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Multiple Solutions of Neumann Systems of Relativistic Type

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Motivated by the existence of radial solutions to the Neumann problem involving the mean extrinsic curvature operator in Minkowski space, we provide a set of conditions under which some relativistic type systems has at least two solutions that appear as global minima of the associated functional. The main tool used is an abstract well-posedness result developed by B. Ricceri.

Keywords: Neumann problem, mean curvature operator, global minimum, well-posedness.

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1. Introduction

This paper is motivated by the existence of radial solutions to the Neumann system involving the mean extrinsic curvature operator in Minkowski space:

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla v_j}{\sqrt{1-|\nabla v_j|^2}}\right) = \partial_{v_j} F(|x|, v_1, \dots, v_m) & \text{in } \mathcal{A} \quad (j \in \{1, \dots, m\}), \\ \dfrac{\partial v_1}{\partial \nu} = \dots = \dfrac{\partial v_m}{\partial \nu} = 0 & \text{on } \partial\mathcal{A}, \end{cases} \tag{1}$$

where $0 \le R_1 < R_2$, $\mathcal{A} = \{x \in \mathbb{R}^N \ : \ R_1 \le |x| \le R_2\}$ and $F\colon [R_1, R_2] \times \mathbb{R}^m \to \mathbb{R}$ verifies:

(H_F) *F is continuous and $\partial_{v_j} F$ exists and is continuous on $[R_1, R_2] \times \mathbb{R}^m$, for all $j \in \{1, \dots, m\}$.*

As usual, for $j \in \{1, \dots, m\}$, we have denoted by $\frac{\partial v_j}{\partial \nu}$ the outward normal derivative of v_j and $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^N$. Setting $r = |x|$ and $v_j(x) = u_j(r)$, the system (1) becomes

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$$\begin{cases} \left[r^{N-1}\left(\dfrac{u_j'}{\sqrt{1-u_j'^2}}\right)\right]' = r^{N-1}\partial_{u_j}F(r,u_1,\ldots,u_m) \quad (j\in\{1,\ldots,m\}), \\ u_1'(R_1)=\ldots=u_m'(R_1)=0=u_1'(R_2)=\ldots=u_m'(R_2) \end{cases} \tag{2}$$

and the solutions of (2) are classical radial solutions of (1). In this paper we deal with the more general system

$$\begin{cases} \left[r^{N-1}\phi(u_j')\right]' = r^{N-1}\partial_{u_j}F(r,u_1,\ldots,u_m) \quad (j\in\{1,\ldots,m\}), \\ u_1'(R_1)=\ldots=u_m'(R_1)=0=u_1'(R_2)=\ldots=u_m'(R_2) \end{cases} \tag{3}$$

where $\phi\colon(-\eta,\eta)\to\mathbb{R}$ is an increasing homeomorphism with $\phi(0)=0$. Such an ϕ is called *singular*. Clearly, (2) is nothing else but a problem of type (3) with

$$\phi(y)=\frac{y}{\sqrt{1-y^2}} \quad (y\in(-1,1)). \tag{4}$$

We provide sufficient conditions under which the problem has at least two solutions that appear as global minima of the associated functional. To this aim we use an abstract well-posedness result due to Ricceri [12] and employed by the same author in [13] to prove a similar result for periodic problems involving a general homeomorphism ψ from B_L $(=\{x\in\mathbb{R}^N : |x|<L\})$ onto $\mathbb{R}^N$ such that $\psi(0)=0$ and $\psi=\nabla\Psi$, where the function $\Psi\colon\overline{B}_L\to(-\infty,0]$ is continuous and strictly convex in $\overline{B}_L$, and of class C^1 in B_L.

