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Solutions of a Class of Discrete Fourth Order Boundary Value Problems

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We study the discrete fourth order boundary value problem

$$\begin{cases} \Delta^4 u(t-2) - \alpha\Delta^2 u(t-1) + \beta u(t) = f(t, u(t)), \quad t \in [1, N]_{\mathbb{Z}}, \\ u(-1) = \Delta u(-1) = 0, \quad u(N+1) = \Delta^2 u(N) = 0, \end{cases}$$

where $N \geq 1$ is an integer, $\alpha, \beta \geq 0$, and $f\colon [1, N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ is continuous in the second argument. We obtain several criteria for the existence of one and multiple solutions of the problem. Our analysis is mainly based on the variational method and critical point theory. Examples are presented to illustrate our results.

Keywords: Discrete boundary value problem, fourth order, solutions, variational methods, local linking, critical points.

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1. Introduction

Throughout this paper, for any integers c and d with $c \leq d$, let $[c, d]_{\mathbb{Z}}$ denote the discrete interval $\{c, c+1, \ldots, d\}$. In this paper, we study the discrete nonlinear fourth order boundary value problem (BVP)

$$\begin{cases} \Delta^4 u(t-2) - \alpha\Delta^2 u(t-1) + \beta u(t) = f(t, u(t)), \quad t \in [1, N]_{\mathbb{Z}}, \\ u(-1) = \Delta u(-1) = 0, \quad u(N+1) = \Delta^2 u(N) = 0, \end{cases} \tag{1}$$

where $N \geq 1$ is an integer, $\alpha, \beta \geq 0$, Δ is the forward difference operator defined by $\Delta u(t) = u(t+1) - u(t)$, $\Delta^0 u(t) = u(t)$, $\Delta^i u(t) = \Delta^{i-1}(\Delta u(t))$ for $i \geq 2$, and $f : [1, N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ is continuous in the second argument. By a *solution* of BVP (1), we mean a function $u\colon [-1, N+2]_{\mathbb{Z}} \to \mathbb{R}$ that satisfies both equations in (1).

We can regard BVP (1) as a discrete analogue of the fourth order problem for a beam equation

$$\begin{cases} u^{(4)}(t) - \alpha u''(t) + \beta u(t) = f(t, u(t)), \quad t \in (0, 1), \\ u(0) = u'(0) = 0, \quad u(1) = u''(1) = 0, \end{cases} \tag{2}$$

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where the boundary conditions correspond to the left end of the beam being clamped, and the right end being hinged when there is no bending moment. In the literature, BVP (2) and its variations have been studied by many researchers using a variety of methods. For a small sample of the work, the reader may refer to [4, 5, 14, 23, 24] and the references therein.

Nonlinear difference equations have been widely used to study discrete models in numerous fields such as statistics, computer science, economics, neural network, biology, and a variety of other fields as can be seen in [1, 18, 19]. In recent years, a lot of research has been carried out by numerous researchers to study the existence and multiplicity of solutions for discrete BVPs. For some recent results, see, for example, [2, 9, 15, 16, 17, 20, 26] for work on fourth order discrete problems with separated BCs and [3, 7, 8, 11, 12, 13] for work on periodic problems. In particular, paper [7] applied the Krasnosel'skii fixed point theorem to study the existence of positive solutions of the periodic BVP

$$\begin{cases} u(t+4) + Mu(t) = \lambda g(t) f(u(t)) + h(t), \ t \in [0, N-1]_{\mathbb{Z}}, \\ u(i) = u(N+i), \ i = 0, 1, 2, 3, \end{cases}$$

where f, g, h are functions satisfying some suitable conditions, $\lambda > 0$ is a parameter, and M is a real parameter; while papers [11, 12, 13] used variational methods and critical point theory to investigate the existence of one and multiple solutions for a class of fourth order periodic problems. Motivated by papers [11, 12, 13], in this paper, we further apply variational methods and some basic theorems in critical point theory to BVP (1) and obtain several criteria for the existence of at least one and multiple solutions. In the statements and proofs of our theorems, we utilize the eigenvalues of a symmetric matrix associated with the problem.

