

A Convex Analysis-Based Optimal control Framework for Industrial Automation Systems

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Abstract

Industrial automated control systems are driven by coupled dynamical systems, actuator constraints, uncertain load, and conflicting objectives of precise tracking, energy conservation, and maintenance of machines. In this paper, a new framework for optimal control using convex analysis theory is developed where bounded uncertainty in operation is modeled as an adversarial disturbance, and the control problem is formulated as a convex-concave minimax problem in infinite dimensions. Through the control-to-state transformation, a saddle point problem in product Hilbert space is obtained from the constrained dynamical system. The solution of this problem is analyzed in terms of a variational inequality, a normal cone inclusion, KKT conditions, and Fenchel duality. Convexity-concavity in its strong form leads to uniqueness, whereas monotonicity in its strong form results in Lipschitz stability with respect to the parameters of the plant and disturbances. A projected extragradient primal-dual algorithm is obtained, and global linear convergence is established under admissible step size choices. Experimental verification on a three-stage interconnected production line process shows that the worst-case cost is 13.84, tracking error is 0.071, the energy index is 7.62, constraint violation is zero, and convergence is achieved in 47 iterations. Compared with other control methods like model predictive control, projected-gradient control, linear-quadratic regulation, and robust H-infinity control, the presented approach outperforms them in feasibility and worst-case performance.

Keywords: Convex analysis, minimax optimal control, variational inequality, industrial automation, primal-dual algorithm, saddle point, convergence analysis.

1. Introduction

Industrial automation in the modern era integrates sensing, actuation, scheduling, and feedback control through physically connected machines (Baggyalakshmi et al., 2023). While such integration leads to increased efficiency and consistency, it also results in a constrained decision-making task, where the tradeoff between tracking, energy consumption, wear and tear, safety, and manufacturing is optimized. Model predictive control has been developed as a crucial industrial tool because it explicitly considers multivariable dynamics and hard constraints on

operations; yet, implementation difficulty, uncertainty handling, and optimization are crucial practical challenges Forbes et al., (2015). Convexity plays a critical role in that it distinguishes tractable control problems from generic nonlinear programs that provide only locally convergent solutions (Lautenschlager et al., 2015).

Optimization in industry usually involves the interplay between real-time control and economics (Gopalakrishnan et al., 2024). The one-level implementation of real-time optimization and predictive control ensures feasibility and reduces the cost of the original nonlinear problem when the convexified update is performed (Alamo et al., 2014). The cooperative economic predictive control framework further extends this idea to coupled subsystems with convex cost functions and offers a way from local to global behavior (Lee & Angeli, 2014). Moreover, economic predictive control shows that terminal, regional, and Lyapunov-type conditions are more than just numeric tricks; they define feasibility, stability, and economic significance of closed-loop systems (Ellis & Christofides, 2015). This suggests a formulation where the geometry of feasible space and the optimality operator are fundamental mathematical constructs.

Further examples of structural requirements can be found in manufacturing systems. Precision machinery involves the use of constraints and coordination based on references where actuators have different bandwidths and ranges. In order for the robust controller to handle uncertainties, convergence and input saturations have to be considered (Oravec & Bakošová, 2015). It is possible to alleviate conservatism using scenarios and adaptive methods, but such methods rely on sampling assumptions or online uncertainty set estimation (Fagiano et al., 2015). In general, a representation of uncertainties by way of a set and optimization against the worst-case disturbances is desirable (Zhang & Rodriguez, 2023).

Convex analytical methods form a sound basis for such computations. Gradient projection and conditional gradient techniques may even achieve linear convergence in strong convexity feasible domains under certain nonconvex objectives, and their utilization on affine optimal-control systems illustrates the concept of feasible domain curvature (Veliov & Vuong, 2020). Maximum principles based on adjoints yield first order conditions in infinite-dimensional distributional control problems (Breitenbach & Borzi, 2020). Modern numerical analysis of control problems within convex domains proves that issues related to the quality of discretization as well as functional regularity should be simultaneously considered (Pfefferer & Vexler, 2025). Problems of infeasibility and critical feasibility point out that actuator constraints and endpoint constraints may contradict each other, necessitating a residual analysis of feasibility (Burachik et al., 2024).

Mathematical research has also been extended to nonlocal, switching, and non-smooth controls. The concept of optimal synthesis for evolution problems involving nonlocal data provides an operator approach in infinite dimensions (Kholova et al., 2025). The solution of switching-constrained parabolic control problems highlights how the discrete constraints affect computation Buchheim et al., (2025), while the use of high-order schemes in bang-bang linear-quadratic control provides accurate approximations around switching points (Scarinci & Veliov, 2018). Differential variational inequalities link dynamic equilibrium and optimal control Friesz & Han, (2019), and the convergence results of variational-hemivariational models highlight the significance of penalization stability. Large scale sequential quadratic programming, soft-constraint coordination techniques, and geometric optimal control strategies offer other ways to achieve separability and feasible trajectory construction (Lu, 2015; Huang et al., 2015).

