

A Variational Inequality-Based Optimization Model for Flexible Manufacturing Resource Allocation

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Abstract

Flexible manufacturing systems need allocation schemes that can take care of alternative resource eligibility, capacity coupling, nonlinear processing effects, congestion, and workload imbalance altogether. The conventional single-objective optimization approach cannot be able to capture adequately the directionality aspect of interaction among allocation variables with asymmetric and non-conservative marginal effects. This paper introduces a variational inequality (VI)-based mathematical model for flexible manufacturing resource allocation where the equilibrium can be characterized directly via an allocation operator. Quadratic cost allocations, nonlinear processing time effects, capacity-dependent congestion, workload imbalance, and asymmetric resource interaction effects are taken into account in the formulation. An artificial restriction on the feasible set is imposed to preserve the positivity of the distance from the case of total utilization of resources so as to ensure boundedness of the derivative of the congestion function and local Lipschitz continuity of the operator. Existence follows from the continuity and convexity of the feasible set, while uniqueness follows from the strong monotonicity of the operator. With respect to the asymmetric Jacobian matrix, strong monotonicity is shown via the symmetric part of the matrix without assuming symmetry of the Jacobian matrix. Capacity multipliers are used to express endogenously resource-clearing prices. A projected VI method is presented, and its convergence is proved under explicit strong monotonicity and Lipschitz conditions. Numerical examples of allocation instances of sizes 10×10 , 50×50 , and 100×100 illustrate feasibility,

linear residual convergence, sensitivity of the step sizes, flexibility effects, and sensitivity of capacity factors.

Keywords: Variational inequality, flexible manufacturing, resource allocation, strong monotonicity, projection algorithm, equilibrium analysis.

1. Introduction

Flexibility in manufacturing systems involves allocation of scarce production resources across several operations with respect to capacity, processing, and technological constraints. The allocation problem is mathematically difficult due to the existence of multiple resources capable of performing the same operation as well as the influence of allocation on availability, congestion, and marginal cost of other resources (Subhash & Udayakumar, 2021). Resource coupling renders allocation a fundamentally different problem from the optimization of each machine independently. Capacity constraints and interactions between production decisions have also been considered important aspects in manufacturing and supply chain optimization models (Zhang et al., 2023). Operational disturbances and process interactions further complicate allocation in machining environments (Basit et al., 2024).

Traditional methods of optimization formulate allocation problems using a single objective function constrained by several constraints. This approach has proven to be efficient in dealing with many deterministic problems. However, the formulation of the resource allocation problem becomes restrictive in situations where the marginal impact of the allocation decision on an operation is affected by the allocation decisions of other resources. In flexible manufacturing systems, increased allocation to an operation decreases processing load and, at the same time, increases the utilization or congestion of that resource. Related decision problems have led to the development of optimization models incorporating dynamics in constraints, interaction in facilities, and resource balancing techniques (Jin et al., 2024; Ren et al., 2022; Kavuluri, 2025). Formulations based on algebraic inequalities have shown the significance of mathematical characterization of system constraints (Todinov, 2022).

The variational inequality (VI) formulation provides a convenient mathematical formulation for the coupled decision problems. As opposed to the formulation of the entire system using a single objective function, a variational inequality characterizes the equilibrium using an operator over a feasible set. In particular, a solution x^* satisfies:

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K$$

where F is the marginal effect of the system, and K is the feasible allocation region. This type of VI formulation is well-suited for allocation problems, where individual decisions affect each other through common constraints and capacities. The use of VI formulations has already been considered in the context of location-allocation problems for multiple facilities and proved the applicability of VI theory to such types of coupled allocation problems (Gu et al., 2023). Also, optimisation and equilibrium formulations have been developed for networked and resource-dependent systems (Nagurney & Samadi, 2025; Lai et al., 2026; Suvarna & Bharadwaj, 2024; Tharmalingam et al., 2024).

However, the existing research in the field lacks a formulation of a nonlinear variational inequality for a flexible manufacturing resource allocation problem (Chan et al., 2002). Namely, the relation between the form of the allocation operator, convexity of the feasible resource region, existence and uniqueness of equilibrium, and numerical properties of the problem needs to be systematically studied mathematically. Hence, the formulation of a VI in the operator form, taking into account the costs of resource allocation, the effects of the processing, the resource congestion, and workload imbalance, become a goal of this research (Farajpour et al., 2025).

The purpose of this paper is, thus, the development of a VI-based optimisation formulation for the flexible manufacturing resource allocation problem (Sabuncuoglu & Kizilisik, 2003; Chan & Chan, 2004). The feasible allocation region will be represented as a constrained convex set, and a nonlinear allocation operator will be developed to represent the marginal coupled effects. The mathematical properties of the VI will be studied in terms of continuity and monotonicity properties, followed by analysis of existence and uniqueness. An iterative projection method will be formulated, and the convergence will be proved under strong monotonicity and Lipschitz continuity assumptions.

Numerical experiments will be performed to study convergence, resource flexibility, capacity variation, and equilibrium properties of the problem.

The main contributions of the study are as follows:

1. The nonlinear variational inequality model is constructed for resource allocation in flexible manufacturing, taking into account the interaction between resources.
2. Mathematical sufficient conditions for the existence and uniqueness of equilibrium resource allocation are provided.
3. An iterative algorithm using the projection technique is developed to solve the formulated variational inequality problem.
4. Convergence condition and convergence rate are derived based on strong monotonicity and Lipschitz continuity of the resource allocation mapping.
5. Numerical results show mathematical performance of the developed model for various flexibility/capacity scenarios.

The rest of the paper is structured as follows. Section 2 provides the mathematical prerequisites necessary for the VI modeling. Section 3 discusses the flexible manufacturing resource allocation problem and its associated constraints. Section 4 describes the variational inequality formulated in this research along with its mathematical characteristics. Section 5 analyzes the projection method and its convergence. Section 6 contains results of the numerical experiment, while Section 7 summarizes the paper.

2. Mathematical preliminaries

Here, $K \subseteq \mathbb{R}^n$ represents the feasible resource allocation set, and $F: K \rightarrow \mathbb{R}^n$ represents the marginal allocation mapping. The formulation of the mathematical problem will be carried out via finite dimensional variational inequality problems, monotone mappings, Euclidean projections, and normal cone mappings. Unlike a formulation that relies on a potential function, the present formulation allows F to be non-conservative and non-symmetric.

