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New Separation Axioms Induced by Nβ*gα Structures in Nano Topology

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Abstract
Nano topology provides a suitable framework for analyzing approximation-based structures under uncertainty. In this paper, new separation axioms induced by Nβ*gα structures are introduced and investigated in N topological spaces. Using Nβ*gα-open and Nβ*gα-cls, we define Nβ*gα-T_0, Nβ*gα-T_1, Nβ*gα-T_2 spaces. Fundamental implications and non-implications among these axioms are established with appropriate examples. The behaviour of these separation properties under Nβ*gα-continuous and Nβ*gα-irresolute mappings is examined. These results extend existing separation concepts in Nano topology.

*Keywords:*Nano topology, Nβ*gα-closed sets, Nβ*gα-open sets, Nβ*gα-separation axioms and Nβ*gα-continuous functions.

1.Introduction

Lellis Thivagar and Richard [5] initially formulated the framework of N topology using upper and lower approximations alongside the boundary region of a subset within an approximation space established by an equivalence relation. This foundational research established N topological spaces by defining core concepts like N-ops, N-cls, N-interior, N-closure, and various generalized variants of N-ops, such as N semi-open, N pre-open, Nα-open, and Nα-ops. Expanding upon these concepts, Nasef et al. [6] explored alternative classes of nearly ops within N topological spaces. Subsequently, Revathy and Gnanambal Ilango [7] introduced the category of Nβ-ops, whereas Sathishmohan et al. [9] developed N neighbourhoods and analyzed their characteristics inside the context of N topology.

2. Preliminaries

Definition 2.1. [9] N-T_0 Space

A NTS U fulfils the N-T_0 separation axiom provided that for any two unique elements $c, d \in U$, a N-ops G exists that includes one point while excluding the other. Stated differently, either $c \in G$ and $d \notin G$, or $d \in G$ and $c \notin G$.

Definition 2.2. [9] Nα-T_0 Space

A NTS (U, τ) qualifies as an Nα-T_0 space provided that a nano α-ops can separate any two unique points in U. In other words, for any elements $c, d \in U$ where $c \neq d$, a nano α-ops G exists such that either $c \in G$ and $d \notin G$, or $d \in G$ and $c \notin G$.

Definition 2.3. [9] Nβ-T_0 Space

A NTS (U, τ) is designated as an Nβ-T_0 space if, given any pair of unique points c and d within U, an Nβ-ops incorporates precisely one of them. Equivalently, $c \in G$ and $d \notin G$, or $d \in G$ and $c \notin G$.

Definition 2.4.[1] Nβ*gα-closed set

Let (U, τ) be a NTS and let $A \subseteq U$.Then A is called a Nβ*gα-closed set, denoted as Nβ*gα-closed, if

$\mathrm{Nspcl}(A) \subseteq V$ whenever$A \subseteq V$and$V$ is $N^{*}g\alpha$-open in U. That is, if $A \subseteq V$ for some $N^{*}g\alpha$-ops V, then the nano semi pre closure of A, denoted $\mathrm{Nspcl}(A)$, is also contained in V.

Definition 2.5. Nβ^{*}gα-neighbourhood

Consider a N topological space U and an element $c \in U$. A subset $M_c \subseteq U$ qualifies as an Nβ^{*}gα-neighbourhood (abbreviated as Nβ^{*}gα-nhd) of point c when there is an Nβ^{*}gα-ops $A \in N\beta^{*}g\alpha O(U)$ such that $c \in A \subseteq M_c$. Under these conditions, point c is designated as an Nβ^{*}gα-neighbourhood point associated with set A.

Definition 2.6. Relative Nβ^{*}gα-Ops

Consider A to be a nano subspace belonging to the NTS (U, τ). A subset $B \subseteq A$ is defined as relative Nβ^{*}gα-open within A when there exists an Nβ^{*}gα-open subset G of U fulfilling $B = A \cap G$. Analogously, a subset $F \subseteq A$ is recognized as relative Nβ^{*}gα-closed within A whenever an Nβ^{*}gα-closed subset H of U exists such that $F = A \cap H$.

