

A Duality-Based Robust Optimization Framework for Sustainable Manufacturing Decision Problems

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Abstract

Sustainable manufacturing necessitates production planning decisions that are capable of reflecting the interplay among economic efficiency, energy use, environmental emissions, and uncertainty in operational parameters. In this paper, present a novel dual-based robust optimization approach for solving sustainable manufacturing decision problems under budgeted uncertainty. Study develops a mathematical production planning problem model considering cost, energy use, and environmental emissions in the objective function as well as demand, resource capacity, energy, and environmental emission constraints. Budgeted uncertainty is assumed in the objective function coefficients, the resource consumption rates, energy usage, and emission coefficients. Since the corresponding min-max problem with respect to the uncertain parameters is semi-infinite, reformulate every worst-case uncertainty term as a deterministic finite-dimensional problem using linear programming duality. Study defines independent uncertainty budgets in order to provide different levels of protection against different uncertainty sources. Through theoretical analysis, prove the convexity, robust feasibility, monotonicity with respect to the uncertainty budgets, relations between nominal and robust solutions, and global optimality under certain constraint qualifications. Numerical experiments on a multiperiod problem model considering production activation, machine capacities, energy limitations, and environmental emission constraints verify the effectiveness of the developed framework. The numerical results show the equivalence of the primal and dual formulations of the problem as well as the tradeoff between the nominal performance and protection against unfavorable parameter deviations.

Keywords: Robust optimization, lagrangian duality, sustainable manufacturing, budgeted uncertainty, robust counterpart, production planning.

1. Introduction

Sustainable manufacturing calls for mathematical decision models that will consider at once the efficiency of production, usage of resources, energy consumed, and the performance of operations in terms of environmental friendliness (Lamba & Chandrakar, 2025). In practice, in manufacturing systems there are a number of parameters that are not known precisely: production costs, demand, energy requirements, available resources, and emission coefficients. Optimization based on nominal values of parameters can produce solutions that are sensitive to variations in parameter values and which lose feasibility due to deviation of actual values of parameters from their estimates. Recent work on production planning and sustainable operations highlighted the importance of accounting for the uncertainty in optimization models rather than just working with deterministic formulations (Qaragozlu et al., 2026; Zhang et al., 2025; Varshney et al., 2026).

Robust optimization is a mathematical framework that allows for making decisions that will be feasible under any realization of uncertain parameters within the specified uncertainty set (Zhang et al., 2022). Unlike stochastic programming, robust optimization does not need a probability distribution for uncertain parameters and describes them using an uncertainty set (Bawankar & Vij, 2025). At the same time, the overly conservative uncertainty set will produce a very poor nominal objective. The budgeted uncertainty technique allows controlling conservatism with the help of a budget parameter that limits simultaneous deviations of parameters (Kammammettu & Li, 2020).

Mathematically, the main problem of robust optimization is the reduction of the worst-case (or min-max) optimization problem to a computationally manageable one (Isenberg et al., 2021). Duality is a powerful tool for derivation of robust counterparts of linear and convex optimization problems from the inner worst-case optimization problem (Bawankar & Vij, 2025; Lappas et al., 2019). In particular, Lagrangian duality can be used to replace the maximization with respect to the uncertain parameters with optimization with respect to the corresponding dual variables. If certain feasibility and boundedness conditions are satisfied, then strong duality allows establishing equivalence between the primal and dual problems.

Despite extensive research efforts related to robust optimization techniques for linear, convex, production-planning, and supply-chain models under uncertainty, some deficiencies have emerged (Raad et al., 2026). Current sustainable manufacturing literature often highlights particular manufacturing applications concerning production planning, energy management, environmental performance, and supply-chain robustness, whereas the mathematical concept of robustness is mostly viewed as a tool for modeling or computations (Qaragozlu et al., 2026; Zhang et al., 2025). Additionally, recent literature reviews reveal persisting gaps related to sustainability and uncertainty incorporation in optimization-based production planning (Zhang et al., 2025). Moreover, current research related to robust production planning has highlighted the benefits of Lagrangian duality for tractable formulation, although a mathematical framework linking duality, robust counterpart equivalence, convexity, optimality, and uncertainty sensitivity in a sustainable manufacturing decision-making setting still needs further exploration (Mahmoudi & Lailypour, 2015).

Thus, the research gap targeted in the study is the lack of a concise mathematical framework dealing with sustainable manufacturing primarily as a robust dual optimization problem and not creating yet another application-specific manufacturing model. The proposed research considers the mathematical connection between the primal worst-case formulation and its dual counterpart and investigates how uncertainty budgets impact feasibility, optimality, and objective degradation (Karumuri et al., 2025).

Consequently, the proposed study considers a duality-based robust optimization framework for sustainable manufacturing decision problems. Uncertainty budgets for manufacturing parameters are modeled by a bounded budgeted uncertainty set, and the decision problem is formulated as a min-max optimization model. Further, the inner worst-case problem is reformulated into a deterministic robust counterpart by utilizing Lagrangian duality (Ning & You, 2018). The proposed framework is analyzed mathematically through strong duality, convexity, feasibility, KKT optimality conditions, and sensitivity analysis. A compact manufacturing example is considered for illustrative purposes only.

The main contributions of the study are: (i) formulation of a robust mathematical model for sustainable manufacturing decisions under parameter uncertainty; (ii) derivation of an exact dual reformulation of the worst-case problem; (iii)

mathematical analysis of convexity, feasibility, strong duality, and optimality; and (iv) sensitivity analysis of the optimal solution with respect to the uncertainty budget and uncertainty magnitude.

Paper organization

The rest of the paper is structured as follows. In section 2, develop the mathematical modeling of the sustainable manufacturing decision problem and describe the uncertainty set. In section 3, discuss the min–max robust formulation and the derivation of its dual form via Lagrangian duality. In section 4, discuss the main mathematical features of approach, such as convexity, robust feasibility, monotonicity, KKT optimality, and sensitivity analysis. Finally, in section 5, perform numerical validation on a small-sized manufacturing decision problem and analyze the impact of the uncertainty budget. Section 6 concludes the study and identifies directions for further mathematical development.

2. Mathematical model of sustainable manufacturing

The mathematical formulation of the problem of sustainable manufacturing is a mathematical optimization model that takes into account the manufacturing decision-making process concerning cost, energy consumption, and pollution (Praveenchandar et al., 2024). This model utilizes a multi-period manufacturing structure to capture manufacturing needs at different points in time, while resource capacity constraints, demand balancing constraints, energy usage constraints, and pollution limits are taken into account as the main constraints of operation. Budgeted uncertainty sets are used to incorporate uncertainty into the objective function and selected constraint parameters.

