

Quasi-Equilibrium Problems and Fixed Point Theorems of the Sum of l.s.c. and u.s.c. Mappings

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We first formulate generalized quasi-equilibrium problems concerning multivalued mappings and give some sufficient conditions to the existence of their solutions. In particular, we establish several results on the existence of fixed points of the sum of l.s.c. and u.s.c. mappings. These generalize some well-known fixed point theorems obtained by previous authors as Ky Fan, F. E. Browder and Ky Fan, X. Wu, L. J. Lin, and Z. T. Yu, etc. Lastly, we apply the obtained results to quasi-equilibrium problems of types I, II and mixed quasi-equilibrium problems of these two types.

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1. Introduction

It is well-known that the fixed point theory plays an important role in applied mathematics. Many results in this theory become a useful tool to show the existence for solutions and to construct algorithms for finding solutions of many mathematical problems as optimization, variational inequality, complementarity, equilibrium... problems. We can shortly describe the main development of the fixed theory concerning continuous mappings as follows. In 1912, L. Brouwer used

combinatorial method to show that a continuous mapping f from a simplex $K \subset R^n$ into itself has a fixed point, i. e, there exists a point $\bar{x} \in K$ such that $\bar{x} = f(\bar{x})$. J. Schauder, 1930, extended this result to the case when K is a nonempty convex compact subset in R^n . In 1941, S. Kakutani generalized to the case when f is an upper semi-continuous mapping with nonempty convex and closed values from K to itself in R^n . In 1952, Ky Fan proved a fixed point theorem of upper semi-continuous mappings with nonempty convex and closed values from a nonempty convex and compact subset K into itself in Hausdorff topological locally convex spaces. In 1968 F.E. Browder and Ky Fan obtained a fixed point theorem of multivalued mapping which have lower open sections. Recently, many authors proved fixed point theorems of lower semi-continuous multivalued mappings with nonempty convex closed values, by using a continuous selection theorem, see for example, N. C. Yannelis and N. D. Prabhakar [18], C. D. Horvath [9], S. Park [12] and many others. In particular, in [16], Tan and Hoa obtain the following result:

Theorem A ([16]). *Let D be a nonempty convex and compact subset of a Hausdorff locally convex topological vector space X and $F: D \rightarrow 2^D$ be a lower semi-continuous multivalued mapping with nonempty values. Then there exists a point $\bar{x} \in D$ such that $\bar{x} \in F(\bar{x})$.*

In this paper, we are interested in the existence of quasi-equilibrium points of multivalued mappings defined on subsets of Hausdorff locally convex topological vector spaces as follows.

Let X, Y and Z be Hausdorff locally convex topological vector spaces over the reals, and $D \subset X$ and $K \subset Z$ nonempty subsets. Given multi-valued mappings $P: D \times K \rightarrow 2^D$, $Q: D \times K \rightarrow 2^K$ and $F: D \times K \rightarrow 2^Y$, we are interested in the problem of finding $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\bar{x} \in P(\bar{x}, \bar{y}), \quad \bar{y} \in Q(\bar{x}, \bar{y}), \quad 0 \in F(\bar{x}, \bar{y}).$$

This problem is called *generalized quasi-equilibrium problem* (in short, GQEP), in which the multivalued mappings P and Q are constraint mappings and F is a utility multivalued mapping that are often determined by equalities and inequalities, or, by inclusions and intersections of other multi-valued mappings, or by general relations in product spaces. The existence of solutions to this problem is studied in [5–7, 15,16] for the case the multivalued mapping P is continuous, the multivalued mapping Q is u.s.c. and the multivalued mapping F is u.s.c. All these mappings P, Q and F need to have nonempty convex and closed values.

It is clear that equilibrium problems are generalizations of variational inequalities and optimization problems, including also optimization-related problems such as fixed point, complementarity problems, Nash equilibrium, minimax problems, etc. For the last decade there have been a number of generalizations of these problems to different directions such as quasi-equilibrium problems with constraint sets depending on parameters, quasi-variational and quasi-equilibrium inclusion problems with multi-valued data (see, for examples, in [5-7], [15, 16]). Problem (GQEP) described above is quite general. It encompasses a large class of

problems of applied mathematics including quasi-optimization problems, quasi-variational inclusion, quasi-equilibrium problems, quasi-variational relation problems etc. Typical instances of (GQEP) are shown in [5, 6, 15, 16] involving upper semi-continuous utility multivalued mappings with nonempty convex closed values. We consider the case the multivalued mapping $F = G + H$, where $G, H: D \times K \rightarrow 2^X, G$ is l.s.c. and H is u.s.c. multivalued mapping. Theorem 3.1 below shows sufficient conditions for the existence of solutions to this problem of concerning the sum of lower and upper semi-continuous utility multivalued mappings. Corollary 3.4 unifies Ky Fan and Browder Ky Fan Theorems together. In particular, we obtain the following theorem: Let D be a nonempty convex and compact subset of Hausdorff locally convex topological vector space X and $G, H: D \rightarrow 2^D$ be multivalued mappings. In addition, assume that G is lower semi-continuous multivalued mapping with nonempty values and H is upper semi-continuous mapping with convex closed values. Then, $G + H$ has a fixed point in D .

2. Preliminaries and definitions

Assume that X, Y are real Hausdorff topological vector spaces and $D \subseteq X$ is a nonempty subset. We consider a multivalued mapping $F: D \rightarrow 2^Y$ and $F^{-1}: Y \rightarrow 2^X$ defined by $x \in F^{-1}(y)$ if and only if $y \in F(x)$. We recall that

(a) The *domain* and the *graph* of F are denoted, respectively, by

$$\text{dom}F = \{x \in D | F(x) \neq \emptyset\}, \quad \text{and} \quad \text{Gr}(F) = \{(x, y) \in D \times Y | y \in F(x)\}.$$

(b) F is said to be a *closed mapping* if the graph $\text{Gr}(F)$ of F is a closed subset in the product space $X \times Y$;

(c) F is said to be a *compact mapping* if the closure $\overline{F(D)}$ of its range $F(D)$ is a compact set in Y ;

(d) $F: D \rightarrow 2^Y$ is said to be *upper semi-continuous* (in short, u.s.c.) at $\bar{x} \in D$ if for each open set V containing $F(\bar{x})$, there exists an open set U of \bar{x} such that for each $x \in U, F(x) \subset V$. F is said to be u.s.c. on D if it is u.s.c. at all $x \in D$;

(e) F is said to be *lower semi-continuous* (in short, l.s.c.) at $\bar{x} \in D$ if for any open set V with $F(\bar{x}) \cap V \neq \emptyset$, there exists an open set U containing \bar{x} such that for each $x \in U, F(x) \cap V \neq \emptyset$. F is said to be l.s.c. on D if it is l.s.c. at all $x \in D$;

(f) F is said to be *continuous* on D if it is at the same time u.s.c. and l.s.c. on D ;

(g) F is said to have *open lower sections* if the inverse mapping F^{-1} is open valued, i.e. for all $y \in Y, F^{-1}(y)$ is open in X .

