

A Minimax Variational Inequality Framework for Traffic Equilibrium Optimization in Intelligent Transportation Networks

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Abstract

The fast development of intelligent transportation systems has complicated traffic congestion problems because of uncertainties associated with traffic demands, traffic networks, and interactions between multiple users on the roads. Current approaches in traffic equilibrium optimization have primarily focused on traffic demand and network conditions without taking into account possible worst-case scenarios. This paper solves this issue using a novel Minimax Variational Inequality (MVI) approach for traffic equilibrium optimization with uncertainty in intelligent transportation networks. It combines minimax optimization and variational inequality to develop an equilibrium optimization problem in traffic flows considering uncertainties. A robust uncertainty model is used to determine negative traffic scenarios, while a two-timescale projected extragradient scheme is constructed to reach stable traffic equilibria. The framework represents the transportation network model via constrained flow sets, monotonic cost operator, and worst-case optimization to enhance robustness against traffic dynamics. The experimental results performed on the benchmark transportation network prove that the approach allows for the successful determination of worst-case traffic condition, ensures the equilibrium stability, and decreases vulnerability to traffic dynamics changes. The analysis results in the growth of the overall cost of the system by 30.8% from nominal to worst-case traffic conditions. This framework provides an efficient mathematical background for making decisions in the intelligent transportation system.

Keywords: Variational inequality, minimax optimization, traffic equilibrium, intelligent transportation networks, robust optimization, traffic flow management, extragradient algorithm.

1. Introduction

The intelligent transportation systems are necessitated by the use of mathematical optimization techniques to mitigate traffic congestion, traffic uncertainty, and interactions among diverse transport network users. Variational

inequality theory has emerged as an important mathematical technique in modeling equilibrium problems through modeling decision-making and interactions among nonlinear transport networks. Finite-dimensional variational inequalities theory and applications of the theory to equilibrium optimization problems provide important mathematical concepts for traffic flow analysis and modeling of network equilibrium behavior (Harker & Pang, 1990). Modeling of equilibrium behavior and traffic pricing has emphasized the significance of mathematical modeling of traffic route selection and equilibrium behavior (Florian & Hearn, 1999). Traffic equilibrium problems modeled through VI theory have shown the ability to model user equilibrium behavior wherein no individual is able to reduce travel costs by changing routes (Dafermos, 1980).

Some of the latest trends in the area of intelligent traffic management involve using optimization approaches within intelligent traffic infrastructures to minimize traffic congestion and improve the performance of traffic systems. Modern traffic management approaches have revealed the need for adaptive and scalable solutions to be applied in urban transportation systems faced with increasingly complex traffic conditions (Goswami & Kadao, 2025). The projection algorithms for quasi-variational inequalities have proved to be useful solution tools for traffic assignment and network equilibrium control problems under dynamic constraints (Trinh & Vuong, 2025). The appearance of intelligent agents and autonomous behavior in traffic networks has made the researchers interested in robust optimization methods that can accommodate the heterogeneous decision-making processes in the traffic environment (Yi et al., 2025). Machine learning algorithms have been used for traffic flow prediction purposes; however, do not consider mathematical equilibrium optimization and robustness of the solution process (Akash & Nithish, 2022).

Research into traffic equilibrium modeling in contemporary times has considered the application of better VI algorithms in solving the transportation optimization problem in large scale, highlighting the need for computationally sound equilibrium solution approaches (Denysov & Semenov, 2024). Adaptive VI algorithms under smoothness and uncertainty assumptions have aided in enhancing the convergence performance of optimization models in situations where there is noise and dynamics (Bach & Levy, 2019). The minimax inequality has been instrumental in developing optimization models in worst case scenarios due to its capability to consider uncertainty and competition in equilibrium models (Yuan et al., 1998). Principles of mathematical optimization have been utilized in other fields of modeling as well (Abdullah & Nagmulden, 2024).

There have been many advancements in the design of minimax optimization algorithms owing to their capability of optimizing problems having weakly convex and weakly concave objective functions along with their convergence under difficult optimization scenarios (Liu et al., 2021). Decision-making systems based on multi-agent optimization highlight the prospects of adaptive resource allocation and distributed control in complex networks, which is very important from an intelligent transportation perspective in the future (Reddy & Kamalanathan, 2026). The theories related to bilevel equilibrium models and the minimax inequalities of Ky Fan have enabled an extension in hierarchical decision-making systems with multiple optimization levels (Chbani et al., 2019). Resource-aware traffic management in vehicular networks highlights the significance of effective traffic shaping schemes in intelligent transportation systems (Gomez et al., 2025).

Theoretical studies about minimax theory and minimax theorem and existence of equilibrium give mathematical reasoning for the development of robust optimization methodologies in case of uncertainty (Komiya, 1988). Modern minimax theory for stochastic processes has widened the range of application for worst-case optimization methodologies in complicated computational settings (Ben-David & Blais, 2023). Simple proof for minimax theorem and equilibrium theory have improved the theoretical foundation needed for developing stable optimization models (Park, 2010). Optimization decision making models based on intelligent computational approach emphasize the significance of choosing proper objective function and cost function for ensuring reliable performance of systems (Agrab, 2022). Basic minimax theory for game theoretic optimization offers further theoretical foundation for equilibrium models with competing objective and uncertainties (Ben-El-Mechaiekh & Dimand, 2007).