Definition 2.7. Cofinite Nano Topology

Let U represent an infinite set, and let the cofinite nano topology on U, designated as τ_{cf}, be defined as $\tau_{cf} = \{\emptyset\} \cup \{G \subseteq U: U \setminus G \text{ is finite}\}$. Consequently, a subset G of U qualifies as a cofinite nano-ops if and only if:

- $G = \emptyset$
- $U \setminus G$ constitutes a finite set

The ordered pair (U, τ_{cf}) is designated as a cofinite NTS.

Definition 2.8. Nβ^{*}gα-T_0 Space

A NTS (U, τ) fulfills the Nβ^{*}gα-T_0 separation condition if an Nβ^{*}gα-ops can distinguish any two unique points of U. Stated differently, given any elements $c, d \in U$ where $c \neq d$, an Nβ^{*}gα-ops G exists such that either $c \in G$ and $d \notin G$, or $d \in G$ and $c \notin G$.

Definition 2.9. Nβ^{*}gα-T_1 Space

A NTS (U, τ) is characterized as an Nβ^{*}gα-T_1 space when Nβ^{*}gα-ops can individually separate any pair of unique points. Specifically, for $c \neq d$, there exist Nβ^{*}gα-ops G and H satisfying $c \in G$, $d \notin G$, alongside $d \in H$ and $c \notin H$.

Definition 2.10. Nβ^{*}gα-T_2 Space

A NTS (U, τ) qualifies as an Nβ^{*}gα-T_2 (alternatively, Nβ^{*}gα-Hausdorff) space if any two distinct points possess disjoint Nβ^{*}gα-open neighbourhoods. Expressly, for $c, d \in U$ with $c \neq d$, there are Nβ^{*}gα-ops G and H such that $c \in G, d \in H$, and $G \cap H = \emptyset$.

3. Nβ^{*}gα-T_0 Space

Theorem 3.1. Every N-T_0-space is an Nβ^{*}gα-T_0 space.

Proof. Let $(U, \tau_R(X))$ represent a nano T_0-space. Following the guidelines of the N-T_0 separation property, given any two unique elements $x, y \in U$, a N-ops G exists containing exactly one of these points. Because all N-ops qualify as Nβ^{*}gα-open, G can be treated as an Nβ^{*}gα-ops. Consequently, an Nβ^{*}gα-ops is capable of separating any pair of unique points. Thus, $(U, \tau_R(X))$ fulfills the criteria for an Nβ^{*}gα-T_0 space, proving that every N-T_0-space is an Nβ^{*}gα-T_0 space.

Example 3.2. Let $U = \{a, b, c\}$ with N-topology $\tau_R(X) = \{\emptyset, \{a\}, U\}$ and collection $N\beta^{*}g\alpha O(U) = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, U\}$. The space $(U, \tau_R(X))$ is not N-T_0 because no N-ops can separate b and c. However, b and c are separated by the Nβ^{*}gα-ops $\{a, b\}$ and $\{a, c\}$, while $\{a\}$ separates a from both b and c. Since every distinct pair of points is separated by an Nβ^{*}gα-ops, $(U, \tau_R(X))$ is an Nβ^{*}gα-T_0 space.

Theorem 3.3. A NTS U is an Nβ^{*}gα-T_0 space $\Leftrightarrow$ $N\beta^{*}g\alpha cl(\{c\}) \neq N\beta^{*}g\alpha cl(\{d\})$, for each pair of non-identical points $c, d \in U$.