The following propositions and theorems are need in this paper. The proofs of them can be found in [16].

Proposition 2.1. $F: D \rightarrow 2^Y$ is l.s.c. at $x \in D, F(x) \neq \emptyset$, if and only if for any net $\{x_\alpha\}$ in D , with $x_\alpha \rightarrow x$ and $y \in F(x)$, there is a net $\{y_\alpha\}$ with $y_\alpha \in F(x_\alpha)$ such that $\{y_\alpha\}$ has a subnet $\{y_{\alpha_\beta}\}, y_{\alpha_\beta} \rightarrow y$.

By X^* we denote the dual space of X i.e.,

$$X^* = \{f: X \rightarrow R \mid f \text{ is a linear and continuous function}\}.$$

The pairing $\langle \cdot, \cdot \rangle$ between elements of $p \in X^*$ and $x \in X$ is defined by $\langle p, x \rangle = p(x)$. We have

Proposition 2.2. If $F: D \rightarrow 2^Y$ is a l.s.c. (u.s.c.) multivalued mapping with nonempty values on D and $p \in Y^*$, then the function $c_p: D \rightarrow R$, defined by $c_p(x) = \inf_{v \in F(x)} \langle p, v \rangle$ ($c_p(x) = \sup_{v \in F(x)} \langle p, v \rangle$) is upper semi-continuous on D .

Proposition 2.3. Let $F: D \rightarrow 2^Y$ be a multivalued mapping with nonempty values on D . Then, if F has open lower sections, then F is l.s.c. on D .

Remark 2.4. It is easy to give examples proving that a continuous mapping may not have open lower sections.

Conversely, we have the following proposition.

Proposition 2.5. Let $F: D \rightarrow 2^Y$ be a l.s.c. multivalued mapping with nonempty values on D . Then, for any open balanced subset V in Y , the multivalued mapping $F_V: D \rightarrow 2^Y$ defined by $F_V(x) = F(x) + V$ has open lower sections.

Remark 2.6. Proposition 2.5 implies that any l.s.c. multivalued mapping F can be approximated by a net $\{F_\alpha\}$ of multivalued mappings having open lower sections in the sense: For any net $\{x_\alpha\}$ in D with $x_\alpha \rightarrow x$, and $y \in F(x)$, one can find a net $y_\alpha \in F_\alpha(x_\alpha)$ with $y_\alpha \rightarrow y$.

Proposition 2.7. Let $F_i: D \rightarrow 2^Y, i = 1, 2$, be a l.s.c. multivalued mapping with nonempty values on D . Then, the multivalued mapping $F: D \rightarrow 2^Y$ defined by

$$F(x) = (F_1 + F_2)(x) = F_1(x) + F_2(x), x \in D,$$

is also l.s.c. on D .

Proposition 2.8. Let $F_i: D \rightarrow 2^Y, i = 1, 2$, be multivalued mappings with nonempty values on D . Assume that F_1 is l.s.c. and F_2 has open lower sections. Then, the multivalued mapping $F: D \rightarrow 2^Y$ defined by

$$F(x) = F_1(x) \cap F_2(x), x \in D,$$

is l.s.c. on D .

Proposition 2.9. Let $F: D \rightarrow 2^Y$ be a multivalued mapping with nonempty values on D . If F has open lower sections, then the multivalued mapping $coF: D \rightarrow 2^Y$, defined by $(coF)(x) = coF(x)$, with $co(A)$ denoting the convex hull of A , also has open lower sections.

Proposition 2.10. *Let $F: D \rightarrow 2^Y$ be a l.s.c. multivalued mapping with nonempty values on D . Then so is the multivalued mapping $coF: D \rightarrow 2^Y$, defined by $(coF)(x) = coF(x)$.*

The following results are very important in the proof of the main result in this paper.

Proposition 2.11. ([1, Lemma 2.92]) *Let $\{V_\alpha\}_{\alpha \in \Lambda}$ be an open cover of a compact Hausdorff space X . Then, there is finite family $\{\psi_\alpha\}_{\alpha \in \Lambda}$, of real functions such that*

- (i) $\psi_\alpha: X \rightarrow [0, 1]$ is continuous for each α ;
- (ii) ψ_i vanishes on $X \setminus V_\alpha$;
- (iii) $\sum_{\alpha \in \Lambda} \psi_\alpha(x) = 1$ for all $x \in X$.

That is, $\{\psi_\alpha\}_{\alpha \in \Lambda}$ is a continuous locally finite partition of unity subordinated to the open cover $\{V_\alpha\}_{\alpha \in \Lambda}$.

Proposition 2.12. (Sion's Minimax Theorem, [14, Corollary 3.3]) *Let M and N be convex spaces one of which is compact, and f a function on $M \times N$, quasi-concave-convex and u.s.c.-l.s.c. Then $\sup \inf f = \inf \sup f$.*

Proposition 2.13. ([5, Corollary 3.4]) *We assume the following conditions:*

- (i) D, K are nonempty convex compact sets;
- (ii) $P: D \times K \rightarrow 2^D$ is a continuous multivalued mapping with nonempty closed convex values, $Q: D \times K \rightarrow 2^K$ is a u.s.c. multivalued mapping with nonempty closed convex values;
- (iii) $\phi: K \times D \times D \rightarrow R$ is a continuous function such that for any fixed $(y, x) \in K \times D$, the function $\phi(y, x, \cdot): D \rightarrow R$ is quasiconvex and $\phi(y, x, x) = 0$.