However, there are three gaps still existing despite all these developments. First, numerous studies of industrial controls provide an expression for the numerical controller but not the corresponding variational inequality and dual problem formulation. Second, uncertainty is always addressed using nominal tuning or specific scenarios, which are based on a convex-concave minimax framework without existence and uniqueness results. Third, convergence analysis is provided without an operator-based convergence rate related to strong monotonicity and Lipschitz continuity. The current study fills these gaps through development of a mathematically rigorous framework, where engineering example is used only for validation purposes.

Contributions are five-fold. First, the industrial automation problem is mathematically formalized on Hilbert spaces, with closed convex state, control and disturbance sets. Second, the control to state map transforms the dynamic problem into a convex-concave saddle-point problem. Third, several equivalent characterizations as saddle, variational, Karush-Kuhn-Tucker, normal cone and Fenchel dual problems are provided. Fourth, existence, uniqueness, parametric stability and convergence results are obtained under explicit assumptions. Fifth, the effectiveness of the projected extragradient primal-dual method is demonstrated on a constrained multi-stage production line with various performance measures.

The rest of this paper is structured as follows. Section 2 presents the mathematical preliminaries and control problem formulation. Section 3 proves optimality and duality characterizations. Section 4 proves theoretical properties and the numerical algorithm. Section 5 describes the industrial application and mathematical discussion. Section 6 concludes this paper.

2. Convex foundations and control problem formulation

2.1. Functional setting and admissible geometry

Let $T > 0$ be a fixed operating horizon. The state, control, and disturbance spaces are the real Hilbert spaces $X = L^2(0, T; \mathbb{R}^n)$, $U = L^2(0, T; \mathbb{R}^m)$, and $W = L^2(0, T; \mathbb{R}^q)$, endowed with their usual inner products. The product space $Z = U \times W$ has norm $\|z\|_Z^2 = \|u\|_U^2 + \|w\|_W^2$. Pointwise industrial limits define nonempty, closed, and convex admissible sets $U_{\text{ad}} \subseteq U$ and $W_{\text{ad}} \subseteq W$. For measurable lower and upper bounds $u_{\min}$ and $u_{\max}$, the control set is

$$U_{\text{ad}} = \{u \in U: u_{\min}(t) \leq u(t) \leq u_{\max}(t) \text{ a.e. on } [0, T]\} \quad (1)$$

The uncertainty set is taken as the intersection of pointwise bounds and an energy ball. This representation captures load changes, unmodelled coupling, and bounded sensor-actuator disturbances while preserving convexity:

$$W_{\text{ad}} = \{w \in W: w_{\min} \leq w(t) \leq w_{\max}, \|w\|_W \leq \rho\} \quad (2)$$

For any nonempty closed convex set C in a Hilbert space, the metric projection $P_C(y) = \arg \min_{v \in C} \|v - y\|$ is single-valued and firmly non-expansive. The normal cone and subdifferential are respectively:

$$N_C(x) = \{\xi: \langle \xi, y - x \rangle \leq 0 \text{ for every } y \in C\} \quad (3)$$

$$\partial f(x) = \{\xi: f(y) \geq f(x) + \langle \xi, y - x \rangle \text{ for every } y\} \quad (4)$$

These objects permit constraints and non-smooth penalties to be incorporated without converting them into differentiable approximations. A map $F: Z \rightarrow Z$ is μ -strongly monotone if $\langle F(z_1) - F(z_2), z_1 - z_2 \rangle \geq \mu \|z_1 - z_2\|^2$ and L -Lipschitz continuous if $\|F(z_1) - F(z_2)\| \leq L \|z_1 - z_2\|$. Strong monotonicity will be the central source of uniqueness, perturbation stability, and linear convergence in (1)-(4).

Weak convergence is the natural compactness tool in this case. Since both U and W are reflexive Hilbert spaces, any bounded sequence contains a weakly convergent subsequence. The closed convex sets are weakly closed; hence, the weak limit of admissible control and disturbance sequences is an admissible pair. The weak lower semi-continuity of any strongly lower semicontinuous convex functional follows from this property. As far as the maximizing function is concerned, the weak upper semi-continuity takes the corresponding position. All these properties enable us to work with minimizing and maximizing sequences without requiring pointwise convergence of industrial command functions.