Proof. Suppose U is an $N\beta^*g\alpha$-T_0 space and let $c, d \in U$ with $c \neq d$. Then $\exists$ an $N\beta^*g\alpha$-ops G containing one of the points and omitting the other. Assume $c \in G$ and $d \notin G$. Since $U \setminus G$ is $N\beta^*g\alpha$-closed, $c \notin N\beta^*g\alpha cl(\{d\})$, whereas $c \in N\beta^*g\alpha cl(\{c\})$. Hence, $N\beta^*g\alpha cl(\{c\}) \neq N\beta^*g\alpha cl(\{d\})$. Conversely, suppose $N\beta^*g\alpha cl(\{c\}) \neq N\beta^*g\alpha cl(\{d\})$ for every $c \neq d$. Then either $c \notin N\beta^*g\alpha cl(\{d\})$ or $d \notin N\beta^*g\alpha cl(\{c\})$. The complement of the corresponding $N\beta^*g\alpha$-closure is an $N\beta^*g\alpha$-ops containing one point but excluding the other. Therefore, U satisfies the $N\beta^*g\alpha$-T_0 separation axiom.

Hence, U is an $N\beta^*g\alpha$-T_0 space $\iff N\beta^*g\alpha cl(\{c\}) \neq N\beta^*g\alpha cl(\{d\})$, for all distinct $c, d \in U$.

Theorem 3.4. Let $(U, \tau_R(X))$ be a NTS. If U is an $N\beta^*g\alpha$-T_0 space and V is a nano-open subspace of U, then V is also an $N\beta^*g\alpha$-T_0 space.

Proof. Let $x, y \in V$ with $x \neq y$. Since U is an $N\beta^*g\alpha$-T_0 space, $\exists$ an $N\beta^*g\alpha$-ops $G \subseteq U$ contains exactly one of the two points. As V is N-open, $G \cap V$ is $N\beta^*g\alpha$-open in the subspace V. Moreover, $G \cap V$ separates x and y. Hence, V satisfies the $N\beta^*g\alpha$-T_0 axiom.

Example 3.5. Let $U = \{a, b, c\}, \tau_R(X) = \{\emptyset, U, \{a\}, \{a, b\}\}$, and $N\beta^*g\alpha O(U) = \{\emptyset, U, \{a\}, \{a, b\}\}$. The space U is an $N\beta^*g\alpha$-T_0 space since every pair of distinct points is separated by an $N\beta^*g\alpha$-ops. Consider the nano-open subspace $V = \{a, b\}$. The induced topology is $\tau_V = \{\emptyset, V, \{a\}\}$. Here, the set $\{a\}$ separates the points a and b. Therefore, V is also an $N\beta^*g\alpha$-T_0 space.

Theorem 3.6. Let $f: U \to V$ be an $N\beta^*g\alpha$-homeomorphism. If U is an $N\beta^*g\alpha$-T_0 space, then V is also an $N\beta^*g\alpha$-T_0 space.

Proof. Let $y_1, y_2 \in V$ with $y_1 \neq y_2$. Since f is bijective, $\exists$ distinct points $x_1, x_2 \in U$ such that $f(x_1) = y_1, f(x_2) = y_2$. As U is an $N\beta^*g\alpha$-T_0 space, there exists an $N\beta^*g\alpha$-ops G containing one of x_1, x_2 but not the other. Assume $x_1 \in G$ and $x_2 \notin G$. Since f is an $N\beta^*g\alpha$-homeomorphism, f(G)is $N\beta^*g\alpha$-open in V. Moreover, $y_1 = f(x_1) \in f(G)$ and $y_2 = f(x_2) \notin f(G)$.Thus, f(G) separates $y_1$and y_2. Hence, V satisfies the $N\beta^*g\alpha$-T_0 separation axiom. Therefore, V is an $N\beta^*g\alpha$-T_0 space.

