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Minimax Inequalities in the Absence of Topological Assumptions

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We state a new minimax inequality without any topological assumptions. Instead, a boundedness condition is required on one of the variables.

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1. Introduction

In order to be precise, let us point out that by a *minimax inequality* we mean a result guaranteeing that, under suitable hypotheses, a function $f\colon X \times Y \longrightarrow \mathbb{R}$ satisfies the inequality

$$\inf_{y\in Y} \sup_{x\in X} f(x,y) \leq \sup_{x\in X} \inf_{y\in Y} f(x,y)$$

and therefore the equality also holds, since the opposite inequality is always true. We refer to [5, 6, 8, 9, 11, 14, 16, 17, 18, 22, 24, 25, 26] for some results on minimax theory and its applications. Almost all minimax inequalities, as the classical ones of K. Fan ([1, Theorem 2]) or M. Sion ([21, Theorem 3.4]), assume some topological conditions. Such topological hypotheses are typically of the type: X is a compact topological space and f is upper semicontinuous on X – although this is not always the case: see, for instance, [19, 10, 3, 7]. If in addition f is continuous on X, then clearly this boundedness condition is fulfilled:

$$y \in Y \ \Rightarrow \ f(\cdot, y) \in \ell^\infty(X), \tag{1}$$

$\ell^\infty(X)$ being the vector space of all real-valued functions defined and bounded on X. This is precisely the assumption we will use, instead of any other one of a topological character.

On the other hand, let $N \geq 1$ and

$$\mathbb{P}_N := \left\{\mathbf{p} \in \mathbb{R}^N : \ p_1, \dots, p_N \geq 0 \text{ and } p_1 + \cdots + p_N = 1\right\}.$$

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Then, a direct application of the Hahn-Banach theorem – see [20, Lemma 9] – yields that given a nonempty convex subset X of a real vector space and a family of convex functions $f_1, \dots, f_N \colon X \longrightarrow \mathbb{R}$, then

$$\text{there exists } \mathbf{p} \in \mathbb{P}_N : \inf_{\mathbf{x} \in X} \max_{j=1,\dots,N} f_j(x) = \inf_{\mathbf{x} \in X} \sum_{j=1}^{N} p_j f_j(x). \tag{2}$$

This property of the convex functions characterizes the *infsup-convexity* of a finite family of functions defined on a nonempty set X, a concept arising naturally in minimax theory ([2, 23, 14]) and that we recall in Definition 2.1 below. More specifically (see [13, Corollary 2.5]), if X is a nonempty set, $N \geq 1$ and $f_1, \dots, f_N \colon X \longrightarrow \mathbb{R}$ are N given functions, then the family $f_1, \dots, f_N$ is infsup-convex on X if, and only if, condition (2) is fulfilled. Both the property (2) for convex functions, and the characterization just mentioned, are basic tools for stating some minimax inequalities that involve a certain kind of topological hypotheses such as compactness or semicontinuity ([20, Theorem 12 and Remark 13], [18, Theorem 3.1]) or some results on nonlinear programming ([13, Theorem 3.3]).

In Section 2 we generalize the characterization of the infsup-convexity [13, Corollary 2.5] when an arbitrary number of functions, satisfying the boundedness condition (1), is considered. In order to establish it, we make use of a theorem of the alternative of the Gordan type [12, Theorem 2.3], which allows one even to consider infinitely many functions. Lastly, Section 3 deals with stating a new minimax theorem where the topological hypotheses are replaced with that of boundedness (1). Furthermore, we show that some recent minimax inequalities with topological assumptions, sharp in a sense, are independent of our minimax inequality without any topological assumption.

2. Characterizing the infsup-convexity

In this section we focus on deriving the aforementioned characterization of the infsup-convexity from the theorem of the alternative [12, Theorem 2.3].