Although in many recent papers a special attention has been paid to qualitative aspects for one equation involving Minkowski operator, there are few ones in which systems with this operator are considered. In [2] the authors obtain existence results concerning radial solutions for Dirichlet problems associated with some systems involving mean curvature operators in Euclidean and Minkowski spaces. Using a Lusternik-Schnirelman type multiplicity result for indefinite functionals, Mawhin prove in [11] the existence of at least $N+1$ geometrically distinct solutions for periodic systems with the homeomorphism ψ and for homogeneous Neumann problems associated to systems of the form

$$\nabla\cdot\left(\frac{\nabla w_i}{\sqrt{1-\sum_{j=1}^N|\nabla w_j|^2}}\right)+\partial_{w_j}G(|x|,w)=h_i(|x|), \quad (i=1,\ldots,N).$$

Also, using the critical point theory for convex, lower semicontinuous perturbations of C^1-functionals, the existence of multiple nontrivial solutions for one parameter potential systems involving the Minkowski operator is obtained in [5]; the solvability of a general non-potential system is also established. For other multiplicity results for systems with singular ϕ-Laplacians we refer the reader to [1], [6]–[10] and the reference therein.

The rest of the paper is organized as follows. In Section 2 we give some preliminary notions and the variational formulation for system (3). Our main multiplicity result will be shown in Section 3.

2. Preliminaries and variational approach

Preliminaries. For the convenience of the reader we briefly present in this paragraph the theory developed by Ricceri in [12] (also see [13, Section 2]). Let X be a Hausdorff topological space and J, Ψ be two real-valued functions defined on X. For $a, b \in [-\infty, +\infty]$ with $a < b$, we denote by M_a (resp. M_b) the set of all global minimum points of the function $J + a\Psi$ (resp. $J + b\Psi$) if $a \in \mathbb{R}$ (resp. $b \in \mathbb{R}$), while if $a = -\infty$ (resp. $b = +\infty$), M_a (resp. M_b) stands for the empty set. Also, we define

$$\alpha := \max\left\{\inf_X \Psi,\ \sup_{M_b} \Psi\right\}, \qquad \beta := \min\left\{\sup_X \Psi,\ \inf_{M_a} \Psi\right\}.$$

In the definition of α (resp. β) we employ the convention $\sup \emptyset = -\infty$ (resp. $\inf \emptyset = +\infty$). On account of [12, Proposition 1] one has that $\alpha \leq \beta$. Next, for a given $f \colon X \to \mathbb{R}$ and a set $D \subseteq X$, the problem of minimizing f over D is said to be *well-posed* if the restriction $f|_D$ has a unique global minimum $\hat{x}$ and every sequence $\{x_n\}$ in D with $\lim_{n\to\infty} f(x_n) = \inf_D f$ converges to $\hat{x}$. A set of the form $\{x \in X : f(x) \leq s\}$, with $s \in \mathbb{R}$, is called *sub-level set* of f.

The following abstract result is proved in [12, Theorem 1].

Theorem 2.1. *Assume that $\alpha < \beta$ and that, for each $\lambda \in (a, b)$, the function $J + \lambda\Psi$ has sequentially compact sub-level sets and admits a unique global minimum in X. Then, for all $s \in (\alpha, \beta)$, the problem of minimizing J over $\Psi^{-1}(s)$ is well-posed. Moreover, denoting by $\hat{x}_s$ the unique global minimum of $J|_{\Psi^{-1}(s)}$ ($s \in (\alpha, \beta)$), the functions $s \mapsto \hat{x}_s$, respectively $s \mapsto J(\hat{x}_s)$ are continuous in (α, β) and, for some $\hat{\lambda}_s \in (a, b)$, $\hat{x}_s$ is the global minimum in X of the function $J + \hat{\lambda}_s\Psi$.*

The variational framework. In what follows, we assume the hypothesis:

(H_Φ) *$\Phi \colon [-\eta, \eta] \to \mathbb{R}$ is continuous, of class C^1 on $(-\eta, \eta)$, $\Phi(0) = 0$ and $\phi := \Phi' \colon (-\eta, \eta) \to \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$.*