Example 3.7. Let $U = \{a, b, c\}, \tau_R(X) = \{\emptyset, U, \{a\}, \{a, b\}\}$& let $N\beta^*g\alpha O(U) = \{\emptyset, U, \{a\}, \{a, b\}\}$. Define a bijection $f: U \to V = \{x, y, z\}$ by $f(a) = x$, $f(b) = y$, $f(c) = z$, where $\tau_R^*(Y) = \{\emptyset, V, \{x\}, \{x, y\}\}$. Since f is an $N\beta^*g\alpha$-homeomorphism and U is an $N\beta^*g\alpha$-T_0 space, Theorem 3.7 implies that V is also an $N\beta^*g\alpha$-T_0 space.

4. Nano β*gα-T₁ Space

Theorem 4.1. Let (U, τ) be a NTS. Every $N\beta^*g\alpha$-T_1 space is an $N\beta^*g\alpha$-T_0 space.

Proof. Assume that (U, τ) is an $N\beta^*g\alpha$-T_1 space. For any distinct points $x, y \in U$, $\exists N\beta^*g\alpha$-ops G and H such that $x \in G, y \notin G, y \in H, x \notin H$. Thus, at least one $N\beta^*g\alpha$-ops separates the points x and y. Hence, the $N\beta^*g\alpha$-T_0 separation axiom is satisfied. Therefore, every $N\beta^*g\alpha$-T_1 space is an $N\beta^*g\alpha$-T_0 space.

Example 4.2. Let $U = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{a, b\}, U\}$, and assume that $N\beta^*g\alpha O(U) = \tau$. The space (U, τ) is an $N\beta^*g\alpha$-T_0 space because every pair of distinct points can be distinguished by an $N\beta^*g\alpha$-ops. For example, {a}separates a and b, while {a,b} separates b and c. However, (U, τ) is not an $N\beta^*g\alpha$-T_1 space. Indeed, there is no $N\beta^*g\alpha$-ops containing b without also containing a. Thus, the pair (a,b) cannot be separated in both directions. Hence, (U, τ) is $N\beta^*g\alpha$-T_0 but not $N\beta^*g\alpha$-T_1.

Theorem 4.3. A NTS (U, τ) is an $N\beta^*g\alpha$-T_1 space $\Leftrightarrow$ every singleton subset $\{x\} \subseteq U$ is $N\beta^*g\alpha$-closed.

Proof. Necessity (⇒) Assume U is an $N\beta^*g\alpha$-T_1 space and let $x \in U$. For each $y \neq x$, $\exists$ an $N\beta^*g\alpha$-open set G_y such that $y \in G_y$ and $x \notin G_y$. Thus, $U \setminus \{x\} = \bigcup_{y \neq x} G_y$. Since arbitrary unions of $N\beta^*g\alpha$-open sets are $N\beta^*g\alpha$-open, the complement $U \setminus \{x\}$ is $N\beta^*g\alpha$-open, making $\{x\}N\beta^*g\alpha$-closed.

Sufficiency (⇐) Conversely, suppose every singleton in U is $N\beta^*g\alpha$-closed. For any two distinct points $x, y \in U$, the sets $U \setminus \{x\}$ and $U \setminus \{y\}$ form $N\beta^*g\alpha$-open nhd containing one point while excluding the other. Hence, U satisfies the $N\beta^*g\alpha$-T_1 separation axiom.

Theorem 4.4. A NTS (U, τ) is an $N\beta^*g\alpha$-T_1 space iff all of its finite subsets are $N\beta^*g\alpha$-closed.

Proof. Necessity (⇒) Assume (U, τ) is an $N\beta^*g\alpha$-T_1 space, meaning every singleton subset is $N\beta^*g\alpha$-closed. Let $F = \{x_1, x_2, \dots, x_n\}$ be any finite subset of U. Since F can be written as the finite union $\bigcup_{i=1}^{n}\{x_i\}$ of $N\beta^*g\alpha$-closed sets, F is likewise $N\beta^*g\alpha$-closed.

