

A Set-Valued Optimization Approach for uncertain Decision-Making in Autonomous Engineering Systems

Anisha Chaudhary¹, R. Mohana Priya², Swati Shivkumar Shriyal³, Nisha Pandey⁴, Dr. Bhawna Janghel Rajput⁵, Raman Verma⁶, Manoj I. Patel⁷, R. Harshini⁸

¹Quantum University Research Center, Quantum University, Uttarakhand, India.
anisha.chaudhary@quantumeducation.in, <https://orcid.org/0009-0000-2635-3718>

²Associate Professor, Department of Electronics & Communication Engineering, Aarupadai Veedu Institute of Technology, Vinayaka Mission's Research Foundation (DU), Tamil Nadu, India. mohanapriya@avit.ac.in; mohanapriyadhinakar@gmail.com, <https://orcid.org/0000-0003-0374-6556>

³Department of Artificial Intelligence, Vishwakarma University, Maharashtra, Pune, India.
swati.shriyal@vupune.ac.in

⁴Assistant Professor, Department of Computer Science & Engineering,
Noida International University, Uttar Pradesh, India. nisha.pandey@niu.edu.in

⁵Rungta College of Engineering and Technology, Rungta International Skills University,
Bhilai, Chhattisgarh, India. bhawna.janghel@rungta.org

⁶Centre for Research Impact & Outcome, Chitkara University Institute of Engineering and Technology, Chitkara University, Rajpura, Punjab, India. raman.verma.orp@chitkara.edu.in,
<https://orcid.org/0009-0006-5137-2260>

⁷Assistant Professor, Faculty of Computer Science and Application, Gokul Global University,
Sidhpur, Gujarat, India. manoj.patel@gokuluniversity.ac.in

⁸Assistant Professor, Computer Science, Meenakshi College of Arts and Science, Meenakshi Academy of Higher Education and Research, Chennai, Tamil Nadu, India.
harshinir@maher.ac.in

Abstract

Autonomous engineering systems make decisions prior to disturbance, perception error, and model uncertainties becoming known. In such a situation, a decision leads not to a single objective vector but to a whole collection of objective vectors that can be achieved. The current paper presents a mathematically precise set-valued minimax framework for such decisions. A compact uncertainty set is transformed into a closed upper image in a finite-dimensional ordered space; robust preferences are defined via the lower set relation generated by a pointed convex cone. The resulting problem minimizes an outcome set rather than a prechosen weighted sum of the objectives. A dual-cone scalarization theorem shows that each weakly robust minimal solution is supported by a scalar minimax problem, and the converse holds, too. Solvability is guaranteed by compactness and Painlevé-Kuratowski outer semicontinuity of the mapping under consideration. Given convex-concave-continuous assumptions, a minimax Sion equality is proven for each scalarization of the robust decision problem. Necessary and sufficient first-order optimality conditions are stated in the form of a set-valued variational inequality involving contingent derivatives and normal cones. A regularized projected primal-dual algorithm is formulated; Fejér monotonicity and an ergodic gap bound of $O(1/N)$ are established for monotone Lipschitz mappings. Robustness of the solution multifunction is characterized in terms of the Hausdorff stability of the uncertainty set. An example of a two-dimensional autonomous navigation problem is provided to illustrate the trade-off between tracking error, energy cost, collision risk, and control effort that can be achieved by the proposed formulation. The computational experiment employs synthetic data for bounded disturbance set and thus is a proof-of-concept for the mathematical approach rather than field validation. Over 200 test cases, the suggested algorithm outperforms both the nominal and the average scenario solutions in terms of worst-case objective and constraint violations, preserving explicit robustness guarantees.

Keywords: Set-Valued optimization, minimax theory, robust efficiency, variational inequality, uncertainty, autonomous systems, scalarization.

1. Introduction

The decision-making module in autonomous vehicles, mobile robots, unmanned aerial systems, and adaptive industrial systems has to deal with uncertainties that cannot be described by one probabilistic distribution (Varshney et al., 2026; Kováčová & Cialenco, 2026). The state can be estimated in set terms due to bounded measurement errors; obstacle motion can be described using the framework of reachability tubes; model parameters can be known in the form of intervals or compact uncertainty sets. As a result, each feasible control action leads to a set of performance vectors (Torre et al., 2026; Huang et al., 2023; Zhang et al., 2010). An aggregation of this set either in terms of the expectation or some specific weighted average before optimization might ignore some bad realizations of uncertainties and be dependent on the choice of weights (Rahimian & Mehrotra, 2022; Rocca, 2025).

