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A Non-Hausdorff Minimax Theorem

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We prove a minimax theorem where one of the spaces is not compact Hausdorff, but merely compact. This is a simple modification of a previous proof by Frenk and Kassay (2002).

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1. Introduction

A large number of minimax theorems are of the following form. Let A be a non-empty Hausdorff topological space satisfying some form of compactness, B be a non-empty set, and a map $f\colon A \times B \to \mathbb{R}$ that enjoys certain convexity-related and continuity-related properties, then:

$$\sup_{b \in B} \inf_{a \in A} f(a, b) = \inf_{a \in A} \sup_{b \in B} f(a, b).$$

The purpose of this paper is to show that Hausdorffness is not needed.

The origin of the present proof is Frenk and Kassay's Theorem 3.1 [2], a variant of Ky Fan's celebrated minimax theorem [1]. A while ago, the author tried to understand what the assumptions on A were in that paper, and notably whether A had to be Hausdorff. A is explicitly assumed to be compact there, but different communities understand "compact" as implying Hausdorffness and some do not. We shall certainly *not* assume that a compact space is Hausdorff: we define a compact space as one that has the Heine-Borel property, namely where every open cover has a finite subcover. Some authors call that notion *quasi*-compactness. Frenk and Kassay do in fact assume Hausdorffness, as their proof relies on the finite intersection property for compact sets, a property that usually fails in non-Hausdorff spaces. (See Equation (10) in [2].) The starting point of the present paper was the realization that taking complements of the sets used in their proof produced a proof where only the Heine-Borel property, and not Hausdorffness, was needed.

For all references to topology, and particularly non-Hausdorff topology, we shall refer to [4]. One should be aware that, in a non-Hausdorff setting, compact

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subsets need not be closed, and the intersection of two compact subsets need not be compact. We return to compactness below.

Every topological space A has a *specialization* quasi-ordering, which we shall simply write as $\leq$, defined by $a \leq a'$ if and only if every open subset that contains a also contains a'. We write $\uparrow K$ for the upward-closure $\{a \in A \mid \exists k \in K, \, k \leq a\}$, and $\downarrow K$ for its downward-closure. If K is compact, so is its upward-closure.

In a Hausdorff space, and more generally in a T_1 space, the specialization quasi-ordering is equality. A source of many non-Hausdorff topological spaces is given by Scott's dcpos, which are certain partially ordered sets X, with the Scott topology. Explicitly, a *directed-complete partial order*, or *dcpo*, is a partially ordered set X in which every directed family $(x_i)_{i \in I}$ as a supremum. As usual, directed means that I is non-empty, and that for any two indices $i, j \in I$, there is a third one $k \in I$ such that $x_i, x_j \leq x_k$. The *Scott-open* subsets of a partially ordered set X are the upwards-closed subsets U such that every directed family $(x_i)_{i \in I}$ that has a supremum in U is such that some x_i is in U already. The Scott-open sets form a topology, the *Scott topology*. We shall write X_σ for X with that topology.

It is also useful to realize that the *Scott-closed* subsets of a partially ordered set X, namely the closed subsets of X_σ, are the downwards-closed subsets C of X such that the supremum of every directed family $(x_i)_{i \in I}$ of elements of C, if it exists, is also in C. For every $x \in X$, $\downarrow x$ is Scott-closed in any partially ordered set X: indeed, this is the closure of the point x in the Scott topology on X.

For every partially ordered set X, the specialization ordering of X_σ coincides with the ordering on X. In particular, none of these spaces are Hausdorff, unless the ordering is equality.

Let us return to general, not necessarily Hausdorff, spaces. Compactness, without Hausdorffness, is an extremely weak property: any space with a least element with respect to the specialization quasi-ordering $\leq$ is compact, for example, and in particular any dcpo with a least element is compact in the Scott topology. The latter are ubiquitous in the denotational semantics of programming languages. However, a number of results still hold. Notably, we have [4, Prop. 4.4.9]: a subset Q of a topological space X is *compact* if and only if, for every filtered family $(C_i)_{i \in I}$ of closed subsets of X that all intersect Q, $\bigcap_{i \in I} C_i$ also intersects Q. A family is *filtered* if and only if it is directed in the superset ordering. This is an immediate consequence of the Heine-Borel property, considering that the complements of closed sets are the open sets. In particular, given a compact subset Q of X, if $(C_\epsilon)_{\epsilon > 0}$ is a chain of closed subsets of X indexed by positive real numbers, and $\bigcap_{\epsilon > 0} C_\epsilon$ does not intersect Q, then some C_ϵ does not intersect Q.