Then, there exists $(\bar{x}, \bar{y}) \in D \times K$ such that $\bar{x} \in P(\bar{x}, \bar{y}), \bar{y} \in Q(\bar{x}, \bar{y})$ and

$$\phi(\bar{y}, \bar{x}, z) \geq 0 \text{ for all } z \in P(\bar{x}, \bar{y}).$$

3. Main results

Let X and Z be Hausdorff locally convex topological vector spaces over reals and $D \subset X, K \subset Z$ be nonempty subsets. Given multi-valued mappings $P: D \times K \rightarrow 2^D, Q: D \times K \rightarrow 2^K$ and $F: D \times K \rightarrow 2^X$, we are interested in sufficient conditions for the existence of solutions of the problem (GQEP): Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\bar{x} \in P(\bar{x}, \bar{y}), \quad \bar{y} \in Q(\bar{x}, \bar{y}), \quad 0 \in F(\bar{x}, \bar{y}).$$

Before proving the main results in this section, we recall the following notions. Let D be a subset in X and $x \in D$. The set

$$T_D(x) = \overline{\{\alpha(y - x), y \in D, \alpha \geq 0\}} = \overline{\{cone(D - x)\}},$$

is called the tangent cone to the set D at x , where $coneM = \{\alpha z, z \in M, \alpha \geq 0\}$.

We now prove the following theorem on the existence for solutions of the above generalized quasi-equilibrium problem.

Theorem 3.1. *We assume the following conditions:*

- (i) D, K are nonempty convex compact sets;
- (ii) $P: D \times K \rightarrow 2^D$ is a continuous multivalued mapping with nonempty closed convex values, $Q: D \times K \rightarrow 2^K$ is a u.s.c. multivalued mapping with nonempty closed convex values;
- (iii) $G: D \times K \rightarrow 2^X$ is a l.s.c. multivalued mapping with nonempty values;
- (iv) $H: D \times K \rightarrow 2^X$ is a u.s.c. multivalued mapping with nonempty closed convex values;
- (v) for any $(x, y) \in P(x, y) \times Q(x, y)$, $H(x, y)$ has nonempty convex closed values and $\emptyset \neq G(x, y) + H(x, y) \cap T_{P(x,y)}(x) \subset T_{P(x,y)}(x)$.

Then there exists $(\bar{x}, \bar{y}) \in D \times K$ such that

$$(1) \bar{x} \in P(\bar{x}, \bar{y}); \quad (2) \bar{y} \in Q(\bar{x}, \bar{y}); \quad (3) 0 \in G(\bar{x}, \bar{y}) + H(\bar{x}, \bar{y}).$$

Proof. We set

$$B = \{(x, y) \in D \times K \mid x \in P(x, y), y \in Q(x, y)\}.$$

Since the multivalued mapping $T: D \times K \rightarrow 2^{D \times K}$, defined by

$$T(x, y) = P(x, y) \times Q(x, y), \quad (x, y) \in D \times K,$$

is upper semi-continuous with nonempty convex and closed values, by using the Ky Fan's fixed point theorem, we conclude that T has a fixed point in $D \times K$. Therefore, B is a nonempty set. The upper semi-continuity and the closedness of values of T imply that B is a closed and then a compact set.

Assume that for any $(x, y) \in B$, $0 \notin G(x, y) + H(x, y)$. Take a point $\bar{v} \in G(x, y)$. Since $H(x, y)$ is a nonempty closed convex, so is $\bar{v} + H(x, y)$ and $0 \notin (\bar{v} + H(x, y))$. By the Hahn-Banach Theorem, there exists $p \in X^*$ such that

$$p(\bar{v}) + \sup_{w \in H(x,y)} p(w) < 0.$$

It follows: $\inf_{v \in \text{co}(G(x,y))} p(v) + \sup_{w \in H(x,y)} p(w) < 0$. Further, we define functions $c_p^1(.,.), c_p^2(.,.): D \times K \rightarrow R$ by

$$c_p^1(x', y') = \inf_{v \in \text{co}(G(x',y'))} p(v) \quad \text{and} \quad c_p^2(x', y') = \sup_{w \in H(x',y')} p(w).$$

By Proposition 2.2, the functions c_p^1 and c_p^2 are u.s.c. on $D \times K$, therefore, the set

$$U_p(x, y) = \{(x', y') \in D \times K \mid c_p^1(x', y') + c_p^2(x', y') < 0\},$$

is open. Since $(x, y) \in U_p(x, y)$, then is a nonempty open neighborhood of (x, y) .

Thus, for any $(x, y) \in B$ there is $p \in X^*$ such that

$$U_p(x, y) = \{(x', y') \in D \times K | c_p^1(x', y') + c_p^2(x', y') < 0\}$$

is nonempty and open and hence $\{U_p\}_{p \in X^*}$ is an open cover of B . Since B is compact, there exist finite $p_1, \dots, p_s \in X^*$ such that

$$B \subseteq \bigcup_{j=1}^s U_{p_j}.$$

Further, since B is closed in $D \times K$, $U_{p_0} = D \times K \setminus B$ is open in $D \times K$ and hence $\{U_{p_0}, U_{p_1}, \dots, U_{p_s}\}$ is an open cover of the compact set $D \times K$. By Proposition 2.11 there exist continuous functions $\psi_i: D \times K \rightarrow R$, $(i = 0, 1, \dots, s)$ such that

- (i) $0 \leq \psi_i(x, y) \leq 1$;
- (ii) $\sum_{i=1}^s \psi_i(x, y) = 1$, for all $(x, y) \in D \times K$;
- (ii) for any $i \in \{0, 1, \dots, s\}$, there exists $j(i) \in \{0, \dots, s\}$ such that $supp \psi_i \subset U_{p_{j(i)}}$.

Further, we define the function $\phi: K \times D \times D \rightarrow R$ by

$$\phi(y, x, t) = \sum_{i=0}^s \psi_i(x, y) \cdot p_{j(i)}(t - x), \text{ for any } t, x \in D, y \in K.$$

Then, ϕ is a continuous function on $K \times D \times D$. Moreover, for any fixed $(x, y) \in D \times K$, $\phi((y, x, \cdot)): D \rightarrow R$ is an affine function and $\phi(y, x, x) = 0$ for all $(x, y) \in D \times K$. Therefore, D, K, P, Q and ϕ satisfy all conditions of Proposition 2.13. It implies that there is $(\bar{x}, \bar{y}) \in D \times K$ such that $(\bar{x}, \bar{y}) \in P(\bar{x}, \bar{y}) \times Q(\bar{x}, \bar{y})$ and $\phi((\bar{y}, \bar{x}, t) \geq 0$, for all $t \in P(\bar{x}, \bar{y})$. This gives

$$\sum_{i=0}^s \psi_i(\bar{x}, \bar{y}) \cdot p_{j(i)}(t - \bar{x}) \geq 0, \text{ for all } t \in P(\bar{x}, \bar{y}). \tag{3.1}$$