Clearly, Φ is strictly convex, $\Phi(y) \geq 0$ for all $y \in [-\eta, \eta]$ and for the homeomorphism ϕ given in (4) one takes

$$\Phi(y) = 1 - \sqrt{1 - y^2}. \tag{5}$$

Using the ideas from [3], a variational approach is introduced for problem (3). With this aim let us denote $C := C[R_1, R_2]$ considered with the usual norm $|v|_\infty = \sup_{[R_1,R_2]} |v|$, $W^{1,\infty} := W^{1,\infty}(R_1, R_2)$ and $\mathcal{C} = \underbrace{C \times C \times \ldots \times C}_{m}$ endowed with the norm $\|u\|_\infty = \sum_{j=1}^{m} |u_j|_\infty$, where $u = (u_1, \ldots, u_m)$. Also, denoting

$$\sigma := \int_{R_1}^{R_2} r^{N-1} dr = \frac{R_2^N - R_1^N}{N}$$

and $L^1_{N-1} := \left\{ v \in (R_1, R_2) \to \mathbb{R} \text{ measurable} : \int_{R_1}^{R_2} r^{N-1} |v(r)| dr < +\infty \right\}$, each $v \in L^1_{N-1}$ can be written in the form $v(r) = \overline{v} + \tilde{v}(r)$, where

$$\overline{v} := \frac{1}{\sigma} \int_{R_1}^{R_2} r^{N-1} v(r) dr, \qquad \int_{R_1}^{R_2} r^{N-1} \tilde{v}(r) dr = 0.$$

If $v \in W^{1,\infty}$, then $\tilde{v}$ vanishes at some $r_0 \in (R_1, R_2)$ and so

$$|\tilde{v}(r)| = |\tilde{v}(r) - \tilde{v}(r_0)| \leq \int_{R_1}^{R_2} |v'(s)| ds \leq (R_2 - R_1)|v'|_\infty. \tag{6}$$

Now, we introduce the convex set $K := \{v \in W^{1,\infty} : |v'|_\infty \leq \eta\}$, which is closed in C and on account of (6) one has

$$|v|_\infty \leq |\overline{v}| + (R_2 - R_1)\eta,$$

for all $v \in K$. Also, it is trivial to see that $|v(r)| \geq |\overline{v}| - \eta(R_2 - R_1)$, for all $v \in K$ and $r \in [R_1, R_2]$. Then, the above two inequalities imply

$$|v(r)| \geq |v|_\infty - 2\eta(R_2 - R_1), \quad \forall\, v \in K,\ r \in [R_1, R_2]. \tag{7}$$

Next, setting $\mathcal{K} = \underbrace{K \times K \times \ldots \times K}_{m}$, we consider the functional $I \colon \mathcal{K} \to \mathbb{R}$ defined by

$$I(u) = \sum_{j=1}^{m} \varphi(u_j) + \mathcal{F}(u), \quad u = (u_1, \ldots, u_m) \in \mathcal{K},$$

where $\varphi \colon K \to \mathbb{R}$ and $\mathcal{F} \colon \mathcal{C} \to \mathbb{R}$ are given by

$$\varphi(v) = \int_{R_1}^{R_2} r^{N-1} \Phi(v')\, dr, \quad \text{and} \quad \mathcal{F}(v) = \int_{R_1}^{R_2} r^{N-1} F(r, v_1, \ldots, v_m)\, dr.$$

We shall need the following result:

Proposition 2.2. *Any minimizer of I in $\mathcal{K}$ is a solution of problem* (3).

Proof. Let $u = (u_1, \ldots, u_m)$ be a minimizer of I in $\mathcal{K}$. Then it is also a minimizer of the functional $\tilde{I} \colon \mathcal{C} \to (-\infty, +\infty]$ defined by

$$\tilde{I}(u) = \sum_{j=1}^{m} \psi(u_j) + \mathcal{F}(u),$$

where $\psi\colon C \to (-\infty, +\infty]$ is given by

$$\psi(v) = \begin{cases} \varphi(v), & \text{if } v \in K, \\ +\infty, & \text{otherwise.} \end{cases}$$

A standard reasoning shows that $\mathcal{F}$ is of class C^1 and according to *Step I* from the proof of [4, Proposition 1], one has that ψ is convex and lower semicontinuous. So, it is easy to see that $\tilde{I}$ has the structure required by Szulkin's critical point theory [14].