Let us consider the following concept of generalized convexity, first appeared as condition (8) in [2, Corollary 3.1] (see also [23, Definition 2.11]). We use this ad hoc name:

Definition 2.1. If X and Y are nonempty sets, a function $f \colon X \times Y \longrightarrow \mathbb{R}$ is said to be *infsup-convex* on Y provided that

$$\left. \begin{array}{l} m \geq 1, \ \mathbf{t} \in \mathbb{P}_m \\ y_1, \dots, y_m \in Y \end{array} \right\} \Rightarrow \inf_{y \in Y} \sup_{x \in X} f(x,y) \leq \sup_{x \in X} \sum_{j=1}^{m} t_j f(x, y_j).$$

In an analogous way, f is *supinf-concave* on X when

$$\left. \begin{array}{l} n \geq 1, \ \mathbf{s} \in \mathbb{P}_n \\ x_1, \dots, x_n \in X \end{array} \right\} \Rightarrow \inf_{y \in Y} \sum_{i=1}^{n} s_i f(x_i, y) \leq \sup_{x \in X} \inf_{y \in Y} f(x,y).$$

It is worth pointing out that this notion of convexity not only properly extends that of classical convexity, but also that of convexlikeness due to K. Fan [1, p. 42], as was shown in [15, Corollary 2.3 and Example 2.4] and [14, Example 2.2].

Let us consider some notations. Let X be a nonempty set and $S\colon \ell^\infty(X) \longrightarrow \mathbb{R}$ be the sublinear functional given for each $h \in \ell^\infty(X)$ as

$$S(h) := \sup_{x\in X} h(x).$$

If we write

$$\mathbb{P}_X := \{L : \ell^\infty(X) \longrightarrow \mathbb{R} : \ L \text{ is linear and } L \leq S\},$$

then, it is a well-known (see, for instance, [4, p. 500]) and easy-to-prove fact that a linear functional $L\colon \ell^\infty(X) \longrightarrow \mathbb{R}$ belongs to $\mathbb{P}_X$ if, and only if, it is positive ($L(h) \geq 0$, whenever $h \in \ell^\infty(X)$ with $h \geq 0$), and $L(\mathbf{1}) = 1$, $\mathbf{1}$ being the constant 1 function. Obviously, when X is a finite set of cardinal N, then $\mathbb{P}_X$ can be identified with the probability simplex $\mathbb{P}_N$.

Now, let us evoke the aforementioned theorem of the alternative [12, Theorem 2.3]:

Theorem 2.2. *If X and Y are nonempty sets and $f\colon X \times Y \longrightarrow \mathbb{R}$ is an infsup-convex function on Y satisfying*

$$y \in Y \ \Rightarrow \ f(\cdot, y) \in \ell^\infty(X),$$

then exactly one of the following alternatives holds:

(i) *There exists $y \in Y$ with $\sup_{x\in X} f(x,y) < 0$.*

(ii) *There exists $L \in \mathbb{P}_X$ such that $\inf_{y\in Y} L\left(f(\cdot, y)\right) \geq 0$.*

As announced above, now we generalize [15, Proposition 2.11] (and therefore [20, Lemma 9]), that is, we state a characterization of the infsup-convexity of a function under a suitable boundedness assumption.

Proposition 2.3. *Let X and Y be nonempty sets and let $f\colon X \times Y \longrightarrow \mathbb{R}$ be a function such that*

$$y \in Y \ \Rightarrow \ f(\cdot, y) \in \ell^\infty(X).$$

Then f is infsup-convex on Y if, and only if, there exists $L \in \mathbb{P}_X$ such that

$$\inf_{y\in Y} \sup_{x\in X} f(x,y) = \inf_{y\in Y} L(f(\cdot, y)).$$

Proof. Let us first assume that f is infsup-convex on Y and define

$$\alpha := \inf_{y\in Y} \sup_{x\in X} f(x,y),$$

which we suppose finite without any lost of generality. Then, Theorem 2.2, when applied to the function $f-\alpha$, yields

$$\text{there exists } L \in \mathbb{P}_X : \inf_{y\in Y} \sup_{x\in X} f(x,y) \le \inf_{y\in Y} L(f(\cdot,y)),$$

since its alternative (i) is not fulfilled, L is linear and $L(\mathbf{1})=1$. The opposite inequality is valid, according to the fact that $L \in \mathbb{P}_X$.

And conversely, the existence of $L \in \mathbb{P}_X$ such that

$$\inf_{y\in Y} \sup_{x\in X} f(x,y) = \inf_{y\in Y} L(f(\cdot,y))$$

implies the infsup-convexity of f on Y, since given $m\ge 1$, $\mathbf{t}\in\mathbb{P}_m$ and $y_1,\ldots,y_m\in Y$, we have

$$\begin{aligned}\inf_{y\in Y}\sup_{x\in X} f(x,y) &= \inf_{y\in Y} L(f(\cdot,y)) \le \min_{j=1,\ldots,m} L(f(\cdot,y_j)) \le \sum_{j=1}^m t_j L(f(\cdot,y_j))\\ &= L\left(\sum_{j=1}^m t_j f(\cdot,y_j)\right) \le \sup_{x\in X}\sum_{j=1}^m t_j f(x,y_j). \qquad \square\end{aligned}$$

The necessary condition for the infsup-convexity of f on Y given in Proposition 2.3 is a proper generalization of [20, Remark 10], where the function f is assumed to be convex on Y.