The projection satisfies $\langle y - P_C(y), v - P_C(y) \rangle \leq 0$ for all $v \in C$. Firm nonexpansiveness gives $\|P_C(a) - P_C(b)\|^2 \leq \langle P_C(a) - P_C(b), a - b \rangle$. Projection onto Ω separates as $P_\Omega(u, w) = (P_{U_{\text{ad}}}(u), P_{W_{\text{ad}}}(w))$. For box-constrained controls, $P_{U_{\text{ad}}}$ is pointwise clipping; for W_{ad} , the box projection is followed by radial scaling whenever the energy constraint is active. This makes the infinite-dimensional geometry consistent with the time-grid implementation.

2.2. Industrial dynamics and performance functional

Consider an interconnected automated system whose dynamics have been linearized about some guaranteed operating point. The state might include buffer levels, machine speeds, temperature differences, and actuator positions. The control could consist of conveyor instructions, machine rate instructions, and robotic actuations, while w is an acceptable disturbance. The evolution equation is

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + D(t)w(t) + b(t), \quad x(0) = x_0 \quad (5)$$

Assume that A is essentially bounded and B, D are square-integrable operator-valued functions of compatible dimensions. Then every $(u, w) \in U \times W$ produces a unique absolutely continuous state. Let x_{ref} denote the desired production trajectory. With positive semidefinite $Q(t)$, positive definite $R(t)$, terminal matrix S , and attenuation $\gamma > 0$, define

$$J(u, w) = \frac{1}{2} \int_0^T (\|x - x_{\text{ref}}\|_Q^2 + \|u\|_R^2 - \gamma^2 \|w\|^2) dt + \frac{1}{2} \|x(T) - x_{\text{ref}}(T)\|_S^2 + \Psi(u) \quad (6)$$

The appropriate lower semi-continuous convex function Ψ may stand for total variation, actuator wear, or any separable convex maintenance cost in (5)-(6). The negative sign of the disturbance energy leads to J being concave

in w as long as γ is large enough in relation to the disturbance state gain. The control system minimizes the performance measure, while the disturbance tries to maximize it.

2.3. Reduced convex-concave minimax problem

Let $S: U \times W \rightarrow X$ be the affine control-to-state operator generated by the dynamics. Substitution $x = S(u, w)$ gives the reduced functional $\tilde{J}(u, w) = J(S(u, w), u, w)$. The proposed robust optimal-control problem in (7) is

$$V = \min_{u \in U_{\text{ad}}} \max_{w \in W_{\text{ad}}} \tilde{J}(u, w) \quad (7)$$

The formulation is not an unconstrained heuristic: it is a saddle problem over $\Omega = U_{\text{ad}} \times W_{\text{ad}}$. To state the analytical conditions transparently, the following assumptions are imposed:

- **(A1)** U_{ad} and W_{ad} are nonempty, closed, convex, and bounded.
- **(A2)** The control-to-state operator is continuous and affine.
- **(A3)** $\tilde{J}$ is proper, lower semicontinuous and σ_u -strongly convex in u for every fixed w .
- **(A4)** $\tilde{J}$ is upper semicontinuous and σ_w -strongly concave in w for every fixed u .
- **(A5)** The smooth part has an L -Lipschitz saddle gradient, and Ψ is proximal.

In finite-horizon industrial systems, (A3) follows from $R(t) \geq rI$ with $r > 0$, whereas (A4) follows when $\gamma^2 I - D^* Q D$ is uniformly positive definite.

$$\sigma_u I \preceq \nabla_{uu}^2 \tilde{J}(u, w), \quad -\nabla_{ww}^2 \tilde{J}(u, w) \preceq \sigma_w I \quad (8)$$

The first-order saddle operator is $F(z) = (\nabla_u \tilde{J}(u, w), -\nabla_w \tilde{J}(u, w))$. Under (A3)–(A5), F is strongly monotone with $\mu = \min\{\sigma_u, \sigma_w\}$, except that the nonsmooth term is handled through $\partial\Psi$. This operator representation separates the physical model in (8), expressed through S and its adjoint, from the universal convex geometry of Ω .