As a final note of this section, we comment that, with little modification of the arguments, the results obtained in this paper can be extended to the more general problem

$$\begin{cases} \Delta^4 u(t-2) - \Delta\big(p(t-1)\Delta u(t-1)\big) + q(t)u(t) = f(t, u(t)), \quad t \in [1, N]_{\mathbb{Z}}, \\ u(-1) = \Delta u(-1) = 0, \quad u(N+1) = \Delta^2 u(N) = 0, \end{cases} \tag{3}$$

where $p\colon [0, N]_{\mathbb{Z}} \to \mathbb{R}$ and $q\colon [1, N]_{\mathbb{Z}} \to \mathbb{R}$ are two nonnegative functions. For the ease of notations in the discussion, we discuss BVP (1) instead of (3) in this paper.

The rest of this paper is organized as follows. Section 2 contains some preliminary results. Section 3 contains the main results of this paper and their proofs, and two examples.

2. Preliminary results

This section presents some background knowledge which is needed in the proofs of our main results. Assume that X is a real Banach space and $J \in C^1(X, \mathbb{R})$.

As usual, J is said to satisfy the *Palais-Smale (PS) condition* if every sequence $\{u_n\} \subset X$, such that $J(u_n)$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence. The sequence $\{u_n\}$ is called a *PS sequence*.

Lemma 2.1 below can be found in [21, 25], Lemma 2.2 in [6], and Lemma 2.3 can be seen in [10, 22].

Lemma 2.1. *Let X be a real reflexive Banach space, and let J be a weakly upper (lower, respectively) semicontinuous functional such that*

$$\lim_{\|u\|\to\infty} J(u) = -\infty \quad \left(\lim_{\|u\|\to\infty} J(u) = \infty, \ \textit{respectively} \right).$$

Then, there exists $u_0 \in X$ such that

$$J(u_0) = \sup_{u\in X} J(u) \quad \left(J(u_0) = \inf_{u\in X} J(u), \ \textit{respectively} \right).$$

Furthermore, if $J \in C^1(X, R)$, then $J'(u_0) = 0$.

Lemma 2.2. *Let X be a real Banach space with a direct sum decomposition $X = X_1 \oplus X_2$ with $\dim X_2 < \infty$. Suppose that $J \in C^1(X, \mathbb{R})$ satisfies the PS condition and is bounded below, $J(0) = 0$, and $\inf_{u\in X} J(u) < 0$. Assume also that J has a local linking at 0, that is, for some $R > 0$,*

$$J(u) \geq 0 \quad \textit{for all } u \in X_1 \textit{ with } \|u\| \leq R,$$
$$J(u) \leq 0 \quad \textit{for all } u \in X_2 \textit{ with } \|u\| \leq R.$$

Then, J has at least two nontrivial critical points.

Lemma 2.3. *Let X be a real Banach space and $J \in C^1(X, R)$ be even, bounded from below, and satisfy the PS condition. Suppose that $J(0) = 0$ and there is a set $K \subset X$ such that K is homeomorphic to S^{n-1} by an odd map and $\sup_{u\in K} J(u) < 0$, where S^{n-1} is the $n-1$ dimensional unit sphere. Then, J has at least n disjoint pairs of nontrivial critical points.*

From now on, let X be the vector space defined by

$$X = \left\{ u \colon [-1, N+2]_{\mathbb{Z}} \to \mathbb{R} \mid u(-1) = \Delta u(-1) = 0, \ u(N+1) = \Delta^2 u(N) = 0 \right\},$$

and for any $u \in X$, define

$$\|u\| = \left(\sum_{t=1}^{N} |u(t)|^2 \right)^{1/2} \quad \text{and} \quad \|u\|_\infty = \max_{t\in[1,N]_{\mathbb{Z}}} |u(t)|.$$

Then, X is a N-dimensional reflexive Banach space equipped with the norm $\|\cdot\|$ or $\|\cdot\|_\infty$.