Setting $p^* = \sum_{i=0}^s \psi_i(\bar{x}, \bar{y}) \cdot p_{j(i)}$, we get from (3.1), $p^*(t - \bar{x}) \geq 0$, for all $t \in P(\bar{x}, \bar{y})$ and hence $p^*(u) \geq 0$, for all $u \in T_{P(\bar{x}, \bar{y})}(\bar{x})$. By assumption (v),

$$\emptyset \neq G(x, y) + H(x, y) \cap T_{P(x,y)}(x) \subset T_{P(x,y)}(x),$$

so we conclude that

$$p^*(v) + p^*(w) \geq 0, \text{ for all } v \in coG(\bar{x}, \bar{y}), w \in H(\bar{x}, \bar{y}) \cap T_{P(\bar{x}, \bar{y})}(\bar{x}).$$

To simplify the presentation we define $A := coG(\bar{x}, \bar{y}) + H(\bar{x}, \bar{y}) \cap T_{P(\bar{x}, \bar{y})}$. This yields

$$\inf_{v+w \in A} p^*(v + w) \geq 0. \tag{3.2}$$

Further, put $I(\bar{x}, \bar{y}) = \{i \in \{0, 1, \dots, s\} \mid \psi_i(\bar{x}, \bar{y}) > 0\}$. Since $\psi_i(\bar{x}, \bar{y}) \geq 0$ and $\sum_{i=0}^s \psi_i(\bar{x}, \bar{y}) = 1$, we deduce $I(\bar{x}, \bar{y}) \neq \emptyset$. So, for any $i \in I(\bar{x}, \bar{y})$, $(\bar{x}, \bar{y}) \in \text{supp } \psi_i \subset U_{p_{j(i)}}$ and

$$c_{p_{j(i)}}^1(\bar{x}, \bar{y}) + c_{p_{j(i)}}^2(\bar{x}, \bar{y}) < 0. \quad (3.3)$$

For any $v \in \text{co}G(\bar{x}, \bar{y})$, $w \in H(\bar{x}, \bar{y}) \cap T_{P(\bar{x}, \bar{y})}(\bar{x})$ we have

$$\begin{aligned} p^*(v+w) &= \sum_i^s \psi_i(\bar{x}, \bar{y}) \cdot (p_{j(i)}(v) + p_{j(i)}(w)) = \sum_{i \in I(\bar{x}, \bar{y})} \psi_i(\bar{x}, \bar{y}) \cdot (p_{j(i)}(v) + p_{j(i)}(w)) \\ &\leq \sum_{i \in I(\bar{x}, \bar{y})} \psi_i(\bar{x}, \bar{y}) \max_{i \in I(\bar{x}, \bar{y})} (p_{j(i)}(v) + p_{j(i)}(w)) = \max_{i \in I(\bar{x}, \bar{y})} (p_{j(i)}(v) + p_{j(i)}(w)). \end{aligned}$$

Hence,

$$\inf_{v+w \in A} (p^*(v+w)) \leq \inf_{v+w \in A} \max_{i \in I(\bar{x}, \bar{y})} (p_{j(i)}(v) + p_{j(i)}(w)). \quad (3.4)$$

Setting $M = \overline{\text{co}}\{p_{j(1)}, \dots, p_{j(s)}\}$, $N = \text{co}G(\bar{x}, \bar{y}) + H(\bar{x}, \bar{y})$, $f(p, u) = p(v) + p(w)$, $u = v + w$ and using the weak* topology on X^* , we can easily verify that all conditions of Proposition 2.12 (Sion's minimax theorem) are satisfied. Therefore, we obtain

$$\inf_{u \in A} \max_{i \in I(\bar{x}, \bar{y})} p_{j(i)}(u) = \max_{i \in I(\bar{x}, \bar{y})} \left\{ \inf_{v+w \in A} (p_{j(i)}(v) + p_{j(i)}(w)) \right\} \quad (3.5)$$

$$\leq \max_{i \in I(\bar{x}, \bar{y})} \left\{ \inf_{v \in \text{co}G(\bar{x}, \bar{y})} p_{j(i)}(v) + \sup_{w \in H(\bar{x}, \bar{y})} p_{j(i)}(w) \right\} \leq \max_{i \in I(\bar{x}, \bar{y})} \{c_{p_{j(i)}}^1(\bar{x}, \bar{y}) + c_{p_{j(i)}}^2(\bar{x}, \bar{y})\} < 0.$$

A combination of (3.4) and (3.5) implies $\inf_{u \in A} p^*(u) < 0$. Thus, we have a contradiction to (3.2). This completes the proof of the theorem. \square

In particular, we obtain the following fixed point result.

Corollary 3.2. *We assume that the following conditions hold:*

- (i) D, K are nonempty convex compact sets;
- (ii) $P: D \times K \rightarrow 2^D$ is a continuous multivalued mapping with nonempty closed convex values;
- (iii) $Q: D \times K \rightarrow 2^K$ is a u.s.c. multivalued mapping with nonempty closed convex values;
- (iv) $G: D \times K \rightarrow 2^X$ is a l.s.c. multivalued mapping with nonempty values;
- (v) $H: D \times K \rightarrow 2^X$ is u.s.c. multivalued mapping with nonempty convex and closed values;
- (vi) $\emptyset \neq G(x, y) - x + H(x, y) \cap T_{P(x, y)}(x) \subset T_{P(x, y)}(x) \quad \forall (x, y) \in P(x, y) \times Q(x, y)$.

Then there exists $(\bar{x}, \bar{y}) \in D \times K$ such that

$$(1) \quad \bar{x} \in P(\bar{x}, \bar{y}), \quad (2) \quad \bar{y} \in Q(\bar{x}, \bar{y}), \quad (3) \quad \bar{x} \in G(\bar{x}, \bar{y}) + H(\bar{x}, \bar{y}).$$

Proof. We define the multivalued mapping $G^*: D \times K \rightarrow 2^X$ by

$$G^*(x, y) = G(x, y) - x, \quad (y, x) \in D \times K.$$

Observe that in consequence of Proposition 2.5 the multivalued mapping G^* is l.s.c. with $G^*(x, y) \neq \emptyset$ for any $(x, y) \in D \times K$. Further, the proof of this corollary follows immediately from Theorem 3.1. \square

Corollary 3.3. *We assume the following conditions:*

- (i) D, K are nonempty convex compact sets;
- (ii) $P: D \times K \rightarrow 2^D$ is a continuous multivalued mapping with nonempty closed convex values;
- (iii) $Q: D \times K \rightarrow 2^K$ is a u.s.c. multivalued mapping with nonempty closed convex values;
- (iv) $G: D \times K \rightarrow 2^X$ is a l.s.c. multivalued mapping with nonempty values;
- (v) $H: D \times K \rightarrow 2^X$ is u.s.c. multivalued mapping with convex and closed values;
- (vi) for any $(x, y) \in P(x, y) \times Q(x, y)$, $x \notin G(x, y) + H(x, y)$, and $\emptyset \neq G(x, y) - x + H(x, y) \cap T_{P(x,y)}(x) \subset T_{P(x,y)}(x)$.

