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A Saddle-Point Variational Model for Dynamic Production Planning Under Uncertainty

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Abstract

For Uncertain dynamic production planning problems, there must be efficient optimization methods that can deal with the changing state of the system, uncertain disturbances, and equilibrium relations. This paper presents a saddle-point variational framework to optimize dynamic production plans based on minimax optimization in combination with variational inequality formulations. In the saddle-point problem formulation, the uncertain production process becomes a convex–concave problem with production strategies minimizing costs and disturbance uncertainties. Equilibrium relations are then converted into the variational inequalities problem using the monotone operator approach. Existence of saddle-point solutions is proved using the minimax theorem, uniqueness follows from strong monotonicity properties, and convergence of the projected iterative algorithm is established via proper operator properties. An example of a dynamic production planning problem with multiple production lines, changing demands, and restrictions is taken into account in order to perform numerical experiments. Numerical results show that the saddle-point variational iteration method converges faster with only 41 iterations and a lower cost value of 104.2 than the gradient descent-ascent method and penalty method.

Keywords*:* Saddle-point optimization, variational inequality, minimax theory, dynamic production planning, monotone operator, uncertainty optimization, equilibrium analysis.

1. Introduction

Modeling of interactions between decision variables and uncertain disturbances using minimax optimization in an uncertain environment can be done using a solid theoretical framework. The extension of minimax optimization to dynamic cases would need saddle point representation to account for dynamic behavior in equilibrium and uncertainties in the system. Mathematical frameworks using variational inequalities have been shown to be powerful tools in analyzing properties like existence of equilibrium, stability, and convergence in such systems (Wood & Dall'Anese, 2023). However, minimax and stochastic optimization models available today deal mainly

with static or particular types of equilibrium cases (Li & Li, 2025; Yue et al., 2023).

Uncertain dynamic production planning requires the application of appropriate mathematical optimization schemes that could accommodate changing demand, resource constraints, and dynamic operational processes. Deterministic mathematical techniques, despite their high efficiency in stable system conditions, often fail to deliver appropriate solutions to the problems related to uncertainty, thus making stochastic and variational mathematical optimization frameworks necessary. Stochastic variational inequality problems have been studied for the formulation of equilibrium problems under uncertainty, offering a solid mathematical basis for modeling interaction among decision variables and uncertainties (Jiang et al., 2019). Recent stochastic equilibrium problems provide further evidence of the applicability of mathematical optimization in uncertain conditions (Li & Li, 2025). Heuristic scheduling methods, although useful for enhancing the performance of practical optimization processes, do not offer theoretical support for convergence and stability (Talwar & Shah, 2023). The formulations of uncertainty-base decision-making processes and robust optimization have been suggested in terms of stochastic saddle-point problems (Wood & Dall'Anese, 2023). The example of variational forecasting reveals the importance of uncertainty modeling in dynamic systems (Li et al., 2021).

The saddle-point dynamics may serve for adaptive resource allocation and dynamic distributed optimization (Yue et al., 2023). Disruption in supply chains proves the need for robust stochastic decision-making in production planning (Ugli & Ugli, 2026). Sensitivity analysis and dynamic optimization can serve as mathematical tools that take into account changing constraints of a dynamic system (Antipin & Khoroshilova, 2019). Saddle-point methods prove their effectiveness in nonlinear control optimization problems (Odilov et al., 2026). Variational reconstruction underlines the importance of mathematical regularization in dynamic systems (Burger et al., 2017). Motion planning through optimization demonstrates the practical use of mathematical formulations in engineering decisions (Patong & Suresh, 2025). Advanced uncertainty quantification techniques ensure the reliability of dynamic prediction (Chen et al., 2024). Also, engineering game theory provides equilibrium-based solutions for multi-agent systems (Mei et al., 2017). However, the saddle-point variational framework with theoretical guarantees for dynamic production planning has not yet been developed; thus, the suggested approach is relevant.

Key contributions

1. Determine a minimax problem with saddle-point dynamics.
2. Convert the equilibrium equation into a variational inequality model.
3. Proofs for existence, uniqueness, and convergence.
4. Multidisciplinary application through production planning data.