3. Connection with minimax inequalities

This section deals with deriving our minimax inequality –Theorem 3.1 below– without any topological assumption.

Proposition 2.3 generalizes the key tool, [15, Proposition 2.11] (or [14, Theorem 2.3]), which yields the minimax theorem [15, Theorem 2.15] ([14, Theorem 3.8]), an extension of the classical Fan's minimax inequality [1, Theorem 2]. Therefore, Proposition 2.3 implies such a recent minimax result. In such a minimax inequality, some compactness and semicontinuity hypotheses are assumed, unlike the following characterization of the minimax identity, where they are replaced with the boundedness condition (1).

Theorem 3.1. *Suppose that X and Y are nonempty sets and that $f\colon X\times Y \longrightarrow \mathbb{R}$ is a function satisfying*

$$y\in Y \;\Rightarrow\; f(\cdot,y)\in \ell^\infty(X).$$

Then f is infsup-convex on Y and for each $L\in\mathbb{P}_X$, the inequality

$$\inf_{y\in Y} L(f(\cdot,y)) \le \sup_{x\in X}\inf_{y\in Y} f(x,y)$$

is valid, if, and only if,

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) = \sup_{x\in X}\inf_{y\in Y} f(x,y).$$

Proof. The minimax identity clearly follows from Proposition 2.3 and the easy, well-known and aforementioned fact in the Introduction that the inequality

$$\sup_{x\in X}\ \inf_{y\in Y} f(x,y) \le \inf_{y\in Y}\ \sup_{x\in X} f(x,y)$$

always holds.

For the converse, it is a basic fact to check that the minimax relation implies the infsup-convexity of f on Y, since given $m \ge 1$, $\mathbf{t} \in \mathbb{P}_m$ and $y_1, \dots, y_m \in Y$,

$$\begin{aligned}\inf_{y\in Y}\ \sup_{x\in X} f(x,y) &= \sup_{x\in X}\ \inf_{y\in Y} f(x,y) &&\text{(validity of the minimax identity)}\\ &\le \sup_{x\in X} \sum_{j=1}^{m} t_j f(x,y_j) &&(\mathbf{t}\in\mathbb{P}_m).\end{aligned}$$

Finally, if $L \in \mathbb{P}_X$, that is, $L \le S$, where $S := \sup\limits_X$ on $\ell^\infty(X)$, then

$$\begin{aligned}\inf_{y\in Y} L(f(\cdot,y)) &\le \inf_{y\in Y}\sup_{x\in X} f(x,y) &&(L\le S)\\ &= \sup_{x\in X}\inf_{y\in Y} f(x,y) &&\text{(validity of the minimax identity).}\end{aligned} \qquad \square$$

As we have checked in the proof of Theorem 3.1 (see also [14, Lemma 3.5]),

$$\inf_{y\in Y}\ \sup_{x\in X} f(x,y) = \sup_{x\in X}\ \inf_{y\in Y} f(x,y) \ \Rightarrow\ f \text{ is infsup-convex on } Y,$$

and reasoning in a similar way,

$$\inf_{y\in Y}\ \sup_{x\in X} f(x,y) = \sup_{x\in X}\ \inf_{y\in Y} f(x,y) \ \Rightarrow\ f \text{ is supinf-concave on } X.$$

However, it is not true that, under the assumption

$$y \in Y \ \Rightarrow\ f(\cdot,y) \in \ell^\infty(X)$$

and

$$f \text{ is infsup-convex on } Y \text{ and supinf-concave on } X,$$

then

$$\inf_{y\in Y}\ \sup_{x\in X} f(x,y) = \sup_{x\in X}\ \inf_{y\in Y} f(x,y),$$

as is shown in Example 3.2 below. In fact, condition

$$L \in \mathbb{P}_X \ \Rightarrow\ \inf_{y\in Y} L(f(\cdot,y)) \le \sup_{x\in X}\ \inf_{y\in Y} f(x,y)$$

in Theorem 3.1 is stronger than the supinf-concavity of f on X. On the one hand, if f satisfies that condition, then it is supinf-concave on X, since given