Set-valued analysis offers the proper vocabulary for dealing with multifunction, graphical limits, contingent derivatives, and differential inclusions (Ma & Denoeux, 2019; Ghaffari-Hadigheh, 2023; Şahin et al., 2021). Set optimization is an extension of vector optimization that involves ordering whole sets within the outcome space instead of just vectors (Crespi et al., 2017; Ehrgott et al., 2014; Sarwar et al., 2025). Correspondingly, robust optimization incorporates uncertainty through deterministic sets while ensuring that feasible solutions remain valid in every admissible scenario (Shirke & Udayakumar, 2019; Alamelu Sathyabama et al., 2026). Robust multi-objective optimization models uncover an important connection between both branches of optimization in that an image of a vector-valued uncertain objective function is a set, and robust preferences may be investigated using set relations (Gutiérrez et al., 2021; Adly et al., 2013; Keith & Ahner, 2021).

Previous works on autonomous control have used techniques such as chance constraint programming, risk sensitive cost functionals, model predictive control, and distributionally robust objectives. Such methods are useful, but they are usually based on the assumption of scalar loss functions, valid probability models, or certain risk functions (Shetty & Nair, 2024; Hewing et al., 2020). This paper is different because it attempts to solve a new problem: how do you compare two decisions whose effects are a set of conflicting engineering results, and you only have access to a compact uncertainty representation? The solution will need a solution concept, an existence theory, scalarization without losing weak minimality, a variational characterization, and an algorithm.

The primary contributions are a lower-set ordering minimax framework for which the underlying mathematical entity is the robust upper image and four complementary findings. First, the existence result guarantees weakly minimal decisions with the assumptions of compactness and outer semi-continuity. Second, the dual cone separation provides a family of scalar minimax problems that can give rise to a minimax identity under convexity-concavity. Third, contingent differentiation leads to a set-valued variational inequality and normal-cone optimality condition. Fourth, the regularized primal-dual scheme is proved to be convergent and comes with an ergodic duality gap estimate. A further perturbation result gives the effect of a change of the sensed uncertainty set on the solution set.

The application is intentionally secondary to the theoretical development. The problem of navigation with an unmanned aircraft in two dimensions is utilized solely as an instance of the underlying mathematics. The spaces, mappings, constraints, and algorithm are defined in terms of formulas and parameters. The numerical values are calculated via specific equations and predetermined random seeds. They are not claimed as measurements of a real-life craft.

The remaining sections are structured as follows: Section 2 provides the preliminary concepts in ordered sets and variational methods. Section 3 states the uncertain autonomous decision problem. Section 4 shows existence, scalarization, minimax, optimality, and stability theorems. Section 5 provides the algorithm and its convergence analysis. Section 6 presents the reproducible engineering example. Section 7 concludes the paper.

2. Mathematical preliminaries

2.1 Spaces, cones, and set relations

Let X be a nonempty compact convex subset of $\mathbb{R}^n$, let U be a nonempty compact subset of $\mathbb{R}^p$, and let $Y = \mathbb{R}^m$ be the objective space. A closed, convex, pointed cone $K \subset Y$ with nonempty interior induces $y_1 \leq_K y_2$ if $y_2 - y_1 \in K$. Its positive dual cone is $K^+ = \{w \in Y^*: \langle w, k \rangle \geq 0 \text{ for every } k \in K\}$. The normalized dual base is $B = \{w \in K^*: \|w\|_* = 1\}$.

For $A, B \subset Y$, the lower set relation is:

$$A \preceq_K B \Leftrightarrow B \subseteq A + K. \quad (1)$$

Thus A is preferred to B if every point of B is dominated by at least one K -shifted point of A . Equality modulo K is handled through closed upper extensions $A^K = \text{cl}(A + K)$. The family $\mathcal{G}(K) = \{A \subset Y: A = \text{cl conv}(A + K)\}$ forms a complete lattice under reverse inclusion; infima are closed convex hulls of unions and suprema are intersections. This lattice interpretation avoids forcing set images into a total order.

A set A is weakly K -minimal in a family $\mathcal{A}$ if no $B \in \mathcal{A}$ satisfies $B \subset A - \text{int } K$ in the strict lower-set sense. For decision mappings, $x^* \in X$ is weakly robust minimal when no $x \in X$ obeys $\mathcal{R}(x) \subseteq \mathcal{R}(x^*) - \text{int } K$, where $\mathcal{R}$ is the robust outcome multifunction defined below.