For any $u \in X$, let the functional J be defined by

$$J(u) = \frac{1}{2}\sum_{t=1}^{N+1}\left(|\Delta^2 u(t-2)|^2 + \alpha|\Delta u(t-1)|^2\right) + \frac{\beta}{2}\sum_{t=1}^{N}|u(t)|^2 - \sum_{t=1}^{N}F(t,u(t)), \quad (4)$$

where $F(t,x) = \int_0^x f(t,s)ds$ for $x \in \mathbb{R}$. Then, it is easy to see that J is continuously differentiable and its derivative $J'(u)$ at $u \in X$ and $v \in X$ is given by

$$\begin{aligned} J'(u)(v) = &\sum_{t=1}^{N+1}\left(\Delta^2 u(t-2)\Delta^2 v(t-2) + \alpha\Delta u(t-1)\Delta v(t-1)\right) \\ &+ \beta\sum_{t=1}^{N}u(t)v(t) - \sum_{t=1}^{N}f(t,u(t))v(t)\,. \end{aligned} \quad (5)$$

Lemma 2.4. *For any $u, v \in X$, we have*

$$\sum_{t=1}^{N+1}\Delta^2 u(t-2)\Delta^2 v(t-2) = \sum_{t=1}^{N}\Delta^4 u(t-2)v(t) \quad (6)$$

and

$$\sum_{t=1}^{N+1}\Delta u(t-1)\Delta v(t-1) = -\sum_{t=1}^{N}\Delta^2 u(t-1)v(t). \quad (7)$$

Proof. We first prove (6). For any $u, v \in X$, applying the summation by parts formula, we have

$$\begin{aligned} &\sum_{t=1}^{N+1}\Delta^2 u(t-2)\Delta^2 v(t-2) = \\ &= \Delta^2 u(N)\Delta v(N) - \Delta^2 u(-1)\Delta v(-1) - \sum_{t=1}^{N+1}\Delta^3 u(t-2)\Delta v(t-1) \\ &= -\sum_{t=1}^{N+1}\Delta^3 u(t-2)\Delta v(t-1) = -\Delta^3 u(N-1)\Delta v(N) - \sum_{t=1}^{N}\Delta^3 u(t-2)\Delta v(t-1) \\ &= \Delta^3 u(N-1)v(N) - \left(\Delta^3 u(N-1)v(N) - \Delta^3 u(-1)v(0) - \sum_{t=1}^{N}\Delta^4 u(t-2)v(t)\right) \\ &= \sum_{t=1}^{N}\Delta^4 u(t-2)v(t), \end{aligned}$$

i.e., (6) holds.

Next, we show (7). Again, the summation by parts formula yields that

$$\begin{aligned}\sum_{t=1}^{N+1}\Delta u(t-1)\Delta v(t-1) &= \Delta u(N)\Delta v(N)+\sum_{t=1}^{N}\Delta u(t-1)\Delta v(t-1)\\ &= -\Delta u(N)v(N)+\left(\Delta u(N)v(N)-\Delta u(0)v(0)-\sum_{t=1}^{N}\Delta^2 u(t-1)v(t)\right)\\ &= -\sum_{t=1}^{N}\Delta^2 u(t-1)v(t),\end{aligned}$$

i.e., (7) holds. This completes the proof of the lemma. □

In view of (5) and Lemma 2.4, $J'(u)$ can be written as

$$J'(u)(v)=\sum_{t=1}^{N}\big(\Delta^4 u(t-2)-\alpha\Delta^2 u(t-1)+\beta u(t)-f(t,u(t))\big)v(t),\quad v\in X.$$

Thus, the following lemma is obviously true.

Lemma 2.5. *A function $u\in X$ is a critical point of J if and only if $u(t)$ is a solution of BVP* (1).

In what follows, we present an equivalent form of the functional J. Let

$$u=(u(1),u(2),\ldots,u(N))^T,$$

$$A=\begin{cases}(5) & \text{if } N=1,\\ \begin{pmatrix}5 & -4\\ -4 & 4\end{pmatrix} & \text{if } N=2,\\ \begin{pmatrix}6 & -4 & 1\\ -4 & 6 & -4\\ 1 & -4 & 5\end{pmatrix} & \text{if } N=3,\\ \begin{pmatrix}6 & -4 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ -4 & 6 & -4 & 1 & 0 & \cdots & 0 & 0 & 0 & 0\\ 1 & -4 & 6 & -4 & 1 & \cdots & 0 & 0 & 0 & 0\\ 0 & 1 & -4 & 6 & -4 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & -4 & 6 & \cdots & 0 & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & 0 & \cdots & 6 & -4 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & \cdots & -4 & 6 & -4 & 1\\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -4 & 6 & -4\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -4 & 5\end{pmatrix}_{N\times N} & \text{if } N\geq 4,\end{cases}$$