This work is one of several bricks needed in a study of quasi-metrics on spaces of continuous valuations (a notion close to measures) and non-linear extensions of the latter, which should appear in a series of papers. (A slightly less general minimax theorem was claimed, with no proof, in [3, Theorem 6], with the same objective.)

In that study, it is necessary to consider functions that can take the value $+\infty$, and we therefore consider functions with values in $\mathbb{R} \cup \{+\infty\}$ instead of $\mathbb{R}$. However benign that may seem, this forbids us to use any of the classical forms of the Hahn-Banach Theorem: we need one that works on $\overline{\mathbb{R}}_+{}^n$, and $\overline{\mathbb{R}}_+{}^n$ not only is not a vector space, but does not even embed into one, since addition there is not cancellative. Hence we shall use a separation theorem due to R. Tix, K. Keimel, and G. Plotkin, which we reproduce as Proposition 2.1 below.

2. Preliminaries

Given two non-empty sets A, B, and a map $f\colon A \times B \to \mathbb{R}$, we clearly have:

$$\sup_{b \in B} \inf_{a \in A} f(a,b) \leq \inf_{a \in A} \sup_{b \in B} f(a,b).$$

Our aim is to strengthen that to an equality, under some extra assumptions. We shall also consider the case of functions with values in $\mathbb{R} \cup \{+\infty\}$, not just $\mathbb{R}$.

Recall that we write $\overline{\mathbb{R}}_{+\sigma}$ for $\overline{\mathbb{R}}_+$ with its Scott topology. The non-trivial (Scott-) open subsets of $\overline{\mathbb{R}}_{+\sigma}$ are the semi-open intervals $]t, +\infty]$, $t \in \mathbb{R}$. Hence a lower semicontinuous map from A to $\overline{\mathbb{R}}_+$ is the same thing as a continuous map from A to $\overline{\mathbb{R}}_{+\sigma}$.

One should note that, given two partially ordered sets X and Y, a function $f\colon X \to Y$ is continuous from X_σ to Y_σ if and only if it is *Scott-continuous*, namely it is monotonic and preserves existing suprema of directed families [4, Proposition 4.3.5].

It turns out that the Scott topology of $\overline{\mathbb{R}}_+{}^n$ coincides with the product topology of n copies of $\overline{\mathbb{R}}_{+\sigma}$, because, as a poset, $\overline{\mathbb{R}}_+$ is a continuous dcpo [4, Proposition 5.1.54]. It follows that the notation $\overline{\mathbb{R}}_{+\sigma}^{\,n}$ is non-ambiguous.

A subset A of $\overline{\mathbb{R}}_+^n$ is *convex* if and only if for all $a_1, a_2 \in A$ and for every $\alpha \in]0,1[$, $\alpha a_1 + (1-\alpha) a_2$ is in A. A map is *linear* if and only if it preserves sums and products by scalars from $\mathbb{R}_+$. Let $\vec{1}$ be the all one vector in $\overline{\mathbb{R}}_+^n$. Then:

Proposition 2.1. ([6], Lemma 3.7) *Let Q be a convex compact subset of $\overline{\mathbb{R}}_{+\sigma}^{\,n}$ disjoint from $\downarrow\vec{1} = [0,1]^n$. Then there is a linear continuous function Λ from $\overline{\mathbb{R}}_{+\sigma}^{\,n}$ to $\overline{\mathbb{R}}_{+\sigma}$ and $b > 1$ such that $\Lambda(\vec{1}) \leq 1$ and $\Lambda(x) > b$ for every $x \in Q$.* □

Let us agree that $0 \cdot (+\infty) = (+\infty) \cdot 0 = 0$; this is the only extension of multiplication to $\overline{\mathbb{R}}_+$ that remains Scott-continuous. Writing e_i for the tuple with a 1 at position i and zeros elsewhere, such a linear continuous map must map every $(x_1, x_2, \cdots, x_n)$ to $\sum_{i=1}^n c_i x_i$, where $c_i = \Lambda(e_i)$. This is indeed certainly true when every x_i is different from $+\infty$, by linearity, and (Scott-)continuity gives us the result for the remaining cases. Scott-continuity is essential here: taking $n = 1$ for simplicity, the map that sends every $x \in \mathbb{R}_+$ to 0 and $+\infty$ to $+\infty$ is linear, monotonic (not Scott-continuous), and cannot be written as $x \mapsto cx$ for any $c \in \overline{\mathbb{R}}_+$.

