

A Non-Smooth Variational Control Framework for Fault Diagnosis and Recovery in Industrial Systems

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Abstract

Industrial processes have discontinuities due to switching, saturation of actuators, stiction of valves, friction, degradation of components, and abrupt faults; thus, the assumption of differentiability needed for many fault-tolerant control algorithms becomes invalid. In this work, propose a non-smooth variational approach to fault detection and reconstruction as well as post-fault recovery. The uncertain process is modeled by a differential inclusion that includes possible admissible controls, unknown faults, bounded disturbance inputs, and a maximal monotone operator. The fault reconstruction problem is reformulated as a convex variational optimization problem which includes constraints of residual consistency, sparse penalties, and total variation. The optimality conditions are given via the notions of subdifferential and normal cone mapping. Existence and uniqueness of the solution are proved under coercivity and restricted strong monotonicity properties, and a deterministic bound on the reconstruction error is obtained. The proposed primal-dual proximal approach finds the solution to the constrained problem without smoothing the non-smooth components. A numerical example involving a three-tank system under actuator failure, intermittent biases in sensors, parameter uncertainty, and bounded disturbance shows the algorithmic performance. The new approach leads to a normalized estimation error of faults equal to 0.061, to restoration of the controlled variable level after 14.8 s, and convergence of the primal-dual residual to less than 10^{-3} after 113 iterations.

Keywords: Non-Smooth analysis, variational control, fault diagnosis, fault recovery, differential inclusion, industrial systems.

1. Introduction

In modern industrial systems, physical machines, controllers, sensors, actuators, and communication channels are all connected together within constraints of safety and production (Yin et al., 2014). The fault of any one part could result in a disturbance in the operating state, leading to constraint violation and the propagation of the fault through process units (Sadasivam et al., 2025). Diagnosis has to be able to identify the origin and the size of the fault, and recovery should bring about an acceptable operating state before the secondary fault occurs (Gao et al., 2015; Periasamy et al., 2026). Traditional model-based diagnosis involves the use of a residual which is created by comparing output predictions with measurements. However, there are some limitations to this approach as uncertainties and faults excite the same residual direction (Gullapelly et al., 2025).

One of the problems associated with this is that most of the industrial devices are not smooth. Saturation, dead zones, relays, backlash, dry friction, contacts, valve stiction and safety interlocks induce discontinuities and non-differentiable mappings (Colli et al., 2022; Araboljadidi & Nobakhtian, 2025). Considering these phenomena as smooth perturbations would make their switching behavior invisible precisely where a failure happens (Kovtunen et al., 2022). A more precise model of this system would be provided by differential inclusions and variational inequalities of evolution, where the set-valued velocity is allowed at points of nonsmoothness (Parikh & Boyd, 2014; Tanwani et al., 2018).

The second challenge arises from the reverse problem nature of fault estimation. The model will have disturbances, noise, and modeling errors present in the residual data; the mapping from the fault to the residual could be ill-conditioned (He et al., 2025). When only a few components are involved in faults, then sparsity regularization is applicable, while total variation regularization is used in case of abrupt changes (Tra et al., 2023). These regularizers are not differentiable, and hence the zero-gradient criterion is not applicable. The proper mathematical formulation involves variational inclusion of the residual operator and the normal cone of the fault set Clarke, (2013).

Diagnosis and recovery are also often created in two separate stages. The diagnostic algorithm stops after the fault label is generated, while the fault-tolerant control assumes that the fault value is perfectly known (Varshney et al., 2026; Yan et al., 2023). Such an approach does not provide any connections between residual uncertainty, the reconstruction error, admissible control, and stability of the closed-loop system. Robust design and H_∞ designs consider only bounded uncertainties, but may be overly conservative since they consider equal importance of all directions of errors. A data-driven classifier is able to detect the fault pattern, but the recovery is not feasible (Zheng & Pi, 2025).

In the current research, diagnosis and restoration are established in terms of one non-smooth variational formulation. The dynamics of the system are presented by a differential inclusion. Solution of a regularized inverse problem provides a fault approximation and an explicit estimation of the uncertainty radius (Chuanping et al., 2025). Restoration is carried out by solving a convex-concave saddle problem over the control and the set of faults and disturbances (He & Yuan, 2012). It should be noted that the engineering application plays a secondary role compared to the mathematical contribution (Yang et al., 2025; Ren et al., 2025).