Then there exists $(\bar{x}, \bar{y}) \in D \times K$ such that

$$(1) \bar{x} \in P(\bar{x}, \bar{y}), \quad (2) \bar{y} \in Q(\bar{x}, \bar{y}), \quad (3) G(\bar{x}, \bar{y}) + H(\bar{x}, \bar{y}) = \emptyset.$$

Proof. We assume that $G(x, y) + H(x, y) \neq \emptyset$, for all $(x, y) \in P(x, y) \times Q(x, y)$. It follows that $G(x, y), H(x, y) \neq \emptyset$ and for any $(x, y) \in P(x, y) \times Q(x, y)$, $G(x, y) - x + H(x, y) \cap T_{P(x,y)}(x) \subset T_{P(x,y)}(x)$. Applying Corollary 3.2, we conclude that there exists a point $(\bar{x}, \bar{y}) \in D \times K$ with $\bar{x} \in P(\bar{x}, \bar{y}), \bar{y} \in Q(\bar{x}, \bar{y})$ and $\bar{x} \in G(\bar{x}, \bar{y}) + H(\bar{x}, \bar{y})$ and we have a contradiction. \square

In particular, we have the fixed point result of lower semi-continuous multivalued mapping with nonempty values. This is a generalization of Browder-Ky Fan's Theorem.

Corollary 3.4. *We assume the following conditions:*

- (i) D is nonempty convex compact subset of X ;
- (ii) $G_0, H_0: D \rightarrow 2^D$ are multivalued mappings with nonempty values; G_0 is a l.s.c. mapping and H_0 is a u.s.c. multivalued mapping with convex closed values.
- (iii) $G_0(x) + H_0(x) \cap T_D(x) - x \subset T_D(x)$, for any $x \in D$.

Then there exists $\bar{x} \in D$ such that $\bar{x} \in G_0(\bar{x}) + H_0(\bar{x})$.

Proof. The proof of this corollary follows immediately from Corollary 3.2 with taking $P(x, y) = Q(x, y) = D$, $G(x, y) = G_0(x), H(x, y) = H_0(x)$, for all $(x, y) \in D \times K$. \square

Remark 3.5. Corollary 3.4 unifies Ky Fan's Theorem and Browder-Ky Fan's Theorem. To get Ky Fan's Theorem, we take $G_0(x) = 0$ for $x \in D$. To get Browder-Ky Fan's Theorem, we take $H_0(x) = 0$ for $x \in D$.

4. Some applications

In what follows, $D \subset X$ and $K \subset Z$ are nonempty subsets. Given multivalued mappings $P: D \times K \rightarrow 2^D$, $Q: D \times K \rightarrow 2^K$ and $F_1: K \times D \times D \times D \rightarrow 2^Y$, we are interested in finding $(\bar{x}, \bar{y}) \in D \times K$ such that

- (i) $\bar{x} \in P(\bar{x}, \bar{y})$;
- (ii) $\bar{y} \in Q(\bar{x}, \bar{y})$;
- (iii) $0 \in F_1(\bar{y}, \bar{x}, \bar{x}, t)$, for all $t \in P(\bar{x}, \bar{y})$.

This problem is called *general quasi-equilibrium problem of type I*, denoted by $(\text{GEP})_{\text{I}}$, in which the multivalued mappings P, Q are called constraint mappings and F_1 is called a utility multivalued mapping.

Next, given multivalued mappings $P, P_0: D \times K \rightarrow 2^D$, $Q: D \times K \rightarrow 2^K$ and $Q_0: D \times D \times K \rightarrow 2^K$ and $F_2: K \times K \times D \times D \rightarrow 2^Y$, we are interested in finding $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\begin{aligned} \bar{x} \in P(\bar{x}, \bar{y}), \quad \bar{y} \in Q(\bar{x}, \bar{y}), \quad \text{and} \\ 0 \in F_2(\bar{y}, v, \bar{x}, t), \text{ for all } t \in P_0(\bar{x}, \bar{y}) \text{ and } v \in Q_0(\bar{x}, t, \bar{y}). \end{aligned}$$

This problem is called *general quasi-equilibrium problem of type II*, denoted by $(\text{GEP})_{\text{II}}$, in which the multivalued mappings P, P_0 and Q, Q_0 are called constraint mappings and F_2 is called a utility multivalued mapping. The problem of finding $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\begin{aligned} \bar{x} \in P(\bar{x}, \bar{y}), \quad \bar{y} \in Q(\bar{x}, \bar{y}), \quad 0 \in F_1(\bar{y}, \bar{x}, \bar{x}, t) \text{ for all } t \in P(\bar{x}, \bar{y}), \text{ and} \\ 0 \in F_2(\bar{y}, v, \bar{x}, t) \text{ for all } t \in P_0(\bar{x}, \bar{y}) \text{ and } v \in Q_0(\bar{x}, t, \bar{y}). \end{aligned}$$

is called *mixed general quasi-equilibrium problem*, in which the multivalued mappings P, P_0 and Q, Q_0 are called *constraint mappings* and F_1, F_2 are called *utility multivalued mappings*.

We apply the results obtained in Section 3 to get the existence of solutions for $(\text{GEP})_{\text{I}}$, $(\text{GEP})_{\text{II}}$ and for mixed general quasi-equilibrium problems. But, we first prove some propositions to be used later.

Proposition 4.1. *We assume the following conditions:*

- (i) D and K are nonempty convex compact subsets;
- (ii) $P: D \times K \rightarrow 2^D$ is a continuous multivalued mapping with nonempty closed values;
- (iii) $Q: D \times K \rightarrow 2^K$ is a u.s.c. multivalued mapping with nonempty closed values;
- (iv) the set $A = \{(y, x, z, t) \in K \times D \times D \times D \mid 0 \in F_1(y, x, z, t)\}$ is closed ;
- (v) for any fixed $(y, x) \in K \times D$, the set

$$B = \{z \in P(x, y) \mid 0 \in F_1(y, x, z, t) \text{ for all } t \in Q(x, y)\}$$

is nonempty and convex.