Write the affine state as $x = x_f + G_u u + G_w w$, where x_f is the free response and G_u, G_w are bounded input-to-state operators. The reduced Hessian has blocks $H_{uu} = R + G_u^* Q G_u$, $H_{ww} = G_w^* Q G_w - \gamma^2 I$, and $H_{uw} = G_u^* Q G_w$. Positive definiteness of H_{uu} and negative definiteness of H_{ww} are sufficient for strong convexity-concavity. This means that the cross-block impacts the Lipschitz constant and the numerical conditioning but not the individual curvature conditions. This shows that γ should be larger than the greatest disturbance amplification in the Q -weighted state norm in (9):

$$\gamma^2 > \|G_w^* Q G_w\| \Rightarrow \sigma_w = \gamma^2 - \|G_w^* Q G_w\| > 0 \quad (9)$$

3. Optimality, duality, and variational characterization

3.1. Adjoint system and saddle-point conditions

For the differentiable part of J , introduce the adjoint $p: [0, T] \rightarrow \mathbb{R}^n$. A standard first variation of the augmented functional gives the backward equation (10):

$$-\dot{p}(t) = A(t)^\top p(t) + Q(t)(x(t) - x_r(t)), \quad p(T) = S(x(T) - x_r(T)) \quad (10)$$

The reduced gradients are obtained without explicitly differentiating the state trajectory with respect to every control component:

$$\nabla_u \hat{J}(u, w) = R(t)u(t) + B(t)^\top p(t) + \partial\Psi(u) \quad (11)$$

$$\nabla_w \hat{J}(u, w) = D(t)^\top p(t) - \gamma^2 w(t) \quad (12)$$

A pair (u^*, w^*) is a saddle point if $\hat{J}(u^*, w) \leq \hat{J}(u^*, w^*) \leq \hat{J}(u, w^*)$ for all admissible u and w in (11) and (12). The corresponding state and adjoint satisfy the dynamics, the backward adjoint equation, and the projected stationarity relations:

$$u^* = P_{U_{\text{ad}}}[u^* - \alpha(Ru^* + B^T p^* + \xi^*)], \quad \xi^* \in \partial\Psi(u^*) \quad (13)$$

$$w^* = P_{W_{\text{ad}}}[w^* + \beta(D^T p^* - \gamma^2 w^*)] \quad (14)$$

The projection equations are not only analytical formulations but also practical control updates in (12) and (13). If the control constraint is a box and Ψ is separable, then the projections and proximations are computed component-wise, which is appropriate for real-time automation.

3.2. Variational inequality and normal-cone formulation

Define the composite monotone operator $T(z) = F(z) + (\partial\Psi(u), 0) + N_{\Omega}(z)$. The optimal solution is characterized by the monotone inclusion $0 \in T(z^*)$. Equivalently, if Ψ is differentiable, z^* solves the variational inequality:

$$\text{Find } z^* \in \Omega \text{ such that } \langle F(z^*), z - z^* \rangle_z \geq 0 \text{ for every } z \in \Omega \quad (15)$$

For non-smooth Ψ , the mixed variational inequality becomes:

$$\langle F_0(z^*), z - z^* \rangle + \Psi(u) - \Psi(u^*) \geq 0 \quad \text{for every } z = (u, w) \in \Omega \quad (16)$$

Where F_0 excludes $\partial\Psi$. The equivalence stems from the sub-gradient inequality together with the definition of the normal cone in (15) and (16). This characterization is essential since it transforms the dynamic control problem into an equilibrium problem for a strongly monotone operator which can then be analyzed uniformly.

A useful error certificate follows directly from strong monotonicity. Let $r_{\alpha}(z) = \alpha^{-1}\|z - P_{\Omega}(z - \alpha F(z))\|$. For α in a bounded admissible interval, metric regularity of the strongly monotone inclusion implies $\text{dist}(z, z^*) \leq c_{\alpha} r_{\alpha}(z)$, where c_{α} depends only on μ, L , and α . Consequently, the small residual that is projected helps in controlling the solution error as opposed to the mere indication of a small difference between two successive iterations.

3.3. KKT system and Fenchel duality

Suppose the industrial constraints are additionally expressed by convex functions $g_i(u) \leq 0$ and affine equalities $Hu = h$. Under a Slater point, the Lagrangian is $L(u, w, \lambda, v) = \hat{J}(u, w) + \sum_i \lambda_i g_i(u) + \langle v, Hu - h \rangle$. Necessary and sufficient saddle KKT conditions in (17) – (20) are:

$$0 \in \partial_u \hat{J}(u^*, w^*) + \sum_i \lambda_i^* \partial g_i(u^*) + H^* v^* + N_{U_{\text{ad}}}(u^*) \quad (17)$$

$$0 \in -\partial_w \hat{J}(u^*, w^*) + N_{W_{\text{ad}}}(w^*) \quad (18)$$

$$g_i(u^*) \leq 0, \quad \lambda_i^* \geq 0, \quad \lambda_i^* g_i(u^*) = 0, \quad Hu^* = h \quad (19)$$

For a decomposition $\min_u f(u) + g(Ku)$ with f incorporating the reduced quadratic control cost and indicator of U_{ad} , the Fenchel conjugate $g^*(y) = \sup_v \{\langle y, v \rangle - g(v)\}$ yields the primal-dual saddle problem

$$\min_u \max_y f(u) + \langle Ku, y \rangle - g^*(y) \quad (20)$$

Fenchel–Rockafellar regularity gives equality of primal and dual values. The computable primal-dual gap G_k is nonnegative and vanishes exactly at a saddle point. Together with the natural residual $r(z) = \|z - P_{\Omega}(z - F(z))\|$, it provides a mathematically meaningful stopping rule that is stronger than monitoring objective differences alone.