$n \geq 1$, $\mathbf{s} \in \mathbb{P}_n$ and $x_1, \ldots, x_n \in X$, let us define the positive and linear functional $L\colon \ell^\infty(X) \longrightarrow \mathbb{R}$ for each $h \in \ell^\infty(X)$ by

$$L(h) := \sum_{i=1}^{n} s_i h(x_i),$$

which also satisfies $L(\mathbf{1}) = 1$, and thus, $L \in \mathbb{P}_X$. Then,

$$\inf_{y\in Y} \sum_{i=1}^{n} s_i f(x_i, y) = \inf_{y\in Y} L(f(\cdot, y)) \leq \sup_{x\in X} \inf_{y\in Y} f(x, y),$$

and so f is supinf-concave on X. On the other hand:

Example 3.2. Let $f\colon (0,1) \times (0,1) \longrightarrow \mathbb{R}$ be the function defined for each $0 < x, y < 1$ as

$$f(x, y) := \left\{ \begin{array}{ll} 0, & \text{if } x \leq y \\ 1, & \text{if } y < x \end{array} \right. .$$

Let us first observe that f is infsup-convex on $Y = (0,1)$. Indeed, if $m \geq 1$, $\mathbf{t} \in \mathbb{P}_m$ and $y_1, \ldots, y_m \in Y$, then

$$\inf_{y\in(0,1)} \sup_{x\in(0,1)} f(x, y) = \sup_{x\in(0,1)} \sum_{j=1}^{m} t_j f(x, y_j),$$

since

$$\inf_{y\in(0,1)} \sup_{x\in(0,1)} f(x, y) = 1,$$

while

$$\sup_{x\in(0,1)} \sum_{j=1}^{m} t_j f(x, y_j) = 1,$$

because if choosing $x_0 \in (0,1)$ such that $x_0 > \max_{j=1,\ldots,m} y_j$, then

$$\sum_{j=1}^{m} t_j f(x_0, y_j) = 1.$$

In a similar fashion we can prove that f also is supinf-concave on X: given $n \geq 1$, $\mathbf{s} \in \mathbb{P}_n$ and $x_1, \ldots, x_n \in X$, then

$$\inf_{y\in(0,1)} \sum_{i=1}^{n} s_i f(x_i, y) = \sup_{x\in(0,1)} \inf_{y\in(0,1)} f(x, y),$$

since

$$\sup_{x\in(0,1)} \inf_{y\in(0,1)} f(x, y) = 0,$$

and if we take $1 > y_0 \geq \max_{i=1,\ldots,n} x_i$, then

$$\sum_{i=1}^{n} s_i f(x_i, y_0) = 0,$$

and thus

$$\inf_{y\in(0,1)} \sum_{i=1}^{n} s_i f(x_i, y) = 0.$$

However, although we have that

$$y \in Y \ \Rightarrow\ f(\cdot, y) \in \ell^\infty(0,1),$$

the previous reasoning yields that f does not satisfy the minimax identity. Moreover, according to Theorem 3.1, there exists a linear functional $L \in \mathbb{P}_{(0,1)}$ such that

$$\inf_{y\in(0,1)} L(f(\cdot, y)) > \sup_{x\in(0,1)} \inf_{y\in(0,1)} f(x, y). \qquad \square$$

As we have mentioned above, Proposition 2.3 for X finite is the main result for establishing the generalization of Fan's minimax inequality [15, Theorem 2.15] (or [14, Theorem 3.8]). Such a minimax inequality is sharp under adequate hypotheses of compactness and semicontinuity (see [15, Theorem 2.20] or [14, Corollary 3.12]). In that sense, Proposition 2.3 is the link between the two results [15, Theorem 2.15] (or [14, Theorem 3.8]) and Theorem 3.1. However, these minimax inequalities are different results, since the type of hypotheses are: topological versus boundedness hypothesis. In fact, the following two examples confirm it:

Example 3.3. Let E and F be nonzero real normed spaces, $x_0^* \in E^*\backslash\{0\}$ and $y_0^* \in F^*\backslash\{0\}$, and define

$$X := \{x \in E :\ x_0^*(x) \geq 0 \text{ and } \|x\| < 1\}, \quad Y := \{y \in F :\ \|y\| < 1\},$$

and $f\colon X \times Y \longrightarrow \mathbb{R}$ by $f(x, y) := x_0^*(x) y_0^*(y),\ (x \in X,\ y \in Y)$.