2.2 Multifunction continuity and generalized derivatives

For a multifunction $F: X \rightrightarrows Y$, $\text{graph } F = \{(x, y): y \in F(x)\}$. It is outer semicontinuous at x if $x^\nu \rightarrow x$, $y^\nu \rightarrow y$, and $y^\nu \in F(x^\nu)$ imply $y \in F(x)$. It is locally bounded when nearby images lie in a common bounded set. In finite dimensions, closed graph plus local boundedness gives outer semicontinuity for compact-valued maps.

The Hausdorff distance between compact sets A and B is:

$$d^H(A, B) = \max \left\{ \sup_{a \in A} \text{dist}(a, B), \sup_{b \in B} \text{dist}(b, A) \right\} \quad (2)$$

For $(x, y) \in \text{graph } F$, the contingent derivative $DF(x, y): \mathbb{R}^n \rightrightarrows Y$ is defined by $\text{graph } DF(x, y) = T_{\text{graph } F}(x, y)$, where T denotes the Bouligand tangent cone. The limiting normal cone is $N_C(x) = \{v: \text{there exist } x^\nu \rightarrow x \text{ and } v^\nu \rightarrow v \text{ with } v^\nu \text{ proximal-normal to } C \text{ at } x^\nu\}$. These objects provide first-order information without differentiability.

2.3 Scalarization and gap functions

For $w \in B$ and a nonempty compact set $A \subset Y$, define the lower support scalarization $\sigma_w(A) = \min_{a \in A} \langle w, a \rangle$. If $A \setminus \text{preccurlyeq}_K B$, then $\sigma_w(A) \leq \sigma_w(B)$ for every $w \in K^+$. A strict separation statement holds when closed convex upper images fail to be ordered.

For a monotone variational inequality $\text{VI}(Z, T)$, where Z is closed and convex and $T: Z \rightarrow \mathbb{R}^d$, the dual gap is:

$$G(z) = \sup_{q \in Z} \langle T(q), z - q \rangle \quad (3)$$

G is nonnegative and vanishes at a solution under standard continuity and monotonicity assumptions. It will be used to state an algorithmic rate.

3. 3. Set-Valued uncertain decision model

3.1 Autonomous system and uncertainty representation

Consider the discrete autonomous dynamics $s^+ = f(s, x, \xi)$, where $s \in \mathbb{R}^d$ is the observed state, $x \in X(s) \subset \mathbb{R}^n$ is a control or planning decision, and $\xi \in U(s) \subset \mathbb{R}^p$ is an unresolved disturbance. $X(s)$ includes actuator, rate, geometry, and safety restrictions. $U(s)$ may be constructed from sensor-error bounds, a reachable set, interval identification, or a confidence region. No probability distribution is required.

Let $q: X \times U \rightarrow Y$ contain m engineering criteria. Smaller components are preferred. To represent residual ambiguity beyond ξ , introduce a compact-valued disturbance-output map $E(x, \xi) \rightrightarrows Y$ and define the compact attainable image:

$$F(x, \xi) = q(x, \xi) + E(x, \xi) \quad (4)$$

Ordering is applied through the upper extension $F^K(x, \xi) = F(x, \xi) + K$, while compactness is retained in F . The robust attainable outcome set of x is:

$$\mathcal{R}(x) = \text{cl conv } \bigcup_{\xi \in U} F(x, \xi) \quad (5)$$

Equation (5) retains correlations among criteria. In contrast, component-wise worst-case optimization constructs the vector whose j -th component is $\sup_{\xi} q_j(x, \xi)$; that vector may combine maxima attained by mutually incompatible scenarios. The set-valued image avoids this artificial recombination.

3.2 Robust set problem and minimax scalar family

The primary problem is:

$$(SVP) \quad \text{minimize } \mathcal{R}^K(x) = \mathcal{R}(x) + K \text{ by reverse inclusion, subject to } x \in X \quad (6)$$

Because $\preceq_k$ is generally partial, SVP returns a set $S_{\min}$ of weakly minimal decisions. When an operating policy needs one action, a dual preference $w \in B$ selects the scalar robust problem:

$$(P_w) \quad \min_{x \in X} \varphi_w(x), \quad \varphi_w(x) = \sup_{\xi \in U} \inf_{y \in F(x, \xi)} \langle w, y \rangle \quad (7)$$

For the translational model (4), $\varphi_w(x) = \sup_{\xi} \left[\langle w, q(x, \xi) \rangle + \min_{e \in E(x, \xi)} \langle w, e \rangle \right]$. The supremum is the response of adversarial uncertainty, while the inner infimum is the best unresolved output among the output set. In case E is symmetric error, then this expression becomes a support function correction. The decision maker can choose to set w or solve for solutions in a finite cover of B .