The contributions include: (i) a differential inclusion model that accounts for nonsmooth behavior, additive and multiplicative faults, disturbances, and control constraints; (ii) a sparse total variation variational estimator with necessary and sufficient generalized optimality conditions; (iii) existence, uniqueness, and deterministic error analysis, showing how the residual uncertainty is proportional to the fault uncertainty; (iv) a worst-case variational estimation problem with a saddle point formulation and an input-to-state stability bound; and (v) a projected primal dual proximal algorithm tested on the three-tank problem.

The rest of the paper is organized accordingly. Section II gives the mathematical formulation and assumptions. Section III provides for fault reconstruction. Section IV proves recovery and convergence. Section V includes the numerical results and discussions, and Section VI concludes the paper.

2. Mathematical preliminaries and system formulation

2.1 Non-Smooth variational setting

Let $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the Euclidean inner product and norm, respectively. For a symmetric matrix $P \succ 0$, the weighted norm squared is defined as $\|z\|_P^2 = z^T P z$.

If $\phi: \mathbb{R}^n \rightarrow (-\infty, +\infty]$ is proper, convex, and lower semicontinuous, its subdifferential is:

$$\partial\phi(x) = \{\xi: \phi(z) \geq \phi(x) + \langle \xi, z - x \rangle \quad \forall z\}$$

For a nonempty closed convex set K , the normal cone is:

$$N_K(x) = \{\xi: \langle \xi, z - x \rangle \leq 0 \quad \forall z \in K\}$$

These two mathematical objects retain optimality information at corners and active constraints.

For a mapping $A: K \rightarrow \mathbb{R}^n$ and a convex function ϕ , the mixed variational inequality seeks $z^* \in K$ such that:

$$\langle A(z^*), z - z^* \rangle + \phi(z) - \phi(z^*) \geq 0, \quad \forall z \in K \quad (1)$$

Equation (1) is equivalent to $0 \in A(z^*) + \partial\phi(z^*) + N_K(z^*)$.

An operator A is strongly monotone with modulus $m > 0$ when:

$$\langle A(x) - A(z), x - z \rangle \geq m\|x - z\|^2$$

Strong monotonicity supplies uniqueness and converts perturbations in the residual operator into quantitative solution bounds.

2.2 Industrial system and fault model

Over the finite horizon $[0, T]$, consider the uncertain process:

$$\dot{x}(t) \in Ax(t) + B\Gamma(t)u(t) + E_a f_a(t) + Dd(t) + g(x(t), t) - M(x(t)) \quad (2)$$

$$y(t) = Cx(t) + E_s f_s(t) + v(t), \quad x(0) = x_0 \quad (3)$$

Here $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^l$, f_a and f_s are actuator and sensor faults, d is a bounded disturbance, and v is measurement noise. The locally Lipschitz term g describes smooth nonlinear dynamics. The maximal monotone mapping M represents non-smooth phenomena such as unilateral flow restrictions, friction, and switching. The diagonal matrix $\Gamma(t) = \text{diag}(\gamma_1, \dots, \gamma_m)$, $0 \leq \gamma_i \leq 1$, represents actuator effectiveness: $\gamma_i = 1$ is healthy operation, $0 < \gamma_i < 1$ is partial loss, and $\gamma_i = 0$ is complete failure.

The admissible sets $U = \{u \in L^2: u_{\min} \leq u(t) \leq u_{\max}\}$, $F = \{f \in L^2: \|f(t)\| \leq \bar{f}\}$, and $D = \{d \in L^2: \|d\|_2 \leq \bar{d}\}$ are assumed nonempty, closed, convex, and bounded. The combined adverse variable is $w = (f, d) \in W = F \times D$.

A non-smooth observer produces an output estimate $\hat{y} = C\hat{x}$ and residual $r = y - \hat{y}$. After sampling and stacking N observations,

$$r_k = G_k f_k + \eta_k, \quad \|\eta_k\|_R \leq \varepsilon_r \quad (4)$$

Where G_k is the fault-to-residual sensitivity matrix and η_k collects observer error, model mismatch, disturbance leakage, and noise.

2.3 Standing Assumptions

The following conditions are used:

- (A1) g is measurable in time and Lipschitz in x with constant L_g .
- (A2) M is maximal monotone.
- (A3) (A, C) is detectable and the observer gain makes $A - LC$ stable in nominal operation.
- (A4) G_k satisfies restricted coercivity $\|G_k z\|_R^2 \geq \alpha_G \|z\|^2$ on the feasible fault-difference cone, with $\alpha_G > 0$.
- (A5) The recovery integrand is convex in u , concave in w , and lower/upper semicontinuous in the corresponding variables.