Sufficiency (⇐) Conversely, assume all finite subsets of U are $N\beta^*g\alpha$-closed. Because every singleton {x} is a finite set, it follows that every singleton is $N\beta^*g\alpha$-closed. By Theorem 4.3, this guarantees that (U, τ) is an $N\beta^*g\alpha$-T_1 space.

Theorem 4.5. A NTS (U, τ) functions as an $N\beta^*g\alpha$-T_1 space iff the cofinite NT σ on U is a subset of τ ($\sigma \subseteq \tau$), where:

$$\sigma = \{\emptyset\} \cup \{U \setminus F \mid F \subseteq U \text{ and } F \text{ is finite}\}$$

Proof. Necessity (⇒) Suppose (U, τ) is an $N\beta^*g\alpha$-T_1 space, which implies every finite subset of U is $N\beta^*g\alpha$-closed. For any set $G \in \sigma$, either $G = \emptyset$ (which trivially belongs to τ) or $G = U \setminus F$ for some finite subset $F \subseteq U$. Since F is $N\beta^*g\alpha$-closed, its complement G is $N\beta^*g\alpha$-open and thus resides in τ. Consequently, $\sigma \subseteq \tau$.

Sufficiency (⇐) Conversely, assume $\sigma \subseteq \tau$. For any arbitrary finite subset $F \subseteq U$, its complement $U \setminus F$ belongs to σ, and therefore also belongs to τ. This means $U \setminus F$ is $N\beta^*g\alpha$-open, making Fan $N\beta^*g\alpha$-closed set. Since all finite subsets are $N\beta^*g\alpha$-closed, the space (U, τ) satisfies the requirement for an $N\beta^*g\alpha$-T_1 space.

Example 4.6. Let$U = \{a, b, c, d\}$ with the NT $\tau = \{\emptyset, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, U\}$.The corresponding cofinite NT is $\sigma = \{\emptyset, U, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$. Since $\sigma \subseteq \tau$, all finite subsets specifically the singletons $\{a\}, \{b\}, \{c\}, \{d\}$ are $N\beta^*g\alpha$-closed. Thus, (U, τ) forms an $N\beta^*g\alpha$-T_1 space.

Theorem 4.7. A NTS (U, τ) is an $N\beta^*g\alpha$-T_1 space if and only if for each point $x \in U$, the intersection of all its $N\beta^*g\alpha$-neighbourhoods equals {x}:

$$\bigcap \{N \mid N \text{ is an } N\beta^*g\alpha\text{-neighbourhood of } x\} = \{x\}$$

Proof. Necessity (⇒) Let (U, τ) be an $N\beta^*g\alpha$-T_1 space and fix $x \in U$. Since every neighbourhood of x contains x, $\{x\} \subseteq \bigcap\{N \mid N \text{ is an } N\beta^*g\alpha\text{-neighbourhood of } x\}$. Conversely, if $y \neq x$, the T_1 property provides an $N\beta^*g\alpha$-open set containing x but not y, meaning y is excluded from at least one neighbourhood of x. Thus, the intersection collapses to {x}.

Sufficiency (⇐) Conversely, assume the intersection of all neighbourhoods for any $x \in U$ reduces to {x}. For distinct points $x, y \in U$, since $y \notin \{x\}$, there exists an $N\beta^*g\alpha$-neighbourhood of x (and similarly for y) that excludes the other point. This guarantees that x and y can be separated by $N\beta^*g\alpha$-open sets, establishing that (U, τ) is an $N\beta^*g\alpha$-T_1 space.

Example 4.8. Let $U = \{a, b, c\}$ with the NT $\tau = \{\emptyset, \{a, b\}, \{a, c\}, \{b, c\}, U\}$, and suppose the family of $N\beta^*g\alpha$-open sets matches τ.The intersection of all $N\beta^*g\alpha$-neighbourhoods for each point yields:For point a: $\{a, b\} \cap \{a, c\} \cap U = \{a\}$, For point b: $\{a, b\} \cap \{b, c\} \cap U = \{b\}$, For point c: $\{a, c\} \cap \{b, c\} \cap U = \{c\}$.Since the intersection of neighbourhoods for every $x \in U$ reduces to the singleton {x}, Theorem 4.7 confirms that (U, τ) is an $N\beta^*g\alpha$-T_1 space.