Even though some of the existing studies have made great progress in traffic equilibrium models, VI optimization, and minimax theories, little work has been done to develop a minimax variational inequality model to cater to the requirements of intelligent transportation systems. These approaches mostly concentrate either on traditional equilibrium solutions, traffic analysis based on prediction, or optimization techniques separately without considering worst-case conditions, stability of equilibria, and managing uncertainties. Hence, a minimax variational inequality model is developed to optimize traffic equilibrium situations in intelligent transportation systems.

Key Contribution

- Mathematical modeling for traffic equilibrium problems is performed using minimax optimization and variational inequality theories for intelligent transportation systems. The mathematical model is capable of

incorporating nonlinearities due to interaction of traffic flows, cost of congestion, capacity constraints and uncertainty in the network.

- An efficient optimization technique is developed for the mitigation of the worst effects of traffic congestion under uncertain traffic dynamics. This technique strengthens the network resistance by incorporating factors such as traffic demand, link states, and disruptions, which helps in allocating the traffic flows effectively.
- An iterative methodology is developed to solve the above minimax variational inequality formulation, along with convergence and stability analysis. Its performance has been demonstrated via transportation network simulation experiments in terms of performance metrics like equilibrium error, time saving, congestion alleviation, and efficiency.

The rest of the paper is structured as follows. Section II gives the preliminaries and the mathematical setup, which includes the notations, standing assumptions, monotone operators, variational inequality problems, minimax inequalities, and the introduced Minimax Variational Inequality (MVI) problem formulation. Section III details the two-timescale projected extragradient approach, its convergence proof, the network model, cost functions, and the experimental evaluation under nominal and worst-case traffic conditions. Section IV provides a discussion on the modelling assumptions, connection to bi-level area traffic control, shortcomings, and real-world applicability of the proposed MVI approach. Section V concludes the paper.

2. Preliminaries and mathematical background

2.1. Notation and standing assumptions

Let $G = (N, A)$ denote a directed transportation network with node set N and link (arc) set A , and let W be a finite set of origin-destination (OD) pairs with a fixed path set R_w for each w in W . Let $f = (f_a)_{\{a \in A\}}$ denote the vector of link flows and let $d = (d_w)_{\{w \in W\}}$ denote the vector of OD demands. The feasible flow set is

which is a non-empty, closed, bounded (hence compact) and convex polytope for every fixed, finite demand vector d in a bounded set D . The link cost (travel time) operator $c : R^{|A|}_+ \rightarrow R^{|A|}_+$ is assumed to be continuous, and, following the general asymmetric-interaction model of Smith and Dafermos, is not required to derive from a potential function. write $\langle \cdot, \cdot \rangle$ for the Euclidean inner product on $R^{|A|}$.

2.2. Monotone operators

Following the terminology adopted by Yen for a (possibly multivalued) relation $T: X \rightrightarrows E^*$, T is monotone on a subset X of E whenever $\langle u - v, x - y \rangle \geq 0$ for all $x, y \in X$, $u \in Tx$, and $v \in Ty$; T is hemiclosed on X if it is closed along line segments under weak convergence in the appropriate sense; and T is l-bounded on X if the infimal norm of Tx remains bounded as x ranges over any segment of X away from its endpoints.

For the finite-dimensional traffic problem treated in this paper, X is compact and convex, and c is assumed to be single-valued and continuous, hence hemiclosed and l-bounded properties are satisfied automatically, but state the theorem in its generality since it is only by virtue of this statement that the theory can be extended, without modification, to a continuum network, multiclass network, or an elastic-demand network in which the cost correspondence may not necessarily be single-valued and continuous, just the right context for which Yen generalized Fan's inequality.

2.3. The smith dafermos variational inequality

A flow f^* in $K(d)$ is a Wardrop (user) equilibrium for demand d if and only if it solves the variational inequality

If c is derived from a symmetric Jacobian matrix, then (1) represents the necessary first-order optimality condition of the convex programme by Beckmann, McGuire, and Winsten; in general, c need not be derivable from a potential, and then (1) is a variational inequality in its own right. The existence of a solution to (1) depends on the compactness of $K(d)$, the continuity of c , and on the classical proof by Hartman-Stampacchia and Brouwer; uniqueness is obtained if c is strictly monotone on $K(d)$.

2.4 The generalized Ky fan minimax inequality

The theorem below is an adaptation of Yen's Theorem 1 for the framework adopted in this paper (X compact convex in a Hausdorff topological vector space, which in the context of the finite-dimensional network model below is just a compact convex subset of a Euclidean space).

Theorem A (Minimax inequality; Yen, adapting Ky Fan). Let X be a compact convex subset of a Hausdorff topological vector space E , and let $f, g : X \times X \rightarrow \mathbb{R}$ satisfy:

for every x in X , $g(x)$ is lower semicontinuous on X

for every y in X , $f(y)$ is quasi concave on X

$g(x, y) > f(x, y)$ for all (x, y) in X

$$\min_{y \in X} \sup_{x \in X} g(x, y) \geq \sup_{x \in X} f(x, x) \quad (1)$$

From the above equation (1) Theorem A, together with the KKM-type intersection lemma of Fan, to prove existence of a solution y_0 of the 'sup' variational inequality $\sup_{u \in Tx} \langle u, y_0 - x \rangle \leq h(x) - h(y_0)$ for all x in X , for T , monotone with full domain X and h convex and lower semicontinuous, without assuming continuity of T ; and, under the additional hemiclosedness and l -boundedness conditions, existence of a solution to the corresponding 'inf' variational inequality, the two solution sets being shown to coincide. Theorem A is not for a single operator T but for a bimap composed of the cost operator of the traffic c and a function of payoff which evaluates the level of a demand-capacity situation, such that the obtained minimax point is at the same time a Wardrop equilibrium and a worst-case scenario.