Now, from [14, Proposition 1.1], we infer that u is a critical point of $\tilde{I}$, which means that it is a solution of the variational inequality

$$\sum_{j=1}^{m} \left[\psi(w_j) - \psi(u_j) + \int_{R_1}^{R_2} r^{N-1} \partial_{u_j} F(r, u_1, \dots, u_m)(w_j - u_j) dr \right] \geq 0, \qquad (8)$$

for all $w = (w_1, \dots, w_m) \in \mathcal{K}$. Let $i \in \overline{1, m}$ be arbitrarily chosen. Taking in (8) $w_j = u_j$ for $j = 1, \dots, m$ with $j \neq i$, one gets

$$\psi(w_i) - \psi(u_i) + \int_{R_1}^{R_2} r^{N-1} \partial_{u_i} F(r, u_1, \dots, u_m)(w_i - u_i) \geq 0,$$

i.e. $u_i \in K$ is a critical point of $\psi(\cdot) + \mathcal{F}(u_1, \dots, u_{i-1}, \cdot, u_{i+1}, u_m)$, which by virtue of [3, Proposition 1] it satisfies

$$\begin{cases} \left[r^{N-1} \phi(u_i') \right]' = r^{N-1} \partial_{u_i} F(r, u_1, \dots, u_m), \\ u_i'(R_1) = 0 = u_i'(R_2) \end{cases}$$

and the proof is complete. □

3. Main result

Our main result is given in the following theorem.

Theorem 3.1. *Assume* (H_Φ), (H_F) *and let* $G\colon \mathbb{R}^m \to \mathbb{R}$ *be of class* C^1, $\mu \in C$ *with* $\mu \geq 0$, $\mu \not\equiv 0$ *and* $\gamma\colon [0, \infty) \to \mathbb{R}$ *be a continuous convex strictly increasing function such that* $\lim_{t \to +\infty} \frac{\gamma(t)}{t} = +\infty$. *Also assume the following hypotheses:*

(H_1) *for all* $r \in [R_1, R_2]$ *and* $x = (x_1, \dots, x_m) \in \mathbb{R}^m$, *one has* $\gamma\left(r^{N-1}|x|\right) \leq r^{N-1} F(r, x_1, \dots, x_m)$;

(H_2) $\liminf_{|x| \to +\infty} \dfrac{G(x_1, \dots, x_m)}{|x|} > -\infty$;

(H_3) *the function* G *has no global minima in* $\mathbb{R}^m$*;*

(H_4) *there are two distinct points* $x^1 = (x_1^1, \ldots, x_m^1)$, $x^2 = (x_1^2, \ldots, x_m^2) \in \mathbb{R}^m$ *such that*

$$s := \max\left\{\int_{R_1}^{R_2} r^{N-1}F(r, x_1^1, \ldots, x_m^1)dr, \int_{R_1}^{R_2} r^{N-1}F(r, x_1^2, \ldots, x_m^2)dr\right\}$$
$$> \inf_{x\in\mathbb{R}^m} \int_{R_1}^{R_2} r^{N-1}F(r, x_1, \ldots, x_m)dr$$

and $G(x_1^1, \ldots, x_m^1) = G(x_1^2, \ldots, x_m^2) = \inf_{\{x\in\mathbb{R}^m : |x|<c\}} G(x_1, \ldots, x_m)$,

where $c = (R_2 - R_1)\left(2m\eta + \frac{\sqrt{m}}{\sigma}\gamma^{-1}\left(\frac{s}{R_2 - R_1}\right)\right)$.

Then, there exists $\lambda^* > 0$ *such that system*

$$\begin{cases} \left[r^{N-1}\phi(u_j')\right]' = r^{N-1}(\partial_{u_j}F(r, u_1, \ldots, u_m) + \lambda^*\mu(r)\partial_{u_j}G(u_1, \ldots, u_m)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (j \in \{1, \ldots, m\}), \\ u_1'(R_1) = \ldots = u_m'(R_1) = 0 = u_1'(R_2) = \ldots = u_m'(R_2) \end{cases} \tag{9}$$

has at least two solutions which are global minima in $\mathcal{K}$ *of the functional*

$$I^*(u) = \sum_{j=1}^m \varphi(u_j) + \mathcal{F}(u) + \lambda^* \int_{R_1}^{R_2} r^{N-1}\mu(r)G(u_1, \ldots, u_m)dr, \tag{10}$$

for $u = (u_1, \ldots, u_m) \in \mathcal{K}$.

