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Duality for an Extended Equilibrium Problem

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We consider an extended equilibrium problem with sum of two functions, one being composed with a linear mapping and we introduce and study a dual problem associated to it. We show that the solutions of the two problems are strictly related to the saddle points of an associated Lagrangian function and, under appropriate conditions, to the solutions of the associated optimization problem and its dual. Among the special instances of our results, we rediscover results obtained for the generalized equilibrium problem considered by Bigi, Castellani and Kassay, and we also prove that for some particular cases the duality scheme considered here become the duality scheme concerning variational inequalities introduced in the literature.

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1. Introduction

The equilibrium problem can be formulated as follows:

$$\text{Find } \overline{x} \in X \;\text{ s.t. }\; \varphi(\overline{x}, y) \geq 0 \;\; \forall y \in X, \tag{EP}$$

where $X \subseteq \mathbb{R}^n$ is a nonempty set and $\varphi\colon X \times X \to \mathbb{R}$ is a bifunction on X.

The term "equilibrium problem" was attributed by Blum and Oettli in [4] where has been shown that equilibrium problem include, as particular cases, scalar and vector optimization problems, saddle point (minimax) problems, variational inequalities, Nash equilibria problems, complementarity problems, etc. This problem has been extensively studied in the past because of its numerous applications in mechanics, electricity, mathematical physics, operational research, game theory, economics etc. In [7] the authors show how image inpainting problems, which aim for recovering lost information, can be efficiently solved by implementing an algorithm for a primal-dual pair of convex optimization problems.

In [2], Bianchi and Schaible derived existence results for (EP) using quasimonotone and pseudomonotone bifunctions. Martínez-Legaz and Sosa established a dual formulation for (EP) using the classical Fenchel conjugation while Bigi,

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Castellani and Kassay [3] introduced the so-called "generalized equilibrium problem":

$$\text{Find } \overline{x} \in \mathbb{R}^n \ \text{ s.t. } \ \varphi(\overline{x}, y) + f(y) \geq f(\overline{x}) \ \ \forall y \in \mathbb{R}^n, \qquad \text{(GEP)}$$

where $f\colon \mathbb{R}^n \to (-\infty, +\infty]$ is a proper, convex and lower semicontinuous function, $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a function satisfying the conditions $\varphi(x, \cdot)$ is convex for all $x \in \operatorname{dom} f$ and $\varphi(x, x) = 0$ for all $x \in \operatorname{dom} f$, and it is been proved that the solution of (GEP) and its dual are strictly related to the saddle points of an associated Lagrangian function. Jacinto and Scheimberg considered in [10] another generalized equilibrium problem and developed a dual scheme for this problem in real topological vector space setting. In [11] Lalitha extended the dual given for equilibrium problem introduced in [12] and established its equivalence with the dual derived in [3].

In this paper we study an extended form of (GEP), a generalized equilibrium problem with sum of two functions, one being composed with a linear continuous mapping and we introduce and study a dual problem associated to it.

This article is organized as follows. Section 2 contain some notions and results from the literature in order to make the paper as self-contained as possible. In Section 3 we give optimality conditions needed in Section 4 where we introduce a dual composed equilibrium problem (DCEP) for the composed equilibrium problem (CEP) and we show that any solution of it provides a solution of its dual and vice versa. We also prove that the solutions of the dual pair of composed equilibrium problems are strictly related to the saddle points of the Lagrangian function associated to these problems. We show that, under appropriate assumptions, the solution set of (CEP) coincide with the union of the solutions sets of the primal optimization problems while the solution set of (DCEP) coincide with the union of the solutions sets of the dual optimization problem. In Section 5 we particularize duality results obtain in this paper and we rediscover the generalized equilibrium problem and its dual one introduced in [3], and the duality theorems for variational inequalities considered in [8] and [13].

2. Preliminaries

In this section we give some notions and results that will be used in the paper. The notations follow [5, 6, 9, 14, 15].

In $\mathbb{R}^n$ we work with the Euclidean topology induced by the Euclidean norm. We denote by $\langle x^*, x\rangle$ (sometimes also by $\langle x, x^*\rangle$) the inner product of the vectors x and x^* in $\mathbb{R}^n$, i.e. if $x = (x_1, ..., x_n)$ and $x^* = (x_1^*, ..., x_n^*)$ then $\langle x^*, x\rangle = x_1 \cdot x_1^* + ... + x_n \cdot x_n^*$.

For a non-empty set $U \subseteq \mathbb{R}^n$, we denote by $\operatorname{aff}(U)$ its *affine hull*. The set $\operatorname{ri}(U)$ denotes the classical *relative interior* of U, that is the interior of U relative to $\operatorname{aff}(U)$.

The *indicator function* of U, $\delta_U\colon \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, is defined as $\delta_U(x) = 0$ if $x \in U$ and $+\infty$ otherwise.

For a function $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ we denote by $\operatorname{dom} f = \{x \in \mathbb{R}^n : f(x) < +\infty\}$ its *domain.* We call f *proper* if $\operatorname{dom} f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in \mathbb{R}^n$. The *Fenchel-Moreau conjugate* of f is the function $f^*\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by $f^*(x^*) = \sup_{x\in\mathbb{R}^n}\{\langle x^*, x\rangle - f(x)\}$ for all $x^* \in \mathbb{R}^n$ and the *biconjugate* function $f^{**}\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ is defined as $f^{**}(x) = \sup_{x^*\in\mathbb{R}^n}\{\langle x^*, x\rangle - f^*(x^*)\}$ for all $x \in \mathbb{R}^n$. Let us mention some properties of the conjugate function. We have the so called *Young-Fenchel inequality*: $f^*(x^*) + f(x) \geq \langle x^*, x\rangle$ for all $x \in \mathbb{R}^n$ and $x^* \in \mathbb{R}^n$. Further, $f^{**} \leq f$ and according to the celebrated *Fenchel-Moreau Theorem*, if f is proper, then f is convex and lower semicontinuous if and only if $f^{**} = f$ (see [5, 6, 9, 14, 15]). Given a linear mapping $A\colon \mathbb{R}^n \to \mathbb{R}^m$, its *adjoint operator* $A^*\colon \mathbb{R}^m \to \mathbb{R}^n$ is defined by $\langle A^*y^*, x\rangle = \langle y^*, Ax\rangle$ for all $y^* \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$.

For $x \in \mathbb{R}^n$ such that $f(x) \in \mathbb{R}$ we define the *sudifferential* of f at x by

$$\partial f(x) = \{x^* \in \mathbb{R}^n : f(y) - f(x) \geq \langle x^*, y - x\rangle \ \ \forall y \in \mathbb{R}^n\}.$$

If $f(x) \in \{\pm\infty\}$ we take by convention $\partial f(x) = \emptyset$. The *normal cone* of U at $x \in \mathbb{R}^n$ is $N_U(x) = \{x^* \in \mathbb{R}^n : \langle x^*, y - x\rangle \leq 0 \ \ \forall y \in U\}$, if $x \in U$ and $N_U(x) = \emptyset$ otherwise.

The following characterization of the sudifferential of a proper function f at $x \in \operatorname{dom} f$ by means of conjugate functions will be useful (see [14]):

$$x^* \in \partial f(x) \Leftrightarrow f(x) + f^*(x^*) \leq \langle x^*, x\rangle. \tag{1}$$

By using this characterization one can prove that

$$x^* \in \partial f(x) \Rightarrow x \in \partial f^*(x^*), \tag{2}$$

and in case f is proper, convex and lower semicontinuous (see [14, Theorem 23.5])

$$x^* \in \partial f\left(x\right) \Leftrightarrow x \in \partial f^*\left(x^*\right). \tag{3}$$

3. Optimality conditions for an optimization problem

In this section we give optimality conditions for an optimization problem which is formed of a sum of three functions, one being composed with a linear operator.

Let us consider the optimization problem

$$\inf_{x\in\mathbb{R}^n}\Big\{g(Ax)\Big\} \tag{P^{A_g}}$$

where $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, $g\colon \mathbb{R}^m \to \overline{\mathbb{R}}$ is a proper function fulfilling $A^{-1}(\operatorname{dom} g) \neq \emptyset$.