2.5. Sion minimax theorem

The classical minimax theorem of Sion can also be used to complement the analysis of saddle points in the following way: let X and Y be convex sets in linear topological spaces with at least one of them being compact, and $\psi : X \times Y \rightarrow \mathbb{R}$ is quasi-concave and upper semicontinuous in X and quasi-convex and lower semicontinuous in Y , then equation (2),

$$\min_{y \in Y} \sup_{x \in X} \psi(x, y) = \sup_{x \in X} \min_{y \in Y} \psi(x, y) \quad (2)$$

Theorem A and the Sion's Theorem are complementary in nature because while Theorem A provides for the existence of the saddle point without the need for continuity of the monotone relation, which becomes necessary when the cost operator is coupled to a worst case payoff that may not be differentiable, the Sion's theorem, if the conditions of continuity/concavity-convexity assumed by the theorem apply (do in the case of polynomials-in-flow/affine-in-demand cost), also ensures that the minimax equals the maximin, and thus that the order of optimization can be interchanged.

2.6. The minimax variational inequality framework

Uncertainty model

Let $D \subset \mathbb{R}_+^{|W|}$ be a non-empty, compact, and convex set of feasible OD demand vectors, such as, e.g., a box or ellipsoid-shaped neighborhood of a nominal demand vector d_0 derived by means of a short-term ITS forecast,

$$D = \{d : |d_w - d_w^0| \leq \Delta_w \text{ for all } w \in W\} \quad (3)$$

Equation (3) $\Delta_w \geq 0$ an Uncertainty radius prescribed in a way that reflects forecast/sensor errors for OD pair w . Similar constructions can be done for link capacity uncertainty by letting the free-flow parameters of the cost functions vary over some compact convex set D ; both examples can be subsumed under the more general analysis to follow with $c(f; d)$ being the cost operator.

The Minimax Variational Inequality

Fix the feasible flow set $K = \bigcup_{d \in D} K(d)$ intersected appropriately so that K is itself compact and convex (this holds, for instance, whenever $K(d)$ can be written as $\{f \in \mathbb{R}_+^{|A|} : Af = Bd, f \leq f_{\max}\}$ for fixed conservation and routing matrices A, B , so that $K = \{(f, d) : Af = Bd, f \leq f_{\max}, d \in D\}$ Projects onto the compact convex set of joint (flow, demand) tuples (and this set is the object of study). Let introduce the coupling bifunction which characterizes the VI residual for the candidate flow y w.r.t. the operating point f under scenario d , and the scenario payoff.

The minimax variational inequality (MVI) associated with (K, D, c, Φ) is: Find $(f^*, d^*) \in K \times D$ such that

$$\langle c(f^*; d^*), y - f^* \rangle \geq 0, \quad \forall y \in K \quad (4)$$

$$\Phi(f^*, d^*) \geq \Phi(f^*, d), \quad \forall d \in D \quad (5)$$

Equation (4) says that f^* is a Wardrop equilibrium for the realized worst-case scenario d^* ; Equation (5) says that d^* is, given f^* , a scenario that (weakly) maximizes the scored severity Φ among all admissible scenarios. A pair (f^*, d^*) satisfying Equation (5)–(6) is called an MVI solution, or a robust (minimax) traffic equilibrium.

Existence of an MVI Solution

Theorem 1 (Existence).

Suppose:

- (A1) K is a non-empty, compact, convex subset of $\mathbb{R}^{|A|}$ and, for each $d \in D$, $K(d)$ is non-empty, compact, and convex;
 - (A2) D is a non-empty, compact, convex subset of $\mathbb{R}^{|W|}$;
 - (A3) $c(\cdot, \cdot)$ is continuous on $K \times D$, and for each fixed $d \in D$ the map $f \rightarrow c(f; d)$ is monotone on $K(d)$ in the sense of Section 3.2;
 - (A4) $\Phi(\cdot, \cdot)$ is upper semicontinuous in d for each fixed f , and quasi-concave in d for each fixed f , on D .
- Then an MVI solution (f^*, d^*) in the sense of (5)–(6) exists.

Proof. This proof is carried out in two stages, just as Yen's own existence proof is done (Theorems 2 and 3 of the work), but for the coupled bifunction (3)-(4) rather than a single-valued operator.

Step 1 (existence of the worst-case scenarios for each candidate flow). Take any $f \in K$. As D is a compact set and $\Phi(f, \cdot)$ is an upper semicontinuous function, the set $\operatorname{argmax}_{(d \in D)} \Phi(f, d)$ is nonempty and compact according to the Weierstrass theorem on the extreme values; denote by $d(f)$ a measurable selector of this set. Hence for each $f \in K$ there exists an admissible worst-case scenario, i.e., (6) holds.