Proof. We shall use similar arguments to those in [13, Theorem 3.1] and apply Theorem 2.1 with the following choices: $a = 0$, $b = +\infty$, $X = \mathcal{K}$ and

$$\Psi(u) = \sum_{j=1}^m \varphi(u_j) + \mathcal{F}(u), \qquad J(u) = \int_{R_1}^{R_2} r^{N-1}\mu(r)G(u_1, \ldots, u_m)dr,$$

for all $u = (u_1, \ldots, u_m) \in \mathcal{K}$. Fix $\lambda > 0$. From (H_2), for a suitable constant $\delta > 0$, one has

$$-\delta(|x| + 1) \le G(x_1, \ldots, x_m), \quad \forall\, x = (x_1, \ldots, x_m) \in \mathbb{R}^m. \tag{11}$$

Also, notice that for any $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$, one has

$$|x| \le \sum_{j=1}^m |x_j| \le \sqrt{m}|x|. \tag{12}$$

Now, for each $u = (u_1, \ldots, u_m) \in \mathcal{K}$, taking into account hypotheses (H_Φ), (H_1) and the convexity of γ and using (11), (12), (7) together with Jensen's inequality, we infer

$$J(u)+\lambda\Psi(u) \geq -\delta \int_{R_1}^{R_2} r^{N-1}\mu(r)(|u(r)|+1)dr + \lambda \int_{R_1}^{R_2} \gamma(r^{N-1}|u(r)|)\, dr$$
$$\geq -\delta(\|u\|_\infty+1)\int_{R_1}^{R_2} r^{N-1}\mu(r)dr + \lambda(R_2-R_1)\gamma\left(\frac{1}{R_2-R_1}\int_{R_1}^{R_2} r^{N-1}|u(r)|dr\right)$$
$$\geq -\delta(\|u\|_\infty+1)\int_{R_1}^{R_2} r^{N-1}\mu(r)dr + \lambda(R_2-R_1)\gamma\left(\frac{\sigma\|u\|_\infty}{\sqrt{m}(R_2-R_1)} - 2\sqrt{m}\eta\sigma\right).$$

Using this, and since

$$\lim_{t\to\infty} \frac{\gamma\left(\frac{\sigma}{\sqrt{m}(R_2-R_1)}t - 2\sqrt{m}\eta\sigma\right)}{t} = +\infty, \tag{13}$$

one gets that, for every $\rho \in \mathbb{R}$, there exists $c_1 > 0$ such that

$$\{u \in \mathcal{K} : J(u)+\lambda\Psi(u) \leq \rho\} \subseteq \{u \in \mathcal{K} : \|u\|_\infty \leq c_1\}. \tag{14}$$

Then, from (14) and since the functions belonging to any sub-level set of $J+\lambda\Psi$ are equicontinuous (since they are in $\mathcal{K}$), by Arzelà-Ascoli theorem, one gets that any sub-level set of $J+\lambda\Psi$ is relatively sequentially compact in $\mathcal{C}$. But, the same set is closed in $\mathcal{C}$ (as $\mathcal{K}$ is closed in $\mathcal{C}$), and so it is sequentially compact in $\mathcal{K}$. Next, notice that J has no global minima in $\mathcal{K}$. Clearly, one has

$$\inf_{\mathcal{K}} J = \inf_{\mathbb{R}^m} G \int_{R_1}^{R_2} r^{N-1}\mu(r)\, dr$$

and hence, if G is unbounded from below, so is J. Thus, suppose that G is bounded from below and, arguing by contradiction, assume that $\hat{u} = (\hat{u}_1, \ldots, \hat{u}_m) \in \mathcal{K}$ is a global minimum of J. Then, we have that

$$\int_{R_1}^{R_2} r^{N-1}\mu(r)\left(G(\hat{u}_1(r),\ldots,\hat{u}_m(r)) - \inf_{\mathbb{R}^m} G\right) dr = 0$$

and as $\mu \geq 0$ and $\mu \not\equiv 0$, it follows that $G(\hat{u}_1(r_0),\ldots,\hat{u}_m(r_0)) = \inf_{\mathbb{R}^m} G$ for some $r_0 \in [R_1, R_2]$, which contradict hypothesis (H_3).

Now, from (H_1) and (13), we obtain that $\sup_{\mathcal{K}} \Psi = +\infty$.