In [14] the following dual optimization problem to (P^{A_g}) is considered

$$\sup_{\substack{y^*\in\mathbb{R}^m\\ A^*y^*=0}}\Big\{ - g^*(y^*)\Big\}. \tag{D^{A_g}}$$

Let $B\colon \mathbb{R}^n \to \mathbb{R}^m$ a linear mapping, $f_1, f_2\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, $f_3\colon \mathbb{R}^m \to \overline{\mathbb{R}}$ proper functions fulfilling $\operatorname{dom} f_1 \cap \operatorname{dom} f_2 \cap B^{-1}(\operatorname{dom} f_3)$. If we consider the function $g\colon \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ defined by $g(x_1, x_2, x_3) = f_1(x_1) + f_2(x_2) + f_3(x_3)$ with $\operatorname{dom} g = \operatorname{dom} f_1 \times \operatorname{dom} f_2 \times \operatorname{dom} f_3$ and $A\colon \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ defined by $Ax = (x, x, Bx)$ the primal optimization problem (P^{A_g}) becomes

$$\inf_{x\in\mathbb{R}^n} \Big\{ f_1(x) + f_2(x) + f_3(Bx) \Big\} \tag{P}$$

Since the adjoint mapping $A^*\colon \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ of A is $A^*(x_1^*, x_2^*, x_3^*) = x_1^* + x_2^* + B^* x_3^*$, we get the dual problem to (P) as

$$\sup_{\substack{x_1^*, x_2^* \in \mathbb{R}^n, x_3^* \in \mathbb{R}^m, \\ x_1^* + x_2^* + B^* x_3^* = 0}} \Big\{ - f_1^*(x_1^*) - f_2^*(x_2^*) - f_3^*(x_3^*) \Big\} \tag{D}$$

or, equivalently

$$\sup_{(x_2^*, x_3^*) \in \mathbb{R}^n \times \mathbb{R}^m} \Big\{ - f_1^*(-x_2^* - B^* x_3^*) - f_2^*(x_2^*) - f_3^*(x_3^*) \Big\}. \tag{D}$$

Let $C\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be defined by $C(x_2^*, x_3^*) = -x_2^* - B^* x_3^*$, where C is a linear mapping and $h\colon \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$, $h(x_2^*, x_3^*) = f_2^*(x_2^*) + f_3^*(x_3^*)$.

In this case (D) can be equivalently written in the following way:

$$\inf_{(x_2^*, x_3^*) \in \mathbb{R}^n \times \mathbb{R}^m} \Big\{ f_1^*(C(x_2^*, x_3^*)) + h(x_2^*, x_3^*) \Big\}. \tag{4}$$

We can observe that $(\overline{x}_2^*, \overline{x}_3^*)$ is a solution of (4) if and only if $0 \in \partial(f_1^* \circ C + h)(\overline{x}_2^*, \overline{x}_3^*)$.

In order to ensure strong duality between the problem (D) and its dual, we need a regularity condition. Using the regularity conditions considered in [6, section 3.2.2] for problems having the composition with a linear continuous mapping in the objective function in finite dimensional case namely (RC_3^A), one has for the problem (D) the following regularity condition: $\operatorname{ri} \operatorname{dom} f_1^* \cap C(\operatorname{ri} \operatorname{dom} h) \neq \emptyset$. Because $\operatorname{dom} h = \operatorname{dom} f_2^* \times \operatorname{dom} f_3^*$ we conclude that for problem (D) the regularity condition is

$$\operatorname{ri} \operatorname{dom} f_1^* \cap (-\operatorname{ri} \operatorname{dom} f_2^* - B^* \operatorname{ri} \operatorname{dom} f_3^*) \neq \emptyset. \tag{RC}$$

Because f_1, f_2, f_3 are proper functions and (RC) is fulfilled we have that f_1^* and h are proper and convex functions and C is a linear mapping such that $\operatorname{dom} h \cap C^{-1} \operatorname{dom} f_1^* \neq \emptyset$. We can apply Theorem 3.3.4 and Remark 3.3.3 in [6] and we can conclude that if $(\overline{x}_2^*, \overline{x}_3^*) \in \mathbb{R}^n \times \mathbb{R}^m$ is a solution of (D) and if the regularity condition $\operatorname{ri} \operatorname{dom} f_1^* \cap C(\operatorname{ri} \operatorname{dom} h) \neq \emptyset$ is fulfilled, then there exists an optimal solution $\overline{x} \in \mathbb{R}^n$ to the dual problem of (D) such that

$$\text{(i)} \quad - C^* \overline{x} \in \partial h(\overline{x}_2^*, \overline{x}_3^*), \qquad \text{and} \qquad \text{(ii)} \;\; \overline{x} \in \partial f_1^*(C(\overline{x}_2^*, \overline{x}_3^*)).$$

Condition (i) is equivalent to $(\overline{x}, B\overline{x}) \in \partial h(\overline{x}_2^*, \overline{x}_3^*)$. Because of $\partial h(x_2^*, x_3^*) = \partial f_2^*(x_2^*) \times \partial f_3^*(x_3^*)$ we have that $\overline{x} \in \partial f_2^*(\overline{x}_2^*)$ and $B\overline{x} \in \partial f_3^*(\overline{x}_3^*)$. Using the definition of the operator C, condition (ii) can be written as $\overline{x} \in \partial f_1^*(-\overline{x}_2^* - B^*\overline{x}_3^*)$.

Now we can formulate the following result:

Theorem 3.1. *Let $B\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping, $f_1, f_2\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, $f_3\colon \mathbb{R}^m \to \overline{\mathbb{R}}$ are proper functions fulfilling* $\operatorname{dom} f_1 \cap \operatorname{dom} f_2 \cap B^{-1}(\operatorname{dom} f_3) \neq \emptyset$. *Let $(\overline{x}_2^*, \overline{x}_3^*) \in \mathbb{R}^n \times \mathbb{R}^m$ be an optimal solution to* (D) *and assume that* (RC) *is fulfilled. Then there exists $\overline{x} \in \mathbb{R}^n$, an optimal solution to the dual optimization problem of* (D), *such that*

(1) $\overline{x} \in \partial f_1^*(-\overline{x}_2^* - B^*\overline{x}_3^*)$, (2) $\overline{x} \in \partial f_2^*(\overline{x}_2^*)$, (3) $B\overline{x} \in \partial f_3^*(\overline{x}_3^*)$.

Remark 3.2. In case the function involved are proper, convex and lower semicontinuous, the dual of (D) is exactly (P). □

Remark 3.3. This result can be extended to the infinite dimensional setting, if we consider X and Y real separated locally convex spaces, $B\colon X \to Y$ a linear and continuous mapping, $f_1, f_2\colon X \to \overline{\mathbb{R}}$, $f_3\colon Y \to \overline{\mathbb{R}}$ proper functions fulfilling $\operatorname{dom} f_1 \cap \operatorname{dom} f_2 \cap B^{-1}(\operatorname{dom} f_3) \neq \emptyset$. There exists in the literature (see [6, 15]) regularity conditions which can be used in infinite dimensional setting (one can use the classical interior point regularity condition, or, regularity conditions that involve the lowersemicontinuity of the functions, generalized interiority notions and in this case the spaces X and Y needs to be Fréchet spaces). □

4. Duality for the equilibrium problem (CEP)

Let us consider now the following equilibrium problem:

$$\text{Find } \overline{x} \in \mathbb{R}^n \text{ s.t. } \varphi(\overline{x}, y) + f(y) + g(Ay) \geq f(\overline{x}) + g(A\overline{x}) \ \ \forall y \in \mathbb{R}^n, \qquad \text{(CEP)}$$

where $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g\colon \mathbb{R}^m \to \overline{\mathbb{R}}$ are proper and convex functions fulfilling $\operatorname{dom} f \cap A^{-1} \operatorname{dom} g \neq \emptyset$, $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, and $\varphi(x, \cdot)$ is a convex function $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$, $\varphi(x, x) = 0$ $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$.

For $\overline{x} \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$ we can rewrite the equilibrium problem (CEP) as an optimization problem

$$\inf_{y \in \mathbb{R}^n} \Big\{ \varphi(\overline{x}, y) + f(y) + g(Ay) \Big\}. \qquad (\mathrm{P}_{\overline{x}})$$

The following theorem establishes the connection between (CEP) and $(\mathrm{P}_{\overline{x}})$.