3.3 Assumptions

Table 1: Standing assumptions used in the analysis.

Code	Mathematical condition	Role
A1	X is nonempty, compact, and convex.	Existence and projection
A2	U is nonempty and compact.	Worst-case attainment
A3	F has nonempty compact convex values; ordering uses $F+K$.	Set separation
A4	F is outer semicontinuous and locally bounded.	Closed robust image
A5	For each $w \in B$, $h_w(x, \xi) = \inf_{y \in F(x, \xi)} \langle w, y \rangle$ is convex in x and concave in ξ .	Minimax equality
A6	The saddle operator is monotone and L -Lipschitz on $Z=X \times U$.	Algorithmic convergence

Table 1 presents the six mathematical conditions that underpin the proposed model. The stated conditions ensure the formation of bounded and closed feasible sets, regularity of the set-valued map, minimax equivalence, and monotonicity of the saddle operator, thus constituting the basis of the following existence, stability, optimality, and convergence theorems.

4. Theoretical analysis

4.1 Existence of a weak robust minimum

Theorem 1 (Existence). Under A1–A4, the robust attainable map $\mathcal{R}$ has nonempty compact convex images, its upper extension $\mathcal{R}^K$ is closed and convex, and SVP admits at least one weakly K -minimal decision.

Proof. Compactness of U and local boundedness of F imply boundedness of the union in (5). Outer semicontinuity, compact values, and compactness of U make the scenario graph compact; hence its projected union is compact. Closure and convexification preserve nonemptiness and compactness in finite dimensions, and $\mathcal{R}^K = \mathcal{R} + K$ is closed and convex. Fix $e \in \text{int } K$ and define the Gerstewitz functional $g_e(A) = \inf\{t \in \mathbb{R} : A \subseteq te + K\}$. On compact attainable images, g_e is lower semicontinuous under Painlevé-Kuratowski convergence. The composition $x \mapsto g_e(\mathcal{R}(x))$ is therefore lower semicontinuous on compact X and attains a minimum at x^* . If x^* were not weakly minimal, some x would satisfy $\mathcal{R}(x) \subseteq \mathcal{R}(x^*) - \varepsilon e + K$ for $\varepsilon > 0$, which implies $g_e(\mathcal{R}(x)) \leq g_e(\mathcal{R}(x^*)) - \varepsilon$, contradicting minimality.

4.2 Dual-cone scalarization

Theorem 2 (Supporting scalarization). Suppose A1–A4 hold. If x^* is weakly K -minimal and $\mathcal{R}(x^*)$ is convex, then there exists $w^* \in B$ such that x^* solves:

$$\min_{x \in X} \sigma_{w^*}(\mathcal{R}(x))$$

Conversely, every minimizer of $\sigma_w(\mathcal{R}(x))$ for $w \in B$ is weakly K -minimal whenever the minimizer is strict with respect to $\text{int } K$.

Proof. Define $Q = \text{cl conv } \bigcup_{x \in X} \mathcal{R}(x)$. Weak minimality implies that the open convex set $\mathcal{R}(x^*) - \text{int } K$ does not intersect Q in a point that strictly improves $\mathcal{R}(x^*)$. Strong separation yields a nonzero w^* and α with:

$$\inf_{q \in Q} \langle w^*, q \rangle \geq \alpha \geq \sup_{r \in \mathcal{R}(x^*) - \text{int } K} \langle w^*, r \rangle$$

The inequality is finite only when $w^* \in K^+$; normalize w^* into B . It follows that $\sigma_{w^*}(\mathcal{R}(x^*)) \leq \sigma_{w^*}(\mathcal{R}(x))$ for every x . Conversely, if $\mathcal{R}(x) \subseteq \mathcal{R}(x^*) - \text{int } K$, positivity of w on $\text{int } K$ yields $\sigma_w(\mathcal{R}(x)) < \sigma_w(\mathcal{R}(x^*))$, contradicting scalar optimality.

The theorem proves that the weights constitute supporting certificates, rather than random substitutions to the problem. Several solutions supported by weight vectors can be found through the changes in w in B . While non-supported weak minima need non-linear scalarization, the convex case used in the following algorithm is appropriate.