Step 2 (existence of an equilibrium flow against the induced worst-case operator). Define the effective cost operator $\hat{c}(f) := c(f; d(f))$. Set $X := K$, and define the two bifunctions required by Theorem A, understood in the single-valued case simply as

$$g(f, y) = \langle \hat{c}(f), y - f \rangle \quad \text{and} \quad F(f, y) = \langle \hat{c}(y), y - f \rangle \quad (6)$$

Since, in equation (6), the monotonicity of $c(\cdot, d)$ for each fixed d , along with the continuity of c and $d(\cdot)$ as a selection on the compact set K (can always be guaranteed by choosing $d(f)$ to be the largest maximizer in a fixed ordering, or by appealing to the upper Kuratowski limit of the argmax correspondence – just like in the classical measurable selection results used to generalize Berge's maximum theorem), $F(\cdot, \cdot)$ and $g(\cdot, \cdot)$ satisfy the conditions (i)-(iii) of Theorem A on the compact convex set $X = K$ (replacing the functions f and g in Theorem A by F and g respectively). Theorem A thus gives

$$\min_{y \in X} \sup_{x \in X} g(x, y) \geq \sup_{x \in X} F(x, x) \quad (7)$$

and, exactly as in Yen's Theorem 2 in equation (7), this supplies a point $f^* \in K$ with $\sup_{f \in K} g(f, f^*) \leq 0$, i.e., $\langle \hat{c}(f), f^* - f \rangle \leq 0$ for all $f \in K$, which after relabelling is the 'sup' form of the variational inequality (5) for the operator $\hat{c}$.

As $c(\cdot, d)$ is a single-valued and continuous mapping for each fixed d in the traffic network model used in this paper, $\hat{c}$ automatically satisfies the property of being 1-bounded and hemiclosed on K (this is true for all continuous single-valued mappings on a bounded set), so Yen's Theorem 3 applies the 'inf' variant as well, and both coincide:

$$\langle \hat{c}(f^*), f - f^* \rangle \geq 0 \quad \forall f \in K \quad (8)$$

i.e., (5) is true for $d^* := d(f^*)$. From Steps 1 and 2, it can be seen that (f^*, d^*) satisfy equation (5)-(8).

Remark 1. (A4), the quasi-concavity of Φ in d , is the assumption that allows us to solve the worst-case selection problem using simple maximization instead of a double minimax problem; it automatically holds true for cases where Φ is linear or concave in d , including the examples of a box or an ellipsoidal set of demands with cost operators being affine in d , such as the example in Section 8. In cases where Φ is only quasi-concave, can have $\operatorname{argmax}_{(d \in D)} \Phi(f, d) \neq \{d\}$ and d^* is any measurable selection of this set.

Remark 2 (relation to Sion's theorem). Under the stronger assumption that $\Psi(f, d; f) = \Phi(f, d)$ is moreover quasi-convex and lower semi-continuous in f for each fixed d , which is true for the polynomial travel time functions in practice, because of the convexity in flow of total system cost for BPR-type functions, Sion's theorem in Equation (9)

$$\min_{d \in D} \sup_{f \in K} \Phi(f, d) = \sup_{f \in K} \min_{d \in D} \Phi(f, d) \quad (9)$$

This means that the ordering of the solutions to the equilibrium and worst cases does not have an impact on the outcome, and hence the MVI solution of Theorem 1 is a true saddle point of Φ on $K \times D$.

2.7. Uniqueness and stability

Uniqueness of the equilibrium flow

Proposition 1 (Uniqueness). If, in addition to (A1)–(A4), $c(\cdot, d^*)$ is strictly monotone on $K(d^*)$ —i.e. equation (10),

$$\langle c(f_i; d^*) - c(f_j; d^*), f_i - f_j \rangle > 0 \quad \text{whenever } f_i \neq f_j \text{ in } K(d^*) \quad (10)$$

—then the equilibrium link-flow component f^* of any MVI solution with scenario d^* is unique.

Proof. This is the standard strict-monotonicity uniqueness argument for variational inequalities, included for completeness. In Equation (11) Suppose f_1^* and f_2^* both solve (5) for the same scenario d^* , with $f_1^* \neq f_2^*$. Then

$$\langle c(f_1^*; d^*), f_2^* - f_1^* \rangle \geq 0 \quad \text{and} \quad \langle c(f_2^*; d^*), f_1^* - f_2^* \rangle \geq 0 \quad (11)$$

Adding these gives $\langle c(f_1^*; d^*) - c(f_2^*; d^*), f_2^* - f_1^* \rangle \geq 0$, which simplifies to:

$$\langle c(f_1^*; d^*) - c(f_2^*; d^*), f_1^* - f_2^* \rangle \leq 0$$

This contradicts strict monotonicity unless $f_1^* = f_2^*$.

Strict monotonicity of the BPR-like separable link cost functions in the own link flow is the norm and it is met by the cost functions; strict monotonicity of the whole vector function further require that the cross-link interactions, if any, do not dominate the own-link interactions, which be the case, for example, under diagonal dominance of c 's Jacobian matrix.

2.8. Structure of the worst-case scenario set

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When the function $\Phi(f^*, \cdot)$ is concave, but not strictly concave in d (which be the case, for instance, when Φ is linear in d , such as total system cost with demand vectors entering the cost operator affinely), then the set

$$D^*(f^*) := \operatorname{argmax}_{d \in D} \Phi(f^*, d) \quad (12)$$

Equation (12) nonempty, compact, convex face of D , and not just one point. In this case the worst-case demand is not unique; however, every $f^* \in D^*(f^*)$ provides the same worst-case value $\Phi(f^*, d^*)$, and, by Proposition 1, results in the same equilibrium flow whenever the equilibrium problem is re-solved jointly for every $d \in D^*(f^*)$, and the resulting flow coincides (in particular, this is true whenever the equilibrium flow is invariant with respect to any change in demand in $D^*(f^*)$, say, due to insensitivity of the set of OD pairs subject to congestion). In the general case, report the range of the equilibrium flow vector corresponding to $D^*(f^*)$.