Strong duality gives the engineering interpretation of the multipliers. Multipliers for the energy budget constraint indicate the marginal rise in optimal worst-case cost due to tighter control of the budget constraint; those multipliers for the actuator constraints correspond to saturating actuators and indicate the greatest value added by any further increases in their authority. Complementarity distinguishes between inactive and effective constraints. These interpretations follow from convex sensitivity analysis without further statistical processing.

4. Theoretical properties and computational method

4.1. Existence, uniqueness, and stability

Theorem 1 (existence). Under assumptions (A1)–(A4), the reduced problem possesses at least one saddle point $z^* = (u^*, w^*) \in \Omega$.

Proof. For each fixed w , strong convexity and lower semicontinuity imply that $\hat{f}(\cdot, w)$ attains a unique minimum on the weakly compact set U_{ad} . For each fixed u , upper semicontinuity and concavity imply attainment of the maximum on W_{ad} . The sets are convex and weakly compact, and $\hat{f}$ is convex-concave; therefore, the minimax equality holds. The attained common value supplies a pair satisfying the two saddle inequalities; hence a saddle point exists in (21).

$$\min_{u \in U_{\text{ad}}} \max_{w \in W_{\text{ad}}} \hat{f}(u, w) = \max_{w \in W_{\text{ad}}} \min_{u \in U_{\text{ad}}} \hat{f}(u, w) \quad (21)$$

Theorem 2 (uniqueness). If F is μ -strongly monotone on Ω , the saddle point is unique.

Proof. Let z_1 and z_2 solve the variational inequality. Testing the inequality for z_1 with $z = z_2$ and that for z_2 with $z = z_1$, then adding, gives $\langle F(z_1) - F(z_2), z_1 - z_2 \rangle \leq 0$. Strong monotonicity gives $\mu \|z_1 - z_2\|^2 \leq 0$. Since $\mu > 0$, $z_1 = z_2$.

Theorem 3 (parameter stability). Let F_η be uniformly μ -strongly monotone and satisfy $\|F_{\eta_1}(z) - F_{\eta_2}(z)\| \leq L_\eta \|\eta_1 - \eta_2\|$. Then the solution map is Lipschitz continuous.

Proof. Let z_1 and z_2 be the solutions for η_1 and η_2 . The two variational inequalities yield $\langle F_{\eta_1}(z_1) - F_{\eta_2}(z_2), z_1 - z_2 \rangle \leq 0$. Add and subtract $F_{\eta_1}(z_2)$, use strong monotonicity for the first difference and Cauchy–Schwarz for the parameter difference. Thus $\mu \|z_1 - z_2\|^2 \leq L_\eta \|\eta_1 - \eta_2\| \|z_1 - z_2\|$. Division by the nonzero norm, with the zero-case immediate, proves the bound in (22):

$$\|z^*(\eta_1) - z^*(\eta_2)\| \leq \left(\frac{L_\eta}{\mu} \right) \|\eta_1 - \eta_2\| \quad (22)$$

The entries in the parameter η can be drawn from A, B, and D, that is, the reference trajectory, uncertainty radius, and quadratic weights. This theorem provides a unified measure of sensitivity of errors in modeling and changes to the operating point. Moreover, it demonstrates the significance of the monotonicity modulus as a robustness measure since a bigger value of μ implies smaller displacement of the solution for the same perturbation. In case μ tends to zero, the solution becomes very sensitive despite the presence of the optimizer.

A related value-function estimate follows when $\hat{f}$ is uniformly Lipschitz in η on Ω . If $|\hat{f}_{\eta_1}(z) - \hat{f}_{\eta_2}(z)| \leq M_\eta \|\eta_1 - \eta_2\|$ for every feasible z , applying the saddle inequalities twice gives $|V(\eta_1) - V(\eta_2)| \leq M_\eta \|\eta_1 - \eta_2\|$. Hence, both the optimization algorithm and the optimal worst-case value change continuously. Numerical spikes with respect to small perturbations in parameters will thus indicate failure of the uncertain model assumption, lack of strong convex-concavity, or lack of numerical convergence.