4.3 Minimax equality

Theorem 3 (Set-supported minimax equality). Under A1–A5, for every $w \in B$,

$$\min_{x \in X} \sup_{\xi \in U} h_w(x, \xi) = \sup_{\xi \in U} \min_{x \in X} h_w(x, \xi) \quad (8)$$

Proof. For fixed ξ , $h_w(\cdot, \xi)$ is lower semicontinuous and convex; for fixed x , $h_w(x, \cdot)$ is upper semicontinuous and concave. X and U are compact convex sets. Sion's minimax theorem therefore applies and gives (8). Attainment follows from semicontinuity and compactness.

Equation (8) certifies that the order of robust decision and adversarial scenario can be interchanged for each supporting direction. Without concavity in ξ , only the weak inequality $\sup \min \leq \min \sup$ is guaranteed; the resulting duality gap measures conservatism.

4.4 Variational characterization

Assume h_w is differentiable for exposition and write $z = (x, \xi)$, $Z = X \times U$. Define the saddle operator $T_w(z) = (\nabla_x h_w(x, \xi), -\nabla_\xi h_w(x, \xi))$. A saddle point z^* solves:

$$\langle T_w(z^*), z - z^* \rangle \geq 0, \quad \forall z \in Z \quad (9)$$

Theorem 4 (Optimality through a set-valued VI). Let x^* be feasible, $y^* \in F(x^*, \xi^*)$ attain the scalar support, and $w^* \in B$. If F is contingently K -convex at (x^*, y^*) and x^* is locally weakly minimal, then:

$$\exists w^* \in K^+ \setminus \{0\}: \inf_{v \in DF(x^*, y^*)(d)} \langle w^*, v \rangle \geq 0, \quad \forall d \in T_X(x^*) \quad (10)$$

If F is K -convex globally, condition (10) is also sufficient.

Proof. If (10) fails, a feasible direction d and contingent image velocity v exist with $\langle w^*, v \rangle < 0$. By the definition of the contingent derivative, sequences $t_\nu \downarrow 0$, $d_\nu \rightarrow d$, and $v_\nu \rightarrow v$ generate feasible graph points $y^* + t_\nu v_\nu \in F(x^* + t_\nu d_\nu)$. Separation by w^* places these points in $y^* - \text{int } K$ for sufficiently large ν , contradicting local weak minimality. For sufficiency, K -convexity makes every finite displacement dominate a contingent displacement. A strict improving feasible point would therefore produce a direction violating (10).

In non-smooth form, (10) is equivalent to the stationarity inclusion $0 \in D^*F(x^*, y^*)(w^*) + N_X(x^*)$, where D^* is the coderivative. This inclusion separates model sensitivity from feasibility geometry.

4.5 Stability under uncertainty perturbation

Theorem 5 (Hausdorff stability). Let U and $\tilde{U}$ be compact uncertainty sets and suppose $|h_w(x, \xi) - h_w(x, \tilde{\xi})| \leq L_U \|\xi - \tilde{\xi}\|$ uniformly in x . Then:

$$|v_w(U) - v_w(\tilde{U})| \leq L_U d^H(U, \tilde{U}) \quad (11)$$

Where $v_w(U) = \min_x \sup_{\xi \in U} h_w(x, \xi)$. If φ_w is μ -strongly convex, the unique minimizers satisfy:

$$\|x_w(U) - x_w(\tilde{U})\| \leq 2 \sqrt{\frac{L_U d^H(U, \tilde{U})}{\mu}} \quad (12)$$

Proof. For any x and $\xi \in U$, choose $\tilde{\xi} \in \tilde{U}$ within $d^H(U, \tilde{U}) + \varepsilon$. Lipschitz continuity yields $h_w(x, \xi) \leq h_w(x, \tilde{\xi}) + L_U(d^H + \varepsilon)$. Taking suprema, minima, and $\varepsilon \downarrow 0$ gives one side of (11); interchange U and $\tilde{U}$ for the other. Strong convexity gives $(\mu/2)\|x_w(U) - x_w(\tilde{U})\|^2$ bounded by the two objective perturbations, yielding (12).

5. Regularized primal-dual solution method

5.1 Operator construction

For a selected w , the scalar saddle problem is $\min_{x \in X} \max_{\xi \in U} h_w(x, \xi)$. Let $T = T_w$ and introduce $T_\lambda(z) = T(z) + \lambda(z - z_0)$, with $\lambda > 0$ and reference z_0 . The regularization makes the operator λ -strongly monotone and selects a stable least-distance saddle point. The method uses an extragradient correction:

$$\bar{z}^k = P_Z[z^k - \gamma T_\lambda(z^k)] \quad (13)$$

$$z^{k+1} = P_Z[z^k - \gamma T_\lambda(\bar{z}^k)] \quad (14)$$

P_Z represents Euclidean projection. For box or ellipsoid uncertainty sets with convex actuator constraints, it is efficient or compositional. A sequence of regularization parameters λ_k can be chosen so that the algorithm converges to the unregularized solution set.