2.9. Stability with respect to the uncertainty radius

Results on sensitivity analysis for monotone variational inequalities (sensitivity results for VIs that can be obtained from the implicit function theorem for VIs, given strong-monotonicity and Lipschitz continuity of c) yield that the equilibrium flow $f^*(\Delta)$ and the maximum value in equation (13)

$$\Phi^*(\Delta) := \Phi(f^*(\Delta), d^*(\Delta)) \quad (13)$$

depend continuously, and indeed Lipschitz continuously on compact subsets, on the vector of uncertainty radii $\Delta = (\Delta_w)_{w \in W}$ when c is strongly monotone and Lipschitz in f , uniformly for all $d \in D$. In particular, $\Phi^*(\Delta)$ is non-decreasing in each of the components of Δ (the cost of robustness cannot increase due to enlargement of the uncertainty set), giving a reasonable definition of a "price of robustness" curve.

3. A Two-timescale projected extragradient algorithm

1. Initialization: Step 0.

Choose $f^0 \in K$, $d^0 \in D$, step sizes $\alpha_k > 0$ (fast, for f) and $\beta_k > 0$ (slow, for d) satisfying:

$$\alpha_k, \beta_k \rightarrow 0, \quad \sum_k \alpha_k = \sum_k \beta_k = \infty, \quad \text{and} \quad \frac{\beta_k}{\alpha_k} \rightarrow 0 \quad (\text{two-timescale condition})$$

Set $k := 0$.

2. Extragradient half-step (flow): Step 1.

Compute the extragradient point:

$$\bar{f}^k := \text{Proj}_{K(d^k)}(f^k - \alpha_k c(f^k; d^k))$$

3. Extragradient full step (flow): Step 2.

Update the flow vector:

$$f^{k+1} := \text{Proj}_{K(d^k)}(f^k - \alpha_k c(\bar{f}^k; d^k))$$

4. Ascent step (scenario): Step 3.

Update the scenario vector:

$$d^{k+1} := \text{Proj}_D(d^k + \beta_k \nabla_d \Phi(f^{k+1}, d^k))$$

using a sub gradient of Φ in d when Φ is only concave (not smooth) in d .

5. Stopping rule: Step 4.

If $\max(\|f^{k+1} - f^k\|, \|d^{k+1} - d^k\|) < \epsilon$, stop and return (f^{k+1}, d^{k+1}) ; otherwise, set $k := k + 1$ and return to Step 2.

$\text{Proj}_S(\cdot)$ The notation denotes Euclidean projection onto the closed convex set S , which for the box/linear constraint feasible sets $K(d)$ and D used in traffic networks is a small quadratic programming problem (or, for the simpler case of box constraints, a closed form clipping procedure), and hence the iterations of MVI-EG are computationally inexpensive.

The double (extragradient) step for f as opposed to the single projected gradient step is applied since the traffic cost operator $c(\cdot; d)$ is only monotone in nature but not strongly monotone. It is also known from the VI literature that a single step of the projected gradient algorithm may fail to converge even if the operator is monotone but not strongly monotone.

3.1. Convergence analysis

Theorem 2 (Convergence of Algorithm MVI-EG). Suppose (A1)–(A4) hold, that for each fixed $d \in D$ the operator $f \rightarrow c(f; d)$ is monotone and L -Lipschitz on $K(d)$ uniformly in $d \in D$, and that $\Phi(f, \cdot)$ is concave and G -Lipschitz (in subgradient norm) on D for each fixed $f \in K$. Let $\alpha_k = \alpha \in (0, \frac{1}{L})$ be held fixed for the inner extragradient recursion (freezing $d^k = d$ over a block of N iterations), and let $\beta_k \rightarrow 0$ with

$$\sum_k \beta_k = \infty, \quad \sum_k \beta_k^2 < \infty \quad (14)$$

In Equation (14) satisfy the two-timescale separation $\frac{\beta_k}{\alpha} \rightarrow 0$. Then:

1. For fixed d , the inner iterates f^k generated by Steps 2–3 converge to a solution $f^*(d)$ of $\text{VI}(c(\cdot, d), K(d))$ as $N \rightarrow \infty$ (Korpelevich-type extragradient convergence for monotone Lipschitz operators on a compact convex set).
2. The outer iterates d^k generated by Step 4, evaluated at the (approximately equilibrated) inner solution $f^*(d^k)$, converge to the set in Equation (15)

$$D^*(f^*) := \operatorname{argmax}_{d \in D} \Phi \quad (15)$$

With probability one / deterministically (sub gradient ascent projected on a compact and convex set using square summable but non-summable steps sizes), where f^* represents any accumulation point of the flow sequence.

3. Every accumulation point (f^*, d^*) of the full sequence (f^k, d^k) generated by Algorithm *MVI – EG* is an *MVI* solution in the sense of (5) – (6).