Theorem 4.1. *Let $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g\colon \mathbb{R}^m \to \overline{\mathbb{R}}$ be proper and convex functions and $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ fulfilling* $\operatorname{dom} f \cap A^{-1} \operatorname{dom} g \neq \emptyset$, *$\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\varphi(x, \cdot)$ a convex function $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$, $\varphi(x, x) = 0$ $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$ and assume that $A(\operatorname{ri} \operatorname{dom} f) \cap \operatorname{ri} \operatorname{dom} g \neq \emptyset$. The following statements are equivalent:*

(i) *$\overline{x}$ is a solution of* (CEP), (ii) *$\overline{x}$ is a solution of* $(\mathrm{P}_{\overline{x}})$, (iii) $\mathcal{D}(\overline{x}) \neq \emptyset$, *where* $\mathcal{D}(x) := \Big\{(u^*, v^*) : u^* \in \partial f(x), v^* \in \partial g(Ax), -u^* - A^*v^* \in \partial\varphi(x,\cdot)(x), \Big\}$, $x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$.

Proof. The equivalence (ii) $\Leftrightarrow$ (i) is trivial.

Let us prove now the equivalence between (ii) and (iii). We take $\overline{x}$ to be a solution for ($P_{\overline{x}}$), which is equivalent to (see [14])

$$0 \in \partial(\varphi(\overline{x},\cdot) + f + g \circ A)(\overline{x}).$$

Since $\operatorname{dom}\varphi(x,\cdot) = \mathbb{R}^n$ one get

$$0 \in \partial\varphi(\overline{x},\cdot)(\overline{x}) + \partial(f + g \circ A)(\overline{x}).$$

Since the regularity condition $A(\operatorname{ri}\operatorname{dom} f) \cap \operatorname{ri}\operatorname{dom} g \neq \emptyset$ is fulfilled, Theorem 23.8 and Theorem 23.9 in [14] ensure that

$$0 \in \partial\varphi(\overline{x},\cdot)(\overline{x}) + \partial f(\overline{x}) + A^*\partial g(A\overline{x}),$$

which means that there exists $-x^* \in \partial\varphi(\overline{x},\cdot)(\overline{x})$ and u^*, v^* such that $u^* + A^*v^* = x^*$ with $u^* \in \partial f(\overline{x})$ and $v^* \in \partial g(A\overline{x})$, which is equivalent to (iii). □

Remark 4.2. Let us underline the fact that the regularity condition is used only for the implication (ii) $\Rightarrow$ (iii). It is proved (see [14]) that the implication $\partial f(x) + A^*\partial g(Ax) \subseteq \partial(f + g \circ A)(x)$ is valid even if the regularity condition is not fulfilled. □

When the regularity condition $A(\operatorname{ri}\operatorname{dom} f) \cap \operatorname{ri}\operatorname{dom} g \neq \emptyset$ is fulfilled, problem (CEP) can be written in the following way:

$$\begin{aligned}&\text{Find } \overline{x} \in \mathbb{R}^n \text{ such that there exists } (\overline{u}^*, \overline{v}^*) \in \mathbb{R}^n \times \mathbb{R}^m \text{ with}\\ &(p1)\quad -\overline{u}^* - A^*\overline{v}^* \in \partial\varphi(\overline{x},\cdot)(\overline{x}),\\ &(p2)\quad \overline{u}^* \in \partial f(\overline{x}),\ \overline{v}^* \in \partial g(A\overline{x}).\end{aligned}$$

Using the characterization of subdifferential of a given function f and its conjugate, (2), we can attach the following dual problem to (CEP):

$$\begin{aligned}&\text{Find } (\overline{u}^*, \overline{v}^*) \in \mathbb{R}^n \times \mathbb{R}^m \text{ such that there exists } \overline{x} \in \mathbb{R}^n \text{ with}\\ &(d1)\quad \overline{x} \in \partial\varphi^*(\overline{x},\cdot)(-\overline{u}^* - A^*\overline{v}^*),\\ &(d2)\quad \overline{x} \in \partial f^*(\overline{u}^*),\ A\overline{x} \in \partial g^*(\overline{v}^*),\end{aligned} \tag{DCEP}$$

where $\varphi^*(x,\cdot)(x^*)$ is the conjugate of φ on its second variable, $\varphi^*(x,\cdot)(x^*) = (\varphi(x,\cdot))^*(x^*)$.

Theorem 4.3. *Let* $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ *and* $g\colon \mathbb{R}^m \to \overline{\mathbb{R}}$ *be proper and convex functions and* $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ *fulfilling* $\operatorname{dom} f \cap A^{-1} \operatorname{dom} g \neq \emptyset$, $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\varphi(x,\cdot)$ *a convex function* $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$, $\varphi(x,x) = 0$, $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$ *and assume that* $A(\operatorname{ri}\operatorname{dom} f) \cap \operatorname{ri}\operatorname{dom} g \neq \emptyset$. *If* $\overline{x} \in \mathbb{R}^n$ *solves* (CEP) *then any element of* $\mathcal{D}(\overline{x})$ *is a solution of* (DCEP).

Proof. Let us suppose that (CEP) is solvable and $\overline{x} \in \mathbb{R}^n$ is a solution of (CEP). This is equivalent, cf. Theorem 4.1, to $\mathcal{D}(\overline{x}) \neq \emptyset$ which means that there exists $(\overline{u}^*, \overline{v}^*) \in \mathbb{R}^n \times \mathbb{R}^m$ such that $\overline{u}^* \in \partial f(\overline{x})$, $\overline{v}^* \in \partial g(A\overline{x})$ and $-\overline{u}^* - A^*\overline{v}^* \in \partial\varphi(\overline{x}, \cdot)(\overline{x})$. Using the characterization (2) we have that $\overline{x} \in \partial f^*(\overline{u}^*)$, $A\overline{x} \in \partial g^*(\overline{v}^*)$ and $\overline{x} \in \partial\varphi^*(\overline{x}, \cdot)(-\overline{u}^* - A^*\overline{v}^*)$ which means that $(\overline{u}^*, \overline{v}^*) \in \mathcal{D}(\overline{x})$ is a solution of (DCEP). □

Remark 4.4. The above theorem tell us that the set $\mathcal{D}(\overline{x})$ is nothing else but the set of solutions of (DCEP) associated to $\overline{x}$. □

We consider the set:

$$\mathcal{P}(u^*, v^*) = \begin{Bmatrix} x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g \ : \\ x \in \partial\varphi^*(x, \cdot)(-u^* - A^*v^*) \cap \partial f^*(u^*) \cap A^{-1}\partial g^*(v^*) \end{Bmatrix}.$$

The next theorem characterizes the solutions of (DCEP) by the set $\mathcal{P}(u^*, v^*)$.

Theorem 4.5. *Let* $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ *and* $g\colon \mathbb{R}^m \to \overline{\mathbb{R}}$ *be proper, convex and lower semicontinuous functions and* $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ *fulfilling* $\operatorname{dom} f \cap A^{-1} \operatorname{dom} g \neq \emptyset$, $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\varphi(x, \cdot)$ *a convex function* $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$, $\varphi(x, x) = 0$, $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$. *If* $(\overline{u}^*, \overline{v}^*)$ *is a solution of* (DCEP) *then any element of* $\mathcal{P}(\overline{u}^*, \overline{v}^*)$ *is a solution of* (CEP).

Proof. Let $(\overline{u}^*, \overline{v}^*)$ be a solution of (DCEP) which means that there exists $\overline{x} \in \mathbb{R}^n$ such that $\overline{x} \in \partial\varphi^*(\overline{x}, \cdot)(-\overline{u}^* - A^*\overline{v}^*)$, $\overline{x} \in \partial f^*(\overline{u}^*)$ and $A\overline{x} \in \partial g^*(\overline{v}^*)$. Using (3) we obtain $-\overline{u}^* - A^*\overline{v}^* \in \partial\varphi(\overline{x}, \cdot)(\overline{x})$, $\overline{u}^* \in \partial f(\overline{x})$ and $\overline{v}^* \in \partial g(A\overline{x})$ that is nothing else but $\overline{x} \in \mathcal{P}(\overline{u}^*, \overline{v}^*)$. We can easily see that (p1) respectively (p2) are fulfilled and in this case $\overline{x} \in \mathcal{P}(\overline{u}^*, \overline{v}^*)$ is a solution of (CEP). □

Remark 4.6. Theorem 4.5 tell us that any element $\overline{x} \in \mathcal{P}(\overline{u}^*, \overline{v}^*)$ generate a solution of (CEP). If we suppose in addition that the regularity condition $A(\operatorname{ri} \operatorname{dom} f) \cap \operatorname{ri} \operatorname{dom} g \neq \emptyset$ is fulfilled then the set $\mathcal{P}(\overline{u}^*, \overline{v}^*)$ is nothing else but the set of solutions of (CEP) associated to $\overline{u}^* + A^*\overline{v}^*$. □

The next two theorems can be proved in the same way as Theorem 4.3 and Theorem 4.5, using the characterizations (2) respectively (3). The details are left to the reader.