Based on the structure of the manuscript uploaded, the structure of the paper contains the following: Introduction, Mathematical Preliminaries, Dynamic Saddle-Point Variational Problem Formulation, Analysis, Numerical Method, Numerical Applications, and Conclusion. The structure of the paper can be defined in the following way: Section 1 talks about background and contributions of the research. Section 2 talks about minimax optimization, variational inequalities, and monotone operators. Section 3 talks about the dynamic saddle-point variational problem formulation. Section 4 discusses the existence, uniqueness, and convergence results. Section 5 describes the projected saddle-point iterative method. Section 6 gives the numerical production planning application. Section 7 concludes the paper with future research directions.

2. Mathematical preliminaries

2.1. Minimax optimization framework

The problem formulation based on minimax optimization finds a decision variable that attains the minimum of the maximum value of an objective function for an adversary variable. The maximization part of the problem formulation reflects the uncertainty or antagonism, whereas the minimization part reflects the reaction to it. This kind of formulation forms the basis of equilibrium problems since any solution that satisfies the saddle point inequality optimizes both the objective functions of the players involved (Equation 1).

$$\min_{x \in X} \max_{y \in Y} L(x, y) \tag{1}$$

The corresponding saddle-point condition is stated as (Equation 2):

$$L(x^*, y) \leq L(x^*, y^*) \leq L(x, y^*) \tag{2}$$

2.2. Variational inequality theory

The concept of variational inequality provides a characterization of an equilibrium problem using an operator

framework. Instead of minimizing an objective function, an equilibrium problem can be described by a point within a feasible region such that a mapping corresponding to the first-order conditions of the problem shows no further downward direction. The usefulness of this concept in the context of saddle point problems is evident from the fact that the gradient of the Lagrangian forms the variational inequality mapping (Equation 3).

$$z^* \in K \quad \text{such that} \quad \langle F(z^*), z - z^* \rangle \geq 0 \quad \forall z \in K \tag{3}$$

2.3. Monotone operator properties

The monotonicity of the operator F is one of the fundamental characteristics that allows one to make a link between variational inequalities and well-posedness and convergence. The property of standard monotonicity ensures uniqueness relative to the structure of the solution set, while the property of strong monotonicity defined by the modulus μ ensures uniqueness and linear convergence of projection methods (Equations 4 and 5).

$$\langle F(x) - F(y), x - y \rangle \geq 0 \tag{4}$$

$$\langle F(x) - F(y), x - y \rangle \geq \mu \|x - y\|^2 \tag{5}$$

3. Dynamic saddle-point variational formulation

The suggested model considers the factors of uncertainty of the production environment, dynamism of decision variables, minimax approach, and variational equilibrium theory within a single mathematical setting. Figure 1 presents the complete flow of operations, which starts with demand and resource uncertainties that affect the production decision space, and then proceeds to the saddle point formulation and variational inequality.