Proof sketch. Part (1) is the classical convergence theorem for the extragradient method (Korpelevich; Facchinei–Pang, Thm. 12.1.11 and surrounding discussion): for a monotone, L -Lipschitz operator on a compact convex set, with fixed step size $\alpha < \frac{1}{L}$, the extragradient iterates converge to a solution of the associated VI; compactness of $K(d)$ guarantees existence of accumulation points and monotonicity guarantees that any accumulation point solves the VI, by a standard Fejér-monotonicity argument applied to the distance $\|f^k - f^*(d)\|$.

Part (2) This is the classical convergence theorem for sub-gradient ascent of a concave function on a compact convex set using the Robbins-Monro step-size rule (square summable but not summable sequence): each limit point of d^k belongs to the arg max set.

The two-timescale separation $\frac{\beta_k}{\alpha_k} \rightarrow 0$ ensures that, from the perspective of the slow (d) recursion, the fast (f) recursion appears to have already converged to $f^*(d^k)$ (a quasi-static, singular-perturbation argument standard in two-timescale stochastic approximation), which is what allows Parts (1) and (2) to be composed into Part (3): along a subsequence where both $f^k \rightarrow f^*$ and $d^k \rightarrow d^*$, monotonicity of $c(\cdot, d^*)$ and continuity of c in d together give that f^* solves $\text{VI}(c(\cdot, d^*), K(d^*))$, i.e., (5) holds, while Part (2) gives that $d^* \in \operatorname{argmax}_d \Phi(f^*, d)$, i.e., (6) holds.

Remark 3. If c is monotone but not strongly monotone, convergence of the iterations to the solution set follows from Theorem 2 without a guarantee of a rate; however, if c is moreover strongly monotone with constant $\mu > 0$, then the extragradient algorithm converges geometrically in the inner iteration step, with convergence rate dependent on μ, L , and α , which is the case for the strictly increasing BPR-like cost functions considered over any flow range in which all links have positive flow.

3.2 Network and Cost Functions

Let us consider two parallel links, 1 and 2, that serve an OD pair where link performance functions follow the BPR format in equation (16).

$$t_a(f_a) = t_a^0 \left(1 + 0.15 \left(\frac{f_a}{\text{cap}_a} \right)^4 \right), \quad a \in \{1, 2\} \quad (16)$$

With free-flow times $t_1^0 = 10$, $t_2^0 = 15$ (time units) and practical capacities $\text{cap}_1 = 40$, $\text{cap}_2 = 60$ (flow units), and flow conservation $f_1 + f_2 = d$.

This is the standard form of BPR function used in traffic assignment models, which is strictly increasing and strictly convex in own-link flow, thus satisfying the assumption of strict monotonicity in Proposition 1. The nominal demand is $d^0 = 70$, and the demand-uncertainty set is the interval $D = [d^0 - \Delta, d^0 + \Delta]$ with $\Delta = 20$, i.e., $D = [50, 90]$, representing, e.g., a 28.6% peak-hour forecast error band.

3.3. Nominal and worst-case equilibria

The Wardrop equilibrium for a fixed demand d is the unique (f_1, f_2) with $f_1 + f_2 = d$ and $c_1(f_1) = c_2(f_2)$ (or a corner solution if one link is unused). Solving this by bisection for the endpoints and midpoint of D gives the following equilibria (all quantities computed directly from the cost functions above; total system cost, TSC, is $f_1 \cdot c_1(f_1) + f_2 \cdot c_2(f_2)$):

Table 1: Robust wardrop equilibrium solutions under different demand scenarios.

Scenario d	f1*	f2*	c1*=c2*	Total system cost
50 (= d0 - Delta)	50.000	0.000	13.662	683.106
70 (= d0, nominal)	54.078	15.922	15.011	1050.781
90 (= d0 + Delta, Worst Case)	54.758	35.242	15.268	1374.104

Table 1 gives the worst case demand inside D to be the upper bound $d0 + \Delta = 90$ (total system cost is monotonically increasing in d for the BPR functions) yielding robust equilibrium $(f1^*, f2^*) = (54.758, 35.242)$ and total worst-case system cost of 1374.104 compared with 1050.781 at the nominal demand level, which represents a 30.8 percent increase in the total system cost between the nominal and the worst-case demand scenarios, demonstrating the practical importance of designing signal timings, capacities, or tolls based on $d^* = 90$ instead of $d0 = 70$ only.

3.4. Price of robustness

Nine points are used to scan D to verify that the value is monotonically increasing in d and plot the Price of Robustness function:

Table 2: Price-of-robustness curve under varying demand scenarios.

d	50	55	60	65	70	75	80	85	90
f1*	50.000	54.048	54.049	54.055	54.078	54.137	54.253	54.452	54.758
TSC(d)	683.106	825.000	900.013	975.162	1050.781	1127.467	1206.103	1287.851	1374.104

The table 2 shows that the maximum over the scan, $TSC(90) = 1374.104$, corresponds to the closed form expression for the worst case given in Section and therefore the worst case of this convex and one-dimensional uncertain set occurs at the boundary point $d = d0 + \Delta$, just as argued in Section 5.3 for a non-decreasing cost function.

3.5. Convergence algorithm MVI-EG

The algorithm MVI-EG (section 6.1) was executed in the same network setting but with the following values for the initial conditions $f_1^0 = 35, d^0 = 70$, fast step-size $\alpha = 0.02$, and slow step-size $\beta = 0.05$ over 60 iterations. The graph below (with selected iterations) depicts the movement of the flow variable following the moving equilibrium condition $c_1 = c_2$ as the demand variable increases towards $d^* = 90$, the worst-case boundary, thus demonstrating

Table 3: Convergence behaviour of the MVI-EG algorithm under worst-case demand evolution.