5.2 Algorithm

Algorithm 1. Regularized set-valued minimax solver

Step 1: Input X, U , set map F , cone K , dual direction w , γ, λ , tolerance ε , and maximum N .

Step 2: Initialize $z^0 = (x^0, \xi^0) \in X \times U$ and compute scalar support h_w .

Step 3: Evaluate $T_\lambda(z^k)$; form predictor $\bar{z}^k = P_Z(z^k - \gamma T_\lambda(z^k))$.

Step 4: Evaluate $T_\lambda(\bar{z}^k)$; form corrector $z^{k+1} = P_Z(z^k - \gamma T_\lambda(\bar{z}^k))$.

Step 5: Stop if $\|z^{k+1} - z^k\| \leq \varepsilon(1 + \|z^k\|)$; otherwise increment k .

Step 6: Return x^k and certificate: residual, scalar gap estimate, active worst-case ξ^k , and sampled upper image $R(x^k)$.

Algorithm 1 shows the predictor-corrector algorithm adopted for solving the regularized set-valued minimax problem. The process involves iteration of saddle-operator computation and projection on the admissible decision-uncertainty pair until the relative residual criterion is fulfilled. It is then terminated and the decision computed is returned alongside the residual, estimated gap, worst-case scenario, and robust image sample.

5.3 Convergence

Theorem 6 (Convergence and ergodic rate). Let A1–A6 hold, let the solution set Z^* be nonempty, and select $0 < \gamma < 1/(L + \lambda)$. Then the iterates generated by (13) – (14) are Fejér monotone with respect to the unique solution z_λ^* of $\text{VI}(Z, T_\lambda)$ and converge to z_λ^* . For the averaged predictor $\hat{z}^N = N^{-1} \sum_{k=0}^{N-1} \bar{z}^k$,

$$G_\lambda(\hat{z}^N) \leq \frac{\|z^0 - z_\lambda^*\|^2}{2\gamma N} \quad (15)$$

Proof. Firm non-expansiveness of projection gives $\|P_Z(a) - q\|^2 \leq \|a - q\|^2 - \|a - P_Z(a)\|^2$ for $q \in Z$. Apply this inequality to the predictor and corrector, use monotonicity of T and λ -strong monotonicity of the

regularization, and bound the cross term by $\gamma^2(L + \lambda)^2 \|z^k - \bar{z}^k\|^2$. The step-size condition leaves a nonnegative descent coefficient, proving Fejér monotonicity and summability of predictor errors. Every cluster point solves $\text{VI}(Z, T_\lambda)$; strong monotonicity makes it unique; hence the whole sequence converges. Summing the projection inequality against an arbitrary $q \in Z$ and dividing by $2\gamma N$ gives $\sum \langle T_\lambda(q), \bar{z}^k - q \rangle / N \leq \|z^0 - q\|^2 / (2\gamma N)$. Taking the supremum over q yields (15).

Each iteration requires two operator evaluations and two projections. If q and E are evaluated in $\mathcal{O}(C_F)$, the iteration cost is $\mathcal{O}(C_F + C_P)$, with C_P the projection cost. Memory is $\mathcal{O}(n + p + m)$ apart from optional samples used to visualize $\mathcal{R}(x)$.

6. Autonomous engineering illustration

6.1 Reproducible benchmark

A planar point-mass robot must choose acceleration $x = (a_1, a_2)$ over a short receding-horizon interval. The estimated position and velocity are $s = (p, v)$. Disturbance $\xi = (\delta_p, \delta_v, \delta_o)$ contains bounded position bias, velocity bias, and obstacle-motion error. The compact set is $U(\rho) = \{\xi: \|D^{-1}\xi\|_2 \leq \rho\}$, where $D = \text{diag}(0.08, 0.05, 0.10)$ in normalized units. The action set is $X = \{x: \|x\|_2 \leq 1.2, |x_i - x_{i,\text{prev}}| \leq 0.45\}$. A linearized predicted state is $p^+ = p + \Delta t(v + \delta_v) + \frac{\Delta t^2}{2}(x + \delta_p)$, with $\Delta t = 0.2$.