Theorem 4.7. *Let* $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ *and* $g\colon \mathbb{R}^m \to \overline{\mathbb{R}}$ *be proper and convex functions and* $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ *fulfilling* $\operatorname{dom} f \cap A^{-1} \operatorname{dom} g \neq \emptyset$, $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\varphi(x, \cdot)$ *a convex function* $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$, $\varphi(x, x) = 0$, $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$. *If* $(\overline{u}^*, \overline{v}^*) \in \mathcal{D}(\overline{x})$ *then* $\overline{x} \in \mathcal{P}(\overline{u}^*, \overline{v}^*)$.

Theorem 4.8. *Let* $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ *and* $g\colon \mathbb{R}^m \to \overline{\mathbb{R}}$ *be proper, convex and lower semicontinuous functions and* $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ *fulfilling* $\operatorname{dom} f \cap A^{-1} \operatorname{dom} g \neq \emptyset$, $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\varphi(x, \cdot)$ *a convex function* $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$, $\varphi(x, x) = 0$, $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$. *If* $\overline{x} \in \mathcal{P}(\overline{u}^*, \overline{v}^*)$ *then* $(\overline{u}^*, \overline{v}^*) \in \mathcal{D}(\overline{x})$.

The solutions of (CEP) respectively (DCEP) are the saddle points of the Lagrangian function:

$$\mathcal{L}_{\overline{x}}(x, y, u^*, v^*) = f(x) - \langle u^*, x\rangle + g(y) - \langle v^*, y\rangle - \varphi^*(\overline{x}, \cdot)(-u^* - A^*v^*)$$

as the following theorem shows. Let us recall the definition of a saddle point.

The point $(\overline{x}, A\overline{x}, \overline{u}^*, \overline{v}^*)$ is called *saddle point* of $\mathcal{L}_{\overline{x}}$ if

$$\mathcal{L}_{\overline{x}}(\overline{x}, A\overline{x}, u^*, v^*) \leq \mathcal{L}_{\overline{x}}(\overline{x}, A\overline{x}, \overline{u}^*, \overline{v}^*) \leq \mathcal{L}_{\overline{x}}(x, y, \overline{u}^*, \overline{v}^*), \tag{5}$$

for all $(x, y, u^*, v^*) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m$.

Theorem 4.9. *Let* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *and* $g : \mathbb{R}^m \to \overline{\mathbb{R}}$ *be proper, convex and lower semicontinuous functions and* $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ *fulfilling* $\operatorname{dom} f \cap A^{-1} \operatorname{dom} g \neq \emptyset$, $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\varphi(x, \cdot)$ *a convex function* $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$, $\varphi(x, x) = 0$, $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$. *The following statements are equivalent:*

(i) $(\overline{u}^*, \overline{v}^*) \in \mathcal{D}(\overline{x})$*;*

(ii) $\overline{x} \in \mathcal{P}(\overline{u}^*, \overline{v}^*)$*;*

(iii) $(\overline{x}, A\overline{x}, \overline{u}^*, \overline{v}^*)$ *is a saddle point of* $\mathcal{L}_{\overline{x}}$.

Proof. The equivalence between (i) and (ii) comes from Theorem 4.7 and Theorem 4.8. We prove the implication (i) $\Rightarrow$ (iii). Let $(\overline{u}^*, \overline{v}^*) \in \mathcal{D}(\overline{x})$, i.e.

$$-\overline{u}^* - A^*\overline{v}^* \in \partial\varphi(\overline{x}, \cdot)(\overline{x}), \tag{6}$$

$$\overline{u}^* \in \partial f(\overline{x}), \tag{7}$$

$$\overline{v}^* \in \partial g(A\overline{x}). \tag{8}$$

Using the definition of the subdifferential of a function at a point one has:

$$(7) \iff f(x) - f(\overline{x}) \geq \langle \overline{u}^*, x - \overline{x}\rangle \quad \forall x \in \mathbb{R}^n,$$

$$(8) \iff g(y) - g(A\overline{x}) \geq \langle \overline{v}^*, y - A\overline{x}\rangle \quad \forall y \in \mathbb{R}^m.$$

Adding now the two inequalities we obtain for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$:

$$f(x) + g(y) - \langle \overline{u}^*, x\rangle - \langle \overline{v}^*, y\rangle \geq f(\overline{x}) + g(A\overline{x}) - \langle \overline{u}^*, \overline{x}\rangle - \langle \overline{v}^*, A\overline{x}\rangle.$$

Adding in both sides $-\varphi^*(\overline{x}, \cdot)(-\overline{u}^* - A^*\overline{v}^*)$ we get

$$\mathcal{L}_{\overline{x}}(x, y, \overline{u}^*, \overline{v}^*) \geq \mathcal{L}_{\overline{x}}(\overline{x}, A\overline{x}, \overline{u}^*, \overline{v}^*),\ \forall (x, y) \in \mathbb{R}^n \times \mathbb{R}^m,$$

and the second inequality of (5) is proved. In order to prove the first inequality we use the relation (2) and we have that relation (6) implies that $\overline{x} \in \partial\varphi^*(\overline{x}, \cdot)(-\overline{u}^* - A^*\overline{v}^*)$. By using again the definition of the subdifferential of the function $\varphi^*(\overline{x}, \cdot)$ at $(-\overline{u}^* - A^*\overline{v}^*)$ we get:

$$\varphi^*(\overline{x}, \cdot)(y^*) - \varphi^*(\overline{x}, \cdot)(-\overline{u}^* - A^*\overline{v}^*) \geq \langle \overline{x}, y^* + \overline{u}^* + A^*\overline{v}^*\rangle,\ \forall y^* \in \mathbb{R}^n,$$

and if we replace y^* with $-u^* - A^*v^*$ in the above inequality we obtain:

$$\varphi^*(\overline{x},\cdot)(-u^* - A^*v^*) - \varphi^*(\overline{x},\cdot)(-\overline{u}^* - A^*\overline{v}^*) \geq \langle \overline{x}, -u^* - A^*v^* + \overline{u}^* + A^*\overline{v}^* \rangle$$

for all $(u^*, v^*) \in \mathbb{R}^n \times \mathbb{R}^m$. Adding to both sides $f(\overline{x}) + g(A\overline{x})$ we get:

$$\mathcal{L}_{\overline{x}}(\overline{x}, A\overline{x}, u^*, v^*) \leq \mathcal{L}_{\overline{x}}(\overline{x}, A\overline{x}, \overline{u}^*, \overline{v}^*), \ \forall (u^*, v^*) \in \mathbb{R}^n \times \mathbb{R}^m,$$

and we showed that the first inequality of (5) is fulfilled, hence $(\overline{x}, A\overline{x}, \overline{u}^*, \overline{v}^*)$ is a saddle point of $\mathcal{L}_{\overline{x}}$.

We only have to prove the implication (iii) $\Rightarrow$ (i). Let $(\overline{x}, A\overline{x}, \overline{u}^*, \overline{v}^*)$ be a saddle point of $\mathcal{L}_{\overline{x}}$. By the first inequality of (5) we have

$$\begin{aligned} f(\overline{x}) - \langle u^*, \overline{x} \rangle + g(A\overline{x}) - \langle v^*, A\overline{x} \rangle - \varphi^*(\overline{x},\cdot)(-u^* - A^*v^*) \leq \\ \leq f(\overline{x}) - \langle \overline{u}^*, \overline{x} \rangle + g(A\overline{x}) - \langle \overline{v}^*, A\overline{x} \rangle - \varphi^*(\overline{x},\cdot)(-\overline{u}^* - A^*\overline{v}^*), \end{aligned}$$

for all $(u^*, v^*) \in \mathbb{R}^n \times \mathbb{R}^m$, which is equivalent to

$$\varphi^*(\overline{x},\cdot)(-u^* - A^*v^*) - \varphi^*(\overline{x},\cdot)(-\overline{u}^* - A^*\overline{v}^*) \geq \langle \overline{x}, -u^* - A^*v^* + \overline{u}^* + A^*\overline{v}^* \rangle,$$

for all $(u^*, v^*) \in \mathbb{R}^n \times \mathbb{R}^m$, hence $\overline{x} \in \partial\varphi^*(\overline{x},\cdot)(-\overline{u}^* - A^*\overline{v}^*)$.