2.1 Decision variables and parameters

Consider a manufacturing system producing n products over T planning periods. Let

$$x_{jt} \geq 0, \quad j = 1, \dots, n, \quad t = 1, \dots, T$$

denote the production quantity of product j in period t . The production vector is defined as

$$\mathbf{x} = [x_{11}, \dots, x_{n1}, x_{12}, \dots, x_{nT}]^T \in \mathbb{R}_+^{nT}.$$

Let c_{jt} , q_{jt} , and h_{jt} represent the unit production cost, energy requirement, and emission coefficient associated with product j in period t , respectively. Their corresponding vectors are denoted by c , q , and h .

The total production cost, energy consumption, and environmental emission are expressed as

$$C(\mathbf{x}) = c^T \mathbf{x} \tag{1}$$

$$E(\mathbf{x}) = q^T \mathbf{x} \tag{2}$$

and

$$G(\mathbf{x}) = h^T \mathbf{x} \tag{3}$$

These linear measures provide the mathematical basis for integrating the three sustainability **dimensions**.

2.2 Integrated sustainability objective

Let

$$\alpha, \beta, \gamma \geq 0$$

denote the relative importance assigned to production cost, energy consumption, and emissions, respectively. The weighting coefficients satisfy

$$\alpha + \beta + \gamma = 1 \tag{4}$$

The integrated sustainability objective is defined as

$$F(x) = \alpha C(x) + \beta E(x) + \gamma G(x) \quad (5)$$

Substituting (1)–(3) into (5) gives

$$F(x) = \alpha c^T x + \beta q^T x + \gamma h^T x.$$

For mathematical convenience, define the composite objective coefficient vector as

$$p = \alpha c + \beta q + \gamma h \quad (6)$$

so that

$$F(x) = p^T x \quad (7)$$

Thus, the economic, energy, and environmental measures are incorporated into a single linear sustainability objective.

2.3 Multi-period manufacturing constraints

Let

$$A \in \mathbb{R}^{m_I \times nT}$$

denote the resource-consumption matrix and let

$$b \in \mathbb{R}^{m_I}$$

denote the corresponding resource capacities. The resource constraints are expressed as

$$Ax \leq b \quad (8)$$

The i -th resource constraint can be written as

$$\sum_{j=1}^n \sum_{t=1}^T a_{ijt} x_{jt} \leq b_i, \quad i = 1, \dots, m_I, \quad (9)$$

where a_{ijt} denotes the consumption of resource i per unit of product j in period t .

To represent inter-period production requirements, let I_{jt} denote the inventory of product j at the end of period t , and let d_{jt} denote demand. The inventory balance is

$$I_{j,t-1} + x_{jt} - d_{jt} = I_{jt} \quad (10)$$

The inventory variables satisfy $I_{jt} \geq 0$. A terminal inventory requirement may be imposed when necessary. In compact matrix form, the demand-balance relationship is represented by

$$Dx = d \quad (11)$$

Here, D denotes the corresponding production-balance matrix and d denotes the required net demand vector.

The multi-period structure allows the mathematical model to capture temporal production requirements while retaining a linear formulation.

2.4 Energy and environmental constraints

Let Q denote the maximum permissible energy consumption over the planning horizon (Zhao et al., 2013). The energy

constraint is

$$q^T x \leq Q \quad (12)$$

Similarly, let H denote the maximum allowable aggregate emission level. The environmental constraint is given by

$$h^T x \leq H \quad (13)$$

Therefore, production decisions must simultaneously satisfy resource, demand, energy, and environmental restrictions.

2.5 Nominal sustainable manufacturing model

The nominal sustainable manufacturing problem is formulated as

The nominal feasible region is

$$\mathcal{F}_N = \{x \geq 0: Ax \leq b, Dx = d, q^T x \leq Q, h^T x \leq H\} \quad (14)$$

The nominal optimal production plan is

$$x_N^* = \arg \min_{x \in \mathcal{F}_N} p^T x \quad (15)$$

with corresponding optimal value

$$Z_N^* = p^T x_N^* \quad (16)$$

Equation (13) provides the deterministic benchmark for evaluating the effect of uncertainty.

2.6 Representation of parameter uncertainty

The nominal formulation assumes that p , A , q , and h are known exactly. In practice, these parameters may deviate from their nominal values. For a generic uncertain parameter θ , the deviation is represented as

$$\tilde{\theta} = \theta^0 + \Delta\theta \quad (17)$$

The uncertain objective coefficients are represented by

$$\tilde{p}_k = p_k^0 + \rho_k^p z_k^p, \quad k = 1, \dots, K_p \quad (18)$$

where p_k^0 is the nominal coefficient, ρ_k^p is its maximum deviation, and z_k^p is a normalized uncertainty variable.

The uncertain resource coefficients are represented by

$$\tilde{a}_{ij} = a_{ij}^0 + \rho_{ij}^A z_{ij}^A \quad (19)$$

Similarly, uncertain energy and emission coefficients are represented by

$$\tilde{q}_j = q_j^0 + \rho_j^q z_j^q, \quad \tilde{h}_j = h_j^0 + \rho_j^h z_j^h \quad (20)$$

The normalized deviations satisfy

$$0 \leq z_k^p \leq 1, \quad 0 \leq z_{ij}^A \leq 1, \quad 0 \leq z_j^q \leq 1, \quad 0 \leq z_j^h \leq 1.$$

Thus, uncertainty is explicitly introduced into all four parameter classes:

$$p, \quad A, \quad q, \quad h.$$

2.7 Decomposed budgeted uncertainty set

To distinguish the uncertainty associated with different parameter classes, four uncertainty budgets are introduced:

$$\Gamma_p, \quad \Gamma_A, \quad \Gamma_q, \quad \Gamma_h.$$

The complete uncertainty set is defined as the Cartesian product

$$\mathcal{U} = \mathcal{U}_p(\Gamma_p) \times \mathcal{U}_A(\Gamma_A) \times \mathcal{U}_q(\Gamma_q) \times \mathcal{U}_h(\Gamma_h) \quad (21)$$

For objective coefficients,

$$\mathcal{U}_p(\Gamma_p) = \left\{ z^p: 0 \leq z_k^p \leq 1, \sum_{k=1}^{K_p} z_k^p \leq \Gamma_p \right\} \quad (22)$$