Theorem 4.9. Let (U, τ) be an $N\beta^*g\alpha$-T_1 space and $A \subseteq U$. A point $x \in U$ is an $N\beta^*g\alpha$-limit point of A iff every $N\beta^*g\alpha$-nhd of x intersects A in infinitely many distinct elements.

Proof. Necessity (⇒) Assume x is an $N\beta^*g\alpha$-limit point of A. Suppose for contradiction that some nhd N contains only a finite intersection $N \cap A = \{a_1, a_2, \dots, a_n\}$. Since U is a T_1 space, finite subsets are $N\beta^*g\alpha$-closed, meaning $F = \{a_1, \dots, a_n\}$ is closed and its complement $U \setminus F$ is open.Since $x \notin F$, $x \in U \setminus F$, making $G = N \cap (U \setminus F)$ a nhd of x. However, $G \cap (A \setminus \{x\}) = \emptyset$, contradicting the fact that x is a limit point. Thus, every nhd of x must contain infinitely many points of A.

Sufficiency (⇐) Conversely, suppose every $N\beta^*g\alpha$-nhd of x contains infinitely many distinct points of A. For any arbitrary nhd N of x, because $N \cap A$ is infinite, it immediately follows that $N \cap (A \setminus \{x\}) \neq \emptyset$. This satisfies the definition of an $N\beta^*g\alpha$-limit point.

Example 4.10. Let $U = \mathbb{N} = \{1,2,3, \dots\}$ carry the cofinite nano topology $\tau = \{\emptyset\} \cup \{U \setminus F \mid F \subseteq U \text{ is finite}\}$. Since singletons are closed, (U, τ) is an $N\beta^*g\alpha$-T_1 space.

Considering the subset of even integers $A = \{2,4,6, \dots\}$ and taking $x = 1$, any nhd of 1 is cofinite and thus misses only finitely many elements from A. Consequently, every nhd of 1contains infinitely many points of A, making 1 an $N\beta^*g\alpha$-limit point of A. By extension, every element in U functions as a limit point of A.

Theorem 4.11. In an $N\beta^*g\alpha$-T_1 space (U, τ), any finite subset of U possesses no $N\beta^*g\alpha$-limit points.

Proof Let $F \subseteq U$ be a finite subset, and assume for contradiction that $x \in U$ is a limit point of F. By Theorem 4.9, every nhd of x must intersect F in infinitely many elements. However, since F is finite, no nhd can contain infinitely many unique points from it. This contradiction proves that F has no limit points.

Theorem 4.12. If (U, τ) is an $N\beta^*g\alpha$-T_1 space, then for any subset $A \subseteq U$, the induced subspace (A, τ_A) also qualifies as an $N\beta^*g\alpha$-T_1 space.

Proof. Let $x, y \in A$ be two distinct elements with $x \neq y$. Since $A \subseteq U$, x and y are also distinct points in the parent space U. Because U is an $N\beta^*g\alpha$-T_1 space, we can find $N\beta^*g\alpha$-open sets $G, H \subseteq U$ such that $x \in G$, $y \notin G$ and $y \in H$, $x \notin H$.

Forming the relative open sets $G_A = G \cap A$ and $H_A = H \cap A$, we obtain subsets in A such that $x \in G_A$ with $y \notin G_A$, and $y \in H_A$ with $x \notin H_A$. This demonstrates that any pair of distinct points in A can be separated by relative open sets, proving that (A, τ_A) is an $N\beta^*g\alpha$-T_1 space.