Corollary 2.2. *Let Q be a non-empty convex compact subset of $\overline{\mathbb{R}}_{+\sigma}^{n}$ disjoint from $\downarrow\vec{1} = [0,1]^n$. There are non-negative real numbers c_1, c_2, …, c_n such that $\sum_{i=1}^n c_i = 1$ and $\sum_{i=1}^n c_i x_i > 1$ for every $(x_1, x_2, \cdots, x_n)$ in Q.*

Proof. Find Λ and $b > 1$ as above. Λ maps every $(x_1, x_2, \cdots, x_n)$ to $\sum_{i=1}^n c_i x_i$, where each c_i is in $\overline{\mathbb{R}}_+$. Since $\Lambda(\vec{1}) \leq 1$, $\sum_{i=1}^n c_i \leq 1$. In particular, no c_i is equal to $+\infty$. Since Q is non-empty, $\sum_{i=1}^n c_i x_i > b$ for some point $(x_1, x_2, \cdots, x_n)$ in Q. This implies that not every c_i is equal to 0, hence $\sum_{i=1}^n c_i \neq 0$. Let $c'_i = c_i / \sum_{i=1}^n c_i$. Then $\sum_{i=1}^n c'_i = 1$, and for every $(x_1, x_2, \cdots, x_n)$ in Q

$$\sum_{i=1}^n c'_i x_i > b / \sum_{i=1}^n c_i \geq b > 1. \qquad \square$$

Following Frenk and Kassay, we say that a map $f\colon A \times B \to \mathbb{R} \cup \{+\infty\}$ is *closely convex in its first argument* if and and only if for all $a_1, a_2 \in A$, for every $\alpha \in]0,1[$, for every $\epsilon > 0$, there is an $a \in A$ such that, for every $b \in B$:

$$f(a,b) \leq \alpha f(a_1,b) + (1-\alpha) f(a_2,b) + \epsilon.$$

We say that f is *closely concave in its second argument* if and only if for all $b_1, b_2 \in B$, for every $\alpha \in]0,1[$, for all $\epsilon, M > 0$, there is a $b \in B$ such that, for every $a \in A$:

$$f(a,b) \geq \min(M, \alpha f(a,b_1) + (1-\alpha) f(a,b_2) - \epsilon).$$

The extra lower bound M serves to handle the cases where $\alpha f(a,b_1) + (1 - \alpha) f(a,b_2)$ is infinite, and is a minor addition to Frenk and Kassay's definition, required to handle $+\infty$.

3. The Proof of the Minimax Theorem

Our theorem will be a consequence of the following two lemmas. We find it practical to call *simple valuation* on a set B any finite non-negative linear combination $\sum_{i=1}^n c_i \delta_{b_i}$ of Dirac measures, that is, any finite family of pairs $(c_i, b_i) \in \mathbb{R}_+ \times B$. It is *normalized* if and only if $\sum_{i=1}^n c_i = 1$.

Lemma 3.1. *Let A be a non-empty compact topological space and B be a non-empty set. Let $f\colon A \times B \to \mathbb{R} \cup \{+\infty\}$ be a map that is closely convex in its first argument, and such that $f(\_, b)$ is lower semicontinuous for every $b \in B$.*

For every $t \in \mathbb{R}$, $\inf_{a \in A} \sup_{b \in B} f(a,b) \leq t$ if and only if, for every normalized simple valuation $\sum_{i=1}^n c_i \delta_{b_i}$ on B, $\inf_{a \in A} \sum_{i=1}^n c_i f(a, b_i) \leq t$.