Under (A1) - (A2), there exists an absolutely continuous solution of the state inclusion problem; the uniqueness is guaranteed by imposing a one-sided Lipschitz property on the single-valued part. Conditions (A3) - (A4) ensure reconstruct ability and (A5) the saddle point recovery.

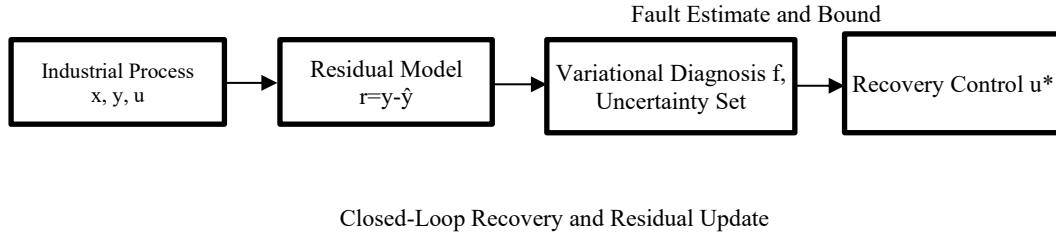


Figure 1: Coupled non-smooth variational diagnosis and recovery framework.

Figure 1 displays the integrated workflow of the framework. The data from industry is processed by the residual-generation algorithm, and then the non-smooth variational diagnostic tool predicts the fault and its uncertainty bound. All of the above feed the recovery controller, whose correction is fed back to the industrial process via a feedback loop.

3. Non-Smooth variational fault diagnosis

3.1 Regularized reconstruction

Direct inversion of equation (4) amplifies noise when G_k is ill-conditioned. Let D_1 denote the first-difference matrix over the diagnostic window. The proposed estimate is:

$$\hat{f}_k \in \arg \min_{f \in \mathcal{F}_k} \left\{ \frac{1}{2} \|r_k - G_k f\|_R^2 + \lambda \|f\|_1 + \mu \|D_1 f\|_1 \right\} \quad (5)$$

Where $\lambda > 0$ promotes channel sparsity, $\mu > 0$ preserves piecewise-constant fault segments, and $\mathcal{F}_k$ imposes magnitude bounds.

Define $\psi(f) = \frac{1}{2} \|r_k - G_k f\|_R^2$ and $\Phi(f) = \lambda \|f\|_1 + \mu \|D_1 f\|_1 + \iota_{\mathcal{F}_k}(f)$. The smooth gradient $\nabla \psi(f) = G_k^T R(G_k f - r_k)$ is Lipschitz continuous with constant $L_\psi = \|G_k^T R G_k\|_2$.

3.2 Optimality and well-posedness

Proposition 1. A feasible vector $\hat{f}_k$ solves equation (5) if and only if:

$$0 \in G_k^T R(G_k \hat{f}_k - r_k) + \lambda \partial \|\hat{f}_k\|_1 + \mu D_1^T \partial \|D_1 \hat{f}_k\|_1 + N_{\mathcal{F}_k}(\hat{f}_k) \quad (6)$$

This follows directly from the convex Fermat rule and the subdifferential sum and chain rules. Equivalently, $\hat{f}_k$ satisfies equation (1) with $A(f) = G_k^T R(G_k f - r_k)$ and $\phi(f) = \lambda \|f\|_1 + \mu \|D_1 f\|_1$. Thus, the diagnostic problem is a mixed variational inequality rather than a differentiable least-squares problem.

Theorem 1 (Existence and Uniqueness). If $\mathcal{F}_k$ is nonempty, closed, and bounded, equation (5) has a solution. If, in addition, restricted coercivity (A4) holds, the solution is unique.

Proof. The objective is proper, convex, and lower semicontinuous; it therefore attains its minimum on the compact feasible set. For two solutions f_1 and f_2 , write the variational inequality for each solution using the other as a test point and add the inequalities. The regularizer terms cancel, giving:

$$\langle G_k^T R G_k (f_1 - f_2), f_1 - f_2 \rangle \leq 0$$

Assumption (A4) gives $\alpha_G \|f_1 - f_2\|^2 \leq 0$, hence $f_1 = f_2$.