By the second inequality of (5) we get

$$\begin{aligned} f(\overline{x}) - \langle \overline{u}^*, \overline{x} \rangle + g(A\overline{x}) - \langle \overline{v}^*, A\overline{x} \rangle - \varphi^*(\overline{x},\cdot)(-\overline{u}^* - A^*\overline{v}^*) \leq \\ \leq f(x) - \langle \overline{u}^*, x \rangle + g(y) - \langle \overline{v}^*, y \rangle - \varphi^*(\overline{x},\cdot)(-\overline{u}^* - A^*\overline{v}^*), \end{aligned}$$

for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ which is equivalent to

$$f(x) - f(\overline{x}) - \langle \overline{u}^*, x - \overline{x} \rangle + g(y) - g(A\overline{x}) - \langle \overline{v}^*, y - A\overline{x} \rangle \geq \ 0$$

for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$. Putting $x = \overline{x}$ in this last inequality we obtain $g(y) - g(A\overline{x}) \geq \langle \overline{v}^*, y - A\overline{x} \rangle$ for all $y \in \mathbb{R}^m$, which is nothing else than $\overline{v}^* \in \partial g(A\overline{x})$ and again, if we put $y = A\overline{x}$, we obtain $f(x) - f(\overline{x}) \geq \langle \overline{u}^*, x - \overline{x} \rangle$ for all $x \in \mathbb{R}^n$, which means that $\overline{u}^* \in \partial f(\overline{x})$. Hence, $(\overline{u}^*, \overline{v}^*) \in \mathcal{D}(\overline{x})$. □

The next results are consequences of Theorem 4.9.

Corollary 4.10. *Let* $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ *and* $g\colon \mathbb{R}^m \to \overline{\mathbb{R}}$ *be proper and convex functions and* $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ *fulfilling* $\operatorname{dom} f \cap A^{-1} \operatorname{dom} g \neq \emptyset$, $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\varphi(x,\cdot)$ *a convex function* $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$, $\varphi(x,x) = 0$, $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$ *and assume that* $A(\operatorname{ri} \operatorname{dom} f) \cap \operatorname{ri} \operatorname{dom} g \neq \emptyset$. $\overline{x} \in \mathbb{R}^n$ *is a solution of* (CEP) *if and only if there exists* $(\overline{u}^*, \overline{v}^*)$ *such that* $(\overline{x}, A\overline{x}, \overline{u}^*, \overline{v}^*)$ *is a saddle point of* $\mathcal{L}_{\overline{x}}$.

Corollary 4.11. *Let* $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ *and* $g\colon \mathbb{R}^m \to \overline{\mathbb{R}}$ *be proper, convex and lower semicontinuous functions and* $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ *fulfilling* $\operatorname{dom} f \cap A^{-1} \operatorname{dom} g \neq \emptyset$, $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\varphi(x,\cdot)$ *a convex function* $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$, $\varphi(x,x) = 0$, $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$. $(\overline{u}^*, \overline{v}^*) \in \mathbb{R}^n \times \mathbb{R}^m$ *is a solution of* (DCEP) *if and only if there exists* $\overline{x} \in \mathbb{R}^n$ *such that* $(\overline{x}, A\overline{x}, \overline{u}^*, \overline{v}^*)$ *is a saddle point of* $\mathcal{L}_{\overline{x}}$.

If we consider the optimization problem

$$\inf_{y\in\mathbb{R}^n}\Big\{\varphi(\overline{x},y)+f(y)+g(Ay)\Big\} \tag{$\mathrm{P}_{\overline{x}}$}$$

we can formulate the Fenchel dual problem to $(\mathrm{P}_{\overline{x}})$ as

$$\sup_{\substack{x^*\in\mathbb{R}^n\\ y^*\in\mathbb{R}^m}}\Big\{-\varphi^*(\overline{x},\cdot)(-x^*)-g^*(y^*)-f^*(x^*-A^*y^*)\Big\} \tag{$\mathrm{D}_{\overline{x}}$}$$

We can observe that problem $(\mathrm{D}_{\overline{x}})$ is not the optimization form of the problem (DCEP), but we can prove the following relation between them.

Theorem 4.12. *Let $f\colon\mathbb{R}^n\to\overline{\mathbb{R}}$ and $g\colon\mathbb{R}^m\to\overline{\mathbb{R}}$ be proper and convex functions and $A\in\mathcal{L}(\mathbb{R}^n,\mathbb{R}^m)$ fulfilling $\operatorname{dom} f\cap A^{-1}\operatorname{dom} g\neq\emptyset$, $\varphi\colon\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$, $\varphi(x,\cdot)$ a convex function $\forall x\in\operatorname{dom} f\cap A^{-1}\operatorname{dom} g$, $\varphi(x,x)=0$, $\forall x\in\operatorname{dom} f\cap A^{-1}\operatorname{dom} g$. If $(\overline{u}^*,\overline{v}^*)$ is a solution of (DCEP) then there exists $\overline{x}\in\mathcal{P}(\overline{u}^*,\overline{v}^*)$ such that $(\overline{u}^*+A^*\overline{v}^*,\overline{v}^*)$ is a solution of $(\mathrm{D}_{\overline{x}})$.*

Proof. Let $(\overline{u}^*,\overline{v}^*)$ be a solution of (DCEP) that means that there exists $\overline{x}\in\mathbb{R}^n$ such that $\overline{x}\in\partial\varphi^*(\overline{x},\cdot)(-\overline{u}^*-A^*\overline{v}^*)$, $\overline{x}\in\partial f^*(\overline{u}^*)$ and $A\overline{x}\in\partial g^*(\overline{v}^*)$ i.e. $\overline{x}\in\mathcal{P}(\overline{u}^*,\overline{v}^*)$. Using the definition of the subdifferential for each of the relations above we obtain:

$$\begin{gathered}\varphi^*(\overline{x},\cdot)(-x^*)-\varphi^*(\overline{x},\cdot)(-\overline{u}^*-A^*\overline{v}^*)\geq\langle\overline{x},\overline{u}^*+A^*\overline{v}^*-x^*\rangle\ \ \forall x^*\in\mathbb{R}^n,\\ g^*(y^*)-g^*(\overline{v}^*)\geq\langle A\overline{x},y^*-\overline{v}^*\rangle=\langle\overline{x},A^*y^*-A^*\overline{v}^*\rangle\ \ \forall y^*\in\mathbb{R}^m,\\ f^*(z^*)-f^*(\overline{u}^*)\geq\langle\overline{x},z^*-\overline{u}^*\rangle\ \ \forall z^*\in\mathbb{R}^n.\end{gathered}$$

If we replace z^* with $x^*-A^*y^*$ in the last inequality we obtain:

$$f^*(x^*-A^*y^*)-f^*(\overline{u}^*)\geq\langle\overline{x},x^*-A^*y^*-\overline{u}^*\rangle\ \ \forall(x^*,y^*)\in\mathbb{R}^n\times\mathbb{R}^m.$$

Adding these inequalities we obtain for all $(x^*,y^*)\in\mathbb{R}^n\times\mathbb{R}^m$

$$-\varphi^*(\overline{x},\cdot)(-\overline{u}^*-A^*\overline{v}^*)-f^*(\overline{u}^*)-g^*(\overline{v}^*)\geq-\varphi^*(\overline{x},\cdot)(-x^*)-f^*(x^*-A^*y^*)-g^*(y^*),$$

Taking the supremum in both sides over $x^*\in\mathbb{R}^n$ and $y^*\in\mathbb{R}^m$ we get

$$\begin{aligned}&\sup_{\substack{x^*\in\mathbb{R}^n\\ y^*\in\mathbb{R}^m}}\Big\{-\varphi^*(\overline{x},\cdot)(-x^*)-g^*(y^*)-f^*(x^*-A^*y^*)\Big\}\leq\\ &\qquad\qquad\leq-\varphi^*(\overline{x},\cdot)(-\overline{u}^*-A^*\overline{v}^*)-g^*(\overline{v}^*)-f^*(\overline{u}^*),\end{aligned}$$

hence $(\overline{u}^*+A^*\overline{v}^*,\overline{v}^*)$ is solution for $(\mathrm{D}_{\overline{x}})$. □

In what follows we give results which guarantees that all the solutions of (CEP) or (DCEP) can be found using the problem $(\mathrm{P}_{\overline{x}})$ respectively $(\mathrm{D}_{\overline{x}})$ if the following property of function φ is fulfilled:

$$\varphi(x,y)\leq\varphi(x,z)+\varphi(z,y),\ \forall x,y,z\in\mathbb{R}^n, \tag{*}$$

property used for the same reason in [1, 3, 4].