Proof. Let $L\colon A \to \mathbb{R} \cup \{+\infty\}$ be defined by $L(a) = \sup_{b \in B} f(a,b)$. Since we have $+\infty$, this supremum always exists. Being a pointwise supremum of lower semicontinuous functions, L is itself lower semicontinuous. Explicitly, $L^{-1}(]t, +\infty]) = \{a \in A \mid \sup_{b \in B} f(a,b) > t\} = \{a \in A \mid \exists b \in B, f(a,b) > t\} = \bigcup_{b \in B} f(\_, b)^{-1}(]t, +\infty])$ is open.

For every normalized simple valuation $\sum_{i=1}^n c_i\delta_{b_i}$ on B, for every $a \in A$, we have $\sum_{i=1}^n c_i f(a, b_i) \leq \sum_{i=1}^n c_i L(a) = L(a)$, so $\inf_{a\in A} \sum_{i=1}^n c_i f(a, b_i) \leq \inf_{a\in A} L(a)$. In particular, if $\inf_{a\in A} L(a) \leq t$, then $\inf_{a\in A} \sum_{i=1}^n c_i f(a, b_i) \leq t$.

In the converse direction, assume that $\inf_{a\in A} L(a) > t$. We wish to show that $\inf_{a\in A} \sum_{i=1}^n c_i f(a, b_i) > t$ for some normalized simple valuation $\sum_{i=1}^n c_i\delta_{b_i}$. To that end, we first pick a real number t' such that $\inf_{a\in A} L(a) > t' > t$, and we shall find a normalized simple valuation $\sum_{i=1}^n c_i\delta_{b_i}$ such that $\sum_{i=1}^n c_i f(a, b_i) \geq t'$ for every $a \in A$.

For every $b \in B$, $U_b = \{a \in A \mid f(a, b) > t'\}$ is open since $f(\_, b)$ is lower semicontinuous, and the sets U_b, $b \in B$, cover A: for every $a \in A$, $L(a) > t'$, so $f(a, b) > t'$ for some $b \in B$. Since A is compact, there are finitely many points b_1, b_2, ..., b_n such that $A = \bigcup_{i=1}^n U_{b_i}$. Note that, since A is non-empty, $n \geq 1$.

Since $f(\_, b_i)$ is lower semicontinuous on the compact space A, it reaches its minimum r_i in $\mathbb{R} \cup \{+\infty\}$. Let $r = \min(r_1, r_2, \cdots, r_n)$, and find $c > 0$ such that $1 + c(r - t') \geq 0$. If $r \geq t'$, we can take any $c > 0$, otherwise any c such that $0 < c < 1/(t' - r)$ will fit. The map $h_i\colon a \mapsto 1 + c(f(a, b_i) - t')$ is then lower semicontinuous from A to $\overline{\mathbb{R}}_+$, and therefore $h(a) = (h_1(a), h_2(a), \cdots, h_n(a))$ defines a continuous map from A to $\overline{\mathbb{R}}_{+\sigma}^{\,n}$.

Let K be the image of A by h. This is a compact subset of $\overline{\mathbb{R}}_{+\sigma}^{\,n}$. It is also non-empty, since A is non-empty. Hence its upward-closure $Q = \uparrow K$ is also non-empty, and is compact.

We observe the following fact: $(*)$ for every $z \in \overline{\mathbb{R}}_+^n$, if $z + \epsilon \cdot \vec{1}$ is in Q for every $\epsilon > 0$, then z is also in Q. Indeed, consider the Scott-closed set $C_\epsilon = \Downarrow(z + \epsilon \cdot \vec{1})$ in $\overline{\mathbb{R}}_+^n$, for every $\epsilon \geq 0$. (Recall that is Scott-closed, being the closure of $(z + \epsilon \cdot \vec{1})$ in the Scott topology.) Then $C_0 = \downarrow z = \bigcap_{\epsilon>0} C_\epsilon$. If z were not in Q, then C_0 would not intersect Q, since Q is upwards-closed. By compactness, C_ϵ would not intersect Q for some $\epsilon > 0$, contradicting the fact that $z + \epsilon \cdot \vec{1}$ is in both sets.