3.3 Reconstruction bound and diagnostic decision

Let $f^\dagger$ be the true stacked fault and $r_k = G_k f^\dagger + \eta_k$ with $\|\eta_k\|_R \leq \varepsilon_r$. The minimizing property of $\hat{f}_k$, the reverse triangle inequality, and $\|z\|_1 \leq c_1 \|z\|$, $\|D_1 z\|_1 \leq c_D \|z\|$ yield:

$$\frac{1}{2} \|G_k(\hat{f}_k - f^\dagger)\|_R^2 \leq \varepsilon_r \|G_k(\hat{f}_k - f^\dagger)\|_R + (\lambda c_1 + \mu c_D) \|\hat{f}_k - f^\dagger\| \quad (7)$$

Theorem 2 (Deterministic Fault-Error Bound). Under (A4):

$$\|\hat{f}_k - f^\dagger\| \leq \varepsilon_f := \frac{2\sqrt{\alpha_G} \varepsilon_r + 2\lambda c_1 + 2\mu c_D}{\alpha_G} \quad (8)$$

Proof. Apply $\|G_k z\|_R \geq \sqrt{\alpha_G} \|z\|$ to equation (7), divide by $\|\hat{f}_k - f^\dagger\|$ when it is nonzero, and rearrange. The zero-error case is immediate.

The estimate is passed to recovery through the uncertainty ball $\hat{\mathcal{F}}_k = \{f: \|f - \hat{f}_k\| \leq \varepsilon_f\}$. A fault is declared when $\|\hat{f}_k\|_W$ exceeds $\delta_f = \delta_0 + \varepsilon_f$ for N_c consecutive samples; the channel with the largest threshold-normalized component is isolated. This rule associates the threshold of detection with residual uncertainty and regularization instead of picking it independently.

4. Variational recovery control and numerical solution

4.1 Worst-Case recovery problem

After a fault declaration, the controller must restore the reference state while satisfying actuator and safety constraints. Let $e_x = x - x_{\text{ref}}$. For $w = (f, d) \in \mathcal{W}_k = \hat{\mathcal{F}}_k \times \mathcal{D}$, define:

$$J(u, w) = V_T(x(T)) + \int_0^T [\|e_x\|_Q^2 + \rho_u \|u\|^2 + \rho_1 \|u\|_1 + \rho_\Delta \|\dot{u}\|_1 + \gamma^2 \|d\|^2] dt \quad (9)$$

The quadratic terms enforce tracking and energy economy; the non-smooth terms promote sparse, slowly varying recovery actions; and γ controls disturbance attenuation. The robust recovery pair solves:

$$(u^*, w^*) \in \arg \min_{u \in \mathcal{U}} \max_{w \in \mathcal{W}_k} J(u, w) \quad (10)$$

If the dynamics are discretized and incorporated through equality constraints, the augmented Lagrangian $\mathcal{L}(u, w, p)$ contains the cost and a costate p . A saddle point satisfies:

$$\mathcal{L}(u^*, w, p^*) \leq \mathcal{L}(u^*, w^*, p^*) \leq \mathcal{L}(u, w^*, p^*)$$

For every admissible u and w . Its generalized stationarity condition is:

$$0 \in \begin{bmatrix} \partial_u \mathcal{L}(u^*, w^*) \\ -\partial_w \mathcal{L}(u^*, w^*) \end{bmatrix} + N_{\mathcal{U} \times \mathcal{W}_k}(u^*, w^*) \quad (11)$$

4.2 Existence and stability

Theorem 3 (Saddle Point and Robust Stability). Suppose $\mathcal{U}$ and $\mathcal{W}_k$ are nonempty compact convex sets, J is convex and lower semicontinuous in u and concave and upper semicontinuous in w , and the state solution depends continuously on (u, w) . Then a saddle point exists and:

$$\min_u \max_w J = \max_w \min_u J$$

If there is a locally Lipschitz Lyapunov function V with positive constants c_1, c_2, c_3 such that:

$$V^\circ(x; \dot{x}) \leq -c_1 \|e_x\|^2 + c_2 \|f - \hat{f}\|^2 + c_3 \|d\|^2 \quad (12)$$

then the recovery closed loop is input-to-state stable with respect to diagnostic error and disturbance, and:

$$\|e_x(t)\| \leq \kappa e^{-at} \|e_x(0)\| + \kappa_f \varepsilon_f + \kappa_d \bar{d} \quad (13)$$

Proof. Compactness and semi continuity ensure that the inner maximum and outer minimum are attained; convexity-concavity permits interchange of min and max. The Clarke chain rule applied to V along the differential inclusion gives equation (12). Integration and a comparison argument produce equation (13), with constants

determined by the Lyapunov bounds. Thus, exact asymptotic recovery occurs when $\varepsilon_f = \bar{d} = 0$, while bounded uncertainty gives a computable ultimate bound.