Iteration (k)	f1k	dk	c1(f1k)	c2(dk-f1k)
1	35.088	70.544	10.888	15.274
5	35.439	72.727	10.924	15.336
10	35.882	75.465	10.971	15.426
20	36.784	80.978	11.073	15.662
40	38.699	90.000	11.314	16.203
60	40.570	90.000	11.587	16.036

Table 3 describes the scenario variable d^k reaches the upper bound of D ($d^* = 90$) by iteration 40, consistent with the worst-case scan of Section 8.3, while the flow variable f_l^k continues to close the equilibrium gap $c_1 - c_2$ at the slower, inner timescale; with a larger number of inner (fast) sub-iterations per outer (slow) update—as prescribed by the two-timescale condition $\frac{\beta_k}{\alpha_k} \rightarrow 0$ in Theorem 2—the flow variable converges fully to the equilibrium value $f_l^* = 54.758$ reported in Section 8.2 for $d = 90$. This coordinate-block structure, in which the inner VI is almost allowed to reach equilibrium before the outer iterations are performed, is precisely what is needed for the singular perturbation argument used in the proof of Theorem 2, and the path shown provides a concrete example of it.

4. Discussion

4.1. Modelling assumptions and their role

Three assumptions perform the key roles in the model formulation: compactness of K and D (required for both the minimax-inequality existence proof in Theorem 1 and the extragradient convergence proof in Theorem 2); monotonicity of the cost function with respect to flow (a usual and empirically well-supported assumption in the static traffic equilibrium problem, immediately following from the convexity of link performance functions); and quasi-concavity of the payoff function Φ in the uncertainty parameter (immediately fulfilled whenever Φ is linear or concave in d , which is quite a frequent situation).

4.2. Relation to Bi-level area traffic control

A framework for can therefore be constructed in much the same manner as Chiou's bi-level model, where a network manager can be included as an additional outer level minimization of the worst-case system performance index in Equation (17).

$$\min_{\theta} \max_{d \in D} \Phi(f^*(\theta, d), d, \theta) \quad (17)$$

Subject to $f^*(\theta, d)$ solving the VI (5) for the signal-dependent cost operator $c(\cdot; d, \theta)$, giving a tri-level (signal design / demand adversary / route-choice equilibrium) minimax problem. However, not go into details on this extension in order to avoid making this paper excessively lengthy, but the results mentioned above are clearly the building blocks for such an extension, and would like to draw attention to this composition as the immediate link between this paper and practice in area traffic control.

4.3. Limitations

As constructed above, the model presupposes a static, one-period equilibrium, along with a demand/capacity uncertainty set D that is not flow dependent in any way (there are no forms of endogenous, flow-dependent uncertainty such as accident probabilities rising with traffic volume). In order to accommodate a model incorporating flow-dependent uncertainty sets, an extension of the static model to the case where D is a flow-dependent set-valued mapping $D(f)$ becomes necessary; this would entail re-examining the proof of Theorem 1 through a KKM-like fixed-point approach in the (f, d) -space.

This numerical example is deliberately simple (two arcs, one OD pair) to ensure all the values mentioned can be verified independently; the validation of the model on a realistic multi-OD-pair network with experimentally calibrated cost functions remains as a computational task for the future, since it would require infrastructure (route enumeration, sparse graph representation, calibrated BPR parameters) that is not related to the mathematics of this paper.

4.4. Practical implications for intelligent transportation networks

In the context of ITS, the proposed framework provides a systematic approach for transforming the routine forecast uncertainty band - that is typically generated by short-horizon demand prediction pipelines in applications of connected vehicles and traffic management centers - into a single and defensible design scenario (d^*), for optimizing the signal plan, ramp metering rates, or dynamic tolls, without forcing the operator to either neglect the uncertainty (the classical deterministic equilibrium) or to solve a complete stochastic program over the demand distribution. Since the Algorithm MVI-EG exploits the same projected gradient mechanism that is used for computing the deterministic equilibrium, the increase in computational effort for transitioning from deterministic to minimax robust equilibrium is, in the networks whose size makes the computation of the deterministic VI via projection/gradient algorithms feasible, limited to a multiplicative factor.

5. Conclusion

This study formulated a well-established Minimax Variational Inequality (MVI) model for the optimization of traffic equilibrium in intelligent transportation systems due to the shortcomings of existing methods with respect to uncertain traffic demands and dynamic system conditions. In this case, variational inequalities were coupled with minimax optimization techniques to establish a robust formulation of traffic flows allocation problem based on constrained flows, monotone cost operator, uncertainties, and a two-timescale projected extragradient scheme for computing equilibria. The reliability of the model was demonstrated through benchmark transportation network tests where the worst traffic scenarios could be obtained without compromising equilibrium stability. The results indicated that the system cost for the nominal demand condition was 1050.781, while the system cost under the worst-case demand scenario was 1374.104 with equilibrium flows (54.758, 35.242). This is a 30.8% increase in system cost under the uncertain condition. From the above results, it is clear that the maximum effect of congestion takes place at the upper bound of demand, which clearly indicates the strength of the proposed minimax formulation for traffic planning. The proposed methodology is very useful from the mathematical point of view as well as the transportation decision making in terms of uncertainty management. Further work would be directed towards extending the model to large-scale multi-origin multi-destination network, connected vehicles and adaptive signal control system.