We claim that Q is convex. Otherwise, there are two points z_1 and z_2 of Q and a real number $\alpha \in]0, 1[$ such that $\alpha z_1 + (1 - \alpha)z_2$ is not in Q. By $(*)$, $\alpha z_1 + (1 - \alpha)z_2 + \epsilon \cdot \vec{1}$ is not in Q for some $\epsilon > 0$. Since z_1 and z_2 are in Q, by definition there are points a_1 and a_2 in A such that $h(a_1) \leq z_1$ and $h(a_2) \leq z_2$. We now use the fact that f is closely convex in its first argument: there is a point $a \in A$ such that, for every i, $1 \leq i \leq n$, $f(a, b_i) \leq \alpha f(a_1, b_i) + (1-\alpha) f(a_2, b_i) + \epsilon/c$. That implies $h(a) \leq \alpha h(a_1) + (1-\alpha)h(a_2) + \epsilon \cdot \vec{1} \leq \alpha z_1 + (1-\alpha)z_2 + \epsilon \cdot \vec{1}$. Since $h(a)$ is in Q and Q is upwards-closed, this contradicts the fact that $\alpha z_1 + (1-\alpha)z_2 + \epsilon \cdot \vec{1}$ is not in Q.

We claim that Q does not intersect $[0, 1]^n$. Assume on the contrary that there is an $a \in A$, and a $z \in [0, 1]^n$ such that $h(a) \leq z$. Since $A = \bigcup_{i=1}^n U_{b_i}$, there is an index i such that $f(a, b_i) > t'$, so $h_i(a) > 1$. However $h(a) \leq z \in [0, 1]^n$ entails that $h_i(a) \leq 1$, contradiction.

Now we use Corollary 2.2 and obtain non-negative real numbers c_1, c_2, ..., c_n such that $\sum_{i=1}^n c_i = 1$, and $\sum_{i=1}^n c_i z_i \geq 1$ for every $(z_1, z_2, \cdots, z_n) \in Q$. This holds for the elements $h(a)$ of $K \subseteq Q$, hence $\sum_{i=1}^{n} c_i(1 + c(f(a, b_i) - t')) \geq 1$ for every $a \in A$. It follows that $\sum_{i=1}^{n} c_i f(a, b_i) \geq t'$, as required. □

Lemma 3.2. *Let A and B be two non-empty sets, and let $f\colon A \times B \to \mathbb{R} \cup \{+\infty\}$ be closely concave in its second argument. For every normalized simple valuation $\sum_{i=1}^{n} c_i \delta_{b_i}$ on B, $\inf_{a \in A} \sum_{i=1}^{n} c_i f(a, b_i) \leq \sup_{b \in B} \inf_{a \in A} f(a, b)$.*

Proof. We first show that, if f is closely concave in its second argument, then for every normalized simple valuation $\sum_{i=1}^{n} c_i \delta_{b_i}$ on B, for all $\epsilon, M > 0$, there exists a $b \in B$ such that, for every $a \in A$:

$$f(a, b) \geq \min(M, \sum_{i=1}^{n} c_i f(a, b_i) - \epsilon).$$

The definition of closely concave is the special case $n = 2$. The case $n = 0$ is vacuous, and the case $n = 1$ is trivial.

We prove the claim by induction on n. Assume $n \geq 3$. If $c_n = 0$, then the result is a direct appeal to the induction hypothesis. If $c_n = 1$, then $c_1 = c_2 = \cdots = c_{n-1} = 0$, and we can take $b = b_n$. Otherwise, let $c'_i = c_i/(1 - c_n)$ for every i, and $\alpha = 1 - c_n$. Note that α is in $]0, 1[$. Also,

$$\sum_{i=1}^{n} c_i f(a, b_i) = \alpha \sum_{i=1}^{n-1} c'_i f(a, b_i) + (1 - \alpha) f(a, b_n).$$

By the induction hypothesis there is a point $b' \in B$ such that, for every $a \in A$:

$$f(a, b') \geq \min\left(M/\alpha, \sum_{i=1}^{n-1} c'_i f(a, b_i) - \epsilon/2\alpha\right).$$

Since f is closely concave in its second argument, there is a point $b \in B$ such that, for every $a \in A$:

$$f(a, b) \geq \min\left(M, \alpha f(a, b') + (1 - \alpha) f(a, b_n) - \epsilon/2\right).$$