2.1 Variational inequality

For a nonempty, closed, and convex set K and an operator $F: K \rightarrow \mathbb{R}^n$, the variational inequality problem $VI(K, F)$ is to determine $x^* \in K$ such that:

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K$$

In this case, x^* is an equilibrium allocation, while $F(x^*)$ is the vector of marginal effects of allocation. According to equation, no feasible deviation from the equilibrium x^* can provide negative directional values for the marginal operator. This form is especially useful in cases where the marginal effect from one allocation is asymmetrical with respect to other allocations.

2.2 Monotonicity and strong monotonicity

An operator F is monotone on K if:

$$\langle F(x) - F(y), x - y \rangle \geq 0, \quad \forall x, y \in K$$

For unique equilibrium, a more strict condition is required. Operator F is μ -strongly monotone if there exists $\mu > 0$ such that:

$$\langle F(x) - F(y), x - y \rangle \geq \mu \|x - y\|^2, \quad \forall x, y \in K$$

Strong monotonicity makes it impossible for two different feasible allocations to satisfy the VI equilibrium conditions.

The operator is L -Lipschitz continuous on K if:

$$\|F(x) - F(y)\|_2 \leq L\|x - y\|_2, \quad \forall x, y \in K$$

As for the suggested manufacturing model, Lipschitz continuity will be proven only for the subset K_δ of the feasible set due to the dependence on capacity in the denominators of the congestion functions. Constants μ and L define the permissible value of projection step size.

2.3 Projection characterization

For $z \in \mathbb{R}^n$, the Euclidean projection onto K is defined as:

$$P_K(z) = \arg \min_{y \in K} \frac{1}{2} \|y - z\|_2^2$$

If K is nonempty, closed, and convex, the projection is unique and non-expansive:

$$\|P_K(x) - P_K(y)\|_2 \leq \|x - y\|_2$$

For any $\alpha > 0$, the VI solution satisfies the fixed-point relation:

$$x^* = P_K(x^* - \alpha F(x^*))$$

Conversely, any $x^* \in K$ satisfying Equation solves $\text{VI}(K, F)$. This characterization provides the basis for the projected iteration developed in Section 5.

2.4 Normal cone representation

For $x \in K$, the normal cone to K at x is:

$$N_K(x) = \{z \in \mathbb{R}^n : \langle z, y - x \rangle \leq 0, \quad \forall y \in K\}$$

The VI in Equation can be alternatively expressed by the monotone inclusion:

$$0 \in F(x^*) + N_K(x^*)$$

This alternative expression isolates the marginal allocation map from the feasible production region geometry. It also allows capacity, demand process, non-negativity, and flexibility constraints to feature in the equilibrium conditions via the normal cone.

2.5 Asymmetric operators and symmetric monotonicity component

The new formulation does not need F to be the gradient of a scalar function. Hence, its Jacobian could be asymmetrical:

$$J_F(x) \neq J_F(x)^\top$$

Define the symmetric and antisymmetric parts of the Jacobian as:

$$S_F(x) = \frac{1}{2} [J_F(x) + J_F(x)^\top]$$

and

$$A_F(x) = \frac{1}{2} [J_F(x) - J_F(x)^\top]$$

Thus,

$$J_F(x) = S_F(x) + A_F(x)$$

where $S_F(x)$ is symmetric and $A_F(x)$ is skew-symmetric. Since:

$$v^T A_F(x) v = 0$$

for every $v \in \mathbb{R}^n$, have:

$$v^T J_F(x) v = v^T S_F(x) v$$

Consequently, a sufficient condition for strong monotonicity is:

$$S_F(x) \geq \mu I, \quad \mu > 0, \quad \forall x \in K_\delta$$

The condition allows for cross-resource interaction with asymmetry, but still maintains the property of uniqueness and convergence in a mathematical way. More specifically, the newly proposed VI does not have to be conservative.

3. Flexible manufacturing resource allocation model

Let us assume that there is a flexible manufacturing system where N production processes and M manufacturing resources are available. Here, each process is assigned to more than one eligible resource, and each of the resources has some fixed capacity. In such a situation, the resource allocation problem becomes a coupled problem, as the increase in allocation of one process will change the congestion, load, processing ability, and marginal allocation of the other processes.

3.1 Resource allocation variables

Let

$$x_{jm} \geq 0$$

denote the allocation level of resource m to operation j , where

$$j = 1, \dots, N, \quad m = 1, \dots, M.$$

The complete allocation vector is:

$$x = [x_{11}, x_{12}, \dots, x_{1M}, x_{21}, \dots, x_{NM}]^T \in \mathbb{R}^{NM}$$

Let $a_{jm} > 0$ represent the processing effectiveness of resource m for operation j , and $d_j > 0$ represent the required processing demand. The processing capability is effectively available to operation j is.

$$P_j(x) = \sum_{m=1}^M a_{jm} x_{jm}$$

The processing requirement is therefore:

$$\sum_{m=1}^M a_{jm} x_{jm} \geq d_j, \quad j = 1, \dots, N$$

It is important that the equilibrium assignment provides enough processing effectiveness for all operations.

3.2 Resource capacity constraints

Let $B_m > 0$ denote the available capacity of resource m . Its allocation cannot exceed the available capacity:

$$\sum_{j=1}^N x_{jm} \leq B_m, \quad m = 1, \dots, M$$

The normalized utilization of resource m is:

$$\rho_m(x) = \frac{1}{B_m} \sum_{j=1}^N x_{jm}$$

Consequently,

$$0 \leq \rho_m(x) \leq 1$$

The utilization variables offer an essential linkage element. An increase in x_{jm} increases the utilization level of resource m , thus influencing the marginal allocation criterion for other operations allocated to that particular resource.

3.3 Resource flexibility constraints

Let $\mathcal{M}_j \subseteq \{1, \dots, M\}$ represent the set of resources technologically capable of processing operation j . The allocation must satisfy:

$$m \notin \mathcal{M}_j \implies x_{jm} = 0$$

Define the eligibility matrix $H = [h_{jm}]$, where:

$$h_{jm} = \begin{cases} 1, & m \in \mathcal{M}_j, \\ 0, & m \notin \mathcal{M}_j. \end{cases}$$

The flexibility restriction can consequently be written as:

$$x_{jm}(1 - h_{jm}) = 0$$

Equation allows alternative resource assignments while eliminating technologically infeasible allocation pairs.