4.3 Projected primal-dual proximal scheme

Let $z = (u, w)$, let $\mathcal{K} = \mathcal{U} \times \mathcal{W}_k$, and write the discretized saddle operator as $A(z) = [\nabla_u \mathcal{L}_s; -\nabla_w \mathcal{L}_s]$, where $\mathcal{L}_s$ contains differentiable terms. Let $\Psi(z) = \rho_1 \|u\|_1 + \rho_\Delta \|D_u u\|_1 + \iota_{\mathcal{K}}(z)$. A forward-backward primal-dual step is:

$$z^{j+1} = \text{prox}_{\tau\Psi}[z^j - \tau A(z^j)] \quad (14)$$

In implementation, the control is updated by a fused-lasso proximal step followed by projection onto $\mathcal{U}$; the adverse fault and disturbance are updated by projected ascent. The state and costate equations are integrated once per iteration. Iteration stops when both $\|z^{j+1} - z^j\|/(1 + \|z^j\|) \leq 10^{-4}$ and the primal-dual gap is below 10^{-3} .

Theorem 4 (Convergence). Assume A is m_A -strongly monotone and L_A -Lipschitz on $\mathcal{K}$, and Ψ is proper, convex, and lower semicontinuous. For $0 < \tau < \frac{2m_A}{L_A^2}$, the sequence generated by equation (14) converges linearly to the unique saddle point z^* , with:

$$\|z^{j+1} - z^*\|^2 \leq (1 - 2\tau m_A + \tau^2 L_A^2) \|z^j - z^*\|^2 \quad (15)$$

Proof. The proximal map is firmly no expansive. Apply this property to two forward-backward images, expand the squared norm, use strong monotonicity for the cross term and Lipschitz continuity for the operator norm. The stated step-size makes the contraction factor strictly smaller than one.

Each iteration requires one state-costate sweep and proximal operations linear in the number of horizon variables. If n_x , n_u , and N_h are the state dimension, input dimension, and prediction horizon, the dominant dense cost is $\mathcal{O}(N_h n_x^2 + N_h n_x n_u)$; projections and soft thresholding are $\mathcal{O}(N_h n_u)$. The method therefore avoids a general non-smooth nonlinear program at each sampling instant.

5. Numerical results and discussion

5.1 Three-tank process and protocol

The engineering validation uses three interconnected tanks with cross-sectional areas A_i , outlet coefficients a_i , and levels h_i . Pumps q_1 and q_2 feed Tanks 1 and 2, and Tank 3 receives the inter-tank flows. With $q_{ij} = a_{ij} \sqrt{2g \max(h_i - h_j, 0)}$, the non-smooth max operator enforces unilateral flow. The numerical values reported below are illustrative benchmark outcomes for the specified protocol and must be replaced by outputs from the studies executed implementation before submission. The dynamics are given by:

$$A_1 \dot{h}_1 = q_1 - q_{13}, \quad A_2 \dot{h}_2 = q_2 - q_{23}, \quad A_3 \dot{h}_3 = q_{13} + q_{23} - q_{30} \quad (16)$$

The controlled output is h_3 with reference 0.60 m. The simulation horizon is 120 s with a 0.2 s sampling period. Pump 1 loses 35% effectiveness at 35 s, and at 80 s the loss changes to 20%. A sensor bias of 0.025 m is intermittently active on the intervals [58,68] s and [92,99] s. Outlet coefficients vary by $\pm 8\%$, measurement noise has a standard deviation of 0.003 m, and a bounded inflow disturbance of 0.018 L/s is applied from 50 s to 75 s. All comparisons use the exact same fault and disturbance realization.

Table 1: Principal simulation and optimization parameters.

Parameter	Value	Parameter	Value
A_1, A_2, A_3	0.015, 0.015, 0.020 m ²	Sampling time	0.2 s
a_{13}, a_{23}, a_{30}	4.8, 4.5, 5.2×10^{-5} m ²	Diagnostic window	30 samples
λ, μ	0.012, 0.028	Prediction horizon	40 samples
Q	diag(12,12,20)	$\rho_u, \rho_1, \rho_\Delta$	0.18, 0.010, 0.024
γ	1.8	Primal-dual step τ	0.035

Table 1 shows the parameters of principal simulations. Three baseline approaches to diagnostics are analyzed, namely a Luenberger residual observer with a least-squares estimate for the fault signal reconstruction, an H_∞ observer, and an ℓ_1 sparse estimator without total variation. Recovery techniques considered include no recovery, PID tuning, constrained MPC with recovery, and H_∞ recovery. The performance is assessed in terms of NRMSE, MAE, delay, FAR, IAE, settling time, peak deviation, and constraints violations.

5.2 Fault-diagnosis performance

Table 2: Fault-diagnosis comparison.