$$B = \begin{cases} (2\alpha) & \text{if } N = 1, \\ \begin{pmatrix} 2\alpha & -\alpha \\ -\alpha & 2\alpha \end{pmatrix} & \text{if } N = 2, \\ \begin{pmatrix} 2\alpha & -\alpha & 0 & \cdots & 0 & 0 \\ -\alpha & 2\alpha & -\alpha & \cdots & 0 & 0 \\ 0 & -\alpha & 2\alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2\alpha & -\alpha \\ 0 & 0 & 0 & \cdots & -\alpha & 2\alpha \end{pmatrix}_{N\times N} & \text{if } N \geq 3, \end{cases}$$

and

$$C = \begin{pmatrix} \beta & 0 & 0 & \cdots & 0 & 0 \\ 0 & \beta & 0 & \cdots & 0 & 0 \\ 0 & 0 & \beta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta & 0 \\ 0 & 0 & 0 & \cdots & 0 & \beta \end{pmatrix}_{N\times N}.$$

Then, A, B, and C are symmetric, and for all $u \in X$, some calculations lead to

$$\sum_{t=1}^{N+1} |\Delta^2 u(t-2)|^2 = \sum_{t=1}^{N+1} \big(u^2(t) + 4u^2(t-1) + u^2(t-2) - $$
$$- 4u(t)u(t-1) - 4u(t-1)u(t-2) + 2u(t)u(t-2)\big) = u^T Au,$$
$$\alpha \sum_{t=1}^{N+1} |\Delta u(t-1)|^2 = \alpha \sum_{t=1}^{N+1} \big(u^2(t) + u^2(t-1) - 2u(t)u(t-1)\big) = u^T Bu,$$

and

$$\beta \sum_{t=1}^{N} |u(t)|^2 = u^T Cu.$$

Thus, in view of (4), J can be rewritten as

$$J(u) = \frac{1}{2} u^T Du - \sum_{t=1}^{N} F(t, u(t)), \tag{8}$$

where

$$D = A + B + C. \tag{9}$$

Remark 2.6. Clearly, A is positive definite, B and C are positive semidefinite. Thus, $D = A + B + C$ is positive definite.

3. Main results

In the sequel, let λ_i, $i = 1, \dots, N$, be the eigenvalues of D ordered as follows

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N,$$

and let ξ_i be an eigenvector of D associated with λ_i such that

$$(\xi_i, \xi_j) = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

Then, for any $u = (u(1), u(2), \dots, u(N))^T \in \mathbb{R}^N$, we see that

$$\lambda_1 \|u\|^2 \leq u^T D u \leq \lambda_N \|u\|^2. \tag{10}$$

For convenience, we use the following notations:

$$f_0 = \liminf_{x \to 0} \min_{t \in [1,N]_{\mathbb{Z}}} \frac{f(t,x)}{x}, \quad f^0 = \limsup_{x \to 0} \max_{t \in [1,N]_{\mathbb{Z}}} \frac{f(t,x)}{x},$$
$$f_\infty = \liminf_{|x| \to \infty} \min_{t \in [1,N]_{\mathbb{Z}}} \frac{f(t,x)}{x}, \quad f^\infty = \limsup_{|x| \to \infty} \max_{t \in [1,N]_{\mathbb{Z}}} \frac{f(t,x)}{x}.$$

The following assumptions are needed in the paper.

(H1) $f_\infty > \lambda_N$;

(H2) $f^\infty < \lambda_1$;

(H3) there exists $m \in \{1, \dots, N-1\}$ such that $\lambda_m < f_0 \leq f^0 < \lambda_{m+1}$;

(H4) $f(t, -x) = -f(t, x)$ for $(t, x) \in [1, N]_{\mathbb{Z}} \times \mathbb{R}$;

(H5) there exists $m \in \{1, \dots, N\}$ such that $f^0 < \lambda_m$;

(H6) there exists $m \in \{1, \dots, N\}$ such that $f_0 > \lambda_m$.