4.2. Projected extragradient primal-dual algorithm

Projection on saddle problems could be oscillatory due to the presence of skewness in the operator resulting from the interaction between the state and disturbance. The presented approach hence employs a predictor-corrector assessment procedure in (23) and (24). Starting from $z^0 \in \Omega$ and choosing $\tau > 0$, compute:

$$\bar{z}^k = \text{prox}_{\tau(\Psi + I_\Omega)}[z^k - \tau F_0(z^k)] \quad (23)$$

$$z^{k+1} = \text{prox}_{\tau(\Psi + I_\Omega)}[z^k - \tau F_0(\bar{z}^k)] \quad (24)$$

This iteration involves one step of forward state integration, backward adjoint integration, operator evaluation, and projection on the admissible boxes and disturbance energy ball. The stopping criterion is a combination of natural residual, primal-dual gap, and relative iterate change:

$$\max\{\tau(z^k), G_k, \|z^k - z^{k-1}\|/(1 + \|z^k\|)\} \leq \varepsilon \quad (25)$$

The algorithm 1 is as follows: Initialization of feasible u^0 , w^0 and parameters τ , ε , K_{max} ; State and adjoint propagation; Evaluate F_0 ; Predict feasible pair; Repeat state adjoint evaluation at the predictor; Correct and Project; Calculate residuals; Terminate when the composite tolerance is met in (25). The warm starting from the previous industrial sampling time will decrease the number of online iterations while maintaining feasibility.

4.3. Convergence and computational complexity

Theorem 4 (linear convergence). Assume F_0 is L -Lipschitz and μ -strongly monotone, and choose $0 < \tau < \min\{1/L, 2\mu/L^2\}$. Then the projected iteration is globally convergent. For the single-correction contraction estimate, $\|z^{k+1} - z^*\| \leq q\|z^k - z^*\|$ with $q = (1 - 2\tau\mu + \tau^2L^2)^{1/2} < 1$.

Proof. Projection and proximal mappings of proper closed convex functions are non-expansive. Since $z^* = \text{prox}_{\tau(\Psi + I_\Omega)}(z^* - \tau F_0(z^*))$, nonexpansiveness gives $\|z^{k+1} - z^*\|^2 \leq \|(z^k - z^*) - \tau(F_0(z^k) - F_0(z^*))\|^2$. Expanding the square, applying strong monotonicity to the inner product and Lipschitz continuity to the operator norm gives $(1 - 2\tau\mu + \tau^2L^2)\|z^k - z^*\|^2$. The step-size condition makes the multiplier strictly smaller than one. The extragradient correction retains this contraction and improves stability for the saddle-induced skew component in (26).

$$k \geq \frac{\log\left(\frac{\varepsilon}{\|z^0 - z^*\|}\right)}{\log(q)} \quad (26)$$

After temporal discretization into N intervals, state and adjoint propagation require $\mathcal{O}(Nn^2)$ operations for dense matrices and less for sparse industrial interconnections. Gradient and projection operations require $\mathcal{O}(N(m + q))$. Thus, the per-iteration cost is $\mathcal{O}(N[n^2 + m + q])$, and storage is $\mathcal{O}(N[n + m + q])$. The iteration count is logarithmic in $1/\varepsilon$ under strong monotonicity. This establishes a direct relation between theoretical conditioning L/μ and online computational effort.

A consistent discretization process is needed for the approximation of the solution of an infinite-dimensional problem. Here, assume that Ω_N consists of piecewise-constant controls and disturbances over a uniform partition while J_N uses a stable state propagator. From the density of the union of Ω_N , uniformly bounded solution operator and lower semi-continuity, can establish the convergence of the convex control part, and also from changing the signs will get the convergence of the concave disturbance part. It follows that any sequence of saddle points in the discrete formulation admits a weakly convergent subsequence whose limit is a continuous saddle point.

For sufficiently regular coefficients, the state discretization error is $\mathcal{O}(\Delta t)$, and strong monotonicity transfers this consistency error to the control-disturbance pair. If F_N approximates F uniformly on Ω with error δ_N , the stability argument of Theorem 3 yields the following estimate. It separates optimization tolerance from time-discretization error: solving the discrete problem substantially below δ_N does not improve approximation of the continuous control in (27).

$$\|z_N^* - z^*\| \leq \|F_N - F\|_\infty / \mu = \mathcal{O}(\Delta t) \quad (27)$$

5. Numerical validation and mathematical discussion

5.1. Industrial case configuration

The framework was tested using a reproducible simulation of a three-level automatic production line with a material feeder, a processing center, and a robotic finishing cell. The six-dimensional state space includes three normalized buffer deviations and three machine-rate deviations. Three control variables control conveyor rate, processing rate, and robot cycle rate. Two disturbance terms correspond to uncertain arrival demand and processing load. The continuous model was discretized with $\Delta t = 0.05$ s over $T = 20$ s. Control bounds were $-1 \leq u_i \leq 1$, disturbance energy radius was $\rho = 1.5$, and the terminal target was the origin. Matrices were selected to give stable individual units with non-negligible upstream-downstream coupling in (28).

$$Q = \text{diag}(8, 7, 6, 3, 3, 2), \quad R = \text{diag}(0.8, 0.9, 0.7), \quad S = 5Q, \quad \gamma = 2.4 \quad (28)$$

Table 1: Simulation model and algorithm parameters.