Theorem 4.13. *Let* $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ *and* $g\colon \mathbb{R}^m \to \overline{\mathbb{R}}$ *be proper and convex functions and* $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ *fulfilling* $\operatorname{dom} f \cap A^{-1} \operatorname{dom} g \neq \emptyset$, $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\varphi(x,\cdot)$ *a convex function* $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$, $\varphi(x,x) = 0$, $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$. *If the function* φ *satisfies property (*) then* $\overline{x}$ *is a solution of the problem* (CEP) *if and only if there exists* $z \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$ *such that* $\overline{x}$ *is a solution of* (P_z).

Proof. Let us consider that $\overline{x}$ is a solution of (CEP). Using Theorem 4.1 we can conclude that there exists $z = \overline{x} \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$ such that $\overline{x}$ is a solution of (P_z). Let us notice that we don't need the regularity condition used in the hypothesis of Theorem 4.1 (see Remark 4.2). For the reverse implication we take $\overline{x}$ to be a solution of (P_z), where $z \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$ which means:

$$\varphi(z,\overline{x}) + f(\overline{x}) + g(A\overline{x}) \leq \varphi(z,y) + f(y) + g(Ay) \, \forall y \in \mathbb{R}^n.$$

Applying now the property (*) we have:

$$\varphi(\overline{x},y) \geq (f + g \circ A)(\overline{x}) - (f + g \circ A)(y) \ \forall y \in \mathbb{R}^n,$$

which is equivalent to

$$\varphi(\overline{x},y) + f(y) + g(Ay) \geq f(\overline{x}) + g(A\overline{x}) \ \forall y \in \mathbb{R}^n,$$

hence $\overline{x} \in \mathbb{R}^n$ is a solution of (CEP). □

Let us consider for problem (D) in Section 3 $f_1^* \mapsto \varphi^*(\overline{x},\cdot)$, $f_2^* \mapsto f^*$, $f_3^* \mapsto g^*$ and $B \mapsto A$. Then, the regularity condition (RC) becomes:

$$\operatorname{ri} \operatorname{dom} \varphi^*(\overline{x},\cdot) \cap (-\operatorname{ri} \operatorname{dom} f^* - A^* \operatorname{ri} \operatorname{dom} g^*) \neq \emptyset \tag{RCC}$$

and Problem (D) becomes Problem $(\mathrm{D}_{\overline{x}})$.

Theorem 4.14. *Let* $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ *and* $g\colon \mathbb{R}^m \to \overline{\mathbb{R}}$ *be proper and convex functions and* $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ *fulfilling* $\operatorname{dom} f \cap A^{-1} \operatorname{dom} g \neq \emptyset$, $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\varphi(x,\cdot)$ *a convex function* $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$, $\varphi(x,x) = 0$, $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$. *Assume that* (RCC) *for all* $x \in \mathbb{R}^n$ *and function* φ *satisfies* (*). *Then,* $(\overline{u}^*, \overline{v}^*)$ *is a solution of* (DCEP) *if and only if there exists* $z \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$ *such that* $(\overline{u}^* + A^*\overline{v}^*, \overline{v}^*)$ *is a solution of* (D_z).

Proof. The necessary condition was proved in Theorem 4.12 if we consider $z = \overline{x}$. For the sufficiency we consider $z \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$ such that $(\overline{u}^* + A^*\overline{v}^*, \overline{v}^*)$ is a solution of (D_z), i.e., for all $x^* \in \mathbb{R}^n$ and $y^* \in \mathbb{R}^m$,

$$\varphi^*(z,\cdot)(-x^*) + g^*(y^*) + f^*(x^* - A^*y^*) \geq \varphi^*(z,\cdot)(-\overline{u}^* - A^*\overline{v}^*) + g^*(\overline{v}^*) + f^*(\overline{u}^*).$$

Making the substitution $-x^* \mapsto -x^* - A^*y^*$ we get for all $x^* \in \mathbb{R}^n$ and $y^* \in \mathbb{R}^m$

$$\varphi^*(z,\cdot)(-x^* - A^*y^*) + g^*(y^*) + f^*(x^*) \geq \varphi^*(z,\cdot)(-\overline{u}^* - A^*\overline{v}^*) + g^*(\overline{v}^*) + f^*(\overline{u}^*),$$

i.e., $(\overline{u}^*, \overline{v}^*)$ is a solution of the problem

$$\sup_{\substack{x^* \in \mathbb{R}^n \\ y^* \in \mathbb{R}^m}} \Big\{ -\varphi^*(z,\cdot)(-x^* - A^*y^*) - g^*(y^*) - f^*(x^*) \Big\} \tag{D'}$$

We can apply Theorem 3.1 to problem (D') and conclude that there exists $\overline{x} \in R^n$ such that

(1) $\overline{x} \in \partial\varphi^*(z,\cdot)(-\overline{u}^* - A^*\overline{v}^*)$, (2) $\overline{x} \in \partial f^*(\overline{u}^*)$, and (3) $A\overline{x} \in \partial g^*(\overline{v}^*)$.

We can observe that the relations (2) and (3) are the same as $(d2)$ in the definition of problem (DCEP). Relation (1) can be written as $(-\overline{u}^* - A^*\overline{v}^*) \in \partial\varphi(z,\cdot)(\overline{x})$, i.e.

$$\varphi(z,\cdot)(x) - \varphi(z,\cdot)(\overline{x}) \geq \langle -\overline{u}^* - A^*\overline{v}^*, x - \overline{x}\rangle \quad \forall x \in \mathbb{R}^n.$$

Using (*) we get for all $x \in \mathbb{R}^n$

$$\varphi(\overline{x},x) - \varphi(\overline{x},\overline{x}) \geq \varphi(\overline{x},x) \geq \varphi(z,x) - \varphi(z,\overline{x}) \geq \langle -\overline{u}^* - A^*\overline{v}^*, x - \overline{x}\rangle,$$

which is equivalent to $(-\overline{u}^* - A^*\overline{v}^*) \in \partial\varphi(\overline{x},\cdot)(\overline{x})$ or, equivalently, to the relation $\overline{x} \in \partial\varphi^*(\overline{x},\cdot)(-\overline{u}^* - A^*\overline{v}^*)$, which is identical to $(d1)$ in (DCEP). □

5. Particular cases

In what follows we particularize the functions considered in the definition of (CEP) and we rediscover some equilibrium problems and variational inequalities introduced in the literature [3, 13, 8].

5.1. Equilibrium problems

Let us particularize the duality statements for the problems (CEP) and (DCEP) to the case when $m = n$, $g(x) = 0, \forall x \in \mathbb{R}^n$, $A\colon \mathbb{R}^n \to \mathbb{R}^n$ is the identity operator. Because in this case $A(\operatorname{ri}\operatorname{dom} g) = \mathbb{R}^n$, the regularity condition used in the theorems hypothesis in Section 4 becomes $\operatorname{ri}\operatorname{dom} f \neq \emptyset$ which is always true in the settings considered for problem (CEP). The equilibrium problem (CEP) becomes:

$$\text{Find } \overline{x} \in \mathbb{R}^n \text{ s.t. } \varphi(\overline{x},y) + f(y) \geq f(\overline{x}) \quad \forall y \in \mathbb{R}^n. \qquad \text{(CEP')}$$

We can notice that the subdifferential of function $g \equiv 0$ is $\partial g(x) = \{0\}$, hence $(d2)$ in formulation of (DCEP) becomes $\overline{v}^* = 0$. So, (DCEP) looks like

$$\begin{array}{c}\text{Find } \overline{u}^* \in \mathbb{R}^n \text{ such that there exists } \overline{x} \in \mathbb{R}^n \text{ with}\\ (d1)\ \ \overline{x} \in \partial\varphi^*(\overline{x},\cdot)(-\overline{u}^*) \ \text{ and } \ (d2)\ \ \overline{x} \in \partial f^*(\overline{u}^*),\end{array} \qquad \text{(DCEP')}$$

which is nothing else but the so called generalized equilibrium problem (GEP) and its dual (DGEP) considered in [3].