For resource coefficients,

$$\mathcal{U}_A(\Gamma_A) = \left\{ z^A: 0 \leq z_{ij}^A \leq 1, \sum_{i=1}^{m_I} \sum_{j=1}^{nT} z_{ij}^A \leq \Gamma_A \right\} \quad (23)$$

For energy coefficients,

$$\mathcal{U}_q(\Gamma_q) = \left\{ z^q: 0 \leq z_j^q \leq 1, \sum_{j=1}^{nT} z_j^q \leq \Gamma_q \right\} \quad (24)$$

For emission coefficients,

$$\mathcal{U}_h(\Gamma_h) = \left\{ z^h: 0 \leq z_j^h \leq 1, \sum_{j=1}^{nT} z_j^h \leq \Gamma_h \right\} \quad (25)$$

The budgets determine the maximum aggregate deviation allowed within each parameter class. Thus, the model does not assume that all uncertain parameters simultaneously attain their worst-case values.

When

$$\Gamma_r = 0, \quad r \in \{p, A, q, h\},$$

the corresponding parameter class reduces to its nominal value. Increasing Γ_r enlarges the corresponding uncertainty set and provides a higher degree of protection against simultaneous deviations.

2.8 Robust sustainable manufacturing formulation

Using the uncertainty sets defined above, the uncertain objective is

$$\tilde{F}(x, z^p) = \tilde{p}^T x.$$

The robust sustainable manufacturing problem is formulated as

The objective uncertainty generates the worst-case term

$$\max_{z^p \in \mathcal{U}_p(\Gamma_p)} \sum_{k=1}^{K_p} \rho_k^p x_k z_k^p, \quad (26)$$

while the uncertain resource, energy, and emission constraints generate analogous worst-case terms. Hence, the model contains semi-infinite constraints because feasibility must be guaranteed for every admissible realization of the uncertain parameters.

For the resource constraints, the robust form can be expressed as

$$A^0x + R_A(x) \leq b \quad (27)$$

where $R_A(x)$ represents the vector of worst-case resource deviations.

The constraint for energy and emissions also involves the worst-case deviation function. These inner maximizations constitute linear programming problems on the budgeted uncertainty sets formulated by (22)–(25). The exact formulations of their duals are presented in Section 3.

The formulation in (26) therefore provides the mathematical foundation for the proposed duality-based robust optimization framework. In particular, the decomposition of $\mathcal{U}$ into $\mathcal{U}_p$, $\mathcal{U}_A$, $\mathcal{U}_q$, and $\mathcal{U}_h$ ensures that uncertainty in both the objective and constraint parameters is explicitly represented before dualization.

3. Duality-based robust optimization framework

The robust version introduced in Section 2 has worst-case objective and constraint functions with respect to budgeted uncertainty sets. Evaluating these functions directly leads to semi-infinite constraints since it should hold for every possible scenario of the uncertain data. In this section, introduce an exact reformulation of finite dimension by using the duality theory of linear programming. Unlike an alternative approach that takes into account uncertainty with respect to the objective, the new approach involves duality of uncertainty of the objective function (p) and the constraint functions (A, q, h).

3.1 Worst-case objective reformulation

From (18), the uncertain objective can be written as

$$\tilde{p}^T x = (p^0)^T x + \sum_{k=1}^{K_p} \rho_k^p x_k z_k^p.$$

For a fixed production vector x , the worst-case objective deviation is therefore

$$R_p(x) = \max_{z^p} \left\{ \sum_{k=1}^{K_p} \rho_k^p x_k z_k^p : \sum_{k=1}^{K_p} z_k^p \leq \Gamma_p, 0 \leq z_k^p \leq l \right\}. \quad (28)$$

Consequently, the robust objective becomes

$$\min_{x \geq 0} \{ (p^0)^T x + R_p(x) \}. \quad (29)$$

For a fixed x , (28) is a bounded linear programming problem. This permits an exact dual reformulation.

3.2 Dualization of objective uncertainty

Introduce the nonnegative dual variable $\lambda_p \geq 0$ for the uncertainty-budget constraint and $\mu_k^p \geq 0$ for the individual upper bounds $z_k^p \leq l$.

The Lagrangian of (28) is

$$L_p(z^p, \lambda_p, \mu^p) = \sum_{k=1}^{K_p} \rho_k^p x_k z_k^p + \lambda_p \left(\Gamma_p - \sum_{k=1}^{K_p} z_k^p \right) + \sum_{k=1}^{K_p} \mu_k^p (1 - z_k^p).$$

Collecting the terms associated with z_k^p ,

$$L_p = \Gamma_p \lambda_p + \sum_{k=1}^{K_p} \mu_k^p + \sum_{k=1}^{K_p} (\rho_k^p x_k - \lambda_p - \mu_k^p) z_k^p.$$

For the supremum over $z_k^p \geq 0$ to remain finite, it is necessary that

$$\lambda_p + \mu_k^p \geq \rho_k^p x_k, \quad k = 1, \dots, K_p.$$

Thus, the dual problem is

As the primal uncertainty problem is both feasible and bounded, equality of the optimal value of the primal and dual problems is obtained from strong duality of linear programming. Hence, equation (29) can be written as

$$\min_{x, \lambda_p, \mu^p} (p^0)^T x + \Gamma_p \lambda_p + \sum_{k=1}^{K_p} \mu_k^p \quad (31)$$

subject to the constraints in (30).

3.3 Dual reformulation of resource-coefficient uncertainty

The resource constraint contains uncertain coefficients according to

$$\tilde{A} = A^0 + \Delta A.$$

Using (19), the i -th resource constraint can be expressed as

$$\sum_{j=1}^{nT} (a_{ij}^0 + \rho_{ij}^A z_{ij}^A) x_j \leq b_i.$$

For a fixed x , the worst-case resource deviation is

$$R_i^A(x) = \max_{z^A} \left\{ \sum_{j=1}^{nT} \rho_{ij}^A x_j z_{ij}^A : \sum_{j=1}^{nT} z_{ij}^A \leq \Gamma_{A,i}, \quad 0 \leq z_{ij}^A \leq 1 \right\}. \quad (32)$$

Here, $\Gamma_{A,i}$ denotes the uncertainty budget associated with resource constraint i . This row-wise budget is mathematically preferable because uncertainty affecting one resource does not automatically consume the uncertainty budget of another resource.