Parameter	Value	Mathematical role
State dimension n	6	Buffers and production-rate deviations
Control dimension m	3	Feeder, processor, and robot commands
Disturbance dimension q	2	Arrival and processing-load uncertainty
Horizon T	20 s	Finite optimal-control interval
Time step Δt	0.05 s	400 temporal intervals
Control bounds	$[-1, 1]^3$	Closed convex actuator set
Uncertainty radius ρ	1.5	Disturbance-energy ball

Attenuation γ	2.4	Strong concavity in disturbance
Step size τ	0.18	Extragradient update
Tolerance ε	10^{-5}	Residual and gap threshold

Table 1 summarizes the essential parameters of the simulation model and numerical algorithm with their respective values and functions.

The wear term was $\Psi(u) = 0.025 \int_0^T \|u(t) - u(t - \Delta t)\|_1 dt$. The proposed method used $\tau = 0.18$, tolerance $\varepsilon = 10^{-5}$, and a maximum of 300 iterations. All methods started from the same feasible zero control and were tested against the same worst-case disturbance budget. The comparison methods were linear-quadratic regulation (LQR), deterministic model predictive control (DMPC), projected gradient control (PGC), and robust H_∞ control. These values presented below are deterministic outcomes of this particular simulation model which are supposed to verify mathematical features rather than generalize plant-wide empirically.

Table 2: Comparative worst-case control performance.

Method	Worst-case cost ↓	Tracking RMSE ↓	Energy index ↓	Violation ↓	Iterations
LQR	21.73	0.124	8.91	0.086	—
DMPC	17.96	0.093	8.14	0.031	38
PGC	15.62	0.082	7.88	0.000	96
H-infinity	14.91	0.078	8.46	0.000	—
Proposed	13.84	0.071	7.62	0.000	47

Table 2 demonstrates that the method proposed reached the minimum worst-case cost and minimum tracking error while keeping the minimum energy index compared to other feasible robust methods. This method had 22.9% lower cost than DMPC and 7.2% lower cost than H-infinity control. Infeasibility occurred in LQR and DMPC since they did not optimize the adversarial disturbance within the whole set of uncertainty. PGC achieved feasibility by performing projection, yet needed more than two times the number of iterations and obtained a larger saddle point residual than the proposed approach.

5.2. Convergence, feasibility, and duality verification

Table 3 gives the iteration-wise convergence analysis for the objective function, natural residual, primal dual gap, and constraint violation till the end of convergence.

Table 3: Iterative convergence diagnostics.

Iteration	Objective	Natural residual	Primal-dual gap	Max. violation
0	28.640	1.84×10^{-1}	3.17×10^{-1}	0
10	16.208	2.71×10^{-2}	4.82×10^{-2}	0
20	14.291	4.38×10^{-3}	7.90×10^{-3}	0
30	13.917	6.94×10^{-4}	1.31×10^{-3}	0
40	13.850	7.62×10^{-5}	1.46×10^{-4}	0
47	13.841	8.91×10^{-6}	9.54×10^{-6}	0

Natural residual and primal-dual gap monotonically decreased after the prediction phase and passed 10^{-5} at iteration 47. Empirical contraction factor averaged over iterations 10 to 40 was 0.822, which is in agreement with theoretical upper bound on contraction factor $q=0.847$ based on local estimates $\mu=0.64$ and $L=2.91$. Projections preserved zero violation at each iterate. The simultaneous decrease of residual and primal-dual gap is the evidence of variational and dual representations of the problem; otherwise, stabilization of the objective value would not imply stationarity.

Table 4: Sensitivity to the disturbance radius.

ρ	Worst-case cost	Tracking RMSE	Energy index	Iterations	Residual
0.5	10.72	0.053	6.98	39	7.8×10^{-6}
1.0	12.18	0.062	7.29	43	8.4×10^{-6}
1.5	13.84	0.071	7.62	47	8.9×10^{-6}
2.0	15.71	0.083	8.05	52	9.2×10^{-6}
2.5	17.93	0.097	8.61	59	9.6×10^{-6}

An increase in the parameter ρ increases W_{ad} and it cannot decrease the maximum value; this is a property of monotonicity verified by table 4. The cost increased linearly from 10.72 to 17.93, while all the scenarios remained feasible. The distance between two consecutive solutions is less than the Lipschitz upper bound shown in equation (25). The increased number of iterations when ρ is large is a result of poor conditioning.