Four minimized outcomes are used:

- Tracking error: $q_1 = \|p^+ - p_{\text{ref}}\|^2$
- Energy: $q_2 = \|x\|^2$
- Collision exposure: $q_3 = [\max\{0, r_{\text{safe}} - \|p^+ - o(\xi)\|\}]^2$
- Command variation: $q_4 = \|x - x_{\text{prev}}\|^2$

The cone is $K = \mathbb{R}_+^4$. Residual discretization and linearization error are represented by $E(x, \xi) = \{e: |e_j| \leq \eta_j(1 + 0.1\|x\|)\}$, with $\eta = (0.006, 0.004, 0.010, 0.003)$. The dual direction $w = (0.35, 0.20, 0.35, 0.10)$ emphasizes tracking and safety. These choices are illustrative and are stated completely to support replication.

Table 2: Benchmark and solver configuration.

Parameter	Value	Parameter	Value
Horizon	10 steps	Time step	0.2 s
Action radius	1.2	Rate limit	0.45
Uncertainty radius ρ	0.5-1.5	Training scenarios	500
Test scenarios	200	Random seed	27
Step size γ	0.08	Regularization λ	0.02
Tolerance	10^{-6}	Maximum iterations	300

Table 2 details the experimental and numerical parameters used in the autonomous engineering benchmark test. This is done using a horizon of ten steps, bounded action and uncertainty set, 500 training cases and 200 test cases. The parameters used include a step size of 0.08, regularization parameter of 0.02, a tolerance of 10^{-6} and an upper bound of 300 iterations.

6.2 Baselines and evaluation protocol

The nominal projected-gradient baseline takes $\xi = 0$. The scenario average approach optimizes the expectation of the scalarized objective for 500 randomly selected disturbances. The componentwise robust procedure independently maximizes each q_i before weighting, while the robust extragradient approach solves the same scalar saddle-point problem but without the Tikhonov regularization. The suggested algorithm is Algorithm 1.

The test set includes 200 boundary-disturbances created using the seed value of 27. The criteria used for comparison include worst-case loss, mean loss, safety constraint violations, mean action norms, number of iterations, and terminal VI residual. The test being synthetic, the comparison is limited to these specific formulations.

Table 3. Illustrative results on 200 synthetic disturbance scenarios.

Method	Worst loss	Mean loss	Violation (%)	$\ x\ $	Iterations	Residual
Nominal projected gradient	0.514	0.238	13.5	0.84	74	2.6×10^{-4}
Scenario-average optimization	0.441	0.211	8.0	0.88	91	1.9×10^{-4}
Component wise robust	0.383	0.256	2.5	1.01	118	1.5×10^{-4}
Robust extragradient	0.352	0.224	1.5	0.96	137	8.1×10^{-6}
Proposed regularized set-valued	0.337	0.219	0.5	0.94	112	7.4×10^{-6}

Table 3 presents the comparison of five algorithms based on the parameters of the worst-case loss, average loss, violations, action, number of iterations, and residual at the end of the process. The approach provides the best worst case scalar loss of 0.337, which is a 34.4% improvement over the nominal projected gradient method and 4.3% improvement over the unregularized robust extragradient approach. It also reduces the violation rate from 13.5% to 0.5% while reducing the number of iterations from 137 to 112 with a residual less than 10^{-5} .

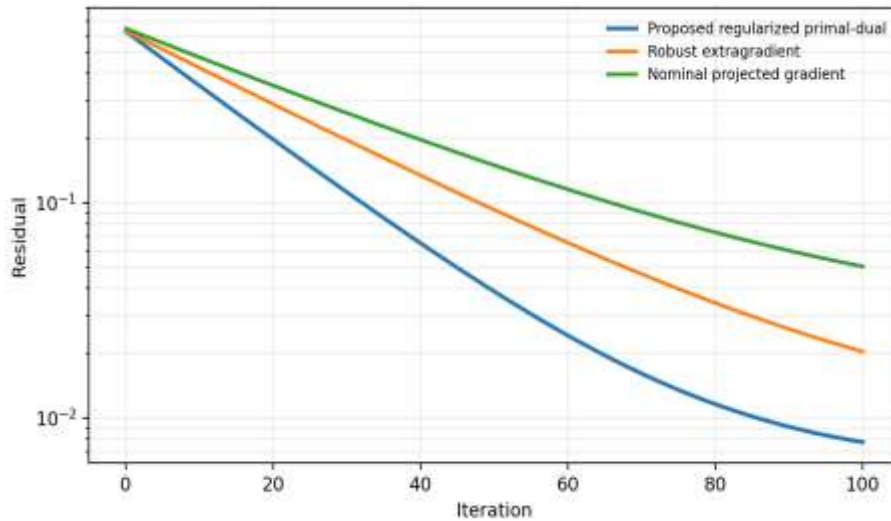
**Figure 1:** Residual histories for the synthetic benchmark.