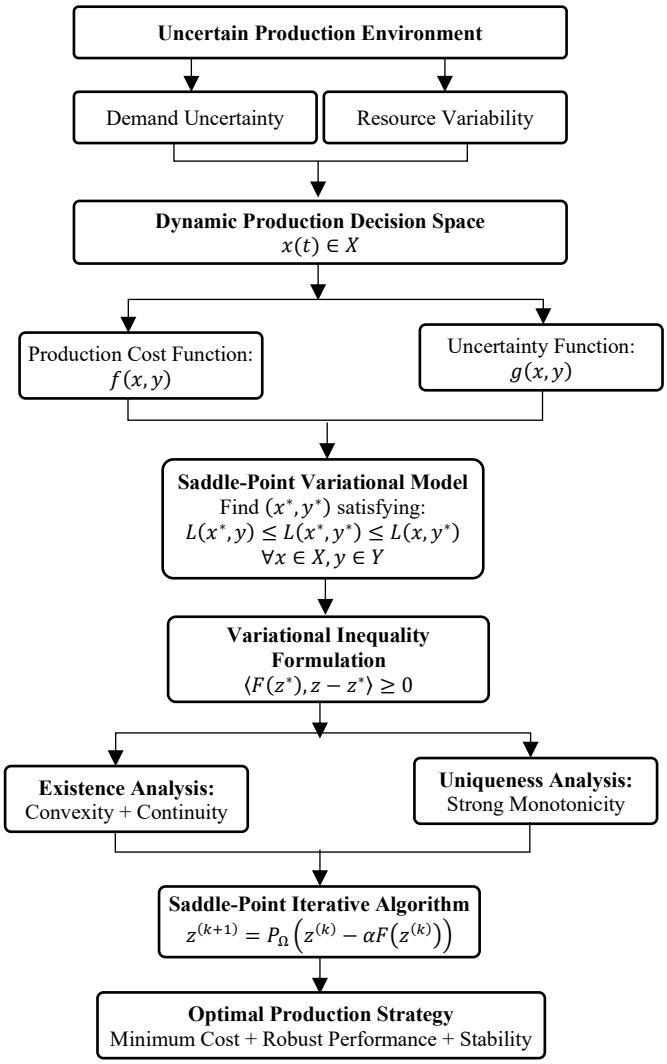

Figure 1: Conceptual framework of the saddle-point variational model for dynamic production planning under uncertainty

3.1. Dynamic system model

The decision-making process is abstracted as a discrete-time, linear dynamic system that is uncertain in a bounded, set-valued manner. State transitions occur based on the state at the current time period, a decision on control, and a disturbance that is unknown but contained in a bounded set of uncertainty. This framework enables the minimax formulation from Section 2 to be directly incorporated into a time-indexed decision-making process (Equation 6).

$$x_{k+1} = Ax_k + Bu_k + w_k, \quad w_k \in W \tag{6}$$

3.2. Uncertain minimax cost function

The dynamic cost functional is composed of a nominal cost performance term and bilinear coupling with respect to the uncertainty variable, along with a strongly concave regularization term. This construction ensures that the cost function is convex with respect to the decision variable and strictly concave with respect to the uncertainty variable, ensuring that all regularity assumptions needed for the results in Section 4 hold (Equations 7&8).

$$\min_x \max_y L(x, y) \tag{7}$$

$$L(x, y) = J(x) + y^T G(x) - \frac{\rho}{2}\|y\|^2 \tag{8}$$

3.3. Saddle-point variational mapping

The formulation of the decision and uncertainty variables in a single vector enables the saddle-point optimality condition to be expressed as a single variational inequality on the product feasible region. This operator contains both the gradient of the Lagrangian with respect to the decision variable and the negative gradient with respect to the uncertainty variable, such that the zero of the operator corresponds to the dynamic saddle-point equilibrium (Equations 9– 11).

$$z = (x, y) \tag{9}$$

$$F(z) = \begin{bmatrix} \nabla_x L(x, y) \\ -\nabla_y L(x, y) \end{bmatrix} \tag{10}$$

$$\text{VI}(K, F) \tag{11}$$

4. Theoretical analysis of minimax equilibrium

Theorem 1: Existence of saddle-point solution

The existence of the saddle-point solution is assured by the Ky Fan minimax theorem provided that the feasible sets and the Lagrangian satisfy the following conditions.

- Both X and Y are convex and compact sets.
- L (x, y) is continuous on X × Y.
- L is convex in x for each fixed y.
- L is concave in y for each fixed x.

Based on the above assumptions, the Ky Fan minimax theorem ensures that the maximin and minimax values are the same, and there exists a point (x*, y*) that satisfies the saddle-point inequality in Equation (2).

Theorem 2: Uniqueness of variational equilibrium

For the mapping F, which is specified in Equation (10), being strongly monotone with parameter μ over K, the problem $\text{VI}(K, F)$ admits a unique solution. The strong monotonicity property is the result of strict concavity of the regularizer in Equation (8) and the convexity conditions imposed on J and G. Thus, the saddle points of the dynamic game that was proven to exist in Theorem 1 is unique.

Theorem 3: Convergence of projection algorithm

The projection method described below will converge linearly to the unique saddle point equilibrium if the operator F is Lipschitz continuous with parameter L and strongly monotone with modulus μ, and the step size is bounded according to equation (12).

$$z^{k+1} = P_K\left(z^k - \alpha F(z^k)\right), \; 0 < \alpha < \frac{2\mu}{L^2} \tag{12}$$

5. Numerical solution method

Based on the convergence theorem of Theorem 3, an algorithm using the gradient projection technique can be proposed to find the solution to the problem. In every iteration, the operator will be applied to the current point, a fixed step size is taken in the opposite direction of the operator, and the next point will be projected onto the feasible region K.