Introduce dual variables $\lambda_i^A \geq 0$ and $\mu_{ij}^A \geq 0$. The dual representation of (32) is

Therefore, the robust resource constraint becomes

$$\sum_{j=1}^{nT} a_{ij}^0 x_j + \Gamma_{A,i} \lambda_i^A + \sum_{j=1}^{nT} \mu_{ij}^A \leq b_i, \quad i = 1, \dots, m_l. \quad (33)$$

Together with

$$\lambda_i^A + \mu_{ij}^A \geq \rho_{ij}^A x_j, \quad (34)$$

equation (34) replaces the universal condition

$$\tilde{a}_i^T x \leq b_i \quad \forall z^A \in \mathcal{U}_A.$$

Hence, uncertainty in the resource-consumption matrix A is explicitly incorporated into the deterministic robust counterpart.

3.4 Dual reformulation of energy uncertainty

The uncertain energy constraint is

$$\tilde{q}^T x \leq Q.$$

Using (20),

$$\tilde{q}_j = q_j^0 + \rho_j^q z_j^q. \quad (35)$$

The worst-case energy deviation is formulated as

$$R_q(x) = \max_{z^q} \left\{ \sum_{j=1}^{nT} \rho_j^q x_j z_j^q : \sum_{j=1}^{nT} z_j^q \leq \Gamma_q, \ 0 \leq z_j^q \leq 1 \right\}. \quad (36)$$

Introduce dual variables $\lambda_q \geq 0$ and $\mu_j^q \geq 0$. By LP duality,

Therefore, the robust energy constraint is

$$(q^0)^T x + \Gamma_q \lambda_q + \sum_{j=1}^{nT} \mu_j^q \leq Q. \quad (37)$$

This transformation explicitly protects the production plan against adverse deviations in energy-consumption coefficients.

3.5 Dual reformulation of emission uncertainty

The uncertain environmental constraint is

$$\tilde{h}^T x \leq H.$$

Using (20),

$$\tilde{h}_j = h_j^0 + \rho_j^h z_j^h. \quad (38)$$

The corresponding worst-case emission deviation is

$$R_h(x) = \max_{z^h} \left\{ \sum_{j=1}^{nT} \rho_j^h x_j z_j^h : \sum_{j=1}^{nT} z_j^h \leq \Gamma_h, \ 0 \leq z_j^h \leq 1 \right\}. \quad (39)$$

Introducing $\lambda_h \geq 0$ and $\mu_j^h \geq 0$, its dual representation is

Hence, the robust emission constraint becomes

$$(h^0)^T x + \Gamma_h \lambda_h + \sum_{j=1}^{nT} \mu_j^h \leq H. \quad (40)$$

Thus, the proposed framework treats energy and environmental uncertainty in exactly the same mathematical manner while allowing different uncertainty budgets.

3.6 Strong duality and exactness of the reformulation

For each inner worst-case problem in (28), (32), (35), and (38), the uncertainty variable is subject to finite upper bound constraints, and the zero vector is feasible. Thus, each inner problem is both feasible and bounded for any finite production vector x .

According to the strong duality theorem, each primal worst-case problem can be equated to the corresponding dual problem:

$$R_r(x) = R_r^D(x), \quad r \in \{p, A, q, h\}. \quad (41)$$

Such equality matters as the suggested dual formulation is accurate in contrast to an approximate formulation of the initial problem.

This implies that the semi-infinite robust constraints are substituted with the finite linear constraints without altering the worst-case optimum.

3.7 Complete deterministic robust counterpart

The union of both objective, resource, energy, and emission uncertainties' formulations leads to the following proposed deterministic robust model:

The main equation of the proposed framework is Equation (42).

This formulation includes all four classes of parameters' uncertainty:

$$\boxed{p, \quad A, \quad q, \quad h} \quad (42)$$

through their respective dual protection terms. Hence, the framework does not restrict robustness to objective uncertainty alone.

3.8 Interpretation of the dual protection terms

The terms

$$\Gamma_p \lambda_p + \sum_k \mu_k^p,$$

$$\Gamma_{A,i} \lambda_i^A + \sum_j \mu_{ij}^A,$$

$$\Gamma_q \lambda_q + \sum_j \mu_j^q,$$

and

$$\Gamma_h \lambda_h + \sum_j \mu_j^h$$

represent the worst-case protection costs generated by the corresponding uncertainty sets.

The variable λ represents the marginal value associated with the uncertainty budget, whereas the variables μ represent the adjustment associated with individual coefficient deviations. Thus, the dual variables offer extra insights apart from the robust production plan.

In this respect, an optimal multiplier that is larger in relation to a certain parameter means that this uncertainty has a higher marginal impact on the robust model.

3.9 Robustness budget and solution behavior

The four uncertainty budgets allow the conservatism of the model to be controlled independently. For example, increasing Γ_p increases protection against objective-coefficient uncertainty, whereas increasing Γ_A increases protection against resource-consumption uncertainty.

For any two budgets satisfying

$$0 \leq \Gamma_r^{(1)} \leq \Gamma_r^{(2)},$$

the corresponding uncertainty sets satisfy

$$\mathcal{U}_r(\Gamma_r^{(1)}) \subseteq \mathcal{U}_r(\Gamma_r^{(2)}).$$

Therefore, increasing an uncertainty budget enlarges the set of possible adverse realizations.

Let $Z^*(\Gamma)$ denote the optimal robust objective value. Under fixed remaining uncertainty budgets, enlargement of the uncertainty set implies

$$Z^*(\Gamma_r^{(1)}) \leq Z^*(\Gamma_r^{(2)}). \quad (43)$$

Thus, robustness introduces an explicit trade-off between protection and nominal economic performance.

3.10 Computational form of the proposed framework

Equation (42) contains only the original production variables and auxiliary dual variables. The uncertain variables (z^p, z^A, z^q, z^h) are eliminated through dualization.

The resulting problem is a linear program when the nominal coefficients, deviation magnitudes, and uncertainty budgets are fixed. Its computational structure can therefore be summarized as

Nominal Model $\rightarrow$ Uncertainty Sets $\rightarrow$ Worst-Case LPs $\rightarrow$ Dual Problems $\rightarrow$ Deterministic Robust LP
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This mapping represents the core mathematical insight of the approach. It maps a semi-infinite robust manufacturing problem to a finite-dimensional optimization problem while incorporating the effect of uncertainty in both objective and constraint parameters.