Also, observe that the absence of global minima for J implies that $\beta = \sup_{\mathcal{K}} \Psi$, since $a = 0$. Moreover, as $b = +\infty$ and $\Phi(0) = 0$, we have

$$\alpha = \inf_{\mathcal{K}} \Psi \leq \inf_{x\in\mathbb{R}^m} \int_{R_1}^{R_2} r^{N-1}F(r, x_1, \ldots, x_m)\, dr.$$

Next, on account of (H_4), it is not difficult to see that $\alpha < s < \beta$. Fix $u \in \Psi^{-1}((-\infty, s])$. From (H_1), the convexity of γ and Jensen's inequality again, one gets

$$s \geq \Psi(u) \geq \int_{R_1}^{R_2} \gamma(r^{N-1}|u(r)|)dr \geq (R_2 - R_1)\gamma\left(\frac{1}{R_2 - R_1}\int_{R_1}^{R_2} r^{N-1}|u(r)|dr\right)$$

and applying γ^{-1}, we obtain

$$\frac{1}{R_2 - R_1}\int_{R_1}^{R_2} r^{N-1}|u(r)|dr \leq \gamma^{-1}\left(\frac{s}{R_2 - R_1}\right).$$

This together with the second inequality in (12) and (7) yields

$$\frac{\sigma\|u\|_\infty}{\sqrt{m}(R_2 - R_1)} - 2\sqrt{m}\eta\sigma \leq \gamma^{-1}\left(\frac{s}{R_2 - R_1}\right),$$

i.e.

$$\|u\|_\infty \leq (R_2 - R_1)\left(2m\eta + \frac{\sqrt{m}}{\sigma}\gamma^{-1}\left(\frac{s}{R_2 - R_1}\right)\right).$$

Then, in view of hypothesis (H_4), one has

$$J(x^1) = J(x^2) \leq J(u)$$

and $x^1, x^2 \in \Psi^{-1}((-\infty, s])$. Consequently, we conclude that x^1, x^2 are two distinct global minima of $J|_{\Psi^{-1}((-\infty,s])}$.

Now, suppose by contradiction that, for every $\lambda > 0$, the functional $J + \lambda\Psi$ has a unique global minimum in $\mathcal{K}$. Then, since the sub-level sets of $J + \lambda\Psi$ are sequentially compact for every $\lambda > 0$, from Theorem 2.1 we would have that there exist $\hat{\lambda}_s > 0$ and $\hat{u}_s \in \Psi^{-1}(s)$ such that $\hat{u}_s$ is the unique minimum of $J + \hat{\lambda}_s\Psi$ in $\mathcal{K}$. For $i = 1, 2$, we would get

$$\inf_{u\in\mathcal{K}}(J(u) + \hat{\lambda}_s\Psi(u)) \leq J(x^i) + \hat{\lambda}_s\Psi(x^i) \leq J(\hat{u}_s) + \hat{\lambda}_s\Psi(\hat{u}_s) = \inf_{u\in\mathcal{K}}(J(u) + \hat{\lambda}_s\Psi(u)),$$

which would imply that $J(x^i) = J(\hat{u}_s)$ and $\Psi(x^i) = \Psi(\hat{u}_s)$. This means that x^1 and x^2 would be two distinct global minima in $\mathcal{K}$ of $J + \hat{\lambda}_s\Psi$, i.e. a contradiction. So, there exists some $\hat{\lambda} > 0$ such that $J + \hat{\lambda}\Psi$ has at least two global minima in $\mathcal{K}$. Taking $\lambda^* = 1/\hat{\lambda}$, the proof is accomplished by virtue of Proposition 2.2. □

Corollary 3.2. *Assume* (H_F) *and let* G, μ *and* γ *be as in Theorem* 3.1. *If* F *and* G *satisfy* $(H_1)-(H_4)$ (*with* $\eta = 1$; Φ *from* (5)), *then there exists* $\lambda^* > 0$ *such that problem*

$$\begin{cases} div\left(\dfrac{\nabla v_j}{\sqrt{1 - |\nabla v_j|^2}}\right) = \partial_{v_j}F(|x|, v_1, \dots, v_m) + \lambda^*\mu(|x|)\partial_{v_j}G(v_1, \dots, v_m) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in } \mathcal{A} \quad (j \in \{1, \dots, m\}), \\ \dfrac{\partial v_1}{\partial \nu} = \dots = \dfrac{\partial v_m}{\partial \nu} = 0 \qquad\qquad\qquad \text{on } \partial\mathcal{A} \end{cases}$$

has two radial solutions which are global minima in $\mathcal{K}$ *of the energy functional* I^* *given in* (10).