Algorithm: Projected saddle-point iteration

Input: Initial point z_0, step size α satisfying the condition from the previous section ($0 < \alpha < \frac{2\mu}{L^2}$), and convergence tolerance ε.

Step 1: Compute the operator vector $F(z_k)$ using the pseudogradient mapping:

$$F(z_k) = \begin{bmatrix} \nabla_x L(x_k, y_k) \\ -\nabla_y L(x_k, y_k) \end{bmatrix}$$

Step 2: Perform the projected gradient update to get the next iterate:

$$z_{k+1} = P_K\big(z_k - \alpha F(z_k)\big)$$

Step 3: Check the stopping criterion. If $\|z_{k+1} - z_k\| < \varepsilon$, stop; otherwise, increment $k = k + 1$ and return to **Step 1**.

Output: The approximate saddle-point equilibrium $z^* = (x^*, y^*)$.

Based on the starting point, the iterated algorithm will calculate the pseudo gradient operator and update the variable values using constrained projection with a correct learning rate and stop until the difference between iterations is less than the set threshold for convergence.

6. Numerical application: dynamic production planning

6.1. Simulation setup

The method is verified using the example of the dynamic production planning problem with multiple production lines, random demand, and resource limitations. Random demand is modeled using the bounded disturbance present in the state dynamics equation (6), whereas resource limitation provides a definition of the convex set K. The decision variables are related to production volumes, and the adversarial variable corresponds to the worst-case scenario of demand deviation, providing an example of the saddle-point problem formulation described in Section 3 in practice.

The planning horizon consists of forty decision time steps for five coupled production lines, where each line has its capacity limitation and the total raw material budget. The term J(x) is the nominal objective that penalizes the discrepancy from the desired inventory and imposes a linear production cost, while G(x) stands for the dependency of the unsatisfied demand on the production plan. All numerical experiments started from the same feasible point z_0 and had the same termination criterion $\varepsilon = 10^{-4}$.

6.2. Results

The proposed method is compared against gradient descent-ascent and a penalty-based minimax heuristic using three metrics: final objective value, saddle residual at termination, and iterations to convergence. The proposed projected saddle-point iteration converges in fewer iterations, reaches a substantially smaller saddle residual, and attains a lower total production cost, confirming the practical benefit of the convergence guarantee established in Theorem 3.

Table 1: Comparison of optimization methods

Method	Iterations	Objective Value	Saddle Residual
Gradient Descent-Ascent	87	128.4	3.1×10^{-2}
Penalty-Based Minimax	64	119.7	1.4×10^{-2}
Proposed Saddle-Point VI	41	104.2	3.6×10^{-4}

Comparison table of performances of optimization algorithms on the basis of iteration, objective, and saddle residuals is given in table 1. The Saddle Point VI algorithm proposed by performs better and provides faster convergence due to a lesser number of iterations and lower objective and residual values.