The optimization problem $(\mathrm{P}_{\overline{x}})$ becomes:

$$\inf_{y \in \mathbb{R}^n} \Big\{\varphi(\overline{x},y) + f(y)\Big\} \qquad (\mathrm{P}'_{\overline{x}})$$

and its dual:

$$\sup_{x^* \in \mathbb{R}^n} \Big\{ -\varphi^*(\overline{x},\cdot)(-x^*) - f^*(x^*)\Big\}, \qquad (\mathrm{D}'_{\overline{x}})$$

which is the dual pair of extremum problems considered in [3].

The Lagrangian function $\mathcal{L}_{\overline{x}}$ becomes in this particular case

$$\mathcal{L}_{\overline{x}}(x,y,u^*,v^*) = f(x) - \langle u^*, x\rangle - \langle v^*, y\rangle - \varphi^*(\overline{x},\cdot)(-u^* - v^*).$$

One can show that if $g \equiv 0$ then its conjugate function is $g^*(x^*) = \delta_{\{0\}}(x^*)$, for all $x^* \in \mathbb{R}^n$, the adjoint operator of A is $A^* = A$. The set $\mathcal{D}$ becomes:

$$\mathcal{D}'(x) = \Big\{(u^*,v^*) : u^* \in \partial f(x), v^* = 0, -u^* \in \partial\varphi(x,\cdot)(x)\Big\},$$

and the set $\mathcal{P}$ looks like:

$$\mathcal{P}'(u^*,v^*) = \Big\{x \in \operatorname{dom} f : x \in \partial\varphi^*(x,\cdot)(-u^* - v^*) \cap \partial f^*(u^*) \cap \partial\delta_{\{0\}}(v^*)\Big\}.$$

We can affirm that if $x \in \mathcal{P}'(u^*,v^*)$, then we have $v^* = 0$.

In this particular case the regularity condition $A(\operatorname{ri}\operatorname{dom} f)\cap\operatorname{ri}\operatorname{dom} g \neq \emptyset$ becomes $\operatorname{ri}\operatorname{dom} f \neq \emptyset$ which is fulfilled when f is proper and convex.

The setting for all the theorems in this particular case becomes: $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous function with $\operatorname{dom} f \neq \emptyset$, $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to \overline{\mathbb{R}}$, $\varphi(x,\cdot)$ is a convex function for all $x \in \operatorname{dom} f$, and $\varphi(x,x) = 0$ for all $x \in \operatorname{dom} f$. This setting is used by the authors of [3] for all the results in their paper.

Theorem 4.1 affirms, in this particular case, that the following statements are equivalent:

(i) $\overline{x}$ is a solution of (CEP'), (ii) $\overline{x}$ is a solution of ($\mathrm{P}'_{\overline{x}}$), (iii) $\mathcal{D}'(\overline{x}) \neq \emptyset$.

The first two conditions are exactly the first ones considered in [3, Theorem 3.1]. We can easily observe that (iii) is equivalent with $v^* = 0$ and $u^* \in \partial f(\overline{x}) \cap -\partial\varphi(\overline{x},\cdot)(\overline{x})$, the last one being exactly the definition of the set $\mathcal{D}$ at $\overline{x}$ considered in [3] and so we rediscover the similar result.

Let us now apply Theorem 4.9 to this particular case. Then we get that the following statements are equivalent:

(i) $(\overline{u}^*,\overline{v}^*)\in\mathcal{D}'(\overline{x})$, (ii) $\overline{x}\in\mathcal{P}'(\overline{u}^*,\overline{v}^*)$, (iii) $(\overline{x},\overline{x},\overline{u}^*,\overline{v}^*)$ is a saddle point of $\mathcal{L}'_{\overline{x}}$.

As previously, (i) is equivalent to $v^* = 0$ and $u^* \in \partial f(\overline{x}) \cap -\partial\varphi(\overline{x},\cdot)(\overline{x})$, which is nothing else but the first statement of [3, Theorem 3.2]. Since $v^* = 0$ we have that $\partial\delta_{\{0\}}(v^*) = \mathbb{R}^n$ and the second statement becomes $\overline{x} \in \partial\varphi^*(\overline{x},\cdot)(-u^*) \cap \partial f^*(u^*)$ which is exactly the second statement in [3, Theorem 3.2]. Let us see how the third statement looks like in this case. We have that $(\overline{x},\overline{x},\overline{u}^*,0)$ is a saddle point of $\mathcal{L}'_{\overline{x}}$, that is

$$\mathcal{L}'_{\overline{x}}(\overline{x},\overline{x},u^*,v^*) \leq \mathcal{L}'_{\overline{x}}(\overline{x},\overline{x},\overline{u}^*,0) \leq \mathcal{L}'_{\overline{x}}(x,y,\overline{u}^*,0) \ \ \forall x,y,u^*,v^* \in \mathbb{R}^n,$$

which means that

$$\begin{aligned} f(\overline{x}) - \langle u^*, \overline{x}\rangle - \langle v^*, \overline{x}\rangle - \varphi^*(\overline{x},\cdot)(-u^* - v^*) &\leq f(\overline{x}) - \langle \overline{u}^*, \overline{x}\rangle - \varphi^*(\overline{x},\cdot)(-\overline{u}^*) \\ &\leq f(x) - \langle \overline{u}^*, x\rangle - \varphi^*(\overline{x},\cdot)(-\overline{u}^*) \end{aligned}$$

for all $x, y, u^*, v^* \in \mathbb{R}^n$. We can see that y plays no role in the above inequality and we can put $u := u^* + v^*$ to get

$$\begin{aligned} f(\overline{x}) - \langle u, \overline{x}\rangle - \varphi^*(\overline{x}, \cdot)(-u) &\leq f(\overline{x}) - \langle \overline{u}^*, \overline{x}\rangle - \varphi^*(\overline{x}, \cdot)(-\overline{u}^*) \\ &\leq f(x) - \langle \overline{u}^*, x\rangle - \varphi^*(\overline{x}, \cdot)(-\overline{u}^*) \end{aligned}$$

for all $x, u \in \mathbb{R}^n$, i.e., $(\overline{x}, \overline{u}^*)$ is a saddle point of the Langrangian function $\mathcal{L}_{\overline{x}}(x, u) = f(x) - \langle u, x\rangle - \varphi^*(\overline{x}, \cdot)(-u)$ considered in [3]. Hence, we rediscover [3, Theorem 3.2].

Suppose now that φ satisfies (*). If we apply Theorem 4.13 we obtain that $\overline{x}$ is a solution of problem (CEP') if and only if there exists $z \in \operatorname{dom} f$ such that $\overline{x}$ is a solution of (P'_z). The property (*) is the same as the authors used in [3] for the Theorem 3.4. and, as we showed before, we have the same problems as in theorem mentioned here and again, we rediscover similar result.

Because $A^*(\operatorname{ri}\operatorname{dom} g^*) = \{0\}$ in this particular case, the regularity condition (RCC) used in Theorem 4.14 becomes $\operatorname{ri}\operatorname{dom}\varphi^*(x, \cdot) \cap -\operatorname{ri}\operatorname{dom} f^* \neq \emptyset$ which is nothing else but the condition used in [3, Theorem 3.5]. In this case Theorem 4.14 affirm that if the condition $\operatorname{ri}\operatorname{dom}\varphi^*(x, \cdot) \cap -\operatorname{ri}\operatorname{dom} f^* \neq \emptyset$ is fulfilled for all $x \in \mathbb{R}^n$ and if function φ satisfies (*) then, $\overline{u}^*$ is a solution of (DCEP') if and only if there exists $z \in \operatorname{dom} f$ such that $\overline{u}^*$ is a solution of (D'_z). This is the same result as Bigi et al. gave in [3, Theorem 3.5].

We can conclude now that if we particularize the results obtain in this paper in Section 4 we obtain similar results as in [3, Section 3].