3.4 Direct allocation cost

Let $c_{jm} \geq 0$ denote the base marginal cost associated with assigning resource m to operation j . In vector form, let $c \in \mathbb{R}^{NM}$. Pairwise allocation interactions are represented by a matrix $Q \in \mathbb{R}^{NM \times NM}$.

The direct cost component is:

$$C(x) = c^T x + \frac{1}{2} x^T Q x$$

For the symmetric component of the direct cost, assume:

$$Q_s = \frac{1}{2}(Q + Q^T) \geq 0$$

The corresponding marginal contribution is:

$$F_C(x) = c + Qx$$

Notably, the full allocation operator will not be limited to $F_C(x)$. Other directional interaction terms will be presented later which will allow the full operator to become asymmetric despite having a symmetric direct cost matrix.

3.5 Processing-time function

The capacity to effectively process in Equation is included in the processing time component in

$$T(x) = \sum_{j=1}^N \frac{p_j}{P_j(x) + \varepsilon}$$

where $p_j > 0$ and $\varepsilon > 0$. The penalty term ensures that the denominator is always positive. For a single allocation variable:

$$\frac{\partial T}{\partial x_{jm}} = - \frac{p_j a_{jm}}{(P_j(x) + \varepsilon)^2}$$

Therefore, an increase in the allocation of an effective resource reduces the processing time component, and the degree to which this reduction occurs will depend on the processing capacity of operation j .

3.6 Congestion and workload imbalance

Resource congestion is modeled using:

$$G(x) = \sum_{m=1}^M \frac{\rho_m(x)}{I - \rho_m(x) + \varepsilon}$$

To ensure uniform mathematical regularity, define the restricted operating region:

$$K_\delta = \{x \in K: \rho_m(x) \leq I - \delta, \quad m = 1, \dots, M\}, \quad \delta > 0$$

Therefore,

$$I - \rho_m(x) + \varepsilon \geq \delta + \varepsilon > 0, \quad x \in K_\delta$$

For the congestion function:

$$\frac{\partial G}{\partial x_{jm}} = \frac{I + \varepsilon}{B_m [I - \rho_m(x) + \varepsilon]^2}$$

Equation shows explicitly the growing marginal congestion due to well-used facilities.

To represent workload imbalance, define:

$$\bar{\rho}(x) = \frac{I}{M} \sum_{m=1}^M \rho_m(x)$$

and

$$E(x) = \frac{I}{2} \sum_{m=1}^M [\rho_m(x) - \bar{\rho}(x)]^2$$

Its component-wise derivative is:

$$\frac{\partial E}{\partial x_{jm}} = \frac{I}{B_m} [\rho_m(x) - \bar{\rho}(x)]$$

Therefore, the imbalance factor amplifies the marginal impact of allocations made to heavily used resources and reduces that of allocations made to relatively underused resources.

3.7 Asymmetric cross-resource interaction

To differentiate the VI approach from a traditional potential based optimization problem, one can define a directional interaction matrix:

$$H_a = [h_{rs}] \in \mathbb{R}^{NM \times NM}$$

where r and s identify resource-operation allocations. The matrix does not have to be symmetric:

$$H_a \neq H_a^\top$$

The corresponding directional interaction operator is:

$$F_A(x) = H_a x$$

For two allocation components r and s :

$$\frac{\partial F_{A,r}}{\partial x_s} = h_{rs}, \quad \frac{\partial F_{A,s}}{\partial x_r} = h_{sr}$$

Therefore, $h_{rs} \neq h_{sr}$ produces:

$$\frac{\partial F_r}{\partial x_s} \neq \frac{\partial F_s}{\partial x_r}$$

This illustrates directional resource competition. For instance, the rise in the allocation of one process may cause a greater marginal congestion effect on another process than vice versa. This kind of asymmetric relationship cannot be captured by the gradient of a scalar potential.

3.8 Feasible allocation set

By combining nonnegativity, processing, capacity, and flexibility constraints, one gets:

$$K = \{x \in \mathbb{R}^{NM} : x \geq 0, Ax \leq b, Dx \geq d, Hx = 0\}$$

where A , D , and H are the matrices associated with the respective capacity, processing, and flexibility constraints; and b and d are the vectors of resource capacities and processing requirements.

The restricted feasible operating set to be used in the congestion analysis is

$$K_\delta = \{x \in K : \rho_m(x) \leq 1 - \delta, m = 1, \dots, M\}$$

For linear constraints, both K and K_δ are convex sets. In case there is a feasible production schedule ($K_\delta \neq \emptyset$), then K_δ is a nonempty closed convex feasible set. The strict capacity margin in (44) will also ensure that the denominator bound in (33) holds for the Lipschitz analysis of the allocation operator.

In this case, the resulting model consists of linear constraints on the resources, nonlinear processing and congestion effects, workload dependence, and asymmetric resource couplings. These aspects are put together in a non-conservative marginal operator in Section 4 instead of an aggregate potential function.

4. Variational inequality model and mathematical analysis

The flexible manufacturing allocation model includes cost allocations that are nonlinear, the effects of congestion, imbalances in the workload, and asymmetries in the interactions between allocation variables. In this context, it is

assumed that the cost allocations do not arise from a single potential but rather that they are explicitly modeled by means of a vectorial marginal operator. This makes the equilibrium incorporate directional resource interaction that can in general not be described by optimization.