Since X does not admit a topology for which X is compact and f is upper semicontinuous, then [15, Theorem 2.15] (or [14, Theorem 3.8]) does not apply, unlike Theorem 3.1. Indeed, f is infsup-convex on Y because it is affine on Y and clearly

$$y \in Y \ \Rightarrow\ f(\cdot, y) \in \ell^\infty(X).$$

In addition, if $L\colon \ell^\infty(X) \longrightarrow \mathbb{R}$ is a positive linear functional (we do not require $L(\mathbf{1}) = 1$), and choosing a $y_0 \in F$ with $y_0^*(y_0) < 0$, then $f(\cdot, y_0) \leq 0$, and, since L is positive,

$$\inf_{y\in Y} L(f(\cdot, y)) \leq L(f(\cdot, y_0)) \leq 0 = \sup_{x\in X} \inf_{y\in Y} f(x, y).$$

Therefore the minimax identity holds. $\square$

For the reverse situation, let us notice that if f is continuous on X and X is compact, then the boundedness hypothesis (1) holds. But if f is only upper semicontinuous, then it does not have to be bounded below, as we now show:

Example 3.4. Let X be the compact –usual topology– interval of $\mathbb{R}$ $[0,1]$, Y be a nonempty set, and define $f\colon X \times Y \longrightarrow \mathbb{R}$ for each $(x,y) \in X \times Y$ as

$$f(x,y) := \begin{cases} \dfrac{1}{x-1}, & \text{if } 0 \le x < 1 \\ 1, & \text{if } x = 1 \end{cases}.$$

Then f is upper semicontinuous on X, but

$$\{f(\cdot,y) : \ y \in Y\} \not\subset \ell^\infty(X);$$

more specifically, given $y \in Y$,

$$f(\cdot,y)(X) = (-\infty,-1] \cup \{1\}.$$

Then Theorem 3.1 does not apply, but [15, Theorem 2.15] (and therefore [14, Theorem 3.8]) does: f is clearly infsup-convex on Y and, furthermore, for any nonempty finite subset Y_0 of Y, we have that $f_{|X \times Y_0}$ is supinf-concave on X, since given $n \ge 1$, $\mathbf{s} \in \mathbb{P}_n$ and $x_1,\dots,x_n \in [0,1]$,

$$\inf_{y \in Y_0} \sum_{i=1}^n s_i f(x_i,y) \le 1 = \inf_{y \in Y_0} f(1,y).$$

So the minimax equality is valid. □

Theorem 3.1 and [15, Theorem 2.15] (or [14, Theorem 3.8]) are not only independent results, but also the main particular case of [15, Theorem 2.15], the Fan minimax inequality [1, Theorem 2], does not admit a bounded version. Such a version should be stated in these terms: if X and Y are nonempty sets and $f\colon X \times Y \longrightarrow \mathbb{R}$ is a function which is concavelike on X and convexlike on Y, and such that

$$y \in Y \ \Rightarrow \ f(\cdot,y) \in \ell^\infty(X),$$

then the minimax relation holds. However, such a result is not true:

Example 3.5. Now we go back to the function in Example 3.2, that is, to $f\colon (0,1) \times (0,1) \longrightarrow \mathbb{R}$ given for each $0 < x,y < 1$ as

$$f(x,y) := \begin{cases} 0, & \text{if } x \le y \\ 1, & \text{if } y < x \end{cases},$$

which clearly satisfies the boundedness assumption (1). Moreover, f is concavelike on its first variable: let $x_1,x_2 \in (0,1)$ and $s \in [0,1]$, and take $x_0 := \max\{x_1,x_2\}$. Then, for all $y \in (0,1)$,

$$f(x_0,y) = \max\{f(x_1,y), f(x_2,y)\}$$

and thus

$$sf(x_1,y)+(1-s)f(x_2,y)\le f(x_0,y).$$

In a similar fashion it is shown that f is convexlike on its second variable (given $y_1,y_2\in(0,1)$ and $t\in[0,1]$ we choose $y_0:=\max\{y_1,y_2\}$ for proving the suitable inequality for all $x\in(0,1)$). But

$$\inf_{y\in(0,1)}\sup_{x\in(0,1)} f(x,y)=1 \quad\text{and}\quad \sup_{x\in(0,1)}\inf_{y\in(0,1)} f(x,y)=0. \qquad \square$$

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