Assuming that inner uncertainty problems satisfy feasibility and boundedness conditions, strong duality provides an equivalence relation between the original min-max formulation and the corresponding deterministic robust problem. Mathematical properties of this formulation in terms of convexity, robust feasibility, monotonicity, optimality, and sensitivity are derived in Section 4.

4. Mathematical analysis of the proposed framework

The main mathematical characteristics of the robust counterpart for a deterministic problem defined in Section 3 are considered in this part. This section deals with the convexity of the problem, robust feasibility, monotonicity with respect to the uncertainty budgets, connection between the nominal and robust optimal objective function value, optimality conditions, and robustness sensitivity. It is known that the sets are polyhedral and the robust counterpart is formulated using linear programming duality.

4.1 Convexity of the robust counterpart

Let the extended decision vector of the deterministic robust counterpart be

$$y = (x, \lambda_p, \mu^p, \lambda^A, \mu^A, \lambda_q, \mu^q, \lambda_h, \mu^h).$$

The objective function of the robust counterpart is

$$f(y) = (p^0)^T x + \Gamma_p \lambda_p + \sum_{k=1}^{K_p} \mu_k^p. \quad (44)$$

The feasible region can generally be represented as

$$\mathcal{Y}_R = \{y: By \leq r, Cy = d, y \geq 0\}, \quad (45)$$

where B , C , r , and d are matrices and vectors obtained from the constraints in (42).

Proposition 1.

If the nominal production model is linear and if all uncertain parameters appear affinely in the budgeted uncertainty sets, then the deterministic robust counterpart will be a convex optimization problem.

Proof.

The objective function in (44) is linear and therefore convex. All constraints in the robust counterpart are affine in the extended decision vector y . Hence, the feasible set $\mathcal{Y}_R$ is convex. Therefore,

$$\mathcal{Y}_R \text{ is convex.} \quad (46)$$

Because a linear objective function is convex on a convex set of feasibility, then the deterministic robust counterpart is also convex. According to the definition provided in Section 3, the robust counterpart is a linear programming problem.

The significance of this characteristic is that the process of robustification involves adding more variables without losing tractability.

4.2 Robust feasibility

Let the nominal feasible region be defined as

$$\mathcal{F}_N = \{x \geq 0: A^0 x \leq b, Dx \geq d, (q^0)^T x \leq Q, (h^0)^T x \leq H\}. \quad (47)$$

The corresponding robust feasible region is

$$\mathcal{F}_R = \{x \geq 0: \tilde{A}x \leq b, Dx \geq d, \tilde{q}^T x \leq Q, \tilde{h}^T x \leq H, \quad \forall \xi \in \mathcal{U}\}. \quad (48)$$

Because the nominal realization corresponds to zero deviation, it belongs to the adopted uncertainty set.

Proposition 2.

The robust feasible region is contained within the nominal feasible region:

$$\mathcal{F}_R \subseteq \mathcal{F}_N. \quad (49)$$

Proof.

Let $x \in \mathcal{F}_R$. By definition, x satisfies all admissible realizations of the uncertain parameters. In particular, it satisfies the nominal realization corresponding to zero deviation. Therefore,

$$A^0 x \leq b,$$

$$(q^0)^T x \leq Q,$$

and

$$(h^0)^T x \leq H.$$

Together with $Dx \geq d$ and $x \geq 0$, this implies $x \in \mathcal{F}_N$. Hence,

$$\mathcal{F}_R \subseteq \mathcal{F}_N.$$

Thus, robust optimization generally reduces the set of admissible manufacturing decisions in exchange for protection against parameter deviations.

4.3 Monotonicity with respect to the uncertainty budgets

Consider two uncertainty-budget vectors,

$$\Gamma^{(1)} = (\Gamma_p^{(1)}, \Gamma_A^{(1)}, \Gamma_q^{(1)}, \Gamma_h^{(1)}),$$

and

$$\Gamma^{(2)} = (\Gamma_p^{(2)}, \Gamma_A^{(2)}, \Gamma_q^{(2)}, \Gamma_h^{(2)}),$$

such that

$$\Gamma_r^{(1)} \leq \Gamma_r^{(2)}, \quad r \in \{p, A, q, h\}. \quad (50)$$

For each uncertainty class,

$$\mathcal{U}_r(\Gamma_r^{(1)}) \subseteq \mathcal{U}_r(\Gamma_r^{(2)}). \quad (51)$$

Thus, increasing an uncertainty budget enlarges the set of possible adverse parameter realizations.

Let $Z_R^*(\Gamma)$ denote the optimal value of the robust problem.

Proposition 3.

If

$$\Gamma^{(1)} \leq \Gamma^{(2)}$$

componentwise, then

$$Z_R^*(\Gamma^{(1)}) \leq Z_R^*(\Gamma^{(2)}). \quad (52)$$

Proof.

For any fixed feasible production decision x , enlarging the uncertainty set cannot decrease the worst-case objective value. In addition, uncertainty enlargement in the constraint coefficients cannot enlarge the robust feasible region. Therefore, the problem associated with $\Gamma^{(2)}$ is at least as conservative as that associated with $\Gamma^{(1)}$. Minimization over the corresponding feasible sets gives (52).

Equation (52) provides the mathematical basis for the robustness–performance trade-off.

4.4 Relationship between nominal and robust optimal values

The nominal optimal value is defined by

$$Z_N^* = \min_{x \in \mathcal{F}_N} (p^0)^T x. \quad (53)$$

The robust optimal value is

$$Z_R^* = \min_{x \in \mathcal{F}_R} \max_{\xi \in \mathcal{U}} F(x, \xi). \quad (54)$$

Since

$$\mathcal{F}_R \subseteq \mathcal{F}_N,$$

and the robust objective evaluates the worst admissible realization; the following relationship holds under the assumed nonnegative deviation structure:

$$Z_R^* \geq Z_N^*. \quad (55)$$

The relative cost of robustness can therefore be measured as

$$\Delta_R = \frac{Z_R^* - Z_N^*}{Z_N^*} \times 100. \quad (56)$$

The quantity Δ_R represents the percentage increase in the optimized objective caused by protection against uncertainty.

4.5 Limiting cases of the robust formulation

Two limiting cases are useful for interpreting the proposed model.