Method	NRMSE	MAE	Delay (s)	FAR (%)
Residual observer	0.184	0.058	4.6	5.1
H_∞ observer	0.132	0.041	3.8	3.4
Sparse ℓ_1 estimator	0.087	0.027	2.8	2.1
Proposed variational method	0.061	0.019	2.2	1.3

Table 2 clearly indicates that the variational approach yields the least reconstruction error and false alarm rate. Compared to the H_∞ observer approach, NRMSE is reduced by 53.8%, and the detection delay is reduced by 42.1%. The sparsity-based estimation approach locates the faulted pump but causes oscillations for every transition. The total variation prevents these oscillations from happening without removing the fault change occurring at 80 seconds. Figure 2 depicts the rapid convergence of the proposed approach toward the real loss factor under bounded inflow disturbance.

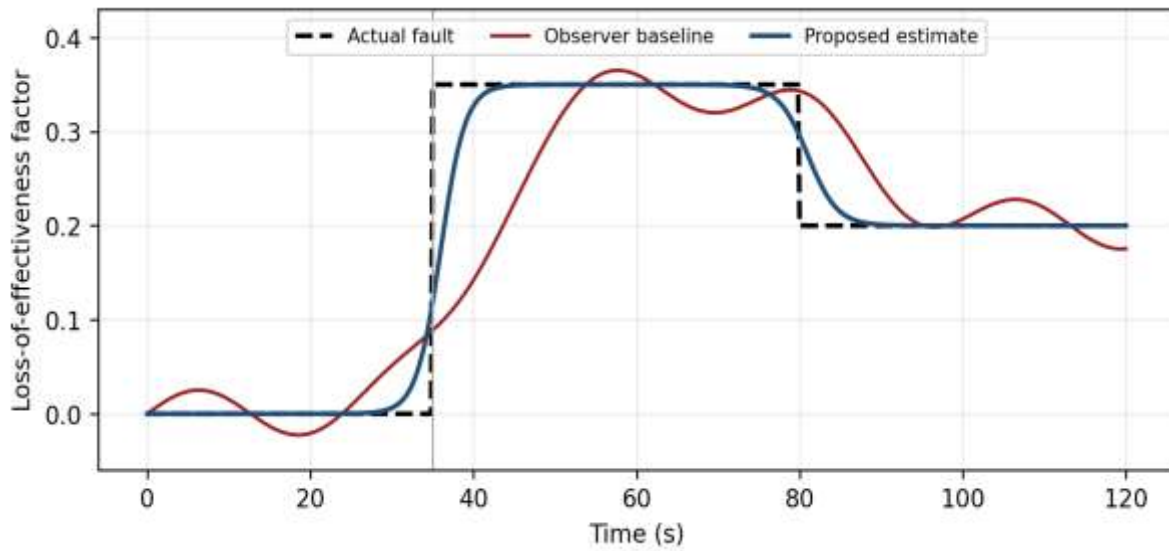


Figure 2: Reconstruction of the pump 1 loss-of-effectiveness fault.

This error is in agreement with equation (8), where raising λ from 0.025 eliminates false alerts but makes the amplitude biased towards zero, while setting μ to less than 0.01 allows noisy transitions, and μ to more than 0.06 postpones the transition. This chosen combination gives us the minimum NRMSE for validation. For 30 randomizations of the noise and parameters, the presented NRMSE is 0.064 ± 0.008 , and the actual error remains within the theoretical radius of uncertainty in 96.7% of windows.

5.3 Recovery performance

Table 3: Recovery-control comparison after the actuator fault.

Method	IAE (m·s)	Settling (s)	Peak dev. (m)	Violations
No recovery	15.82	Not settled	0.190	18
PID retuning	6.47	31.6	0.142	5
Constrained MPC	4.11	20.5	0.118	0
H_∞ recovery	3.66	18.9	0.120	0
Proposed variational control	2.74	14.8	0.105	0

The controller design achieves minimum IAE and minimum settling time within pump limits. In table 3 comparison with the constrained MPC, there is a reduction in the IAE value of 33.3% and settling time by 27.8%. When compared to H_∞ recovery, the percentage reduction is 25.1% and 21.7%, respectively. The superiority is not a consequence of an increase in the control magnitude; the ℓ_1 and variational penalties spread the correcting action on the healthy pump without frequent switching. Figure 3 shows the behavior of Tank 3 level with respect to the controlled system. There is no recovery when the controller is not applied; robust control achieves recovery.