Then, the multivalued mapping $H: D \times K \rightarrow 2^D$ defined by

$$H(x, y) = \{z \in P(x, y) \mid 0 \in F_1(y, x, z, t) \text{ for all } t \in P(x, y)\}$$

is u.s.c. with nonempty convex closed values on $D \times K$.

Proof. Condition (v) implies that $H(x, y) \neq \emptyset$ and nonempty convex for any $(x, y) \in D \times K$. Now, we show that H is u.s.c. on $D \times K$. Since D is compact and H has closed values, therefore, we need only to prove that H is a closed mapping. Indeed, let (x_α, y_α) be a net converging to (x, y) and z_α be a net with $z_\alpha \in H(x_\alpha, y_\alpha), z_\alpha \rightarrow z$. We have to show $z \in H(x, y)$. Indeed, we can see $0 \in F_1(y_\alpha, x_\alpha, z_\alpha, t_\alpha)$, for all $t_\alpha \in P(x_\alpha, y_\alpha)$. Let $v \in P(x, y)$ be arbitrary. Since P is l.s.c., there is $t_\alpha \in P(x_\alpha, y_\alpha), t_\alpha \rightarrow t$. Hence, we get $0 \in F_1(y_\alpha, x_\alpha, z_\alpha, t_\alpha)$. For $(y_\alpha, x_\alpha, z_\alpha, t_\alpha) \rightarrow (y, x, z, t)$ and the set A is closed, we deduce $(y, x, z, t) \in A$. Hence, $0 \in F_1(y, x, z, t)$, for all $t \in P(x, y)$. This shows that the multivalued mapping H is closed, and then H is u.s.c. on $D \times K$. \square

The following corollary on the existence to solutions of $(\text{GEP})_I$ generalizes Theorems 3.1 in [5].

Corollary 4.2. $(\text{GEP})_I$ has a solution if D, K, P, Q , and F_1 satisfy all conditions of Proposition 4.1.

Proof. It follows immediately from Corollary 3.2 with $G = 0$ that there exists $(\bar{x}, \bar{y}) \in D \times K$ such that (1) $\bar{x} \in P(\bar{x}, \bar{y})$, (2) $\bar{y} \in Q(\bar{x}, \bar{y})$, and (3) $\bar{x} \in H(\bar{x}, \bar{y})$. This gives $0 \in F_1(\bar{y}, \bar{x}, \bar{x}, t)$ for all $t \in P(\bar{x}, \bar{y})$. \square

Proposition 4.3. We assume the following conditions:

- (i) D is a nonempty convex compact subset;
- (ii) $P: D \times K \rightarrow 2^D$ is a multivalued mapping with a nonempty closed fixed set $B = \{(x, y) \in D \times K \mid x \in P(x, y), y \in Q(x, y)\}$;
- (iii) $P_0: D \times K \rightarrow 2^D$ is a multivalued mapping having open lower sections and $\text{co}(P_0(x, y))$ is contained in $P(x, y)$ for each $(x, y) \in D \times K$;
- (iv) for any fixed $t \in D$ the set

$$A = \{(x, y) \in D \times K \mid 0 \notin F_2(y, v, x, t) \text{ for some } v \in Q_0(x, t, y)\}$$

is open in D .

Then, the multivalued mapping $G: D \times K \rightarrow 2^D$ defined by

$$G(x, y) = \{t \in P_0(x, y) \mid 0 \notin F_2(y, v, x, t) \text{ for some } v \in Q_0(x, t, y)\},$$

where $(x, y) \in D \times K$, is l.s.c. on $D \times K$.

Proof. We can write

$$G(x, y) = \{t \in D \mid 0 \notin F_2(y, v, x, t) \text{ for some } v \in Q_0(x, t, y)\} \cap P_0(x, y).$$

For any $t \in D$ the set

$$\begin{aligned} G^{-1}(t) &= \{(x, y) \in D \times K \mid 0 \notin F_2(y, v, x, t) \text{ for some } v \in Q_0(x, t, y)\} \cap P_0^{-1}(t) \\ &= A \cap P_0^{-1}(t), \end{aligned}$$

is open. Hence, G has open lower sections, so G is l.s.c. \square

Now, we assume that for any fixed $(x, y) \in D \times K$ the multivalued mapping $Q_0(x, \cdot, y): D \rightarrow 2^K$ is l.s.c. and the multivalued mapping $F_2(\cdot, \cdot, \cdot, t): D \times K \rightarrow 2^Y$ is closed.

Proposition 4.4. *We assume the following conditions:*

- (i) D is a nonempty convex compact subset;
- (ii) $P: D \times K \rightarrow 2^D$ is a multivalued mapping with a nonempty closed fixed set $B = \{(x, y) \in D \times K \mid x \in P(x, y), y \in Q(x, y)\}$;
- (iii) $P_0: D \times K \rightarrow 2^D$ is a multivalued mapping having open lower sections and $\text{co}(P_0(x, y))$ is contained in $P(x, y)$ for each $(x, y) \in D \times K$;
- (iv) for any fixed $t \in D$, the multivalued mapping $Q_0(\cdot, t, \cdot): D \times K \rightarrow 2^K$ is l.s.c. and the multivalued mapping $F_2(\cdot, \cdot, \cdot, t): K \times K \times D \rightarrow 2^Y$ is closed.

Then, the multivalued mapping $G: D \times K \rightarrow 2^D$ defined by

$$G(x, y) = \{t \in P_0(x, y) \mid 0 \notin F_2(y, v, x, t) \text{ for some } v \in Q_0(x, t, y)\},$$

where $(x, y) \in D \times K$, is l.s.c. on $D \times K$.