5.3. Robustness and mathematical discussion

Table 5: Step-size and initialization robustness.

τ	Initial point	Converged	Iterations	Final gap	Final cost
0.08	Zero	Yes	102	9.7×10^{-6}	13.841
0.12	Zero	Yes	69	9.4×10^{-6}	13.841
0.18	Zero	Yes	47	9.5×10^{-6}	13.841
0.22	Random A	Yes	53	9.8×10^{-6}	13.841
0.22	Random B	Yes	55	9.2×10^{-6}	13.841
0.40	Random B	No	>300	2.6×10^{-2}	14.006

Table 5 gives the effect of step sizes τ and initial conditions on convergence robustness, iterations, gap, and cost.

All admissible values of step size and three initial guesses led to the same cost, thus confirming uniqueness. Lower step sizes were stable but slower. The optimal combination was $\tau=0.18$. The value of $\tau=0.40$ is not admissible since it is greater than the sufficient one and did not meet the tolerance condition, showing what the convergence theorem means in practice.

Table 6: Component ablation of the proposed formulation.

Configuration	Worst cost ↓	RMSE ↓	Violation ↓	Iterations
Full framework	13.84	0.071	0.000	47
Without extragradient	15.62	0.082	0.000	96
Without disturbance maximization	17.96	0.093	0.031	38
Without projection	13.31	0.069	0.074	42
Without wear penalty	13.52	0.070	0.000	45

Ablation of components isolates their contribution to the framework as shown in table 6. Ablation of the extragradient step leads to slower projected gradient dynamics and an increase in the objective value for a fixed number of iterations. Ablation of disturbance maximization leads to optimistic nominal control and a non-zero violation under worst-case analysis. Projection ablation decreases the objective value while making the trajectory infeasible, illustrating why comparisons on unconstrained objectives are not appropriate. Ablation of the wear term slightly reduces the numerical objective but increases control variation by 31.6%, which is equivalent to a more aggressive actuator movement. Only the complete framework guarantees robust cost, feasibility, smooth actuation, and a certificate.

From a convex analysis viewpoint, there are four crucial observations that can be made. Firstly, convex uncertainty leads to a saddle formulation of robustness that is achieved. Secondly, control regularization and disturbance suppression lead to a strongly monotone saddle operator; thus, the solution is unique instead of belonging to an uncontrolled set of equilibria. Thirdly, projection enforces actuator and disturbance feasibility at each step of the algorithm, which is an indispensable condition for a receding horizon implementation. Finally, the primal-dual gap and variational residual are two independent numerical measures of optimality. In consequence, the engineering system is a case of the underlying mathematical construction.

The proposed theory has certain drawbacks. Dynamics are assumed to be linearized around the region of operation, the uncertainty set needs to be defined reliably, and the numbers are computed via simulations. Convexity-concavity may not hold when the control cost term is singular or the attenuation cost is too weak. Existence will still hold true, but there will be no guarantees of uniqueness and linear convergence. Mismatch of models other than W_{ad} is also not safeguarded by the minimax result. These drawbacks point towards possible future developments in the direction of non-linear monotone inclusions and time varying feasible sets.

6. Conclusion

A convex analysis approach to optimal control under uncertainty and constraints in industrial automation is proposed. The dynamic process was simplified with an affine control to state operator and the problem is written as a convex-concave minimax problem on a product Hilbert space. It is shown that saddle-point inequalities, adjoint problems, variational inequalities, normal-cone inclusions, KKT conditions, and Fenchel duality all

characterize the same optimal solution. Existence follows from convex feasibility and semi-continuity, while strong convexity-concavity gives uniqueness and a strongly monotone operator. The stability constant measures how the optimal control depends on changes in the plant and uncertainty. A projected extragradient primal-dual algorithm maintains feasibility and allows a global linear convergence rate under a step size condition. The proposed algorithm in the simulated three-stage production line achieved a minimum cost of 13.84, RMSE of 0.071, energy index of 7.62, zero constraint violations, and convergence in 47 iterations. The natural residual and the primal-dual gap were smaller than 10^{-5} , and multiple starting points converged to the same result. The above achievements prove that convex analysis provides guarantees of feasibility and computational procedures for control design. Further research should focus on nonlinear process dynamics, non-smooth state constraints, distributed controllers, asynchronous industrial networks, and uncertainty set from data while maintaining the saddle-point and convergence properties.

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