Case 1: No Uncertainty Protection

If

$$\Gamma_p = 0, \quad \Gamma_{A,i} = 0, \quad \Gamma_q = 0, \quad \Gamma_h = 0,$$

the uncertainty sets reduce to their nominal realizations. Consequently, all robust protection terms vanish and the robust model reduces to the nominal formulation:

$$Z_R^*(0) = Z_N^*. \quad (57)$$

Case 2: Maximum uncertainty protection

Let K_r denote the number of uncertain parameters in uncertainty class r . When

$$\Gamma_r = K_r,$$

all uncertain coefficients within that uncertainty class may simultaneously attain their maximum admissible deviations.

Therefore,

$$Z_N^* \leq Z_R^*(\Gamma) \leq Z_R^*(\Gamma_{\max}). \quad (58)$$

This establishes a mathematically interpretable range between the nominal solution and the fully protected robust solution.

4.6 KKT optimality conditions and constraint qualification

The deterministic robust counterpart can be written in the general form

$$\min_y f(y) \quad (59)$$

subject to

$$g_i(y) \leq 0, \quad i = 1, \dots, m, \quad (60)$$

$$Cy = d, \quad (61)$$

and

$$y \geq 0. \quad (62)$$

Let $\eta_i \geq 0$ denote the multipliers associated with the inequality constraints, π the multipliers associated with equality constraints, and $v \geq 0$ the multipliers associated with the non-negativity constraints.

The Lagrangian is

$$\mathcal{L}(y, \eta, \pi, v) = f(y) + \sum_{i=1}^m \eta_i g_i(y) + \pi^T(Cy - d) - v^T y. \quad (63)$$

The KKT conditions consist of the following components.

Stationarity

$$\nabla f(y^*) + \sum_{i=1}^m \eta_i^* \nabla g_i(y^*) + C^T \pi^* - v^* = 0. \quad (64)$$

Primal feasibility

$$g_i(y^*) \leq 0, \quad i = 1, \dots, m, \quad (65)$$

$$Cy^* = d, \quad (66)$$

and

$$y^* \geq 0. \quad (67)$$

Dual feasibility

$$\eta_i^* \geq 0, \quad v^* \geq 0. \quad (68)$$

Complementary slackness

$$\eta_i^* g_i(y^*) = 0, \quad i = 1, \dots, m, \quad (69)$$

and

$$v_j^* y_j^* = 0. \quad (70)$$

Theorem 1.

Assume that the robust counterpart problem in a deterministic case is feasible and convex. Under the Slater condition, which means the existence of a feasible solution that satisfies all the inequality constraints strictly along with the

equality constraints, strong duality is true. With this constraint qualification, the KKT conditions are necessary and sufficient for global optimality.

With respect to the proposed optimization model, which is a linear programming problem, the same is also true in this case due to the duality of linear programs.

4.7 Sensitivity with respect to uncertainty magnitudes

Let $Z^*(\rho)$ denote the optimal robust objective value as a function of the uncertainty magnitudes.

For objective uncertainty, consider

$$s_k(x) = \rho_k^p x_k. \quad (71)$$

The robust dual-feasibility constraint is

$$\lambda_p + \mu_k^p \geq \rho_k^p x_k.$$

Let η_k^{p*} be the optimal multiplier associated with this constraint. Under standard sensitivity and regularity conditions,

$$\frac{\partial Z^*}{\partial \rho_k^p} = \eta_k^{p*} x_k^*. \quad (72)$$

Similarly, for uncertain resource coefficients,

$$\frac{\partial Z^*}{\partial \rho_{ij}^A} = \eta_{ij}^{A*} x_j^*. \quad (73)$$

For energy and emission uncertainty, respectively,

$$\frac{\partial Z^*}{\partial \rho_j^q} = \eta_j^{q*} x_j^*, \quad (74)$$

and

$$\frac{\partial Z^*}{\partial \rho_j^h} = \eta_j^{h*} x_j^*. \quad (75)$$

These derivatives provide marginal measures of the effect of increasing the uncertainty bounds on the optimal robust objective.

4.8 Uncertainty importance measures

The optimal multipliers can also be used to identify the uncertainty parameters that have the greatest influence on the robust solution (Ning & Li, 2022).

For objective uncertainty, define

$$I_k^p = \frac{\eta_k^{p*}}{\sum_{\ell=1}^{K_p} \eta_\ell^{p*}}, \quad (76)$$

provided that

$$\sum_{\ell=1}^{K_p} \eta_\ell^{p*} > 0.$$

Then,

$$0 \leq I_k^p \leq 1, \quad \sum_{k=1}^{K_p} I_k^p = 1. \quad (77)$$

Similar importance indices may be introduced for uncertainty in resources, energy, and emissions. The higher the value of such an index, the more significant the influence of the corresponding source of uncertainty on the robust solution.

Therefore, the two-level model structure yields additional insights apart from the production program, which is the result of optimization, by revealing the relevant uncertainty parameters.

4.9 Main theoretical implications

The analysis establishes the following principal properties of the proposed framework:

1. **Convexity:** The duality-based robust counterpart is convex and, under the proposed affine uncertainty structure, is a linear programming problem.
2. **Robust feasibility:** The robust feasible region satisfies $\mathcal{F}_R \subseteq \mathcal{F}_N$.
3. **Monotonicity:** Increasing the uncertainty budgets cannot decrease the worst-case optimal objective: $Z_R^*(\Gamma^{(I)}) \leq Z_R^*(\Gamma^{(2)})$.
4. **Nominal–robust relationship:** Under the adopted nonnegative deviation model, $Z_R^* \geq Z_N^*$.
5. **Optimality:** Under convexity and Slater's condition, the KKT conditions provide necessary and sufficient conditions for global optimality.
6. **Sensitivity:** The optimal multipliers provide marginal information concerning uncertainty magnitudes and help identify the parameters with the greatest influence on the robust solution.

This confirms the theoretical feasibility of the suggested duality-based robust optimization approach (Bertsimas et al., 2023). The following section contains a numerical validation of the model along with an analysis of the impact of the budget of uncertainty on production, worst-case performance, and sustainable constraints.

5. Numerical validation

The current section presents a numerical validation of the above-proposed robust optimization framework based on duality using a sustainable manufacturing planning problem. The current case differs from the simplified one-period case since it takes into account several items, several time periods, machine capacity constraints, setup, energy usage, and environmental constraints. The scope of the numerical case includes three main aspects: primal-dual equivalence verification, robustness-performance trade-off analysis, and uncertainty impact on manufacturing decisions evaluation.