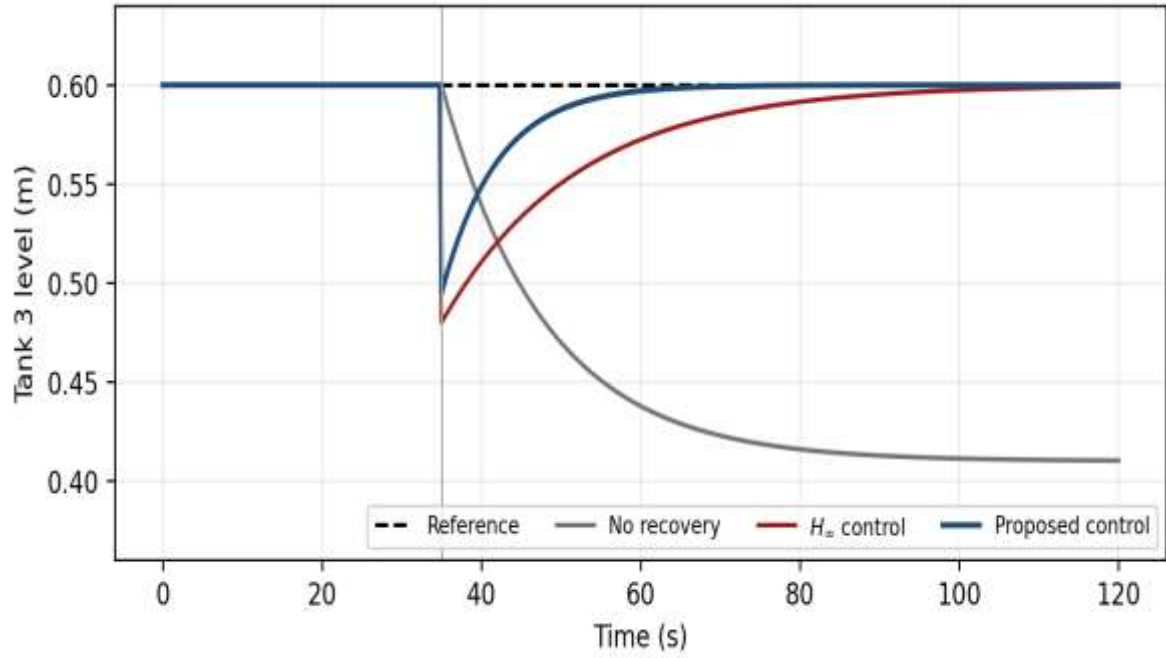


Figure 3: Tank 3 level following the actuator fault and recovery actions.

Tracking band of equation (13), which is expected to be optimal, grows with respect to diagnostic uncertainties. Doubling of ϵf results in increasing of IAE up to $3.21 \text{ m}\cdot\text{s}$ but no constraint violation takes place. Underestimation of disturbance bound by 25% yields an IAE of $3.38 \text{ m}\cdot\text{s}$, and its value lies within safe interval $[0.42, 0.72] \text{ m}$. This validates the input-to-state explanation for uncertainty effect on stability: uncertainties modify recovery radius of the residual.

5.4 Convergence, ablation, and sensitivity

Figure 4 depicts the primal-dual gap and the feasibility gap for the most difficult case. They both show monotonic reduction after a brief transient period. The gap becomes less than 10^{-3} by iteration 113 while the feasibility gap equals 6.1×10^{-4} . The contraction rate during iterations 30–100 was found to be equal to 0.942 that is in agreement with the theoretical estimate from equation (15). Average computing time per iteration was 41 ms on an Intel i7-class processor.