Example 3.3. Setting

$$\theta := 2m\eta(R_2 - R_1) + \sqrt{\frac{mR_2^{N-1}}{\sigma(R_2 - R_1)}} + 1,$$

hypotheses $(H_1)-(H_4)$ are satisfied by

$$F(r, x_1, \dots, x_m) = |x|^2 \quad \left(\text{with } \gamma(t) = \frac{t^2}{R_2^{N-1}}\right)$$

and

$$G(x_1, \dots, x_m) = \begin{cases} 1, & |x| \leq \theta, \\ e^{-(|x|-\theta)^2}, & |x| > \theta; \end{cases}$$

notice that (H_4) is satisfied by any pair of distinct points $x^1, x^2 \in \mathbb{R}^m$ with $|x^1| = |x^2| \leq 1$.

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References

[1] C. Bereanu, P. Jebelean: *Multiple critical points for a class of periodic lower semi-continuous functionals*, Discrete Contin. Dyn. Syst. 33 (2013) 47–66.

[2] C. Bereanu, P. Jebelean, J. Mawhin: *Radial solutions for systems involving mean curvature operators in Euclidean and Minkowski spaces*, in: Mathematical Models in Engineering, Biology and Medicine, AIP Conf. Proc. 1124, A. Cabada, E. Liz and J. J. Nieto (eds.), Amer. Inst. Phys., Melville (2009) 50–59.

[3] C. Bereanu, P. Jebelean, J. Mawhin: *Variational methods for non-linear perturbations of singular ϕ-Laplacian*, Rend. Lincei Mat. Appl. 22 (2011) 89–111.

[4] C. Bereanu, P. Jebelean, J. Mawhin: *Radial solutions of Neumann problems involving mean extrinsic curvature and periodic nonlinearities*, Calc. Var. 46 (2013) 113–122.

[5] D. Gurban, P. Jebelean, C. Şerban: *Nontrivial solutions for potential systems involving the mean curvature operator in Minkowski space*, Adv. Nonlinear Stud. 17(4) (2017) 769–780.

[6] P. Jebelean, J. Mawhin, C. Şerban: *Multiple periodic solutions for perturbed relativistic pendulum systems*, Proc. Amer. Math. Soc. 143 (2015) 3029–3039.

[7] P. Jebelean, J. Mawhin, C. Şerban: *Periodic solutions for discontinuous perturbations of the relativistic operator*, Bull. Sci. Math. 40 (2016) 99–117.

[8] P. Jebelean, J. Mawhin, C. Şerban: *Multiplicity results for some quasilinear differential systems with periodic nonlinearities*, Minimax Theory Appl. 2 (2017) 69–78.

[9] P. Jebelean, J. Mawhin, C. Şerban: *A vector p-Laplacian type approach to multiple periodic solutions for the p-relativistic operator*, Commun. Contemp. Math. 19 (2017), 16 pages.

[10] P. Jebelean, C. Şerban: *Boundary value problems for discontinuous perturbations of singular ϕ-Laplacian operator*, J. Math. Anal. Appl. 431 (2015) 662–681.

[11] J. Mawhin: *Multiplicity of solutions of variational systems involving ϕ-Laplacians with singular ϕ and periodic nonlinearities*, Discrete Contin. Dyn. Syst. 32 (2012) 4015–4026.

[12] B. Ricceri: *Well-posedness of constrained minimization problems via saddle-points*, J. Global Optim. 40 (2008) 389–397.

[13] B. Ricceri: *Multiple periodic solutions of Lagrangian systems of relativistic oscillators*, arXiv: 1608.05903.

[14] A. Szulkin: *Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems*, Ann. Inst. H. Poincaré Anal. Non Linéaire 3 (1986) 77–109.