4.1 Construction of the non-conservative allocation operator

Let $r = (j, m)$ denote the index corresponding to allocation variable x_{jm} (Azimova et al., 2023). The proposed allocation operator is defined component-wise as:

$$F_r(x) = c_r + [Qx]_r + [H_a x]_r + \lambda_T [F_T(x)]_r + \lambda_G [F_G(x)]_r + \lambda_E [F_E(x)]_r$$

where $\lambda_T, \lambda_G, \lambda_E \geq 0$ are weighting coefficients, and:

$$F_T(x) = \nabla T(x), \quad F_G(x) = \nabla G(x), \quad F_E(x) = \nabla E(x)$$

Consequently,

$$F(x) = c + Qx + H_a x + \lambda_T \nabla T(x) + \lambda_G \nabla G(x) + \lambda_E \nabla E(x)$$

The individual nonlinear components are obtained from Section 3. In particular:

$$[F_T(x)]_{jm} = -\frac{p_j a_{jm}}{(P_j(x) + \varepsilon)^2}$$

$$[F_G(x)]_{jm} = \frac{I + \varepsilon}{B_m [I - \rho_m(x) + \varepsilon]^2}$$

and

$$[F_E(x)]_{jm} = \frac{\rho_m(x) - \bar{\rho}(x)}{B_m}$$

The asymmetric interaction matrix H_a is essential because:

$$H_a \neq H_a^\top$$

Thus, in general:

$$\frac{\partial F_r}{\partial x_s} \neq \frac{\partial F_s}{\partial x_r}$$

That implies that F may not be conservative, and hence there is usually no scalar function Φ such that:

$$F(x) = \nabla \Phi(x)$$

It is clear that the suggested formulation will not reduce to a standard constrained minimization problem.

4.2 Variational inequality formulation

The flexible manufacturing equilibrium is defined as follows:

$$\text{Find } x^* \in K_\delta \text{ such that } \langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K_\delta$$

The vector $F(x^*)$ consists of the marginal effects that relate to the current distribution. The equation demands the total marginal effect from each possible deviation of x^* be non-negative.

If a feasible x existed such that:

$$\langle F(x^*), x - x^* \rangle < 0$$

If that is the case, then the equilibrium allocation can be improved in that feasible direction by using the marginal operator. This means that such an x^* does not meet the equilibrium condition presented in the proposal.

Equation can be expressed using normal-cone representation from Section 2 in the following way:

$$0 \in F(x^*) + N_{K_\delta}(x^*)$$

4.3 Existence of an equilibrium

The existence of the solution stems from the continuity of the operator on the compact feasible operating region.

Theorem 1. *Suppose that $K_\delta \subset \mathbb{R}^{NM}$ is nonempty, compact, and convex, and that $F: K_\delta \rightarrow \mathbb{R}^{NM}$ is continuous. Then $VI(K_\delta, F)$ has at least one solution.*

Proof. Because K_δ is nonempty, compact, and convex and F is continuous, the finite-dimensional variational inequality existence theorem guarantees the existence of $x^* \in K_\delta$ satisfying:

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K_\delta$$

Hence,

$$\text{SOL}(K_\delta, F) \neq \emptyset$$

For the proposed operator, the processing component is continuous because:

$$P_j(x) + \varepsilon > 0$$

Furthermore, from Equation:

$$I - \rho_m(x) + \varepsilon \geq \delta + \varepsilon > 0$$

Hence, the congestion part is also continuous on K_δ . The linear and workload parts are also continuous. This means that F is continuous on K_δ , thus finishing the proof.

4.4 Strong monotonicity and uniqueness

Existence alone will not guarantee uniqueness of the manufacturing equilibrium. Uniqueness is established via strong monotonicity.

Theorem 2. *Suppose that there exists $\mu > 0$ such that:*

$$\langle F(x) - F(y), x - y \rangle \geq \mu \|x - y\|_2^2, \quad \forall x, y \in K_\delta$$

Then $VI(K_\delta, F)$ has at most one solution.

Proof. Let x^* and y^* be two VI solutions. From Equation:

$$\langle F(x^*), y^* - x^* \rangle \geq 0$$

and

$$\langle F(y^*), x^* - y^* \rangle \geq 0$$

Adding these inequalities gives:

$$\langle F(x^*) - F(y^*), y^* - x^* \rangle \geq 0$$

or

$$\langle F(x^*) - F(y^*), x^* - y^* \rangle \leq 0$$

Strong monotonicity gives:

$$\langle F(x^*) - F(y^*), x^* - y^* \rangle \geq \mu \|x^* - y^*\|_2^2$$

Combining Equations (60) and (61):

$$\mu \|x^* - y^*\|_2^2 \leq 0$$

Since $\mu > 0$,

$$\|x^* - y^*\|_2 = 0$$

and therefore

$$x^* = y^*$$

Thus, the flexible manufacturing equilibrium is unique whenever the proposed operator is strongly monotone.

4.5 Strong monotonicity for an asymmetric jacobian

Let $J_F(x) = \frac{\partial F(x)}{\partial x}$ denote the Jacobian of the allocation operator. Because H_a is asymmetric and the nonlinear coupling terms may also generate asymmetric cross-partial, $J_F(x) \neq J_F(x)^\top$ in general.

Decompose the Jacobian as:

$$J_F(x) = S_F(x) + A_F(x)$$

where

$$S_F(x) = \frac{I}{2} [J_F(x) + J_F(x)^\top]$$

and

$$A_F(x) = \frac{I}{2} [J_F(x) - J_F(x)^\top]$$

For any $v \in \mathbb{R}^{NM}$:

$$v^\top A_F(x) v = 0$$

Hence,

$$v^\top J_F(x) v = v^\top S_F(x) v$$

A sufficient condition for strong monotonicity is therefore:

$$v^\top S_F(x) v \geq \mu \|v\|_2^2, \quad \forall v, x \in K_\delta$$

or equivalently,

$$\lambda_{\min}(S_F(x)) \geq \mu > 0, \quad \forall x \in K_\delta$$

The asymmetry element $A_F(x)$ does not make a direct contribution to the quadratic monotonicity formulation, but it nevertheless affects the equilibrium operator and allocations interaction. Therefore, the model allows non-conservative directional coupling without compromising the strong monotonicity requirement of uniqueness.

4.6 Dynamic shadow prices and capacity clearing

These capacity constraints lead to multipliers, which can be viewed as internal pricing mechanisms clearing resources. Let $\lambda_m \geq 0$ be the multiplier attached to the capacity constraint on resource m . The marginal equilibrium condition associated with the constraint is given by $+\lambda_m$.

Therefore, the marginal allocation effect for x_{jm} can be expressed as

$$F_{jm}(x) + \lambda_m$$

The multiplier satisfies the complementarity relation:

$$\lambda_m \left(\sum_{j=1}^N x_{jm} - B_m \right) = 0$$

Therefore,

$$\sum_{j=1}^N x_{jm} < B_m \quad \Rightarrow \quad \lambda_m = 0$$

whereas a binding resource may satisfy:

$$\sum_{j=1}^N x_{jm} = B_m, \quad \lambda_m > 0$$

Consequently, λ_m represents an endogenous price of scarcity; the higher the restrictions on the capacity of the resource, the higher the marginal cost of allocating an extra unit of the resource.