Proof. For given $t \in D$, we set

$$C = \{(x, y) \in D \times K \mid 0 \in F_2(y, v, x, t) \text{ for all } v \in Q_0(x, t, y)\}$$

and we show that this set is closed in $D \times K$. Indeed, let (x_α, y_α) be a net in C , with $x_\alpha \rightarrow x, y_\alpha \rightarrow y$. Take $v \in Q_0(x, t, y)$ arbitrary. Since $Q_0(\cdot, t, \cdot)$ is l.s.c. at (x, y) , we can find $v_\alpha \in Q_0(x_\alpha, t, y_\alpha)$ such that $v_\alpha \rightarrow v$. Since $(x_\alpha, y_\alpha) \in C$, we have $0 \in F_2(y_\alpha, v_\alpha, x_\alpha, t)$. The closedness of $F_2(\cdot, \cdot, \cdot, t)$ follows $0 \in F_2(y, v, x, t)$. Therefore, $0 \in F_2(y, v, x, t)$ for all $v \in Q_0(x, t, y)$ and so $(x, y) \in C$ which shows that C is a closed set. This implies that the set

$$A = D \setminus C = \{(x, y) \in D \times K \mid 0 \notin F_2(y, v, x, t) \text{ for some } v \in Q_0(x, t, y)\}$$

is open in D . Thus, Condition (iv) of Proposition 4.3 is satisfied. Therefore, to complete the proof of the corollary, it remains to apply Proposition 4.3. \square

The following corollary extends Theorem 2.1 in [6].

Corollary 4.5. *(GEP)_{II} has a solution if D, K, P, P_0, Q, Q_0 , and F_2 satisfy all conditions of Proposition 4.3 or Proposition 4.4 and $0 \in F_2(y, v, x, x)$ for any $y \in K$ and $v \in Q(x, x, y)$.*

Proof. It follows immediately from Corollary 3.3, with $H = 0$ and G defined from Proposition 4.3 or Proposition 4.4, that there exists $(\bar{x}, \bar{y}) \in D \times K$ such that

(1) $\bar{x} \in P(\bar{x}, \bar{y})$, (2) $\bar{y} \in Q(\bar{x}, \bar{y})$ and (3) $G(\bar{x}, \bar{y}) = \emptyset$. This gives $0 \in F_2(\bar{y}, v, \bar{x}, t)$ for all $t \in P_0(\bar{x}, \bar{y})$ and $v \in Q_0(\bar{x}, t, \bar{y})$. \square

Next, we apply Corollary 3.5 to prove the existence of solutions to mixed general quasi-equilibrium problems of $(\text{GEP})_I$ and $(\text{GEP})_{II}$. The following theorem is a generalization of Theorem 3.1 in [4].

Theorem 4.6. *We assume the following conditions:*

- (i) D and K are nonempty convex compact subsets;
- (ii) $P: D \times K \rightarrow 2^D$ is a continuous multivalued mapping with nonempty convex closed values;
- (iii) $Q: D \times K \rightarrow 2^K$ is a u.s.c. multivalued mapping with nonempty convex closed values;
- (iv) the set $A = \{(y, x, z, t) \in K \times D \times D \times D \mid 0 \in F_1(y, x, z, t)\}$ is closed ;
- (v) for any fixed $(y, x) \in K \times D$, the set

$$B = \{z \in P(x, y) \mid 0 \in F_1(y, x, z, t) \text{ for all } t \in Q(x, y)\}$$

is nonempty and convex;

- (vi) $P_0: D \times K \rightarrow 2^D$ is a multivalued mapping having open lower sections and for each $(x, y) \in P_0(x, y) \times Q(x, y)$, one has

$$0 \in F_1(y, x, x, t) \text{ for all } t \in P(x, y);$$

- (vii) $Q_0: D \times D \times K \rightarrow 2^K$ and $F_2: K \times K \times D \times D \rightarrow 2^Y$ are multivalued mappings such that for any fixed $t \in D$ the set

$$C = \{(x, y) \in D \times K \mid 0 \notin F_2(y, v, x, t) \text{ for some } v \in Q_0(x, t, y)\}$$

is open in D .

- (viii) $0 \in F_2(y, v, x, x)$ for all $(x, y) \in P(x, y) \times Q(x, y), v \in Q_0(x, x, y)$.

Then there exists $(\bar{x}, \bar{y}) \in D \times K$ such that

- (1) $\bar{x} \in P(\bar{x}, \bar{y})$, (2) $\bar{y} \in Q(\bar{x}, \bar{y})$, (3) $0 \in F_1(\bar{y}, \bar{x}, \bar{x}, t)$ for all $t \in P(\bar{x}, \bar{y})$, and (4) $0 \in F_2(\bar{y}, v, \bar{x}, t)$ for all $t \in P_0(\bar{x}, \bar{y})$ and $v \in Q_0(\bar{x}, t, \bar{y})$.

Proof. We define the multivalued mappings $H, G: D \times K \rightarrow 2^D$ by

$$H(x, y) = \{z \in P(x, y) \mid 0 \in F_1(y, x, z, t) \text{ for all } t \in P(x, y)\},$$

$$G(x, y) = \{t \in P_0(x, y) \mid 0 \notin F_2(y, v, x, t) \text{ for some } v \in Q_0(x, t, y)\},$$

for $(x, y) \in D \times K$. Propositions 4.1 and 4.2 show that H is u.s.c. with nonempty convex closed values and G is l.s.c. on $D \times K$. Observe that if for some $(\bar{x}, \bar{y}) \in H(\bar{x}, \bar{y}) \times Q(\bar{x}, \bar{y})$ we have $G(\bar{x}, \bar{y}) \cap P_0(\bar{x}, \bar{y}) = \emptyset$, then (i) $\bar{x} \in P(\bar{x}, \bar{y})$, (ii) $\bar{y} \in Q(\bar{x}, \bar{y})$, (iii) $0 \in F_1(\bar{y}, \bar{x}, \bar{x}, t)$ for all $t \in P(\bar{x}, \bar{y})$, and $0 \in F_2(\bar{y}, v, \bar{x}, t)$ for all

$t \in P_0(\bar{x}, \bar{y})$ and $v \in Q_0(\bar{x}, t, \bar{y})$ – and hence the proof of the theorem is complete. Thus, our aim is to show the existence of such a point (\bar{x}, \bar{y}) .