Therefore, for every $a \in A$,

$$\begin{aligned} f(a, b) &\geq \min\left(M, \alpha \min\left(M/\alpha, \sum_{i=1}^{n-1} c'_i f(a, b_i) - \epsilon/2\alpha\right) + (1 - \alpha) f(a, b_n) - \epsilon/2\right) \\ &= \min\left(M, \alpha \sum_{i=1}^{n-1} c'_i f(a, b_i) - \epsilon/2 + (1 - \alpha) f(a, b_n) - \epsilon/2\right) \\ &= \min(M, \sum_{i=1}^{n} c_i f(a, b_i) - \epsilon). \end{aligned}$$

We now prove the lemma by showing that for every real number t satisfying $t < \inf_{a\in A} \sum_{i=1}^{n} c_i f(a, b_i)$ there is an element $b \in B$ such that, for every $a \in A$, $f(a, b) \geq t$. For that, we pick $\epsilon > 0$ such that $t + \epsilon \leq \inf_{a\in A} \sum_{i=1}^{n} c_i f(a, b_i)$, and we let M be any positive number larger than t. The above generalization of the notion of closely concave then yields the existence of a point b such that for every $a \in A$, $f(a, b) \geq \min(M, \sum_{i=1}^{n} c_i f(a, b_i) - \epsilon) \geq t$. □

Theorem 3.3 (Minimax). *Let A be a non-empty compact topological space and B be a non-empty set. Let $f\colon A \times B \to \mathbb{R} \cup \{+\infty\}$ be a map that is closely convex in its first argument, closely concave in its second argument, and such that $f(\_, b)$ is lower semicontinuous for every $b \in B$. Then:*

$$\sup_{b\in B} \inf_{a\in A} f(a, b) = \inf_{a\in A} \sup_{b\in B} f(a, b),$$

and the infimum on the right-hand side is attained.

Proof. The $\leq$ direction is obvious. If $\sup_{b\in B} \inf_{a\in A} f(a, b) = +\infty$, then the equality is clear. Otherwise, let $t = \sup_{b\in B} \inf_{a\in A} f(a, b)$. This is a real number that is larger than or equal to $\inf_{a\in A} \sum_{i=1}^{n} c_i f(a, b_i)$ for every normalized simple valuation $\sum_{i=1}^{n} c_i \delta_{b_i}$ on B, by Lemma 3.2. Lemma 3.1 then tells us that $\inf_{a\in A} \sup_{b\in B} f(a, b) \leq t$, which gives the $\geq$ direction of the inequality.

The fact that the infimum on the right-hand side is attained is due to the fact that A is compact, and that $a \mapsto \sup_{b\in B} f(a, b)$ is lower semicontinuous, as a pointwise supremum of lower semicontinuous maps. □

Remark 3.4. Among the closely convex functions (in the first argument), one finds the *convex* functions in the sense of Ky Fan [1], that is, the maps such that for all $a_1, a_2 \in A$, for every $\alpha \in]0, 1[$, there is a point $a \in A$ such that for every $b \in B$, $f(a, b) \leq \alpha f(a_1, b) + (1 - \alpha) f(a_2, b)$. Similarly with closely concave functions and concave functions in the sense of Ky Fan, which satisfy that for all $b_1, b_2 \in B$, for every $\alpha \in]0, 1[$, there is a point $b \in B$ such that for every $a \in A$, $f(a, b) \geq \alpha f(a, b_1) + (1 - \alpha) f(a, b_2)$.

Among the convex functions (in the first argument) in the sense of Ky Fan, one simply finds the functions f such that $f(\alpha a_1 + (1 - \alpha) a_2, b) \leq \alpha f(a_1, b) + (1 - \alpha) f(a_2, b)$, namely those that are convex in their first argument, in the ordinary sense, provided one can interpret scalar multiplication and addition on A, for example when A is a convex subset of a topological real vector space, or more generally of a semitopological cone in the sense of Keimel [5].

One also finds more unusual cases of Ky Fan convexity. Consider for example a family of functions $f_i\colon A \to \mathbb{R} \cup \{+\infty\}$, $i \in I$. We can see them as one function $f\colon A \times I \to \mathbb{R} \cup \{+\infty\}$ by letting $f(a, i) = f_i(a)$. If the family $(f_i)_{i\in I}$ is directed, then f is Ky Fan concave: for all $i, j \in I$, find $k \in I$ such that $f_i, f_j \leq f_k$, then for every $\alpha \in [0, 1]$, for every $a \in A$, $f_k(a) \geq \alpha f_i(a) + (1 - \alpha) f_j(a)$.

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