5.2. Variational inequalities

We now discuss the composed equilibrium problem (CEP) for the case of variational inequalities. If we consider $\varphi(x, y) = \langle F(x), y - x\rangle$, where $F\colon \mathbb{R}^n \to \mathbb{R}^n$, then we get $\varphi^*(x, \cdot)(y^*) = \delta_{\{F(x)\}}(y^*) + \langle F(x), x\rangle$ and

$$\partial\varphi^*(x, \cdot)(y^*) = \begin{cases} \mathbb{R}^n, & y^* = F(x) \\ \emptyset, & y^* \neq F(x) \end{cases}.$$

In this case (CEP) and (DCEP) become:

$$\text{Find } \overline{x} \in \mathbb{R}^n \text{ s.t. } \langle F(\overline{x}), y - \overline{x}\rangle + f(y) + g(Ay) \geq f(\overline{x}) + g(A\overline{x}) \ \ \forall y \in \mathbb{R}^n, \quad \text{(CVI)}$$

respectively,

$$\begin{aligned} &\text{Find } \ (\overline{u}^*, \overline{v}^*) \in \mathbb{R}^n \times \mathbb{R}^m \ \text{ such that there exists } \ \overline{x} \in \mathbb{R}^n \ \text{ with} \\ &(d1)\ \ \overline{x} \in \partial\varphi^*(\overline{x}, \cdot)(-\overline{u}^* - A^*\overline{v}^*) = \begin{cases} \mathbb{R}^n, & -\overline{u}^* - A^*\overline{v}^* = F(\overline{x}) \\ \emptyset, & -\overline{u}^* - A^*\overline{v}^* \neq F(\overline{x}) \end{cases}; \qquad \text{(DCVI)} \\ &(d2)\ \ \overline{x} \in \partial f^*(\overline{u}^*) \ \text{ and } \ A\overline{x} \in \partial g^*(\overline{v}^*), \end{aligned}$$

which is equivalent to:

$$\begin{array}{ll} \text{Find } (\overline{u}^*,\overline{v}^*)\in\mathbb{R}^n\times\mathbb{R}^m \text{ such that there exists } \overline{x}\in F^{-1}(-\overline{u}^*-A^*\overline{v}^*) \text{ with} & \\ (e1)\ \ \langle \overline{x}, x^*-\overline{u}^*\rangle \leq f^*(x^*)-f^*(\overline{u}^*)\ \ \forall x^*\in\mathbb{R}^n, & \text{(DCVI)} \\ (e2)\ \ \langle A\overline{x}, y^*-\overline{v}^*\rangle \leq g^*(y^*)-g^*(\overline{v}^*)\ \ \forall y^*\in\mathbb{R}^m. & \end{array}$$

In [8] the authors considered the following dual scheme concerning ε-variational inequalities, where $\varepsilon, \varepsilon_1, \varepsilon_2 \geq 0$:

$$\begin{array}{ll} (\mathrm{VI})_{\varepsilon}^{g,f,A} & \text{Find } \overline{x}\in X \text{ such that there exists } v\in F(\overline{x}), \\ & \text{s.t. } \langle v, x-\overline{x}\rangle \geq (g+f\circ A)(\overline{x})-(g+f\circ A)(x)-\varepsilon,\ \forall x\in X, \end{array} \quad (9)$$

$$\begin{array}{ll} (\mathrm{DVI})_{\varepsilon_1,\varepsilon_2}^{g,f,A} & \text{Find } (\overline{x}^*,\overline{y}^*)\in X^*\times Y^* \text{ such that there exists} \\ & (w_1,w_2)\in\big(F^{-1}(-\overline{x}^*-A^*\overline{y}^*)\times A(F^{-1}(-\overline{x}^*-A^*\overline{y}^*))\big)\cap\Delta_X^A, \\ & \text{s.t. } \left\{\begin{array}{l} \langle w_1, x^*-\overline{x}^*\rangle \leq g^*(x^*)-g^*(\overline{x}^*)+\varepsilon_1,\ \ \forall x^*\in X^*, \\ \langle w_2, y^*-\overline{y}^*\rangle \leq f^*(y^*)-f^*(\overline{y}^*)+\varepsilon_2,\ \ \forall y^*\in Y^*, \end{array}\right. \end{array} \quad (10)$$

where $\Delta_X^A = \{(x, Ax) : x\in X\}$, X, Y are real separated locally convex spaces, $f\colon Y\to\overline{\mathbb{R}}$, $g\colon X\to\overline{\mathbb{R}}$ are proper, convex functions and $A\colon X\to Y$ is a linear operator, fulfilling $\operatorname{dom} g\cap A^{-1}(\operatorname{dom} f)\neq\emptyset$..

If we consider $X = R^n$ and $Y = R^m$, $\varepsilon = \varepsilon_1 = \varepsilon_2 = 0$, $g\mapsto f$ and $f\mapsto g$ we obtain the following variational inequality and its dual:

$$\begin{array}{ll} (\mathrm{VI})^{f,g,A} & \text{Find } \overline{x}\in\mathbb{R}^n \text{ such that} \\ & \langle F(\overline{x}), x-\overline{x}\rangle \geq (f+g\circ A)(\overline{x})-(f+g\circ A)(x),\ \ \forall x\in\mathbb{R}^n, \end{array} \quad (11)$$

$$\begin{array}{ll} (\mathrm{DVI})^{f,g,A} & \text{Find } (\overline{x}^*,\overline{y}^*)\in\mathbb{R}^n\times\mathbb{R}^m \text{ such that there exists} \\ & (w_1,w_2)\in\big(F^{-1}(-\overline{x}^*-A^*\overline{y}^*)\times A(F^{-1}(-\overline{x}^*-A^*\overline{y}^*))\big)\cap\Delta_{R^n}^A, \\ & \text{s.t. } \left\{\begin{array}{l} \langle w_1, x^*-\overline{x}^*\rangle \leq f^*(x^*)-f^*(\overline{x}^*),\ \ \forall x^*\in\mathbb{R}^n, \\ \langle w_2, y^*-\overline{y}^*\rangle \leq g^*(y^*)-g^*(\overline{y}^*),\ \forall y^*\in\mathbb{R}^m, \end{array}\right. \end{array} \quad (12)$$

The relation $(w_1, w_2)\in\Delta_{R^n}^A$ is equivalent with $w_2 = Aw_1$ and we can conclude that (CVI) and (DCVI) are equivalent with (11) respectively (12).

In this particular case the set $\mathcal{D}$ becomes

$$\mathcal{D}(\overline{x}) = \Big\{(u^*, v^*) : u^*\in\partial f(\overline{x}),\ v^*\in\partial g(A\overline{x}),\ -u^*-A^*v^* = F(\overline{x})\Big\}.$$

The regularity condition $A(\operatorname{ri}\operatorname{dom} f)\cap\operatorname{ri}\operatorname{dom} g$ used in Theorem 4.3 is nothing else but $(\mathrm{RC}_4^{g,f,A})$ used for the duality statement between $(\mathrm{VI})_{\varepsilon}^{g,f,A}$ and its dual $(\mathrm{DVI})_{\varepsilon}^{g,f,A}$. If we apply Theorem 4.3 we obtain that if $\overline{x}\in\mathbb{R}^n$ solves (CVI) then any $u^*\in\partial f(\overline{x})$, $v^*\in\partial g(A\overline{x})$, with $-u^*-A^*v^* = F(\overline{x})$ is a solution of (DCVI).

The set $\mathcal{P}$ becomes

$$\mathcal{P}(u^*, v^*) = \Big\{x\in\operatorname{dom} f\cap A^{-1}\operatorname{dom} g : x\in F^{-1}(-u^*-A^*v^*)\cap\partial f^*(u^*)\cap A^{-1}\partial g^*(v^*)\Big\}.$$

If we apply Theorem 4.5 we obtain that if $(\overline{u}^*, \overline{v}^*)$ is a solution of (DCVI) then any $x \in F^{-1}(-u^* - A^*v^*) \cap \partial f^*(u^*) \cap A^{-1}\partial g^*(v^*)$ is a solution of (CVI). Furthermore, if we consider $n = m$, $g \equiv 0$, A is the identical operator and if F is an injective mapping, (DCVI) becomes:

$$\text{Find } \overline{x}^* \in \mathbb{R}^n \text{ such that } \langle -F^{-1}(-\overline{x}^*), y^* - \overline{x}^* \rangle + f^*(y^*) \geq f^*(\overline{x}^*) \quad \forall y^* \in \mathbb{R}^n,$$

which is nothing else but the dual variational inequality introduced by Mosco [13].

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