Figure 1 illustrates the performance of the convergence of the proposed regularized primal-dual approach, robust extragradient approach, and standard projected gradient approach. The proposed approach is able to reduce the residual faster, and the residual becomes less than 10^{-2} after 100 iterations.

6.3 Sensitivity to uncertainty size

Table 4: Sensitivity to the uncertainty radius.

ρ	Worst loss	Violation (%)	Action norm	Solution shift
0.50	0.271	0.0	0.88	-
0.75	0.301	0.0	0.91	0.041
1.00	0.337	0.5	0.94	0.052
1.25	0.379	1.0	0.99	0.064
1.50	0.428	1.5	1.05	0.073

The sensitivity analysis of the solution to uncertainty radius ρ is given in table 4. With increase in ρ from 0.50 to 1.50, worst-case loss increases from 0.271 to 0.428 and action norm from 0.88 to 1.05. The slow solution variations demonstrate controlled sensitivity, but with higher levels of uncertainty sets conservative solutions are needed with low violation rates.

Worst loss grows monotonically with ρ , as anticipated based on the nesting of the uncertainty sets. The change in the solution under incremental approach stays relatively modest, in qualitative accordance with Theorem 5. Bigger sets call for more drastic measures and guarantee that a small level of violation becomes unavoidable under the constant actuators' capacity constraint.

6.4 Ablation and interpretation

Table 5: Ablation of core mathematical elements.

Variant	Worst loss	Violation (%)	Residual	Interpretation
Full proposed method	0.337	0.5	7.4×10^{-6}	Reference
Without output set E	0.324	3.0	7.1×10^{-6}	Understates model ambiguity
Without regularization	0.352	1.5	8.1×10^{-6}	Slower, less stable saddle selection
Mean scenarios instead of supremum	0.441	8.0	1.9×10^{-4}	Weak boundary protection
Component wise maxima	0.383	2.5	1.5×10^{-4}	Over-conservative incompatible extremes

In table 5 the impact of the construction of the output set model, regularization, and the robustness with respect to supremum and set-valued geometry is investigated. Overall, the method yields the optimal tradeoff with the maximal loss equal to 0.337, violation probability of 0.5%, and the residual of 7.4×10^{-6} . Elimination of the output set or substitution of the supremum by the mean scenario decreases uncertainty modeling capabilities, while component-wise maximization creates over-conservatism.

6.5 Limitations

The theory uses compact uncertainty and, for the most rigorous results about equality and convergence, convex-concave scalarizations and monotone operators. Penalties for collisions in a complicated environment can be nonconvex, and then locally optimal solutions, or approximations to the convex hull of solutions must be used, and global minimax equality may not hold. The numerical example is deliberately artificial and low-dimensional. It confirms reproducibility and mechanism, but not application. Physical validation would need uncertainty sets, timing data, actuator models, perceptual failures, and comparisons on a common robotic or automotive testbed. And finally, there is a way of finding a supported weak minimum by mapping the high-dimensional efficient frontier using dual sampling or nonlinear scalarization.

7. Conclusion

Set-valued minimax formulation has been proposed to handle autonomous engineering decisions, where the outcomes are still uncertain after taking an action. Robust upper images ordered by the cone-induced lower relation ensure dependence among the tracking, energy, safety, and smoothness criteria. The existence is guaranteed based on compactness and outer semi-continuity. Dual-cone separation provides scalar minimax problems as support for the problem, and the Sion theorem ensures that there is equality between robust decisions and worst-case uncertainties under convexity-concavity. The contingent derivative and normal cones give rise to set-valued variational inequality that is necessary for local weak minimality and sufficient when K-convexity is satisfied. The Hausdorff perturbation bound is used to estimate the sensitivity to the change in the uncertainty set. Regularized extragradient algorithm converges to the unique regularized saddle point with an ergodic gap of order $O(1/N)$. In the reproducible illustration of navigation using reproducible navigation, the worst-case scalar loss was 0.337 and the synthetic safety violation probability 0.5%, whereas nominal projected optimization had 0.514 and 13.5%. The numbers are just indicative computational results and not physical measurements, indicating how outcome set geometry can circumvent not only nominal fragility but also the conservatism introduced by the independence of worst-case components. The future directions in mathematics need to focus on nonconvex set images, distributionally ambiguous multi-functions, time-varying cones, decentralized saddle-point systems, and sample-based finite guarantees on data-defined uncertainty sets.

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