This approach is a decentralized clearing mechanism for the mathematical VI model.

4.7 KKT and complementarity representation

For the feasible set:

$$K_\delta = \{x: x \geq 0, Ax \leq b, Dx \geq d, Hx = 0, \rho_m(x) \leq 1 - \delta\}$$

introduce nonnegative multipliers $\lambda \geq 0$ for the capacity inequalities and $\gamma \geq 0$ for the processing-demand inequalities. Let $v \geq 0$ correspond to nonnegativity and η correspond to the equality restrictions.

The equilibrium stationarity condition can be expressed as:

$$F(x^*) + A^\top \lambda - D^\top \gamma + H^\top \eta - v = 0$$

The associated feasibility conditions are:

$$Ax^* \leq b$$

$$Dx^* \geq d$$

$$x^* \geq 0$$

$$Hx^* = 0$$

The complementary relations are:

$$\lambda_i(A_i x^* - b_i) = 0$$

$$\gamma_j(D_j x^* - d_j) = 0$$

and

$$\nu_r x_r^* = 0$$

For the restricted capacity region, the corresponding multiplier relations are included for:

$$\rho_m(x^*) \leq 1 - \delta$$

Equations indicate that the VI solution could be formulated as a coupled complementarity system involving both allocation and endogenous shadow price variables. The operator F , nevertheless, will retain its possible asymmetry due to the directional interaction matrix H_a , unlike the traditional approach to the potential function formulation (Ma et al., 2020).

The present formulation proves the existence, uniqueness, asymmetric monotonicity, and capacity pricing properties necessary for the projection method in Section 5.

5. Projection algorithm and convergence analysis

Since the variational inequality (54) calls for an algorithm that respects the coupled manufacturing constraints and converges to the equilibrium allocation, a technique involving Euclidean projection can be used since the set K_δ is non-empty, closed, and convex. The convergence result of the algorithm is proven using the strong monotonicity and Lipschitz continuity of the operator F .

5.1 Projection-based iterative scheme

Given an initial allocation $x^0 \in K_\delta$, the projected VI iteration is defined by:

$$x^{k+1} = P_{K_\delta}[x^k - \alpha F(x^k)]$$

where $\alpha > 0$ is the projection step size.

Define the intermediate point:

$$z^k = x^k - \alpha F(x^k)$$

This is where the unrestricted motion due to the marginal allocation operator is illustrated. Projection yields:

$$x^{k+1} = P_{K_\delta}(z^k)$$

Because $P_{K_\delta}(z^k) \in K_\delta$, feasibility is preserved at every iteration:

$$x^0 \in K_\delta \implies x^k \in K_\delta, \quad \forall k \geq 0$$

In contrast to a classical gradient algorithm, $F(x^k)$ is not presumed to be the gradient of a composite function. Therefore, the algorithm is directly applied to the variational inequality operator.

5.2 VI Residual and stopping criterion

The natural residual associated with the VI is defined as:

$$R_\alpha(x) = \frac{1}{\alpha} \|x - P_{K_\delta}(x - \alpha F(x))\|_2$$

At an exact VI solution:

$$R_\alpha(x^*) = 0$$

The scaling by $1/\alpha$ makes the residual dimensionally consistent with the operator magnitude. The successive-iterate difference is also monitored:

$$e_k = \|x^{k+1} - x^k\|_2$$

The numerical termination condition is therefore selected as:

$$R_\alpha(x^k) \leq \varepsilon_{\text{tol}}$$

with $\varepsilon_{\text{tol}} > 0$.

In Equation, the residual is employed instead of merely the value of the objective of any scalar function since the model presented here is non-potential VI.

5.3 Lipschitz continuity on the restricted operating region

The congestion component requires special treatment because:

$$G(x) = \sum_m \frac{\rho_m(x)}{1 - \rho_m(x) + \varepsilon}$$

contains a denominator that becomes small as resource utilization approaches its capacity limit.

The analysis is therefore performed on:

$$K_\delta = \{x \in K : \rho_m(x) \leq 1 - \delta, \forall m\}, \quad \delta > 0$$

From Equation:

$$1 - \rho_m(x) + \varepsilon \geq \delta + \varepsilon$$

For the congestion derivative:

$$\frac{\partial G}{\partial x_{jm}} = \frac{1 + \varepsilon}{B_m[1 - \rho_m(x) + \varepsilon]^2}$$

and consequently:

$$\left| \frac{\partial G}{\partial x_{jm}} \right| \leq \frac{1 + \varepsilon}{B_m(\delta + \varepsilon)^2}$$

Similarly, the second derivative is bounded by:

$$\left| \frac{\partial^2 G}{\partial x_{jm} \partial x_{is}} \right| \leq \frac{2(1 + \varepsilon)}{B_m B_s (\delta + \varepsilon)^3}$$

whenever the corresponding variables belong to the same congestion-coupled resources.

Because the remaining components of F are continuously differentiable on the compact set K_δ , the complete Jacobian is bounded:

$$\sup_{x \in K_\delta} \|J_F(x)\|_2 < \infty$$

Define:

$$L = \sup_{x \in K_\delta} \|J_F(x)\|_2$$

By the mean-value property:

$$\|F(x) - F(y)\|_2 \leq L\|x - y\|_2, \quad \forall x, y \in K_\delta$$

Thus, the Lipschitz constant is finite on K_δ , even though an unrestricted domain approaching $\rho_m = l$ could cause the congestion derivative to become unbounded.

5.4 Convergence condition

Assume that F is μ -strongly monotone and L -Lipschitz continuous on K_δ . Let x^* be the unique solution of $\text{VI}(K_\delta, F)$.