Theorem 4.13. Let $f: (U, \tau) \to (V, \sigma)$ be a bijective, $N\beta^*g\alpha$-open mapping from an $N\beta^*g\alpha$-T_1 space U onto V. Then (V, σ) is also an $N\beta^*g\alpha$-T_1 space.

Proof. Let $x, y \in V$ be distinct points with $x \neq y$. Since f is a bijection, there exist unique preimages $a, b \in U$ such that $f(a) = x$ and $f(b) = y$.Because U is an $N\beta^*g\alpha$-T_1 space, we can find open sets $G, H \subseteq U$ containing one point while excluding the other ($a \in G, b \notin G$ and $b \in H, a \notin H$). Since f is an open map, their images f(G) and f(H) are $N\beta^*g\alpha$-open in V.Furthermore, $x = f(a) \in f(G)$ while $y = f(b) \notin f(G)$, and similarly $y \in f(H)$ while $x \notin f(H)$. Thus, (V, σ) satisfies the T_1 separation axiom.

5. $N\beta^*g\alpha$-T_2 Space

Theorem 5.1. Every $N\beta^*g\alpha$-T_2 space qualifies as an $N\beta^*g\alpha$-T_1 space, though the reverse implication fails.

Proof. Given distinct points x, y in an $N\beta^*g\alpha$-T_2 space U, there exist disjoint $N\beta^*g\alpha$-open sets G, H containing x and y respectively. Because $G \cap H = \emptyset$, it follows that $x \in G \setminus H$ and $y \in H \setminus G$, ensuring each point is contained in a neighborhood excluding the other. Hence, the space is $N\beta^*g\alpha$-T_1.

Theorem 5.2. Any finite $N\beta^*g\alpha$-T_2 space is discrete.

Proof. Let (U, τ) be a finite $N\beta^*g\alpha$-T_2 space. Because U is finite, every subset $A \subseteq U$ is finite and therefore $N\beta^*g\alpha$-closed. Consequently, the complement $U \setminus A$ is also finite and $N\beta^*g\alpha$-closed, implying that A is $N\beta^*g\alpha$-open. Since every subset is both closed and open, the topology τ equals the power set $\mathcal{P}(U)$, making the space discrete.

Theorem 5.3. Every subspace of an $N\beta^*g\alpha$-T_2 space is also an $N\beta^*g\alpha$-T_2 space.

Proof. Let (U, τ) be an $N\beta^*g\alpha$-T_2 space and (A, τ_A) be a subspace with $A \subseteq U$. For any distinct points $x, y \in A$, they are also distinct in U. Thus, there exist disjoint $N\beta^*g\alpha$-open sets $G, H \subseteq U$ containing x and y respectively. Form the subspace sets $G_A = G \cap A$ and $H_A = H \cap A$, which are $N\beta^*g\alpha$-open in A such that $x \in G_A$ and $y \in H_A$. Their intersection yields $G_A \cap H_A = (G \cap H) \cap A = \emptyset \cap A = \emptyset$, confirming that x and y are separated by disjoint nhds in the subspace. Hence, (A, τ_A) is an $N\beta^*g\alpha$-T_2 space.

Theorem 5.4. The $N\beta^*g\alpha$-T_2 property is invariant under bijective $N\beta^*g\alpha$-open maps.

Proof. Let $f: (U, \tau) \to (V, \sigma)$ be a bijective $N\beta^*g\alpha$-open function where U is an $N\beta^*g\alpha$-T_2 space. For any distinct points $x, y \in V$, find their unique preimages $a, b \in U$ via f. Since U is $N\beta^*g\alpha$-T_2, there exist disjoint $N\beta^*g\alpha$-open sets $G, H \subseteq U$ containing a and b. The images f(G) and f(H) are $N\beta^*g\alpha$-open in V (as f is open) and remain disjoint because f is injective. Since $x \in f(G)$ and $y \in f(H)$ with empty intersection, V satisfies the $N\beta^*g\alpha$-T_2 condition.