Theorem 3.1. (a) *Assume that either* (H1) *or* (H2) *holds. Then,* BVP (1) *has at least one solution.*
(b) *Assume that either* (H1) *and* (H3) *or* (H2) *and* (H3) *hold. Then,* BVP (1) *has at least two nontrivial solutions.*

Proof. We first prove part (a). Assume that (H1) holds. Then, there exists $\epsilon \in (0, f_\infty - \lambda_N)$ and $c_1 > 0$ such that

$$\frac{f(t,x)}{x} \geq \lambda_N + \epsilon \quad \text{for all } (t, |x|) \in [1, N]_{\mathbb{Z}} \times (c_1, \infty),$$

which in turn implies that there exists $c_2 \in \mathbb{R}$ such that

$$F(t,x) \geq \frac{\lambda_N + \epsilon}{2} x^2 + c_2 \quad \text{for all } (t,x) \in [1, N]_{\mathbb{Z}} \times \mathbb{R}. \tag{11}$$

For any $u \in X$, from (8), (10), and (11), we have

$$J(u) \leq \frac{1}{2}\lambda_N \|u\|^2 - \frac{\lambda_N + \epsilon}{2} \sum_{t=1}^{N} |u(t)|^2 - c_2 N = -\frac{1}{2}\epsilon\|u\|^2 - c_2 N. \qquad (12)$$

Thus, $J(u) \to -\infty$ as $\|u\| \to \infty$. Hence, from Lemma 2.1, there exists $u_0 \in X$ such that $J'(u_0) = 0$. By Lemma 2.5, $u_0(t)$ is a solution of BVP (1).

Assume now that (H2) holds. Using (H2), similar to (11), there exist $\epsilon \in (0, \lambda_1 - f^\infty)$ and $c_3 \in \mathbb{R}$ such that

$$F(t,x) \leq \frac{\lambda_1 - \epsilon}{2} x^2 + c_3 \quad \text{for all } (t,x) \in [1,T]_{\mathbb{Z}} \times \mathbb{R}.$$

Combining this with (8) and (10) yields that for $u \in X$,

$$J(u) \geq \frac{1}{2}\lambda_1 \|u\|^2 - \frac{\lambda_1 - \epsilon}{2} \sum_{t=1}^{N} |u(t)|^2 - c_3 N = \frac{1}{2}\epsilon\|u\|^2 - c_3 N.$$

Thus, $J(u) \to \infty$ as $\|u\| \to \infty$. Again, applying Lemmas 2.1 and 2.5, BVP (1) has at least one solution. This shows part (a).

To prove part (b), first assume that (H1) and (H3) hold. Let $H(u) = -J(u)$. Then, $H(0) = 0$, and from (12),

$$H(u) \geq \frac{1}{2}\epsilon\|u\| + c_2 N \quad \text{for all } u \in X.$$

Then, H is bounded below and satisfies the PS condition. In fact, any PS sequence $\{u_n\} \subset X$ must be bounded, and since the dimension X is finite, $\{u_n\}$ has a convergent subsequence.

Define X_1 and X_2 as follows: $X_1 = \text{span}\{\xi_1, \ldots, \xi_m\}$, $X_2 = \text{span}\{\xi_{m+1}, \ldots, \xi_N\}$. Clearly, $X = X_1 \oplus X_2$. By (H3), we see that there exists $c_4 > 0$ such that

$$\lambda_m < \frac{f(t,x)}{x} < \lambda_{m+1} \quad \text{for all } (t, |x|) \in [1,N]_{\mathbb{Z}} \times [0, c_4]. \qquad (13)$$