From the projection characterization:

$$x^* = P_{K_\delta}[x^* - \alpha F(x^*)]$$

Using the non-expansiveness of projection:

$$\|x^{k+1} - x^*\|_2 \leq \|(x^k - \alpha F(x^k)) - (x^* - \alpha F(x^*))\|_2$$

Squaring both sides gives:

$$\|x^{k+1} - x^*\|_2^2 \leq \|x^k - x^*\|_2^2 - 2\alpha \langle F(x^k) - F(x^*), x^k - x^* \rangle + \alpha^2 \|F(x^k) - F(x^*)\|_2^2$$

By strong monotonicity:

$$\langle F(x^k) - F(x^*), x^k - x^* \rangle \geq \mu \|x^k - x^*\|_2^2$$

and by Lipschitz continuity:

$$\|F(x^k) - F(x^*)\|_2^2 \leq L^2 \|x^k - x^*\|_2^2$$

Hence:

$$\|x^{k+1} - x^*\|_2^2 \leq (1 - 2\alpha\mu + \alpha^2 L^2) \|x^k - x^*\|_2^2$$

For contraction:

$$1 - 2\alpha\mu + \alpha^2 L^2 < 1$$

which gives:

$$0 < \alpha < \frac{2\mu}{L^2}$$

This condition guarantees convergence of the projected iteration.

5.5 Linear convergence

Define the contraction factor:

$$q = \sqrt{1 - 2\alpha\mu + \alpha^2 L^2}$$

Under Equation:

$$0 \leq q < 1$$

Equation consequently yields:

$$\|x^{k+1} - x^*\|_2 \leq q \|x^k - x^*\|_2$$

Applying Equation recursively:

$$\|x^k - x^*\|_2 \leq q^k \|x^0 - x^*\|_2$$

Therefore, the projected VI sequence converges linearly to the unique equilibrium.

Theorem 3. *If F is μ -strongly monotone and L -Lipschitz continuous on K_δ , and $0 < \alpha < \frac{2\mu}{L^2}$, then the sequence generated by Equation converges linearly to the unique solution of $\text{VI}(K_\delta, F)$.*

This comes from the fact that Euclidean projection is non-expansive and the contraction property stated in Equations.

5.6 Step-size selection

The contraction factor squared is:

$$q^2(\alpha) = 1 - 2\alpha\mu + \alpha^2 L^2$$

Differentiating with respect to α :

$$\frac{dq^2}{d\alpha} = -2\mu + 2\alpha L^2$$

Setting Equation equal to zero gives:

$$\alpha_{\text{opt}} = \frac{\mu}{L^2}$$

The corresponding contraction factor is:

$$q_{\text{opt}} = \sqrt{1 - \frac{\mu^2}{L^2}}$$

Therefore, step size may be selected based on the estimates of monotonicity and Lipschitz constants instead of selecting it randomly.

5.7 Computational procedure

The complete numerical procedure is summarized as follows:

1 Initialize System Parameters

Select initial allocation $x^0 \in K_\delta$, step size $\alpha > 0$, convergence tolerance $\varepsilon_{\text{tol}} > 0$, and maximum iterations K_{max} . Set iteration index $k = 0$.

2 Evaluate Marginal Operator

Compute the non-conservative allocation operator evaluation $F(x^k)$ at the current allocation state x^k .

3 Compute Unconstrained Movement

Determine the intermediate unconstrained allocation vector:

$$z^k = x^k - \alpha F(x^k)$$

4 Project Feasible Allocation

Enforce capacity and operational restrictions by projecting z^k onto the restricted operating region:

$$x^{k+l} = P_{K_\delta}(z^k)$$

5 Evaluate VI Equilibrium Residual

Calculate the dimensionally consistent residual:

$$R_\alpha(x^{k+l}) = \frac{l}{\alpha} \left\| x^{k+l} - P_{K_\delta}(x^{k+l} - \alpha F(x^{k+l})) \right\|_2$$

6 Check Stopping Conditions

If $R_\alpha(x^{k+l}) \leq \varepsilon_{\text{tol}}$, terminate and return $x^* = x^{k+l}$. If $k = K_{\text{max}}$, terminate with the current iterate. Otherwise, set $k \leftarrow k + l$ and return to Step 2.

Simultaneously, the projection method takes care of meeting the requirements of processing requirements satisfaction, resource capacity, non-negativity, feasibility, and the particular interior capacity slack. The convergence conditions mentioned above are tested numerically in Section 6 in terms of residual reduction, iteration counts, CPU time, and α sensitivity.

6. Numerical experiments

The numerical analysis considers the mathematical aspects of the non-conservative variational inequality approach instead of applying the manufacturing example as a predictive data model. The experiments are carried out to check whether four criteria are satisfied, namely, feasibility of the projection solution, convergence of the VI residual, dependence on the step size of the projection, and scalability concerning the number of operations and resources. The benchmark examples are created directly from the mathematical model introduced in sections 3-5.

Unless specified otherwise, the numerical tolerance is set to 10^{-6} , the regularization parameter is set to $\varepsilon = 10^{-3}$, and the interior capacity margin is set to $\delta = 0.05$. The weight parameters are chosen as

$$\lambda_T = 0.20, \quad \lambda_G = 0.35, \quad \lambda_E = 0.15.$$

The asymmetric matrix H_a is constructed using the elements that are bounded off-diagonal, while the symmetric part is chosen in such a way that the operator remains strictly monotone. The experiments employ the Projected VI approach (Algorithm 1).

6.1 Benchmark instance construction

Three problem dimensions are considered:

$$(N, M) \in \{(10, 10), (50, 50), (100, 100)\}.$$

Thus, the corresponding numbers of allocation variables are 100, 2500, and 10000, respectively.

For each instance, the processing coefficients are generated within $a_{jm} \in [0.6, 1.4]$ for eligible operation-resource pairs, while:

$$p_j \in [8, 20], \quad d_j \in [10, 25].$$

The resource capacity values are chosen in accordance with the total process requirement such that the eligible set is not empty. The eligibility matrix is created with about 40% of eligible operator-resource pairs in the baseline scenario.

The allocation cost coefficients satisfy $c_{jm} \in [0.5, 2.0]$. The interaction matrix is constructed as:

$$Q = Q_s + Q_d$$

where $Q_s = Q_s^\top \geq 0$ and Q_d contains bounded asymmetric interaction terms. The resulting operator is numerically checked through the minimum eigenvalue of:

$$S_F(x) = \frac{I}{2} [J_F(x) + J_F(x)^\top].$$

For each benchmark problem, the computed lower bound is positive throughout all considered feasible solutions.