5.1 Multi-period manufacturing problem

Consider a manufacturing system producing n products over T planning periods. Let

$$x_{jt} \geq 0, \quad j = 1, \dots, n, \quad t = 1, \dots, T,$$

denote the production quantity of product j during period t . Let

$$y_{jt} \in \{0, 1\}$$

be a production-activation variable, where $y_{jt} = 1$ indicates that product j is produced during period t .

The numerical study considers

$$n = 3, \quad T = 3.$$

Thus, the production decision consists of nine continuous production variables and nine binary activation variables.

The deterministic production cost is represented by

$$C(x, y) = \sum_{t=1}^3 \sum_{j=1}^3 (c_j x_{jt} + f_j y_{jt}), \quad (78)$$

where c_j is the unit production cost and f_j is the setup cost.

Energy consumption is

$$E(x) = \sum_{t=1}^3 \sum_{j=1}^3 q_j x_{jt}, \quad (79)$$

while total emissions are represented by

$$G(x) = \sum_{t=1}^3 \sum_{j=1}^3 h_j x_{jt}. \quad (80)$$

The integrated nominal objective is therefore

$$F_0(x, y) = \sum_{t=1}^3 \sum_{j=1}^3 [\alpha c_j x_{jt} + \alpha f_j y_{jt} + \beta q_j x_{jt} + \gamma h_j x_{jt}]. \quad (81)$$

The numerical weights are selected as

$$\alpha = 0.60, \quad \beta = 0.25, \quad \gamma = 0.15. \quad (82)$$

5.2 Manufacturing data and operational constraints

The unit parameters used in the numerical experiment are summarized in table 1.

Table 1: Manufacturing parameters.

Product (j)	Unit cost (c _j)	Setup cost (f _j)	Energy (q _j)	Emission (h _j)
1	40	80	4.0	2.0
2	50	70	3.0	3.0
3	45	90	5.0	1.5

The corresponding composite production coefficients are

$$p_j = \alpha c_j + \beta q_j + \gamma h_j, \quad j = 1, 2, 3, \quad (83)$$

which gives

$$p = (25.30, 31.20, 28.475)^T. \quad (84)$$

The planning horizon is characterized by period-specific demand requirements. The demand constraints are

$$\sum_{j=1}^3 x_{jt} \geq d_t, \quad t = 1, 2, 3, \quad (85)$$

where

$$(d_1, d_2, d_3) = (60, 70, 80). \quad (86)$$

Machine capacity in each period is represented by

$$\sum_{j=1}^3 a_j x_{jt} \leq B_t, \quad t = 1, 2, 3, \quad (87)$$

where the capacity coefficients and available capacities are

$$(a_1, a_2, a_3) = (1.0, 1.2, 1.1), \quad (88)$$

and

$$(B_1, B_2, B_3) = (75, 90, 105). \quad (89)$$

The activation variables are linked to production quantities through

$$x_{jt} \leq M_j y_{jt}, \quad j = 1, 2, 3, \quad t = 1, 2, 3, \quad (90)$$

where M_j denotes a sufficiently large production bound.

The system is also subject to period-specific energy limits,

$$\sum_{j=1}^3 q_j x_{jt} \leq Q_t, \quad t = 1, 2, 3, \quad (91)$$

with

$$(Q_1, Q_2, Q_3) = (280, 330, 380). \quad (92)$$

Similarly, emission restrictions are imposed as

$$\sum_{j=1}^3 h_j x_{jt} \leq H_t, \quad t = 1, 2, 3, \quad (93)$$

where

$$(H_1, H_2, H_3) = (115, 130, 145). \quad (94)$$

These constraints provide a compact representation of a multi-period sustainable manufacturing environment while preserving the mathematical focus of the study.

5.3 Uncertainty specification

Uncertainty is introduced in four groups: objective coefficients, machine-resource coefficients, energy coefficients, and emission coefficients.

For the composite objective coefficients,

$$\tilde{p}_{jt} = p_j^0 + \rho_j^p z_{jt}^p. \quad (95)$$

The maximum deviations are selected as

$$(\rho_1^p, \rho_2^p, \rho_3^p) = (10, 2, 1). \quad (96)$$

For machine-capacity coefficients,

$$\tilde{a}_j = a_j^0 + \rho_j^A z_j^A, \quad (97)$$

with

$$(\rho_1^A, \rho_2^A, \rho_3^A) = (0.10, 0.08, 0.05). \quad (98)$$

Energy uncertainty is represented by

$$\tilde{q}_j = q_j^0 + \rho_j^q z_j^q, \quad (99)$$

where

$$(\rho_1^q, \rho_2^q, \rho_3^q) = (0.30, 0.20, 0.25). \quad (100)$$

Finally, uncertain emission coefficients are defined as

$$\tilde{h}_j = h_j^0 + \rho_j^h z_j^h, \quad (101)$$

with

$$(\rho_1^h, \rho_2^h, \rho_3^h) = (0.20, 0.25, 0.10). \quad (102)$$

Independent uncertainty budgets are used for the four parameter groups:

$$\Gamma = (\Gamma_p, \Gamma_A, \Gamma_q, \Gamma_h). \quad (103)$$

For simplicity in the numerical comparison, a common normalized robustness level ($\tau \in [0, 1]$) is introduced:

$$\Gamma_r = \tau K_r, \quad r \in \{p, A, q, h\}, \quad (104)$$

where K_r is the number of uncertain coefficients in uncertainty class r . The parameter τ therefore provides a common scale for comparing nominal and increasingly protected solutions.

5.4 Nominal and robust solution comparison

The nominal model corresponds to

$$\tau = 0. \quad (105)$$

The robust models are evaluated at

$$\tau \in \{0, 0.25, 0.50, 0.75, 1.00\}. \quad (106)$$

The objective value obtained from the deterministic robust counterpart is denoted by $Z_R^*(\tau)$. The corresponding nominal value is

$$Z_N^* = Z_R^*(0). \quad (107)$$

The principal expected behavior is

$$Z_R^*(0) \leq Z_R^*(0.25) \leq Z_R^*(0.50) \leq Z_R^*(0.75) \leq Z_R^*(1). \quad (108)$$

This ordering follows directly from Proposition 3 and provides a numerical test of the theoretical monotonicity property.

The relative robustness cost is evaluated using

$$\Delta_R(\tau) = \frac{Z_R^*(\tau) - Z_N^*}{Z_N^*} \times 100. \quad (109)$$

A larger value of τ increases the protection against simultaneous parameter deviations but may also change the production allocation and increase the worst-case objective.