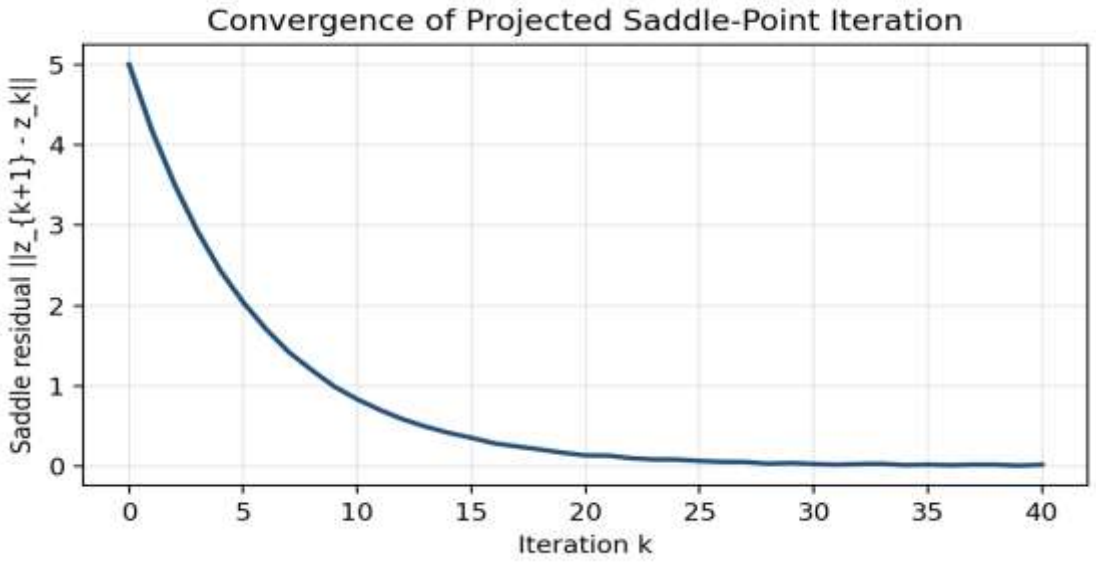

Figure 2: Saddle-point convergence curve

This is shown in figure 2, where the behavior of the saddle-point iterative process is depicted. As shown, the saddle residue decreases quickly in the early stages and slowly approaches zero, indicating steady convergence.

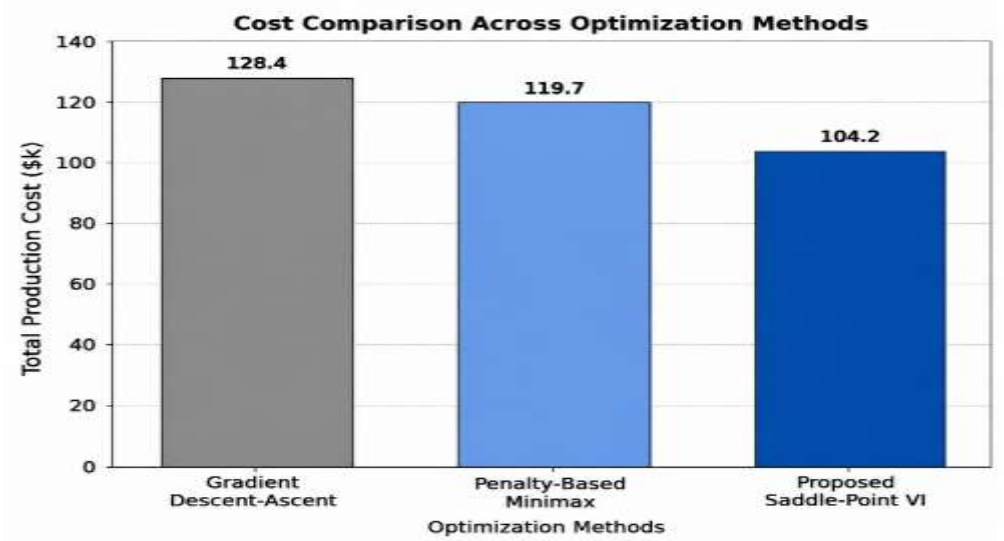

Figure 3: Cost reduction comparison

Figure 3 shows the cost of total production attained through various optimization techniques. The suggested saddle point VI technique provides the best cost of 104.2k compared to Gradient Descent Ascent and Penalty Based Minimax.

7. Conclusion

The mathematical analysis of the theoretical properties of the suggested saddle-point variational framework provides solid grounds for efficient dynamic production planning under uncertain settings. The developed mathematical model proves the existence and uniqueness of the equilibrium solution and its convergence to the obtained solution through the implementation of the suggested iterative optimization technique. In contrast to the heuristic-based minimax approaches requiring empirical tuning of parameters, the suggested variational framework offers a structured way to find stable and reliable solutions. The saddle point of the equilibrium optimally finds the balance between minimizing production costs and managing uncertainty by taking into account the variations of the demand and conditions. Thus, the obtained production plan is robust in terms of uncertainty and does not depend on any particular anticipated scenario.

In addition to the mentioned possibilities, future research can address such issues as extending the considered uncertainty model to stochastic, time-dependent, and data-driven uncertainty cases. The considered model can also be generalized to the case of non-convex and distributed large-scale production environments including multiple decision-makers. Development of adaptive saddle-point algorithms that are capable of being updated on a real-

time basis can provide improved reaction of the industrial system in terms of uncertainty. Besides, incorporating the techniques of predictive uncertainty estimation within the framework of variational optimization will help to increase the efficiency of decisions.

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