6.2 Feasibility and equilibrium verification

The first test concerns the final solution derived by using Algorithm 1. The following values are computed:

$$V_B = \max_m \left[\sum_j x_{jm}^* - B_m \right]_+$$

$$V_D = \max_j \left[d_j - \sum_m a_{jm} x_{jm}^* \right]_+$$

$$V_x = \max_{j,m} [-x_{jm}^*]_+$$

The VI residual is evaluated using:

$$R_\alpha(x^*) = \frac{I}{\alpha} \|x^* - P_{K_\delta}(x^* - \alpha F(x^*))\|_2$$

The results are shown in table 1.

Table 1: Feasibility and equilibrium verification.

Instance	Variables	V_B	V_D	V_x	$R_\alpha(x^*)$	Iterations
(10 × 10)	100	3.1×10^{-9}	2.7×10^{-9}	0	7.8×10^{-7}	86
(50 × 50)	2500	6.4×10^{-9}	5.9×10^{-9}	0	8.6×10^{-7}	104
(100 × 100)	10000	9.2×10^{-9}	8.7×10^{-9}	0	9.4×10^{-7}	119

All constraint violations are strictly less than 10^{-8} , whereas the VI residual stays within the pre-specified limit of 10^{-6} . This indicates that the projection operation preserves feasibility at all points in the numerical solution process. Even more important, feasibility is accomplished without switching the VI problem to scalar objective minimization.

6.3 Monotonicity verification

The strong-monotonicity condition is evaluated from the symmetric part of the Jacobian:

$$S_F(x) = \frac{1}{2}[J_F(x) + J_F(x)^\top].$$

For sampled feasible points $x \in K_\delta$, the minimum eigenvalue:

$$\mu_{\text{obs}} = \min_x \lambda_{\min}(S_F(x))$$

is calculated. The largest observed Jacobian norm provides an estimate of the Lipschitz constant:

$$L_{\text{obs}} = \max_x \|J_F(x)\|_2$$

The resulting constants are summarized in table 2.

Table 2: Numerical verification of operator properties.

Instance	μ_{obs}	L_{obs}	$2\mu/L^2$	$\alpha_{\text{opt}} = \mu/L^2$
(10 × 10)	0.842	3.164	0.1683	0.0841
(50 × 50)	0.801	3.287	0.1481	0.0741
(100 × 100)	0.774	3.416	0.1326	0.0663

Nevertheless, in all three cases, the minimum eigenvalues remain strictly positive. This implies that $S_F(x) \geq \mu I$ is numerically validated in the considered operational domain. However, the Jacobian matrix is not symmetric due to the asymmetric interaction element. The above result implies that the numerical model has the desired non-conservative VI framework.

For instance, in the case of a (100×100) matrix, it holds that $\mu_{\text{obs}} = 0.774 > 0$ and $J_F(x) \neq J_F(x)^\top$. As such, the uniqueness of solutions is derived using the symmetric Jacobian matrix element.

6.4 Residual convergence

The convergence behavior is evaluated using $R_k = R_\alpha(x^k)$ and $e_k = \|x^{k+1} - x^k\|_2$. For the (50 × 50) benchmark, the step size is selected as $\alpha = 0.0741I$, which corresponds to the estimated theoretical optimum.

Table 3 gives representative residual values during the iteration.

Table 3: Residual progression for the (50 × 50) instance.

Iteration (k)	$R_\alpha(x^k)$	e_k
0	1.84×10^{-1}	1.96×10^{-1}
10	4.37×10^{-2}	4.81×10^{-2}
20	1.31×10^{-2}	1.44×10^{-2}
40	1.86×10^{-3}	2.03×10^{-3}
60	2.74×10^{-4}	3.01×10^{-4}
80	4.21×10^{-5}	4.58×10^{-5}
100	4.17×10^{-6}	4.53×10^{-6}
104	8.60×10^{-7}	9.31×10^{-7}

The residual gets reduced by more than five orders of magnitude throughout the iterative process. The successive iterates show the same reducing trend. This trend corroborates with the results on the contraction obtained in section 5.

The contraction factor empirically is:

$$\hat{q}_k = \frac{R_{k+1}}{R_k}$$

For the transient initial iterations, the ratio stabilizes at 0.82- 0.87, thus indicating geometrical rather than just sub-linear convergence.

6.5 Step-size sensitivity

The influence of α is analyzed by means of the (50×50) instance. According to table 2, the theoretically allowable range of α is roughly $0 < \alpha < 0.1481$. In table 4, a numerical experiment on the values of α less than, around, and greater than

Table 4: Step-size sensitivity.

α	Theoretical Status	Iterations	Final Residual
0.025	Stable	247	9.2×10^{-7}
0.050	Stable	143	8.9×10^{-7}
0.0741	Near optimal	104	8.6×10^{-7}
0.100	Stable	116	9.1×10^{-7}
0.125	Stable	168	9.5×10^{-7}
0.145	Near upper boundary	392	9.8×10^{-7}
0.160	Outside theoretical interval	No reliable convergence	$> 10^{-3}$

The smallest number of iterations is achieved near the value $\alpha_{\text{opt}} = \mu/L^2 = 0.0741$. Small steps lead to stability, but slow convergence since the operator shift is small. On the contrary, when approach $2\mu/L^2$, the contraction coefficient increases, thus the number of iterations increases as well.

6.6 Resource flexibility analysis

Three flexibility levels are considered:

- **Low flexibility:** approximately 20% of operation-resource pairs are eligible;
- **Moderate flexibility:** approximately 40% are eligible;
- **High flexibility:** approximately 70% are eligible.

The (50 × 50) instance is used while keeping the processing demand and total resource capacity unchanged.

Table 5: Effect of resource flexibility.

Flexibility	$C(x^*)$	$T(x^*)$	$G(x^*)$	$E(x^*)$	Maximum ρ_m
Low	1842.7	5.83	31.46	2.18	0.931
Moderate	1716.4	5.21	27.83	1.61	0.904
High	1648.9	4.96	25.71	1.37	0.881

As table 5 shows, greater flexibility will reduce the processing time and congestion terms of equilibrium due to the ability of operations to reallocate their allocations to different resources. The workload imbalance will also drop from 2.18 to 1.37.

This is not because of the increase in capacity, but rather the total capacity of the resources is still the same. The reason for that is the increase in allocation directions of the VI equilibrium.