Note that X is finite dimensional. Then, there is $R > 0$ such that $\|u\|_\infty < c_4$ for $u \in X$ with $\|u\| \leq R$. From (8) and (13) we get for $u \in X_1$ with $\|u\| \leq R$,

$$\begin{aligned} H(u) &= -\frac{1}{2}u^T D u + \sum_{t=1}^{N} F(t,u(t)) \geq -\frac{1}{2}\lambda_m \|u\|^2 + \sum_{t=1}^{N} \int_0^{u(t)} f(t,s)ds \\ &\geq -\frac{1}{2}\lambda_m \|u\|^2 + \frac{1}{2}\lambda_m \|u\|^2 = 0, \end{aligned}$$

and for $u \in X_2$ with $\|u\| \le R$,

$$H(u) = -\frac{1}{2}u^T Du + \sum_{t=1}^{N} F(t, u(t)) \le -\frac{1}{2}\lambda_{m+1}\|u\|^2 + \sum_{t=1}^{N} \int_0^{u(t)} f(t, s)ds$$
$$\le -\frac{1}{2}\lambda_{m+1}\|u\|^2 + \frac{1}{2}\lambda_{m+1}\|u\|^2 = 0,$$

i.e., H has a local linking at 0.

If $\inf_{u \in X} H(u) < 0$, then Lemma 5 implies that H has at least two nontrivial critical points, and so does J. Hence, by Lemma 2.5, BVP (1) has at least two nontrivial solutions. If $\inf_{u \in X} H(u) \ge 0$, then $H(u) = \inf_{u \in X_2} H(u) = 0$ for any $u \in X_2$ with $\|u\| \le R$. This shows that H has infinitely many critical points, and so does J. From Lemma 2.5, BVP (1) has infinitely many nontrivial solutions.

For the case when (H2) and (H3) hold, we apply Lemma 2.2 to J directly with X_1 and X_2 defined by $X_1 = \operatorname{span}\{\xi_{m+1}, \dots, \xi_N\}$ and $X_2 = \operatorname{span}\{\xi_1, \dots, \xi_m\}$.

The rest of the proof is similar to the case when (H1) and (H3) hold, and hence is omitted. This completes the proof of the theorem. □

Theorem 3.2. *Assume that* (H1), (H4), *and* (H5) *hold. Then,* BVP (1) *has at least* $2(N - m + 1)$ *nontrivial solutions.*

Proof. Let $H(u) = -J(u)$. Then, $H(0) = 0$ and H is even by (H4). Since (H1) holds, from the proof of Theorem 3.1, H is bounded below and satisfies the PS condition. If $f^0 > -\infty$, we let $0 < \mu < \lambda_m - f^0$. Then, by (H5), there exists $c_5 > 0$ such that

$$\frac{f(t, x)}{x} \le f^0 + \mu \quad \text{for all } (t, |x|) \in [1, N]_{\mathbb{Z}} \times [0, c_5].$$

Consequently,

$$F(t, x) = \int_0^x f(t, s)ds \le \frac{1}{2}(f^0 + \mu)x^2 \quad \text{for all } (t, |x|) \in [1, T]_{\mathbb{Z}} \times [0, c_5]. \tag{14}$$

Define a set $K \subset X$ by $K = \left\{u = \sum_{i=m}^{N} d_i\xi_i \;:\; \sum_{i=m}^{N} d_i^2 = c_5^2\right\}$.

Let S^{N-m} be the unit sphere in $\mathbb{R}^{N-m+1}$. Let $T\colon K \to S^{N-m}$ be defined by

$$T(u) = \frac{1}{c_5}\left(d_m, d_{m+1}, \dots, d_N\right).$$

Obviously, T is an odd homeomorphism between K and S^{N-m}. For any $u \in K$, from (8) and (14), it follows that

$$H(u) = -\frac{1}{2}u^T Du + \sum_{t=1}^{N} F(t, u(t)) \le -\frac{1}{2}\lambda_m\|u\|^2 + \frac{1}{2}(f^0 + \mu)\sum_{t=1}^{N} |u(t)|^2$$
$$= \frac{1}{2}(f^0 + \mu - \lambda_m)c_5^2 < 0.$$

Then, $\sup_{u\in K} H(u) < 0$. When $f^0 = -\infty$, we can similarly show that this inequality also holds. Hence, we have shown that all the conditions of Lemma 2.3 are satisfied. Thus, H has at least $2(N-m+1)$ nontrivial critical points. Applying Lemma 2.5 then completes the proof of the theorem. □