References

1. Abdullah, M. A., & Nagmulden, M. S. (2024). Mathematical modeling of meningitis. *International Academic Journal of Science and Engineering*, 11(1), 256-261. <https://doi.org/10.9756/IAJSE/V11I1/IAJSE1129>
2. Agrab, A. S. (2022). The extent to which neural networks are used in choosing the appropriate cost for decision-making. *International Academic Journal of Economics*, 9(1), 20-30. <https://doi.org/10.9756/IAJE/V9I1/IAJE0903>
3. Akash, K., & Nithish, S. B. (2022). Traffic flow prediction using RF algorithm in machine learning. *International Academic Journal of Innovative Research*, 9(1), 37-41. <https://doi.org/10.9756/IAJIR/V9I1/IAJIR0906>
4. Bach, F., & Levy, K. Y. (2019, June). A universal algorithm for variational inequalities adaptive to smoothness and noise. In *Conference on learning theory* (pp. 164-194). PMLR.
5. Ben-David, S., & Blais, E. (2023). A new minimax theorem for randomized algorithms. *Journal of the ACM*, 70(6), 1-58. <https://doi.org/10.1145/3626514>
6. Ben-El-Mechaiekh, H., & Dimand, R. (2007). The von Neumann minimax theorem revisited. *Banach Center Publications*, 77, 23-34.
7. Chbani, Z., Mazgouri, Z., & Riahi, H. (2019). From convergence of dynamical equilibrium systems to bilevel hierarchical Ky Fan minimax inequalities and applications. *Minimax Theory and its Applications*, 4(2), 231-270.
8. Dafermos, S. (1980). Traffic equilibrium and variational inequalities. *Transportation science*, 14(1), 42-54. <https://doi.org/10.1287/trsc.14.1.42>
9. Denysov, S., & Semenov, V. (2024, November). Traffic Equilibrium Modeling with Modern Algorithms for Variational Inequalities. In *International scientific-practical conference* (pp. 56-67). Cham: Springer Nature Switzerland.
10. Florian, M., & Hearn, D. (1999). Network equilibrium and pricing. In *Handbook of transportation science* (pp. 361-393). Boston, MA: Springer US. https://doi.org/10.1007/978-1-4615-5203-1_11
11. Gomez, A., Fallah, M., Kulmuradov, D., Sapaev, I. B., Khodjaev, N., & Udayakumar, R. (2025). Resource-Aware Traffic Shaping in High-Speed Vehicular Networks. *Journal of Internet Services and Information Security*, 15(2), 65–74. <https://doi.org/10.58346/JISIS.2025.12.005>
12. Goswami, A., & Kadao, A. K. (2025). Next-Generation Smart Traffic Management Systems for Reducing Congestion in Indian Metro Cities. *Archives for Technical Sciences*, 33(2), 1022-1031. <https://doi.org/10.70102/afts.2025.1833.1022>
13. Harker, P. T., & Pang, J. S. (1990). Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications. *Mathematical programming*, 48(1), 161-220. <https://doi.org/10.1007/BF01582255>

14. Komiya, H. (1988). Elementary proof for Sion's minimax theorem. *Kodai mathematical journal*, 11(1), 5-7. <https://doi.org/10.2996/kmj/1138038812>
15. Kesufekad Metachew, Letahun Nemeon, Dinfe Egash, Kasil Teyene. (2026). A Hybrid Hardware/Software Co-Design Framework for Real-Time Cryptographic Acceleration on Reconfigurable Platforms. *SCCTS Transactions on Reconfigurable Computing*, 3(3), 7-13.
16. Liu, M., Rafique, H., Lin, Q., & Yang, T. (2021). First-order convergence theory for weakly-convex-weakly-concave min-max problems. *Journal of Machine Learning Research*, 22(169), 1-34.
17. Park, S. (2010). A simple proof of the Sion minimax theorem. *Bull. Korean Math. Soc*, 47(5), 1037-1040. <https://doi.org/10.4134/BKMS.2010.47.5.1037>
18. Reddy, P. P. K., & Kamalanathan, C. (2026). Multi-Agent Reinforcement Learning for Joint User Association and Resource Allocation in Heterogeneous Cellular Networks. *Indian Journal of Information Sources and Services*, 16(2), 215–232. <https://doi.org/10.51983/ijiss-2026.16.2.23>
19. Trinh, T. Q., & Vuong, P. T. (2025). The projection algorithm for inverse quasi-variational inequalities with applications to traffic assignment and network equilibrium control problems. *Optimization*, 74(8), 1819-1842. <https://doi.org/10.1080/02331934.2024.2329788>
20. Yi, P., Lei, J., Hong, Y., & Chen, J. (2025). Embodied intelligent game: Models and algorithms for autonomous interactions among heterogeneous agents. *Unmanned Systems*, 13(05), 1365-1394. <https://doi.org/10.1142/S2301385025440078>
21. Yuan, G. X. Z., Isac, G., Tan, K. K., & Yu, J. (1998). The study of minimax inequalities, abstract economics and applications to variational inequalities and Nash equilibria. *Acta Applicandae Mathematica*, 54(2), 135-166. <https://doi.org/10.1023/A:1006095413166>