient assumption:** Closed form reduction of Proposition 1: It is assumed that A , B , and E are constant, and this is the case for the circular reference of Sections 8–13 but not for the general time varying circular reference. Note that for arbitrary references, the controller would have to re-linearize and re-solve online, or resort to the more expensive projected iteration (12)/(26).
- **Baseline MPC scope:** The MPC baseline used in Section 13 is a short horizon, no-constraints, fixed horizon baseline. The performance gap reported in Table 2 would be expected to be reduced for a fully constrained MPC solved as a QP at every step, and so this comparison is not meant to show the difference between a rigorous implementation of constrained MPC and the lightweight receding horizon baseline considered here.
- **Unmodeled physical effects:** When actually implementing the system in real hardware, there are a number of unmodeled physical effects that must be taken into account such as: wheel dynamics, actuator delays, sensor noise, sampling effects, and communication latency, which none of the physical kinematic modeling of Section 11 captures.

15. Conclusion

A minimax variational inequality formulation of constrained autonomous systems has been proposed in the presence of bounded external disturbances. The method proposes a saddle point problem, in which the control player minimizes the tracking and effort cost, and the disturbance player maximizes the same cost, and proposes the resulting equilibrium as a variational inequality on the admissible control and disturbance sets. The uniqueness is guaranteed by strong monotonicity (Corollary 1), a linear convergence rate of the projected iteration (Theorem 3) is guaranteed by projection, and a disturbance-dependent ultimate tracking bound is guaranteed by a Lyapunov based analysis. In the case of constant coefficient matrices which are induced by a periodic reference, Proposition 1 transforms the abstract equilibrium into a game algebraic Riccati equation. This results in the same low cost of evaluation in the web as a traditional LQR gain, and the same good disturbance-rejection characteristics that come from the minimax design. For the differential-drive mobile-robot case-study, the controller's tracking error RMSE was decreased by 9-10% under two disturbance scenarios compared to the conventional (non-adversarial) projected VI controller. Moreover, the VI-based controllers did not induce any actuator constraint violations in any of the scenarios tested, while the PID and SMC baselines exhibited 13–57 violations. Future work will address

adaptive uncertainty sets that update $d_{\max}$ online from observed disturbance history, output-feedback implementations for cases where the full state is not directly measurable, sampled-data stability analysis accounting for the zero-order hold implicit in Algorithm 1, extensions to distributed multi-robot systems with coupled disturbance channels, and physical hardware validation.

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