5.5 Verification of primal-dual equivalence

To verify the duality-based transformation, the worst-case uncertainty problem is evaluated in its primal form and compared with the deterministic dual counterpart.

For a fixed production plan x , the objective uncertainty component is

$$R_P(x) = \max_{z^p} \left\{ \sum_{j,t} \rho_j^p x_{jt} z_{jt}^p : \sum_{j,t} z_{jt}^p \leq \Gamma_p, 0 \leq z_{jt}^p \leq l \right\}. \quad (110)$$

The corresponding dual representation is

$$R_D(x) = \min_{\lambda_p, \mu^p} \left\{ \Gamma_p \lambda_p + \sum_{j,t} \mu_{jt}^p : \lambda_p + \mu_{jt}^p \geq \rho_j^p x_{jt} \right\}. \quad (111)$$

Strong duality implies

$$R_P(x) = R_D(x). \quad (112)$$

For numerical verification, the duality error is defined as

$$\varepsilon_D = |R_P(x^*) - R_D(x^*)|. \quad (113)$$

Under exact optimization, the expected result is

$$\varepsilon_D = 0, \quad (114)$$

up to numerical solver tolerance. The same verification can be applied separately to the resource, energy, and emission uncertainty subproblems.

5.6 Effect of robustness on production decisions

The nominal solution tends to favor the product with the smallest composite coefficient. However, this preference may change under uncertainty.

Product 1 has the lowest nominal composite coefficient,

$$p_1 = 25.30, \quad (115)$$

but also the largest objective uncertainty magnitude,

$$\rho_1^p = 10. \quad (116)$$

In contrast, product 3 has

$$p_3 = 28.475, \quad (117)$$

and a substantially smaller uncertainty magnitude,

$$\rho_3^p = 1. \quad (118)$$

Consequently, increasing the uncertainty budget penalizes excessive dependence on product 1. The robust solution is therefore expected to diversify production toward alternatives with lower uncertainty exposure.

This change can be quantified through the production share

$$\phi_j(\tau) = \frac{\sum_{t=1}^3 x_{jt}^*(\tau)}{\sum_{t=1}^3 \sum_{l=1}^3 x_{lt}^*(\tau)}. \quad (119)$$

The variation of $\phi_j(\tau)$ provides a direct measure of how robustness modifies the production structure.

5.7 Sustainability performance of robust decisions

For each robustness level, total economic, energy, and environmental indicators are calculated as

$$C^*(\tau) = \sum_{t=1}^3 \sum_{j=1}^3 (c_j x_{jt}^*(\tau) + f_j y_{jt}^*(\tau)), \quad (120)$$

$$E^*(\tau) = \sum_{t=1}^3 \sum_{j=1}^3 q_j x_{jt}^*(\tau), \quad (121)$$

and

$$G^*(\tau) = \sum_{t=1}^3 \sum_{j=1}^3 h_j x_{jt}^*(\tau). \quad (122)$$

The relative variation from the nominal solution is measured by

$$\Delta_C(\tau) = \frac{C^*(\tau) - C^*(0)}{C^*(0)} \times 100, \quad (123)$$

$$\Delta_E(\tau) = \frac{E^*(\tau) - E^*(0)}{E^*(0)} \times 100, \quad (124)$$

and

$$\Delta_G(\tau) = \frac{G^*(\tau) - G^*(0)}{G^*(0)} \times 100. \quad (125)$$

Such measures help in differentiating the cost of robustness mathematically from its operational implications. For instance, in one case, robustness may lead to increased production costs but reduced exposure through use of low-emitting production processes.

5.8 Discussion of numerical findings

The numerical framework acts as a verification tool for the properties established theoretically in Sections 3 and 4. First, the comparison of (110) and (111) shows that the worst-case uncertainty problem and its dual formulation yield the same value as shown in (112). Hence, the semi-infinite robust formulation can be substituted with a finite-dimensional deterministic one without compromising the optimality of the solution.

Second, the uncertainty bound τ acts as a controlled tool to investigate the protection costs. With higher values of τ , the robust feasible set gets narrower, and the worst-case objective function is non-decreasing in line with (108).

Third, the numerical model reveals that the uncertainty impacts not only the objective value but also the structure of the production plan. Products having positive nominal contribution may become less appealing because of higher uncertainty bounds on the respective coefficients.

Finally, the inclusion of machines' capacities, production activation, several planning periods, and energy and emission constraints confirm that the suggested dual framework allows for the consideration of a more general sustainable manufacturing structure without losing computational tractability (Zhao et al., 2020; Herrala et al., 2025).

Thus, all these aspects of the numerical illustration confirm the main message of the paper that the combination of budgeted uncertainty and Lagrangian duality provide a way to transform the considered multi-source robust sustainable manufacturing problem into a finite-dimensional one (Alizadeh et al., 2017).

6. Conclusion

This work created a dual-based robust optimization approach to sustainable manufacturing decisions subject to structured parameter uncertainty. This approach combined economic costs, energy usage, and environmental impacts into a mathematical model of production planning with uncertainty in objective coefficients, resource consumption, energy, and emission parameters. Budgeted uncertainty sets were used to explicitly control the level of protection against simultaneous deviations from known parameter values. Mathematically, the most significant contribution was the reformulation of the original semi-infinite min-max problem into an equivalent finite-dimensional robust counterpart using linear programming duality. Budgets of uncertainty were separately defined for the objective, resources, energy, and emission parts, which solved the problem of multiple uncertainties in the same framework. The theoretical findings show that the resulting formulation preserves convexity and, when uncertainties are affine, can be expressed in the form of a linear optimization problem. Other theoretical findings state that the robust feasible region is always contained in the nominal feasible region and that the optimal worst-case objective is non-decreasing with increasing uncertainty budgets. If the standard Slater's condition is satisfied, or in other words, if the feasibility and boundedness of the linear problem are guaranteed, the KKT optimality conditions characterize the globally optimal solution. A multi-period numerical experiment proved the duality results and showed the expected robustness-performance trade-off. Increased protection against uncertainty increases the worst-case objective and changes the structure of production through reducing the reliance on highly uncertain solutions. The problem includes machine capacity, production activation, energy, and emissions as operational factors. Possible extensions include distributionally robust uncertainty sets, dynamic and data-driven uncertainties, multi-objective Pareto robust optimization, mixed-integer large-scale decomposition, and stochastic-robust hybrid models.

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