Arguing by contradiction, we assume that $(x, y) \in H(x, y) \times Q(x, y)$ implies $G(x, y) \cap P_0(x, y) \neq \emptyset$. We consider the multivalued mapping $T: D \times K \rightarrow 2^{D \times K}$ defined by

$$T(x, y) = \begin{cases} G(x, y) \cap P_0(x, y) \times \{y\}, & \text{if } x \in H(x, y), \\ P_0(x, y) \times \{y\}, & \text{else.} \end{cases}$$

We show that T verifies the hypotheses of Theorem A of Section 1. Apply this corollary, we conclude that there is a point $(\bar{x}, \bar{y}) \in D \times K$ such that $\bar{x} \in P(\bar{x}, \bar{y})$, $\bar{y} \in Q(\bar{x}, \bar{y})$ and $(\bar{x}, \bar{y}) \in T(\bar{x}, \bar{y})$. If $\bar{x} \in H(\bar{x}, \bar{y})$, then $\bar{x} \in G(\bar{x}, \bar{y}) \cap P_0(\bar{x}, \bar{y})$ and hence $0 \notin F_2(\bar{y}, v, \bar{x}, \bar{x})$ for some $v \in Q_0(\bar{x}, \bar{x}, \bar{y})$, which stands in contradiction to Condition (viii). If $\bar{x} \notin H(\bar{x}, \bar{y})$, then $(\bar{x}, \bar{y}) \in T(\bar{x}, \bar{y}) = P_0(\bar{x}, \bar{y}) \times \{\bar{y}\}$ and then by Condition (vi), $\bar{x} \in H(\bar{x}, \bar{y})$, which is also a contradiction. \square

Remark 4.7. We can understand Theorem 4.6 as solving $(\text{GEP})_{\text{II}}$ on the set of solutions of $(\text{GEP})_{\text{I}}$.

In particular, for the scalar case, we consider the functions $\phi_1: K \times D \times D \times D \rightarrow R$ and $\phi_2: K \times K \times D \times D \rightarrow R$.

Corollary 4.8. *We assume the following conditions:*

- (i) D and K are nonempty convex compact subsets;
- (ii) $P: D \times K \rightarrow 2^D$ is a continuous multivalued mapping with nonempty convex closed values;
- (iii) $Q: D \times K \rightarrow 2^K$ is a u.s.c. multivalued mapping with nonempty convex closed values;
- (iv) the set $A = \{(y, x, z, t) \in K \times D \times D \times D \mid \phi_1(y, x, z, t) \leq 0\}$ is closed;
- (v) for any fixed $(y, x) \in K \times D$ the set $B = \{z \in P(x, y) \mid \phi_1(y, x, z, t) \leq 0 \text{ for all } t \in Q(x, y)\}$ is nonempty and convex;
- (vi) $P_0: D \times K \rightarrow 2^D$ is a multivalued mapping having open lower sections and for each $(x, y) \in P_0(x, y) \times Q(x, y)$, one has $\phi_1(y, x, x, t) \geq 0$ for all $t \in P(x, y)$;
- (vii) $Q_0: D \times D \times K \rightarrow 2^K$ and $\phi_2: K \times K \times D \times D \rightarrow 2^Y$ are mappings such that for any fixed $t \in D$ the set

$$C = \{(x, y) \in D \times K \mid \phi_2(y, v, x, t) < 0 \text{ for some } v \in Q_0(x, t, y)\}$$

is open in D .

- (viii) $\phi_2(y, v, x, x) \geq 0$ for all $(x, y) \in P(x, y) \times Q(x, y)$ and $v \in Q_0(x, x, y)$.

Then there exists $(\bar{x}, \bar{y}) \in D \times K$ such that

- (1) $\bar{x} \in P(\bar{x}, \bar{y})$, (2) $\bar{y} \in Q(\bar{x}, \bar{y})$, (3) $\phi_1(\bar{y}, \bar{x}, \bar{x}, t) \leq 0$ for all $t \in P(\bar{x}, \bar{y})$,
and (4) $\phi_2(\bar{y}, v, \bar{x}, t) \geq 0$ for all $t \in P_0(\bar{x}, \bar{y})$ and $v \in Q_0(\bar{x}, t, \bar{y})$.

Proof. We define the mappings $F_1: K \times D \times D \times D \rightarrow 2^R$ and $F: K \times K \times D \times D \rightarrow 2^R$ by

$$F_1(y, x, z, t) = \phi_1(y, x, z, t) + R_+ \quad \text{and} \quad F_2(y, v, x, t) = \phi(y, v, x, t) - R_+.$$

We see that the set

$$\begin{aligned} A &= \{(y, x, z, t) \in K \times D \times D \times D \mid 0 \in F_1(y, x, z, t)\} \\ &= \{(y, x, z, t) \in K \times D \times D \times D \mid \phi_1(y, x, z, t) \leq 0\} \end{aligned}$$

is closed and the set

$$\begin{aligned} B &= \{z \in P(x, y) \mid 0 \in \phi_1(y, x, z, t) \leq 0 \text{ for all } t \in Q(x, y)\} \\ &= \{z \in P(x, y) \mid 0 \in F_1(y, x, z, t) \text{ for all } t \in Q(x, y)\} \end{aligned}$$

is nonempty and convex. $(x, y) \in P_0(x, y) \times Q(x, y)$, $0 \in F_1(y, x, x, t)$ for all $t \in P(x, y)$ if and only if $(x, y) \in P_0(x, y) \times Q(x, y)$ and one has $\phi_1(y, x, x, t) \leq 0$ for all $t \in P(x, y)$. Further, the set

$$\begin{aligned} C &= \{(x, y) \in D \times K \mid 0 \notin F_2(y, v, x, t) \text{ for some } v \in Q_0(y, x, t)\} \\ &= \{(x, y) \in D \times K \mid \phi_2(y, v, x, t) < 0 \text{ for some } v \in Q(x, y)\} \end{aligned}$$

is open in $D \times K$ and $\phi_2(y, v, x, x) \geq 0$ for all $(x, y) \in P(x, y) \times Q(x, y)$, $v \in Q_0(x, x, y)$ implies that $0 \in F_2(y, v, x, x)$ for all $(x, y) \in P(x, y) \times Q(x, y)$, $v \in Q_0(x, x, y)$. Therefore, to complete the proof of the corollary, it remains to apply Theorem 4.6. \square

It follows from the following remark that Corollary 4.8 generalizes Ky Fan's result in [8] and Minty's result in [11].

Remark 4.9. (1) In the case when ϕ_1 is a lower semi-continuous function and for any fixed $(y, x, t) \in K \times D \times D$ the function $\phi_1(y, x, \cdot, t): D \rightarrow R$ is a quasi-convex function, then conditions (vi) and (v) are satisfied.

(2) In the case when $Q_0: D \times D \times K \rightarrow 2^K$ is a lower semi-continuous mapping and $\phi_2: K \times K \times D \times D \rightarrow 2^Y$ is a function such that for any fixed $t \in D$ the function $\phi_2(\cdot, \cdot, \cdot, t): K \times D \times D \rightarrow R$ is upper semi-continuous, then condition (vii) is satisfied.

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