Theorem 3.3. *Assume that* (H2), (H4), *and* (H6) *hold. Then,* BVP (1) *has at least* $2m$ *nontrivial solutions.*

Proof. It is clear that $J(0) = 0$ and J is even by (H4). Since (H2) holds, from the proof of Theorem 3.1, we know that J is bounded below and satisfies the PS condition. Assume first that $f_0 < \infty$. Let $0 < \nu < f_0 - \lambda_m$. Then, from (H6), there exists $c_6 > 0$ such that

$$\frac{f(t,x)}{x} \geq f_0 - \nu \quad \text{for all } (t,|x|) \in [1,N]_{\mathbb{Z}} \times [0,c_6],$$

and so

$$F(t,x) = \int_0^x f(t,s)ds \geq \frac{1}{2}(f_0-\nu)x^2 \quad \text{for all } (t,|x|) \in [1,N]_{\mathbb{Z}} \times [0,c_6]. \tag{15}$$

Let a set $K \subset X$ be defined by $K = \{u = \sum_{i=1}^m d_i\xi_i \ : \ \sum_{i=1}^m d_i^2 = c_6^2\}$.
Let S^{m-1} be the unit sphere in $\mathbb{R}^m$. Let $T\colon K \to S^{m-1}$ be defined by

$$T(u) = \frac{1}{c_6}\left(d_1, d_2, \ldots, d_m\right).$$

Clearly, T is an odd homeomorphism between K and S^{m-1}. For any $u \in K$, (8) and (15) imply that

$$\begin{aligned} J(u) &= \frac{1}{2}u^T Du - \sum_{t=1}^N F(t,u(t)) \leq \frac{1}{2}\lambda_m\|u\|^2 - \frac{1}{2}(f_0-\nu)\sum_{t=1}^N |u(t)|^2 \\ &= \frac{1}{2}(\lambda_m - f^0 + \tau)c_6^2 < 0. \end{aligned}$$

Hence, $\sup_{u\in K} J(u) < 0$. If $f_0 = \infty$, it can be shown that this inequality also holds. Thus, all the conditions of Lemma 2.3 are satisfied. Now, Lemmas 2.3 and 2.5 imply that BVP (1) has at least $2m$ nontrivial solutions. This completes the proof of the theorem. □

To end this paper, we provide the following two examples.

Example 3.4. In BVP (1), let $N = 8$, $\alpha = 2$, $\beta = 3$, and

$$f(t,x) = \frac{5(1+6x^3)}{1+x^3}x \quad \text{for all } (t,x) \in [1,8]_{\mathbb{Z}} \times \mathbb{R}.$$

Then, we claim that BVP (1) has at least two nontrivial solutions.

In fact, with the above N, α, and β, let the matrix D be defined by (9). Then, using MATLAB, we find that the eigenvalues of D are given by

$$\lambda_1 \approx 3.2767, \quad \lambda_2 \approx 4.2343, \quad \lambda_3 \approx 6.1555, \quad \lambda_4 \approx 9.2506,$$
$$\lambda_5 \approx 13.4278, \quad \lambda_6 \approx 18.1781, \quad \lambda_7 \approx 22.6399, \quad \lambda_8 \approx 25.8371.$$

Moreover, $f^0 = f_0 = 5$ and $f^\infty = f_\infty = 30$. Thus, (H1) and (H3) with $m = 2$ hold. The claim then follows from Theorem 3.1.

Example 3.5. Let N, α, and β be given as in Example 3.4. Define $f(t, x)$ by

$$f(t,x) = \frac{5(1+6x^2)}{1+x^2}x \quad \text{for all } (t,x) \in [1,8]_{\mathbb{Z}} \times \mathbb{R}.$$

Then, we claim that BVP (1) has at least 12 nontrivial solutions.

In fact, it is obvious that $f(t, x)$ is odd in x, $f^0 = f_0 = 5$, and $f^\infty = f_\infty = 30$. Also, the eigenvalues λ_i, $i = 1, \ldots, 8$, of D, defined by (9), are given as in Example 3.4. Thus, (H1), (H4), and (H5) with $m = 3$ hold. The claim then follows from Theorem 3.2.

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