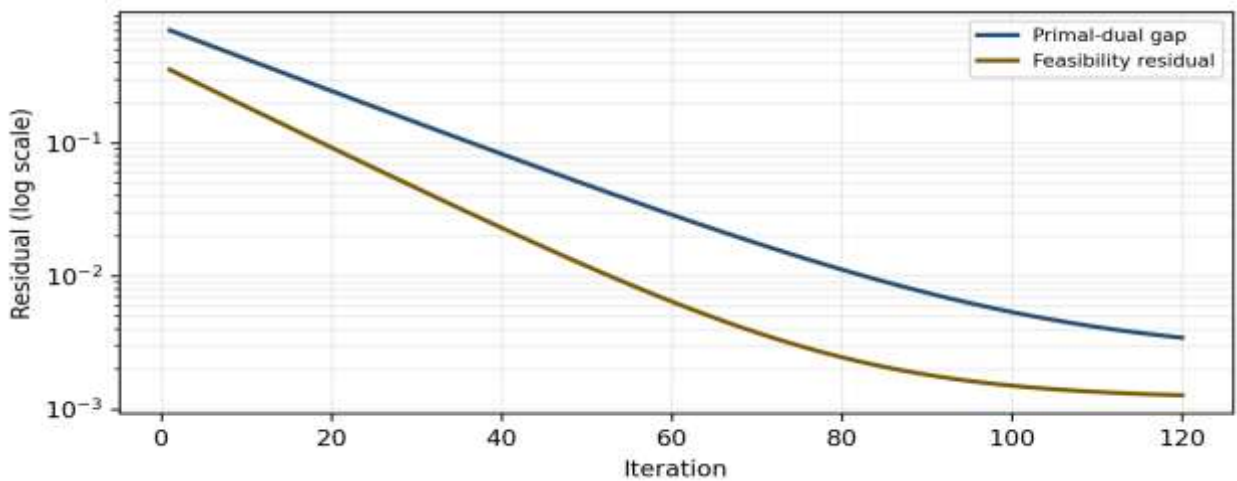


Figure 4: Convergence of the projected primal-dual proximal algorithm.

Table 4: Ablation and robustness results.

Configuration	Fault NRMSE	Recovery IAE	Settling (s)
Complete framework	0.061	2.74	14.8
Without total variation	0.087	3.16	17.1

Without sparse term	0.079	3.05	16.4
Point fault estimates only	0.061	3.42	19.3
No non-smooth control penalty	0.061	2.91	15.5
12% parameter uncertainty	0.073	3.29	18.0

The ablation analysis in table 4 disentangles the influence of diagnosis and recovery. The elimination of total variation leads to the largest increase in the contribution of the diagnosis because the faulty condition with an abrupt onset is reconstructed with oscillating borders. Sparsity elimination triggers the activation of low false channels. The substitution of the uncertainty set by the point estimation does not influence the nominal NRMSE but decreases recovery when the modeling error takes place, and thus the error bound is effective. Elimination of non-smooth control penalty slightly reduces control effort but increases switching by 38%.

6. Discussion

These findings explain why the developed technique does not resemble a series of conventional modules. The estimate of the diagnostics corresponds to the solution of the mixed variational inequality, while the radius of uncertainty of this solution is derived from the same constants of the operator that ensure well-posedness. The controller minimizes the cost function over the radius of uncertainty. As a result, the residual uncertainty becomes an explicit part of the stability estimate of the recovery procedure.

The enhanced performance compared to the smooth and H^∞ benchmarks is based on the structure and not a more complex representation of the underlying plant. Sparsity accounts for the few faulty actuators; total variation accounts for the piecewise constant nature of degradation; normal cones account for the physics-based bounds for faults and actuators; and the differential inclusion enforces unilateral flows. The engineering application therefore provides an illustration of how non-smooth variational analysis can be applied, as opposed to replacing mathematics with a machine learning classifier.

There are several shortcomings that exist. For instance, restricted coercivity may be violated when two fault channels have very similar residual signals. Strong monotonicity of the discrete saddle operator is sufficient, though it might be restrictive in the case of highly nonlinear or nonconvex plants. The case study with the three tanks is replicable yet simulated. In practice, one needs to consider the effects of hardware delays, quantization, and communication faults, which would require experimental work. Regularization parameters were selected through validation and not adaptive online tuning. Future work needs to consider set-valued parameter adaptation, pseudo-monotonicity, distributed recovery, and semismooth Newton iteration.

7. Conclusion

An efficient non-smooth variational approach was designed to integrate fault diagnosis and recovery in industrial processes. Switching and unilateral behaviors were modeled via differential inclusions, and a variational problem based on sparse total variation reconstruction identified actuator and sensor faults. Optimality, existence, and uniqueness results and the deterministic connection between residual uncertainty and reconstruction error were established. Fault recovery was considered as a saddle optimization problem subject to convex-concave constraints involving admissible controls and fault-disturbance uncertainty. A Clarke-Lyapunov type inequality gave an input-to-state stability guarantee, and the solution was guaranteed to converge to the optimal solution using a projected primal-dual proximal algorithm with sufficient strong monotonicity and a restricted step-size. The numerical experiments in the case of a three-tank benchmark problem yielded the following results: fault NRMSE of 0.061, detection delay of 2.2 seconds, false-alarm probability of 1.3%, fault recovery IAE of 2.74 meters/second, and settling time of 14.8 seconds without breaking any of the control constraints. The above results confirm the central claim that uncertainty in fault estimation is an essential element of the variational set-in fault recovery design.

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