6.7 Capacity sensitivity and shadow prices

The capacity sensitivity experiment varies individual resource capacities according to:

$$B_m(\delta_c) = (1 + \delta_c)B_m$$

where $\delta_c \in \{-0.15, -0.10, 0, 0.10, 0.15\}$. The resulting equilibrium changes are measured using:

$$S_x(\delta_c) = \frac{\|x^*(\delta_c) - x^*(0)\|_2}{\|x^*(0)\|_2}$$

Table 6: Capacity sensitivity.

Capacity Variation	$C(x^*)$	$G(x^*)$	Maximum ρ_m	S_x
-15%	1927.3	35.81	0.950	0.183
-10%	1859.6	32.74	0.941	0.121
0%	1716.4	27.83	0.904	0
+10%	1682.1	25.94	0.865	0.084
+15%	1661.8	24.97	0.841	0.119

As presented in table 6, the capacity constraint is more influential when capacity is reduced rather than increased by the same proportion. The reason behind this difference is that of the nonlinear congestion function.

The multipliers on the capacity of heavily used resources become larger in value too. In fact, for the resource that faces the highest constraint in the base case, the equilibrium shadow price is about $\lambda_m^* = 0.84$. When capacity is increased by 15%, the multiplier decreases to approximately $\lambda_m^* = 0.37$, which shows less scarcity.

6.8 Computational comparison

To examine the numerical superiority of the VI formulation, the proposed approach is benchmarked against three traditional constrained optimization algorithms: the Frank-Wolfe approach, an interior point method (IPM), and sequential quadratic programming (SQP). The comparison utilizes the same allocation constraints as well as the same benchmark problem instances.

Since the proposed operator is non-conservative, the comparison is done using the residual of VI instead of objective values alone.

Table 7: Computational comparison.

Instance	Method	Iterations	CPU Time (s)	VI Residual	Constraint Violation
(10 × 10)	Projected VI	86	0.18	7.8×10^{-7}	3.1×10^{-9}
	Frank-Wolfe	214	0.31	4.6×10^{-5}	8.2×10^{-8}
	IPM	19	0.42	2.1×10^{-6}	1.7×10^{-8}
	SQP	31	0.37	3.4×10^{-6}	2.4×10^{-8}
(50 × 50)	Projected VI	104	2.73	8.6×10^{-7}	6.4×10^{-9}
	Frank-Wolfe	487	8.94	7.3×10^{-5}	1.2×10^{-7}
	IPM	42	5.87	3.8×10^{-6}	3.5×10^{-8}
	SQP	67	7.26	5.1×10^{-6}	4.7×10^{-8}
(100 × 100)	Projected VI	119	11.84	9.4×10^{-7}	9.2×10^{-9}
	Frank-Wolfe	812	39.67	1.1×10^{-4}	2.4×10^{-7}
	IPM	76	28.51	6.2×10^{-6}	5.8×10^{-8}
	SQP	103	34.83	7.4×10^{-6}	6.9×10^{-8}

Table 7 shows the projected VI algorithm converges to the required residual accuracy for all three problem sizes. Its number of iterations is slightly incremented from 86 to 119 as the number of variables changes from 100 to 10000.

The comparative analysis also highlights one more significant difference in methodology. The algorithms of Frank-Wolfe, IPM, and SQP are oriented mainly toward constrained optimization problems, while the suggested approach deals with the equilibrium operator directly (Mahmoudi & Lailypour, 2015; Wang & Huang, 2026). Thus, the residual of VI is more relevant as a criterion of the result quality.

6.9 Numerical interpretation

The experiments provide four principal observations:

1. **Feasibility Retention:** The predicted model iteration retains the full feasible allocation structure. The capacity, processing demand, non-negativity, and flexibility constraint violations remain negligible numerically.
2. **Structural Validity:** The positive lower bound for the symmetric part of the Jacobian matrix validates the strong monotonicity assumption even though the full Jacobian matrix is asymmetric. Uniqueness does not depend on a mathematically incorrect symmetry assumption.
3. **Contraction Dynamics:** The remaining error converges at a geometric rate if the step size satisfies $0 < \alpha < \frac{2\mu}{L^2}$. The optimal step size is given by $\alpha_{\text{opt}} = \frac{\mu}{L^2}$, as expected.
4. **Equilibrium Sensitivity:** Resource flexibility and capacity directly affect the VI equilibrium. Higher flexibility alleviates congestion and workload imbalance while higher capacity constraints raise the endogenously determined shadow prices of scarce resources. The conclusions imply that the model correctly incorporates coupling of resources via the operator and its complementary structure rather than a scalar potential function.

Overall, the numerical evidence supports the mathematical claims of Sections 4 and 5:

$$x^* \in K_\delta, \quad R_\alpha(x^*) \rightarrow 0, \quad \|x^{k+l} - x^k\|_2 \rightarrow 0$$

while the asymmetric interaction structure remains explicitly represented in the equilibrium operator.

7. Conclusion

The study introduced a mathematically oriented variational inequality model of flexible manufacturing resource allocation with coupled, nonlinear, and asymmetric resource interactions. The main contribution of this study is the mathematical modeling of the resource allocation problem as an equilibrium problem driven by the non-conservative operator as opposed to a classical scalar potential minimization problem. The presented operator includes the cost of allocation, the nonlinear effect on processing time, congestion caused by resource capacity, workload imbalance, and asymmetric resource interaction. The above construction enables the marginal impact of one allocation variable on another to have different directions; hence, it provides a mathematical model that cannot be represented as a symmetric gradient field. The theoretical analysis proved the existence of the equilibrium under continuity and compact convexity of the restricted feasible set. Uniqueness was obtained using strong monotonicity. It should be noted that the theoretical analysis took into account the most general case $J_F \neq J_F^T$ and used the symmetric part of the Jacobian to prove strong monotonicity. The set of capacity margin K_δ was used to avoid unboundedness of the congestion derivative in case of close-to-full utilization, so the finite Lipschitz constant L could be obtained. The corresponding projection algorithm converges linearly if $0 < \alpha < \frac{2\mu}{L^2}$ with $\alpha_{\text{opt}} = \frac{\mu}{L^2}$ being the theoretical optimal step size. The numerical simulations verified the feasibility of the solution, residual convergence, step size properties, flexibility sensitivity, and capacity effects for increasing problem dimensions up to (100×100) operations and resources (10000 variables). The associated multipliers provided the intrinsic shadow price interpretation of the congested resources.

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