Theorem 5.5. Let $f, g: (U, \tau) \to (V, \sigma)$ be $N\beta^*g\alpha$-continuous functions, with (V, σ) being an $N\beta^*g\alpha$-T_2 space. Then the coincidence set $F = \{x \in U \mid f(x) = g(x)\}$ is $N\beta^*g\alpha$-closed in U.

Proof. For any point $x \notin F$, we have $f(x) \neq g(x)$. Since V is $N\beta^*g\alpha$-T_2, there exist disjoint $N\beta^*g\alpha$-open sets $G, H \subseteq V$ such that $f(x) \in G$ and $g(x) \in H$. By $N\beta^*g\alpha$-continuity, $W = f^{-1}(G) \cap g^{-1}(H)$ is an $N\beta^*g\alpha$-open nhd containing x. For any $y \in W$, $f(y) \in G$ and $g(y) \in H$. Because $G \cap H = \emptyset$, it follows that $f(y) \neq g(y)$, meaning

W ⊆ U \ F. Since every point outside F has an Nβ*gα-open nhd entirely contained in U \ F, the complement U \ F is Nβ*gα-open, making F an Nβ*gα-closed set.

Logical Implications among Nβ*gα-Separation Axioms

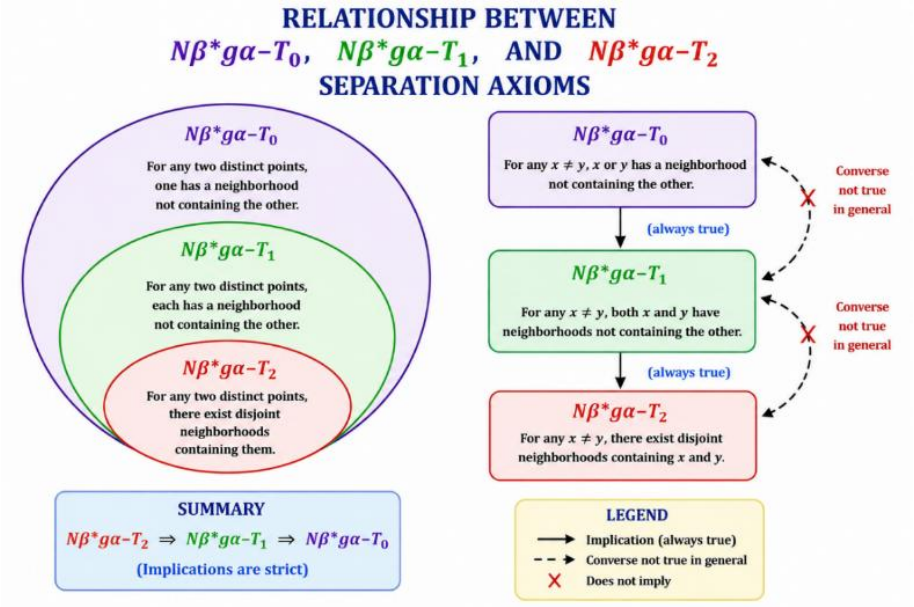


Figure 1

6. Conclusion

In this paper, we have developed a systematic study of separation axioms generated by Nβ*gα-structures in NTS. By employing Nβ*gα-open and Nβ*gα-closed sets, we introduced and analyzed the classes Nβ*gα-T_0, Nβ*gα-T_1, Nβ*gα-T_2. The relationships among these newly defined axioms were thoroughly examined, and both implication and non-implication results were established with suitable examples, demonstrating that the proposed framework forms a proper and non-trivial extension of existing nano separation theories. The study further investigated the preservation and transfer of these separation properties under Nβ*gα-continuous and Nβ*gα-irresolute mappings, thereby strengthening the functional aspects of the theory. Future research may explore compactness, connectedness, metrization-type results, and potential applications of Nβ*gα-structures in graph theory, information systems, and related mathematical models.

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